

The Expectation Monad in Quantum Foundations

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The expectation monad is introduced abstractly via two composable adjunctions, but concretely captures measures. It turns out to sit in between known monads: on the one hand the distribution and ultrafilter monad, and on the other hand the continuation monad. This expectation monad is used in two probabilistic analogues of fundamental results of Manes and Gelfand for the ultrafilter monad: algebras of the expectation monad are convex compact Hausdorff spaces, and are dually equivalent to so-called Banach effect algebras. These structures capture states and effects in quantum foundations, and the duality between them. Moreover, the approach leads to a new re-formulation of Gleason's theorem, expressing that effects on a Hilbert space are free effect modules on projections, obtained via tensoring with the unit interval.

1 Introduction

Techniques that have been developed over the last decades for the semantics of programming languages and programming logics gain wider significance. In this way a new interdisciplinary area has emerged where researchers from mathematics, (theoretical) physics and (theoretical) computer science collaborate, notably on quantum computation and quantum foundations. The article [6] uses the phrase “Rosetta Stone” for the language and concepts of category theory that form an integral part of this common area.

The present article is also part of this new field. It uses results from programming semantics, topology and (convex) analysis, category theory (esp. monads), logic and probability, and quantum foundations. The origin of this article is an illustration of the connections involved. Previously, the authors have worked on effect algebras and effect modules [21, 19, 20] from quantum logic, which are fairly general structures incorporating both logic (Boolean and orthomodular lattices) and probability (the unit interval $[0, 1]$ and fuzzy predicates). By reading completely different work, on formal methods in computer security (in particular the thesis [35]), the expectation monad was noticed. The monad is used in [35, 9] to give semantics to a probabilistic programming language that helps to formalize (complexity) reduction arguments from security proofs in a theorem prover. In [35] (see also [5, 33]) the expectation monad is defined in a somewhat *ad hoc* manner (see Section 10 for details). Soon it was realized that a more systematic definition of this expectation monad could be given via the (dual) adjunction between convex sets and effect modules (elaborated in Subsection 2.4). Subsequently the two main parts of the present paper emerged.

1. The expectation monad turns out to be related to several known monads as described in the fol-

lowing diagram.

$$\begin{array}{ccc}
 (\text{distribution } \mathcal{D}) & \searrow & \\
 & & (\text{expectation } \mathcal{E}) \longrightarrow (\text{continuation } \mathcal{C}) \\
 (\text{ultrafilter } \mathcal{UF}) & \nearrow &
 \end{array} \tag{1}$$

The continuation monad \mathcal{C} also comes from programming semantics. But here we are more interested in the connection with the distribution and ultrafilter monads \mathcal{D} and \mathcal{UF} . Since the algebras of the distribution monad are convex sets and the algebras of the ultrafilter monad are compact Hausdorff spaces (a result known as Manes theorem) it follows that the algebras of the expectation monad must be some subcategory of convex compact Hausdorff spaces. One of the main results in this paper, Theorem 4, makes this connection precise. It can be seen as a probabilistic version of Manes theorem. It uses basic notions from Choquet theory, notably barycenters of measures.

2. The adjunction that gives rise to the expectation monad \mathcal{E} yields a (dual) adjunction between the category $\text{Alg}(\mathcal{E})$ of algebras and the category of effect modules. By suitable restriction this adjunction gives rise to an equivalence between “observable” \mathcal{E} -algebras and “Banach” (complete) effect modules, see Theorem 5.

These two parts of the paper may be summarized as follows. There are classical results:

$$\text{Alg}(\mathcal{UF}) \xrightarrow{[\text{Manes}]} (\text{compact Hausdorff spaces}) \xrightarrow{[\text{Gelfand}]} (\text{comm. } C^*\text{-algebras})^{\text{op}}$$

Here we give the following “probabilistic” analogues:

$$\text{Alg}_{\text{obs}}(\mathcal{E}) \simeq (\text{convex compact Hausdorff spaces})_{\text{obs}} \simeq (\text{Banach effect modules})^{\text{op}}$$

The subscript ‘obs’ refers to a suitable observability condition, see Section 7. The role played by the two-element set $\{0, 1\}$ in these classical results—*e.g.* as “schizophrenic” object—is played in our probabilistic analogues by the unit interval $[0, 1]$.

