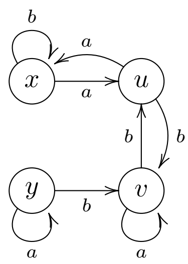
Coalgebra: homework assignment 2

January 23, 2017

If you have any questions, send me an email: `jrot@cs.ru.nl`. The deadline is January 13th, 2017. You can hand it in by email, or hand it to me during the exercise class.

1. Give a bisimulation up to equivalence on the following automaton, which relates (x, y) and contains at most three pairs.



2. In the lecture, we've talked a lot about defining basic operations on streams, and how to complicate that using distributive laws. Jurriaan is quite happy with this complicated approach (that's why he showed it), and decides to define an operation $s\colon \mathcal{P}_f(\mathbb{N}^\omega) \to \mathbb{N}^\omega$ in this way, which should take the pointwise sum of a finite set of streams: $s(S)(n) = \sum_{\sigma \in S} \sigma(n)$. His idea is to use the following setup:

   - $B\colon \mathsf{Set} \to \mathsf{Set}$, $B(X) = \mathbb{N} \times X$,
   - $\mathcal{P}_f\colon \mathsf{Set} \to \mathsf{Set}$ the finite powerset functor (defined on functions by direct image),
   - the distributive law $\lambda\colon \mathcal{P}_f B \Rightarrow B\mathcal{P}_f$, given on a component X by

$$\lambda_X\colon \mathcal{P}_f(\mathbb{N} \times X) \to \mathbb{N} \times \mathcal{P}_f(X)$$
$$S \mapsto \left(\sum_{(n,x)\in S} n, \{x \mid (n, x) \in S\} \right)$$

   He proudly shows this to Joshua. But Joshua frowns, and asks: is λ really a natural transformation? It's up to you to find out whether Joshua's worries are justified. Give either a proof or a counterexample for naturality of λ.

3. A *deformed stream system* is a coalgebra of the form $\langle o, f\rangle\colon X \to \mathbb{N} \times (X + 2)$, where $2 = \{*, \dagger\}$. We would like each element $x \in X$ to represent a stream in $\mathbb{N}^\omega$, by just reading the numbers that we encounter in the usual way; so start with $o(x)$, then $o(f(x))$ if $f(x) \in X$, $o(f(f(x))$ if $f(f(x)) \in X$, and so on. When we hit $*$, we continue with the stream $(42, 42, 42, \ldots)$ and when we hit $\dagger$, we continue with $(37, 37, 37, \ldots)$.

For instance, if we take $X = \{x, y, z\}$, $o(x) = 1$, $o(y) = 2$, $o(z) = 3$, $f(x) = x$, $f(y) = \dagger$, $f(z) = y$ then x represents the stream $(1, 1, 1, \ldots)$, y represents the stream $(2, 37, 37, 37, \ldots)$ and z represents the stream $(3, 2, 37, 37, 37, \ldots)$.

(a) Give a concrete example of a deformed stream system with two states x, y such that x and y represent the same stream, but x and y are not bisimilar (in the coalgebraic sense).

(b) Define a determinisation procedure which yields the representation of streams that we have in mind. That is, for each deformed stream system $\langle o, f\rangle\colon X \to \mathbb{N} \times (X + 2)$, define a stream system $\langle o^\sharp, f^\sharp\rangle\colon (X + 2) \to \mathbb{N} \times (X + 2)$ and a map $\eta_X\colon X \to X + 2$, such that the triangle on the left below commutes and, given the unique coalgebra morphism $\mathsf{beh}\colon X + 2 \to \mathbb{N}^\omega$ to the final coalgebra $(\mathbb{N}^\omega, z)$, $\mathsf{beh} \circ \eta_X\colon X \to \mathbb{N}^\omega$ yields the representation of streams that we have in mind (you don't need to prove that $\mathsf{beh} \circ \eta_X$ really is the intended representation, but give an informal explanation why your construction does the job).

