Coalgebra: homework assignment 1

October 21, 2016

If you have any questions, send me an email: `jrot@cs.ru.nl`. The deadline is Monday 7 November. You can hand it in by email, or in the lecture.

1. Let A be a fixed set. For a function $h \colon X \to Y$, we define $h^A \colon X^A \to Y^A$ by composition: $h^A(t) = h \circ t$. Show that the mapping of a set X to X^A and from h to h^A yields a functor from Set to Set. Describe and prove all the necessary requirements.

2. A *strange binary system* (over a set of labels A) consists of a set of states X and, for each state $x \in X$ and each $a \in A$, a transition $x \xrightarrow{a} (y, z)$ to a pair of states $(y, z) \in X \times X$. Moreover, every state $x \in X$ has a color: it is either red, green or blue.

   (a) Define a functor on Set whose coalgebras are strange binary systems.

   (b) Define a concrete notion of homomorphism between strange binary systems, based on the coalgebraic notion of homomorphism for your functor.

3. Recall that A^ω denotes the set of streams over a set A.

   (a) Consider the function $\mathsf{alt} \colon A^\omega \times A^\omega \to A^\omega$ given by $\mathsf{alt}(\sigma, \tau) = (\sigma(0), \tau(1), \sigma(2), \tau(3), \ldots)$. Describe alt in terms of initial value and derivative.

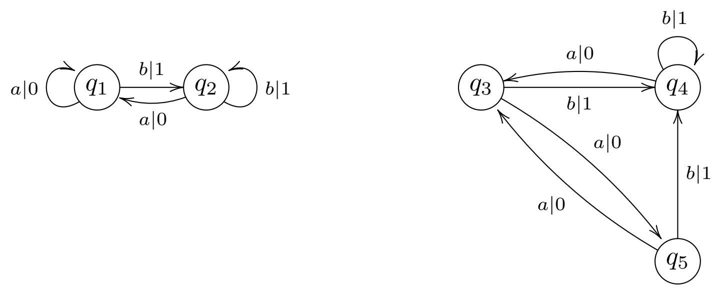
   (b) Prove, by constructing a suitable bisimulation, that $\mathsf{alt}(\sigma, \mathsf{alt}(\tau, \phi)) = \mathsf{alt}(\sigma, \mathsf{alt}(\chi, \phi))$ for all $\sigma, \tau, \phi, \chi \in A^\omega$.

4. Consider the functor $M \colon \mathsf{Set} \to \mathsf{Set}$ defined on sets by $M(X) = (B \times X)^A$ and on functions by $M(h) = (\mathsf{id}_B \times h)^A$. Coalgebras for M are so-called *Mealy machines*. Thus, a Mealy machine is a pair (S, f) where S is a set of states and $f \colon S \to (B \times S)^A$. Typically, transitions are represented by

$$x \xrightarrow{a|b} y \iff f(x)(a) = (b, y).$$

   (a) Instantiate the coalgebraic definition of bisimulation for M to obtain a concrete description of bisimulations $R \subseteq S \times S$ for Mealy machines. Use thereby the notation $x \xrightarrow{a|b} y$ we introduced above.

(b) Consider the following two Mealy machines over the input alphabet $A = \{a, b\}$ and output $B = \{0, 1\}$. Show that states q_1 and q_3 are bisimilar, using the answer to the previous question.



(c) Given streams σ and τ, we say σ and τ are equal up to n, denoted $\sigma \equiv_n \tau$, if for all i with $0 \leq i < n$: $\sigma(i) = \tau(i)$. A map $\varphi\colon A^\omega \to B^\omega$ is called *causal* if for all $\sigma, \tau \in A^\omega$: $\sigma \equiv_n \tau$ implies $\varphi(\sigma) \equiv_n \varphi(\tau)$. Give an example of a function that is causal, and one that is not.

(d) Let Z be the set of all causal functions from A^ω to B^ω. Define the Mealy machine $z\colon Z \to (B \times Z)^A$ as follows, for all $\varphi \in Z$ and $a \in A$:

$$z(\varphi)(a) = (\varphi[a], \psi)$$

where $\varphi[a] = \varphi(a : \sigma)(0)$ for some stream $\sigma \in A^\omega$, and $\psi\colon A^\omega \to B^\omega$ is defined by $\psi(\tau) = \varphi(a : \tau)'$, i.e., $\psi(\tau)(n) = \varphi(a : \tau)(n+1)$.

The choice of σ in the definition of $\varphi[a]$ does not matter; why not?

(e) Prove that (Z, z) is a final coalgebra.