Quantum mechanics is notoriously non-intuitive. Hence a proper mathematical understanding of the relevant phenomena is important, certainly within the emerging field of quantum computation. It seems fair to say that such an all-encompassing understanding of quantum mechanics does not exist yet. For instance, the categorical analysis in [1, 2] describes some of the basic underlying structure in terms of monoidal categories, daggers, and compact closure. However, an integrated view of logic and probability is still missing. Here we certainly do not provide this integrated view, but possibly we do contribute a bit. The states of a Hilbert space \mathcal{H} , described as density matrices $\text{DM}(\mathcal{H})$, fit within the category of convex compact Hausdorff spaces investigated here. Also, the effects $\text{Ef}(\mathcal{H})$ of the space fit in the associated dual category of Banach Hausdorff spaces. The duality we obtain between convex compact Hausdorff spaces and Banach effect algebras precisely captures the translations back and forth between states and effects, as expressed by the isomorphisms:

$$\text{Hom}(\text{Ef}(\mathcal{H}), [0, 1]) \cong \text{DM}(\mathcal{H}) \qquad \text{Hom}(\text{DM}(\mathcal{H}), [0, 1]) \cong \text{Ef}(\mathcal{H}).$$

These isomorphisms (implicitly) form the basis for the quantum weakest precondition calculus described in [13].

In this context we shed a bit more light on the relation between quantum logic—as expressed by the projections $\text{Pr}(\mathcal{H})$ on a Hilbert space—and quantum probability—via its effects $\text{Ef}(\mathcal{H})$. In Section 9

it will be shown that Gleason's famous theorem, expressing that states are probability measures, can equivalently be expressed as an isomorphism relating projections and effects:

$$[0, 1] \otimes \text{Pr}(\mathcal{H}) \cong \text{Ef}(\mathcal{H}).$$

This means that the effects form the free effect module on projections, via the free functor $[0, 1] \otimes (-)$. More loosely formulated: quantum probabilities are freely obtained from quantum predicates.

We briefly describe the organization of the paper. It starts with a quick recap on monads in Section 2, including descriptions of the monads relevant in the rest of the paper. Section 3 gives a brief introduction to effect algebras and effect modules. It also establishes equivalences between (Banach) order unit spaces and (Banach) Archimedean effect modules. In Section 4 we give several descriptions of the expectation monad in terms of effect algebras and effect modules. We also describe the map between the expectation monad and the continuation monad here. Sections 5 and 6 deal with the construction of the other two monad maps from Diagram (1): those from the ultrafilter and distribution monads to the expectation monad. Here we also explore some of the implications of these maps. Next, in Section 7, we study the algebras of the expectation monad. We prove that the category of \mathcal{E} -algebras is equivalent to the category compact convex sets with continuous affine mappings. In Section 8 we establish a dual adjunction between \mathcal{E} -algebras and effect modules. We prove that when restricted to so-called observable \mathcal{E} -algebras and Banach effect modules this adjunction becomes an equivalence. In Section 9 we apply this duality to quantum logic. We prove that the isomorphism $[0, 1] \otimes \text{Pr}(\mathcal{H}) \cong \text{Ef}(\mathcal{H})$ is an algebraic reformulation of Gleason's theorem. Finally in Section 10 we examine how the expectation monad has appeared in earlier work on programming semantics. We also suggest how it might be used to capture both non-deterministic and probabilistic computation simultaneously, although the details of this are left for future work.

2 A recap on monads

This section recalls the basics of the theory of monads, as needed here. For more information, see *e.g.* [29, 8, 28, 10]. Some specific examples will be elaborated later on.

A monad is a functor $T: \mathbf{C} \rightarrow \mathbf{C}$ together with two natural transformations: a unit $\eta: \text{id}_{\mathbf{C}} \Rightarrow T$ and multiplication $\mu: T^2 \Rightarrow T$. These are required to make the following diagrams commute, for $X \in \mathbf{C}$.

$$\begin{array}{ccc} T(X) & \xrightarrow{\eta_{T(X)}} & T^2(X) & \xleftarrow{T(\eta_X)} & T(X) \\ & \searrow & \downarrow \mu_X & \swarrow & \\ & & T(X) & & \end{array} \qquad \begin{array}{ccc} T^3(X) & \xrightarrow{\mu_{T(X)}} & T^2(X) \\ T(\mu_X) \downarrow & & \downarrow \mu_X \\ T^2 & \xrightarrow{\mu_X} & T(X) \end{array}$$

Each adjunction $F \dashv G$ gives rise to a monad GF .

