

Exercises Coalgebra for Lecture 6

The exercises labeled with (*) are optional and more advanced.

1. Consider the following stream system over some set A with $a, b \in A$:

$$x_1 \xrightarrow{a} x_2 \xrightarrow{b} x_3 \xrightarrow{a} x_4 \xrightarrow{b} x_5 \xrightarrow{a} x_6 \xrightarrow{b} \dots$$

What is the largest bisimulation on it? What is the smallest?

2. Let A, B be sets, and define $F: \mathbf{Set} \rightarrow \mathbf{Set}$ by $F(X) = A \times X + B$ and $F(f) = \text{id}_A \times f + \text{id}_B$. For an F -coalgebra (X, g) , $x, x' \in X$, $a \in A$, $b \in B$, we write $x \xrightarrow{a} x'$ if $g(x) = ((a, x'), 1)$, and $x \downarrow b$ if $g(x) = (b, 2)$.
 - (a) Instantiate the abstract notion of bisimulation between F -coalgebras to the above functor F , and spell out the details to obtain a concrete notion of bisimulation, formulated in terms of the notation $x \xrightarrow{a} x'$ and $x \downarrow b$.
 - (b) Give an example of such an F -coalgebra, with two states that are bisimilar (show it with a suitable bisimulation) and two states that are not.
3. Let (X, f) be an F -coalgebra, for some functor $F: \mathbf{Set} \rightarrow \mathbf{Set}$. Show that the diagonal relation $\Delta_X = \{(x, x) \mid x \in X\}$ is a bisimulation on (X, f) . Hint: use that $\text{id}_X: X \rightarrow X$ is a homomorphism, and that there is a bijection $\Delta_X \cong X$.
4. (*) In the lecture, we have stated that homomorphisms preserve bisimilarity. In this exercise, we will investigate the converse. Concretely, for $F: \mathbf{Set} \rightarrow \mathbf{Set}$ a functor, we would like to prove (under certain conditions) that, if $h: (X, f) \rightarrow (Y, g)$ is a homomorphism of F -coalgebras, then the kernel relation

$$\ker(h) = \{(x, y) \in X \times X \mid h(x) = h(y)\}$$

is a bisimulation on (X, f) . Below, we denote the projections of this relation by $\pi_1: \ker(h) \rightarrow X$ and $\pi_2: \ker(h) \rightarrow X$.

- (a) Prove the statement for stream systems: coalgebras for the functor F given by $F(X) = A \times X$. (You can also skip directly to the next exercise, where we prove this more generally.)
- (b) Let F be an arbitrary functor, and $h: (X, f) \rightarrow (Y, g)$ a homomorphism of F -coalgebras. Let

$$\ker(F(h)) = \{(u, v) \in F(X) \times F(X) \mid F(h)(u) = F(h)(v)\},$$

with projections $\pi'_1: \ker(F(h)) \rightarrow F(X)$ and $\pi'_2: \ker(F(h)) \rightarrow F(X)$.

Suppose there is a map $i: \ker(F(h)) \rightarrow F(\ker(h))$ such that the following diagram commutes:

$$\begin{array}{ccccc}
 & & \ker(F(h)) & & \\
 & \swarrow \pi'_1 & \downarrow i & \searrow \pi'_2 & \\
 F(X) & \xleftarrow{F(\pi_1)} & F(\ker(h)) & \xrightarrow{F(\pi_2)} & F(X)
 \end{array}$$

Show that $\ker(h)$ is a bisimulation on (X, f) .

(c) Consider the functor $F: \mathbf{Set} \rightarrow \mathbf{Set}$ defined on a set X by

$$F(X) = \{(x, y, z) \mid \text{card}\{x, y, z\} \leq 2\}$$

where $\text{card}\{x, y, z\}$ is the number of elements of $\{x, y, z\}$; $F(X)$ consists of all triples of elements of X where at least two elements are equal. On a function f , the functor is defined by $F(f)(x, y, z) = (f(x), f(y), f(z))$. Every two states of a coalgebra for this functor are behaviourally equivalent.

Consider the coalgebra $c: 2 \rightarrow F2$ (where $2 = \{0, 1\}$) defined by $c(0) = (0, 1, 1)$ and $c(1) = (1, 1, 0)$. Show that there is no bisimulation $R \subseteq 2 \times 2$ on the coalgebra $(2, c)$ such that $(0, 1) \in R$.