$$
\begin{tikzcd}
X \arrow[r, "\eta_X"] \arrow[d, "\langle o,f\rangle"'] & X+2 \arrow[r, "\mathsf{beh}"] \arrow[dl, "\langle o^\sharp, f^\sharp\rangle"] & \mathbb{N}^\omega \arrow[d, "z"] \\
\mathbb{N} \times (X+2) \arrow[rr, "\mathsf{id}\times\mathsf{beh}"'] & & \mathbb{N} \times \mathbb{N}^\omega
\end{tikzcd}
$$

4. Let $(X, \to)$ be a labelled transition system over A. Let $A^\infty = A^\omega \cup A^*$ be the set of streams and (finite) words over A. The empty word is denoted by $\langle\rangle$. Consider the following rules, involving a relation $\downarrow\, \subseteq X \times A^\infty$.

$$
\frac{}{x \downarrow \langle\rangle} \qquad \frac{x \xrightarrow{a} y \quad y \downarrow w}{x \downarrow aw} \tag{1}
$$

(for all $a \in A$, $w \in A^\infty$).

Let $\mathsf{Rel}_{X,A^\infty}$ be the set of relations of the form $R \subseteq X \times A^\infty$, partially ordered by subset inclusion $\subseteq$. This partial order is a complete lattice.

Given $x \in X$ and $w \in A^\infty$, we say w is a *trace* of x if there is a path from x labelled by w, that is, a path $x_1 \xrightarrow{a_1} x_2 \xrightarrow{a_2} x_3 \xrightarrow{a_3} \ldots$ such that $x_1 = x$ and $w = a_1 a_2 a_3 \ldots$. If $w \in A^*$ then we call this a *finite* trace, if $w \in A^\omega$ we call it an *infinite* trace.

(a) Describe the least upper bound $\bigvee S$ and greatest lower bound $\bigwedge S$ of an arbitrary set $S \subseteq \mathsf{Rel}_{X,A^\infty}$, and give the top and bottom elements of the lattice (you don't have to give a proof).

(b) Formulate the rules (1) in terms of a function $b\colon \mathsf{Rel}_{X,A^\infty} \to \mathsf{Rel}_{X,A^\infty}$. Show that your function is monotone.

(c) What is a pre-fixed point of b? And what is the least fixed point? Give a concrete description, in terms of the transition system and elements of A^∞.

(d) What is a post-fixed point of b? And what is the greatest fixed point? Give a concrete description, in terms of the transition system and elements of A^∞.

(e) Use your answer to one of the previous questions to show that, in the transition system below, every finite trace of x is a prefix of the stream $ababab\ldots$.

$$
\begin{tikzcd}
x \arrow[r, bend left, "a"] & y \arrow[l, bend left, "b"]
\end{tikzcd}
$$

(f) Use your answer to one of the previous questions to show that, in the transition system above, the stream $ababab\ldots$ is an infinite trace of x.

(g) Suppose that our transition system is finitely branching, meaning that for each $x \in X$, the set $\{y \mid x \xrightarrow{a} y \text{ for some } a\}$ is finite. Consider the relation $\| \subseteq X \times A^\infty$, given by: $x \| w$ iff there are infinitely many prefixes that are finite traces of x. Prove that for all $x \in X$ and $w \in A^\infty$: if $x \| w$, then w is an infinite trace of x.

5. We define a functor $\mathcal{M} \colon \mathsf{Set} \to \mathsf{Set}$ by

$$\mathcal{M}(X) = \left\{ m \colon X \to \mathbb{N} \;\middle|\; |\{x \in X \mid m(x) \neq 0\}| \text{ is finite} \right\}$$

on sets, and on functions by

$$\begin{aligned} \mathcal{M}(f \colon X \to Y) \colon \mathcal{M}(X) &\to \mathcal{M}(Y) \\ \mathcal{M}(f)(m)(y) &= \sum_{x \in f^{-1}(y)} m(x)\,. \end{aligned}$$

(a) What is the difference between $\mathsf{List}(Y)$ and $\mathcal{M}(Y)$?

(b) Define a non-trivial function from $\mathsf{List}(Y)$ to $\mathcal{M}(Y)$, using initiality of $\mathsf{List}(Y)$ with respect to $F_Y(X) := 1 + Y \times X$.

(c) Define a monad structure on $\mathcal{M}$. You don't have to prove the equations or naturality (though doing so could result in bonus points!).