Given a monad T one can form a category $\text{Alg}(T)$ of so-called (Eilenberg-Moore) algebras. Objects of this category are maps of the form $a: T(X) \rightarrow X$, making the first two squares below commute.

$$\begin{array}{ccc} X & \xrightarrow{\eta} & TX \\ & \searrow & \downarrow a \\ & & X \end{array} \qquad \begin{array}{ccc} T^2X & \xrightarrow{T(a)} & TX \\ \mu \downarrow & & \downarrow a \\ TX & \xrightarrow{a} & X \end{array} \qquad \begin{array}{ccc} TX & \xrightarrow{T(f)} & TY \\ a \downarrow & & \downarrow b \\ X & \xrightarrow{f} & Y \end{array}$$

A homomorphism of algebras $(X, a) \rightarrow (Y, b)$ is a map $f: X \rightarrow Y$ in \mathbf{C} between the underlying objects making the diagram above on the right commute. The diagram in the middle thus says that the map a is a homomorphism $\mu \rightarrow a$. The forgetful functor $U: \mathbf{Alg}(T) \rightarrow \mathbf{C}$ has a left adjoint, mapping an object $X \in \mathbf{X}$ to the (free) algebra $\mu_X: T^2(X) \rightarrow T(X)$ with carrier $T(X)$.

Each category $\mathbf{Alg}(T)$ inherits limits from the category \mathbf{C} . In the special case where $\mathbf{C} = \mathbf{Sets}$, the category of sets and functions (our standard universe), the category $\mathbf{Alg}(T)$ is not only complete but also cocomplete (see [8, § 9.3, Prop. 4]).

A map of monads $\sigma: T \Rightarrow S$ is a natural transformation that commutes with the units and multiplications, as in:

$$\begin{array}{ccc} X & \xlongequal{\quad} & X \\ \eta_X \downarrow & & \downarrow \eta_X \\ T(X) & \xrightarrow{\sigma_X} & S(X) \end{array} \qquad \begin{array}{ccc} T^2(X) & \xrightarrow{\sigma_{TX}} & S(T(X)) \xrightarrow{S(\sigma_X)} S^2(X) \\ \mu_X \downarrow & & \downarrow \mu_X \\ T(X) & \xrightarrow{\sigma_X} & S(X) \end{array} \quad (2)$$

Such a map of monads $\sigma: T \Rightarrow S$ induces a functor $(-) \circ \sigma: \mathbf{Alg}(S) \rightarrow \mathbf{Alg}(T)$ between categories of algebras that commutes with the forgetful functors.

Lemma 1. *Assume a map of monads $\sigma: T \Rightarrow S$.*

1. *There is a functor $(-) \circ \sigma: \mathbf{Alg}(S) \rightarrow \mathbf{Alg}(T)$ that commutes with the forgetful functors.*
2. *If the category $\mathbf{Alg}(S)$ has sufficiently many coequalizers—like when the underlying category is \mathbf{Sets} —this functor has a left adjoint $\mathbf{Alg}(T) \rightarrow \mathbf{Alg}(S)$; it maps an algebra $a: T(X) \rightarrow X$ to the following coequalizer a_σ in $\mathbf{Alg}(S)$.*

$$\left(\begin{array}{c} S^2(TX) \\ \downarrow \mu \\ S(TX) \end{array} \right) \xrightarrow{\mu \circ S(\sigma)} \left(\begin{array}{c} S^2(X) \\ \downarrow \mu \\ S(X) \end{array} \right) \xrightarrow{c} \left(\begin{array}{c} S(X_\sigma) \\ \downarrow a_\sigma \\ X_\sigma \end{array} \right) \quad \square$$

Proof We need to establish a bijective correspondence between algebra maps:

$$\frac{\left(\begin{array}{c} S(X_\sigma) \\ \downarrow a_\sigma \\ X_\sigma \end{array} \right) \xrightarrow{f} \left(\begin{array}{c} S(Y) \\ \downarrow b \\ Y \end{array} \right)}{\left(\begin{array}{c} T(X) \\ \downarrow a \\ X \end{array} \right) \xrightarrow{g} \left(\begin{array}{c} T(Y) \\ \downarrow b \circ \sigma \\ Y \end{array} \right)}$$

This works as follows. Given f , one takes $\bar{f} = f \circ c \circ \eta: X \rightarrow Y$. And given g one obtains $\bar{g}: X_\sigma \rightarrow Y$ because $b \circ T(g): S(X) \rightarrow Y$ coequalizes the above parallel pair $\mu \circ S(\sigma)$ and $S(a)$. Remaining details are left to the interested reader. \square

2.1 The Distribution monad

We shall write \mathcal{D} for the discrete probability distribution monad on \mathbf{Sets} . It maps a set X to the set of formal convex combinations $r_1 x_1 + \cdots + r_n x_n$, where $x_i \in X$ and $r_i \in [0, 1]$ with $\sum_i r_i = 1$. Alternatively,

$$\mathcal{D}(X) = \{ \varphi: X \rightarrow [0, 1] \mid \text{supp}(\varphi) \text{ is finite, and } \sum_x \varphi(x) = 1 \},$$

where $\text{supp}(\varphi) \subseteq X$ is the support of φ , containing all x with $\varphi(x) \neq 0$. The functor $\mathcal{D} : \mathbf{Sets} \rightarrow \mathbf{Sets}$ forms a monad with the Dirac function as unit in:

$$\begin{array}{ccc} X & \xrightarrow{\eta} & \mathcal{D}X \\ x \mapsto & 1x = \lambda y. \begin{cases} 1 & \text{if } y = x \\ 0 & \text{if } y \neq x \end{cases} & \end{array} \qquad \begin{array}{ccc} \mathcal{D}\mathcal{D}X & \xrightarrow{\mu} & \mathcal{D}X \\ \Psi \mapsto & \lambda y. \sum_{\varphi \in \mathcal{D}X} \Psi(\varphi) \cdot \varphi(y). & \end{array}$$

