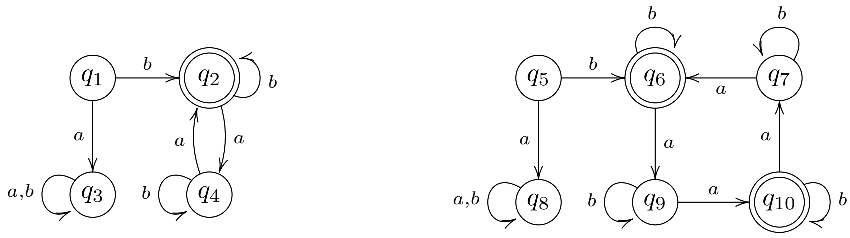
Exercises Coalgebra for Lecture 7

The exercises labeled with **(*)** are optional and more advanced.

1. Consider the following two deterministic automata. Show that states q_1 and q_5 accept the same language, by proving that they are bisimilar.



2. Prove the following equalities over languages $L, K, M \in 2^{A^*}$ using bisimulations:

   (a) $L + K = K + L$

   (b) $L \cdot 1 = L = 1 \cdot L$

   (c) $LL^* + 1 = L^*$

   (d) Let $A = \{a, b\}$; prove that $(a + 1)^* = a^*$.

3. In the lecture, we have characterized the output and derivative of $L + K$, LK and L^* in terms of the output and derivatives of L and K.

   (a) Give a similar characterization for intersection $L \cap K$ and complement $\overline{L}$, with the latter defined by $\overline{L} = \{w \mid w \notin L\}$.

   (b) **(*)** The *shuffle* operation is defined on words w, v inductively as follows: $w \circledcirc \langle\rangle = \langle\rangle \circledcirc w = w$ and $aw \circledcirc bv = a(w \circledcirc bv) + b(aw \circledcirc v)$ for any alphabet letters a, b. This is extended to languages L, K as $L \circledcirc K = \sum_{w \in L, v \in K} w \circledcirc v$. Give a characterization of output and derivative for the shuffle.

   (c) **(*)** Use your result in the previous exercise to prove, by coinduction, that $L \circledcirc K = K \circledcirc L$ for all $L, K \in 2^{A^*}$.

4. In the lecture, we defined, for any automaton $(S, \langle \epsilon, \delta \rangle)$, a homomorphism $\mathsf{beh}\colon S \to 2^{A^*}$ to the automaton of languages $(2^{A^*}, \langle e, d \rangle)$. Finish the proof that $(2^{A^*}, \langle e, d \rangle)$ is a final coalgebra (what is the missing step?).

5. **(*)** Prove that for all languages $L, K \in 2^{A^*}$ and all $a \in A$:

$$(L + K)_a = L_a + K_a$$

$$(LK)_a = \begin{cases} L_a K + K_a & \text{if } L \downarrow \\ L_a K & \text{otherwise} \end{cases}$$

$$(L^*)_a = L_a L^*$$