[Here we use the ‘‘lambda’’ notation from the lambda calculus [7]: the expression $\lambda x. \dots$ is used for the function $x \mapsto \dots$. We also use the associated application rule $(\lambda x. f(x))(y) = f(y)$.]

Objects of the category $\text{Alg}(\mathcal{D})$ of (Eilenberg-Moore) algebras of this monad \mathcal{D} can be identified as *convex sets*, in which sums $\sum_i r_i x_i$ of convex combinations exists. Morphisms are so-called affine functions, preserving such convex sums, see [19]. Hence we also write $\text{Alg}(\mathcal{D}) = \mathbf{Conv}$, where \mathbf{Conv} is the category of convex sets and affine functions.

The prime example of a convex set is the unit interval $[0, 1] \subseteq \mathbb{R}$ of probabilities. Also, for an arbitrary set X , the set of functions $[0, 1]^X$, or fuzzy predicates on X , is a convex set, via pointwise convex sums.

2.2 The ultrafilter monad

A particular monad that plays an important role in this paper is the ultrafilter monad $\mathcal{UF} : \mathbf{Sets} \rightarrow \mathbf{Sets}$, given by:

$$\begin{aligned} \mathcal{UF}(X) &= \{ \mathcal{F} \subseteq \mathcal{P}(X) \mid \mathcal{F} \text{ is an ultrafilter} \} \\ &\cong \{ f : \mathcal{P}(X) \rightarrow \{0, 1\} \mid f \text{ is a homomorphism of Boolean algebras} \} \end{aligned} \tag{3}$$

Such an ultrafilter $\mathcal{F} \subseteq \mathcal{P}(X)$ satisfies, by definition, the following three properties.

- It is an upset: $V \supseteq U \in \mathcal{F} \Rightarrow V \in \mathcal{F}$;
- It is closed under finite intersections: $X \in \mathcal{F}$ and $U, V \in \mathcal{F} \Rightarrow U \cap V \in \mathcal{F}$;
- For each set U either $U \in \mathcal{F}$ or $\neg U = \{x \in X \mid x \notin U\} \in \mathcal{F}$, but not both. As a consequence, $\emptyset \notin \mathcal{F}$.

For a function $f : X \rightarrow Y$ one obtains $\mathcal{UF}(f) : \mathcal{UF}(X) \rightarrow \mathcal{UF}(Y)$ by:

$$\mathcal{UF}(f)(\mathcal{F}) = \{V \subseteq Y \mid f^{-1}(V) \in \mathcal{F}\}.$$

Taking ultrafilters is a monad, with unit $\eta : X \rightarrow \mathcal{UF}(X)$ given by so-called principle ultrafilters:

$$\eta(x) = \{U \subseteq X \mid x \in U\}.$$

The multiplication $\mu : \mathcal{UF}^2(X) \rightarrow \mathcal{UF}(X)$ is:

$$\mu(\mathcal{A}) = \{U \subseteq X \mid D(U) \in \mathcal{A}\} \quad \text{where} \quad D(U) = \{\mathcal{F} \in \mathcal{UF}(X) \mid U \in \mathcal{F}\}.$$

The set $\mathcal{UF}(X)$ of ultrafilters on a set X is a topological space with basic (compact) clopens given by subsets $D(U) = \{\mathcal{F} \in \mathcal{UF}(X) \mid U \in \mathcal{F}\}$, for $U \subseteq X$. This makes $\mathcal{UF}(X)$ into a compact Hausdorff space. The unit $\eta : X \rightarrow \mathcal{UF}(X)$ is a dense embedding.

The following result shows the importance of the ultrafilter monad, see *e.g.* [27], [22, III.2], or [10, Vol. 2, Prop. 4.6.6].

Theorem 1 (Manes). $\text{Alg}(\mathcal{UF}) \simeq \mathbf{CH}$, i.e. the category of algebras of the ultrafilter monad is equivalent to the category \mathbf{CH} of compact Hausdorff spaces and continuous maps.

The proof is complicated and will not be reproduced here. We only extract the basic constructions. For a compact Hausdorff space Y one uses denseness of the unit η to define a unique continuous extensions $f^\#$ as in:

$$\begin{array}{ccc} X & \xrightarrow{\eta} & \mathcal{UF}(X) \\ & \searrow f & \downarrow f^\# \\ & & Y \end{array} \quad (4)$$

One defines $f^\#(\mathcal{F})$ to be the unique element in $\bigcap \{\overline{V} \mid V \subseteq Y \text{ with } f^{-1}(V) \in \mathcal{F}\}$. This intersection is a singleton precisely because Y is a compact Hausdorff space. In such a way one obtains an algebra $\mathcal{UF}(Y) \rightarrow Y$ as extension of the identity.

Conversely, assuming an algebra $\text{ch}_X: \mathcal{UF}(X) \rightarrow X$ one defines $U \subseteq X$ to be closed if for all $\mathcal{F} \in \mathcal{UF}(X)$, $U \in \mathcal{F}$ implies $\text{ch}(\mathcal{F}) \in U$. This yields a topology on X which is Hausdorff and compact. There can be at most one such algebra structure $\text{ch}_X: \mathcal{UF}(X) \rightarrow X$ on a set X , corresponding to a compact Hausdorff topology, because of the following standard result.

Lemma 2. Assume a set X carries two topologies $\mathcal{O}_1(X), \mathcal{O}_2(X) \subseteq \mathcal{P}(X)$ with $\mathcal{O}_1(X) \subseteq \mathcal{O}_2(X)$, $\mathcal{O}_1(X)$ is Hausdorff and $\mathcal{O}_2(X)$ is compact, then $\mathcal{O}_1(X) = \mathcal{O}_2(X)$. \square

Proof If U is closed in $\mathcal{O}_2(X)$, then it is compact, and, because $\mathcal{O}_1(X) \subseteq \mathcal{O}_2(X)$, also compact in $\mathcal{O}_1(X)$. Hence it is closed there. \square

We can apply this result to the space $\mathcal{UF}(X)$ of ultrafilters: as described before Theorem 1, $\mathcal{UF}(X)$ carries a compact Hausdorff topology with sets $D(U) = \{\mathcal{F} \in \mathcal{UF}(X) \mid U \in \mathcal{F}\}$ as clopens. Also, it carries a compact Hausdorff topology via the (free) algebra $\mu_X: \mathcal{UF}^2(X) \rightarrow \mathcal{UF}(X)$. It is not hard to see that the subsets $D(U)$ are closed in the latter topology, so the two topologies on $\mathcal{UF}(X)$ coincide by Lemma 2. Later we shall use a similar argument.

Example 1. The unit interval $[0, 1] \subseteq \mathbb{R}$ is a standard example of a compact Hausdorff space. Its Eilenberg-Moore algebra $\text{ch}: \mathcal{UF}([0, 1]) \rightarrow [0, 1]$ can be described concretely on $\mathcal{F} \in \mathcal{UF}([0, 1])$ as:

$$\text{ch}(\mathcal{F}) = \inf\{s \in [0, 1] \mid [0, s] \in \mathcal{F}\}. \quad (5)$$

For the proof, recall that $\text{ch}(\mathcal{F})$ is the (sole) element of the intersection $\bigcap \{\overline{V} \mid V \in \mathcal{F}\}$. Hence if $[0, s] \in \mathcal{F}$, then $\text{ch}(\mathcal{F}) \in \overline{[0, s]} = [0, s]$, so $\text{ch}(\mathcal{F}) \leq s$. This establishes the (\leq) -part of (5). Assume next that $\text{ch}(\mathcal{F}) < \inf\{s \mid [0, s] \in \mathcal{F}\}$. Then there is some $r \in [0, 1]$ with $\text{ch}(\mathcal{F}) < r < \inf\{s \mid [0, s] \in \mathcal{F}\}$. Then $[0, r]$ is not in \mathcal{F} , so that $\neg[0, r] = (r, 1] \in \mathcal{F}$. But this means $\text{ch}(\mathcal{F}) \in \overline{(r, 1]} = [r, 1]$, which is impossible.

Notice that (5) can be strengthened to: $\text{ch}(\mathcal{F}) = \inf\{s \in [0, 1] \cap \mathbb{Q} \mid [0, s] \in \mathcal{F}\}$.

The second important result about compact Hausdorff spaces is as follows.

Theorem 2 (Gelfand). $\mathbf{CH} \simeq C^*\text{-Alg}^{\text{op}}$, i.e. the category \mathbf{CH} of compact Hausdorff spaces is equivalent to the opposite of the category of commutative C^* -algebras.

This paper presents probabilistic analogues of these two basic results (Theorems 1 and 2), involving convex compact Hausdorff spaces (see Theorem 5).

2.3 The continuation monad

The so-called continuation monad is useful in the context of programming semantics, where it is employed for a particular style of evaluation. The monad starts from a fixed set C and takes the “double dual” of a set, where C is used as dualizing object. Hence we first form a functor $\mathcal{C}: \mathbf{Sets} \rightarrow \mathbf{Sets}$ by:

$$\mathcal{C}(X) = C^{(C^X)} \quad \text{and} \quad \mathcal{C}(X \xrightarrow{f} Y) = \lambda h \in C^{(C^X)}. \lambda g \in C^Y. h(g \circ f).$$

This functor \mathcal{C} forms a monad via:

$$\begin{array}{ccc} X & \xrightarrow{\eta} & C^{(C^X)} \\ x \mapsto & \lambda g \in C^X. & g(x) \end{array} \quad \begin{array}{ccc} C^{(C^{(C^X)})} & \xrightarrow{\mu} & C^{(C^X)} \\ H \mapsto & \lambda g \in C^X. & H(\lambda k \in C^{(C^X)}. k(g)). \end{array}$$

The following folklore result will be useful in the present context.

Lemma 3. *Let $T: \mathbf{Sets} \rightarrow \mathbf{Sets}$ be an arbitrary monad and $\mathcal{C}(X) = C^{(C^X)}$ be the continuation monad on a set C . Then there is a bijective correspondence between:*

$$\begin{array}{ccc} T(C) \xrightarrow{a} C & \text{Eilenberg-Moore algebras} \\ \hline T \xRightarrow{\sigma} \mathcal{C} & \text{maps of monads.} \end{array}$$

Proof First, given an algebra $a: T(C) \rightarrow C$ define $\sigma_X: T(X) \rightarrow C^{(C^X)}$ by:

$$\sigma_X(u)(g) = a(T(g)(u)).$$

Conversely, given a map of monads $\sigma: T \Rightarrow C^{(C^(-))}$, define as algebra $a: T(C) \rightarrow C$,

$$a(u) = \sigma_C(u)(\text{id}_C). \quad \square$$

Taking $C = 2 = \{0, 1\}$ to be the two-element set, yields as associated continuation monad $\mathcal{C}(X) = 2^{(2^X)} \cong \mathcal{P}(\mathcal{P}(X))$, the double-powerset monad. For a function $f: X \rightarrow Y$ we have a map $\mathcal{P}^2(X) \rightarrow \mathcal{P}^2(Y)$, by functoriality, given by double inverse image: $U \subseteq \mathcal{P}(X) \mapsto (f^{-1})^{-1}(U) = \{V \subseteq Y \mid f^{-1}(V) \subseteq U\}$.

It is not hard to see that the inclusion maps:

$$\mathcal{UF}(X) \xrightarrow[\cong]{(3)} \mathbf{BA}(2^X, 2) \hookrightarrow 2^{(2^X)}$$

form a map of monads, from the ultrafilter monad to the continuation monad (with constant $C = 2$).

2.4 Monads from composable adjunctions

It is well-known, see *e.g.* [29, Ch. VI] that each adjunction $F \dashv G$ gives rise to a monad GF . The expectation monad arises from a slightly more complicated situation, involving two composable adjunctions. This situation is captured abstractly in the following result.

The composable adjunctions that form the basis of the expectation monad are:

$$\begin{array}{ccc}
 \mathbf{Sets} & \begin{array}{c} \xrightarrow{\mathcal{D}} \\ \perp \\ \xleftarrow{U} \end{array} & \mathbf{Alg}(\mathcal{D}) \\
 & & \parallel \\
 & & \mathbf{Conv}
 \end{array}
 \begin{array}{c}
 \xrightarrow{\mathbf{Conv}(-,[0,1])} \\ \perp \\ \xleftarrow{\mathbf{EMod}(-,[0,1])}
 \end{array}
 \mathbf{EMod}^{\text{op}}
 \quad (8)$$

The adjunction on the left is the standard adjunction between a category of algebras $\mathbf{Alg}(\mathcal{D})$ of the distribution monad (see Subsection 2.1) and its underlying category. The adjunction on the right will be described in the next section.

3 Effect modules

This section introduces the essentials of effect modules and refers to [19, 21] for further details. Intuitively, effect modules are vector spaces, not with the real or complex numbers as scalars, but with scalars from the unit interval $[0, 1] \subseteq \mathbb{R}$. Also, the addition operation $+$ on vectors is only partial; it is written as \oplus . These effect modules occur in [32] under the name ‘convex effect algebras’.

More precisely, an effect module is an *effect algebra* E with an action $[0, 1] \otimes E \rightarrow E$ for scalar multiplication. An effect algebra E carries both:

- a partial commutative monoid structure $(0, \oplus)$; this means that \oplus is a partial operation $E \times E \rightarrow E$ which is both commutative and associative, taking suitably account of partiality, with 0 as neutral element;
- an orthosupplement $(-)^{\perp}: E \rightarrow E$. One writes $x \perp y$ if the sum $x \oplus y$ is defined; x^{\perp} is then the unique element with $x \oplus x^{\perp} = 1$, where $1 = 0^{\perp}$; further $x \perp 1$ holds only for $x = 0$.

These effect algebras carry a partial order given by $x \leq y$ iff $x \oplus z = y$, for some element z . Then $x \perp y$ iff $x \leq y^{\perp}$ iff $y \leq x^{\perp}$. The unit interval $[0, 1]$ is the prime example of an effect algebra with partial sum $r \oplus s = r + s$ if $r + s \leq 1$; then $r^{\perp} = 1 - r$.

A homomorphism $f: E \rightarrow D$ of effect algebras satisfies $f(1) = 1$ and: if $x \perp x'$ in E , then $f(x) \perp f(x')$ in D and $f(x \oplus x') = f(x) \oplus f(x')$. It is easy to deduce that $f(x^{\perp}) = f(x)^{\perp}$ and $f(0) = 0$. This yields a category, written as \mathbf{EA} . It carries a symmetric monoidal structure \otimes with the 2-element effect algebra $\{0, 1\}$ as tensor unit (which is at the same time the initial object). The usual multiplication of real numbers (probabilities in this case) yields a monoid structure on $[0, 1]$ in the category \mathbf{EA} . An *effect module* is then an effect algebra with an $[0, 1]$ -action $[0, 1] \otimes E \rightarrow E$. Explicitly, it can be described as a scalar multiplication $(r, x) \mapsto rx$ satisfying:

$$\begin{array}{ll}
 1x = x & (r+s)x = rx + sx \quad \text{if } r+s \leq 1 \\
 (rs)x = r(sx) & r(x \oplus y) = rx \oplus ry \quad \text{if } x \perp y.
 \end{array}$$

In particular, if $r + s \leq 1$, then a sum $rx \oplus sy$ always exists (see [32]).

Example 2. The unit interval $[0, 1]$ is again the prime example, this time for effect modules. But also, for an arbitrary set X , the set $[0, 1]^X$ of all functions $X \rightarrow [0, 1]$ is an effect module, with structure inherited pointwise from $[0, 1]$. Another example, occurring in integration theory, is the set $[X \rightarrow_s [0, 1]]$ of *simple* functions $X \rightarrow [0, 1]$, having only finitely many output values (also known as ‘step functions’).

A morphism $E \rightarrow D$ in the category \mathbf{EMod} of such effect modules is a function $f: E \rightarrow D$ between the underlying sets satisfying:

$$f(rx) = rf(x) \quad f(1) = 1 \quad f(x \otimes y) = f(x) \otimes f(y) \quad \text{if } x \perp y.$$

We collect some of the results we use about effect modules (see [21] for more information).

Proposition 5. 1. For each effect module E the homset $\mathbf{EMod}(E, [0, 1])$ is a convex set. In the other direction, each convex set X gives rise to an effect module $\mathbf{Conv}(X, [0, 1])$. This gives the adjunction on the right in (8), with $[0, 1]$ as dualizing object

2. Let $\mathbf{poVectu}$ be the category of ordered vector spaces with a strong unit u , and with linear, monotone mappings that preserve the unit as morphisms. The “partialization” mapping $\hat{\mathcal{F}}_a: \mathbf{poVectu} \rightarrow \mathbf{EMod}$ that takes a space V to the interval $[0, u] = \{x \in V \mid 0 \leq x \leq u\}$ is an equivalence of categories.

The effect algebra structure on the set $\mathbf{Conv}(X, [0, 1])$ of affine maps to $[0, 1]$ is obtained pointwise: $f \otimes g$ is defined if $f(x) + g(x) \leq 1$ for all $x \in X$, and in that case $f \otimes g$ at $x \in X$ is $f(x) + g(x)$. The orthosupplement is also obtained pointwise: $(f^\perp)(x) = 1 - f(x)$. Scalar multiplication is done similarly $(rf)(x) = r(f(x))$. In the reverse direction, each effect module E gives rise to a convex set $\mathbf{EMod}(E, [0, 1])$ of homomorphisms, with pointwise convex sums. The adjunction $\mathbf{Conv}(-, [0, 1]) \dashv \mathbf{EMod}(-, [0, 1])$ arises in the standard way, with unit and counit given by evaluation.

That the element u in point 2 of the proposition is a strong unit means that for each x there is an $n \in \mathbb{N}$ with $nu \geq x$. We write $\hat{\mathcal{F}}_o: \mathbf{EMod} \rightarrow \mathbf{poVectu}$ for the “totalization” part of the equivalence. We will not describe $\hat{\mathcal{F}}_o$ in detail here (see [21] for details) but roughly $\hat{\mathcal{F}}_o(E)$ is constructed by adding undefined sums and differences such as $1 \otimes 1$ and $0 \oplus 1$ to E . The essential part is that $\hat{\mathcal{F}}_o(E)$ is an ordered vector space with a strong unit whose unit interval is isomorphic to E . For example $\hat{\mathcal{F}}_o([0, 1]) \cong \mathbb{R}$ and more generally if E is a collection of a certain type of functions (e.g. affine) $X \rightarrow [0, 1]$ then $\hat{\mathcal{F}}_o(E)$ is usually the collection of those same type of functions only from X to \mathbb{R} .

If $V \in \mathbf{poVectu}$ and the unit u is Archimedean in the sense that $x \leq ru$ for all $r > 0$ implies $x \leq 0$ then V is called an *order unit space*. The Archimedean property of the unit can be used to define a norm $\|x\| = \inf\{r \in [0, 1] \mid -ru \leq x \leq ru\}$. We denote by \mathbf{OUS} the full subcategory of $\mathbf{poVectu}$ consisting of all order unit spaces.

This Archimedean property can also be expressed on the effect module level but some caution is required as effect modules contain no elements less than 0 and sums may not be defined. The following formulation works: an effect module is said to be Archimedean if $x \leq y$ follows from $\frac{1}{2}x \leq \frac{1}{2}y \otimes \frac{r}{2}1$ for all $r \in (0, 1]$. All Archimedean effect modules form a full subcategory $\mathbf{AEMod} \hookrightarrow \mathbf{EMod}$. Of course with this definition comes a theorem.

Proposition 6. The equivalence $\mathbf{poVectu} \simeq \mathbf{EMod}$, between partially ordered vector spaces with a strong unit and effect modules, restricts to an equivalence $\mathbf{OUS} \simeq \mathbf{AEMod}$, between order unit spaces and Archimedean effect modules.

Proof We only check that if $E \in \mathbf{AEMod}$ then its totalization satisfies $\hat{\mathcal{F}}_o(E) \in \mathbf{OUS}$; the rest is left to the reader. Suppose $E \in \mathbf{AEMod}$ and $x \in \hat{\mathcal{F}}_o(E)$ is such that $x \leq ru$ for all $r \in (0, 1]$. The trick is to transform x into an element in the unit interval $[0, u] \cong E$. Since u is a strong unit we can find a natural number n such that $x + nu \geq 0$, and again using the fact that u is a strong unit we can find a positive real number $s < 1$ such that $sx + nsu \leq u$. Hence $sx + nsu \in [0, u] \cong E$. Now, for $r \in (0, 1]$ we have $sx \leq x \leq ru$ and so $\frac{s}{2}x + \frac{ns}{2}u \leq \frac{ns}{2}u + \frac{r}{2}u$. Thus, by the Archimedean property of E , we get $sx + nsu \leq nsu$. Hence $sx \leq 0$ and therefore $x \leq 0$. \square

Since $E \in \mathbf{AEMod}$ is isomorphic to the unit interval of its totalization $\hat{\mathcal{T}}_o(E)$, E inherits a metric from the normed space $\hat{\mathcal{T}}_o(E)$. This metric can be described wholly in terms of E . However the partial addition does force us into a somewhat awkward definition: for $x, y \in E$ their distance $d(x, y) \in [0, 1]$ can be defined as:

$$d(x, y) = \max \left(\inf \{ r \in (0, 1] \mid \frac{1}{2}x \leq \frac{1}{2}y \odot \frac{r}{2}1 \}, \inf \{ r \in (0, 1] \mid \frac{1}{2}y \leq \frac{1}{2}x \odot \frac{r}{2}1 \} \right). \quad (9)$$

A trivial consequence is the following lemma.

Lemma 7. *A map of effect modules $f: M \rightarrow M'$ between Archimedean effect modules M, M' is automatically non-expansive: $d'(f(x), f(y)) \leq d(x, y)$, for all $x, y \in M$. \square*

Of particular interest later in this paper are Archimedean effect modules that are complete in their metric. We call these *Banach effect modules* and denote by \mathbf{BEMod} the full subcategory of all Banach effect modules. The previous lemma implies that each map in \mathbf{BEMod} is automatically continuous.

Since an order unit space is complete in its metric if and only if its unit interval is complete we get the following result.

Proposition 8. *The equivalences from Proposition 6 restrict further to an equivalence between Banach effect modules and the full subcategory $\mathbf{BOUS} \hookrightarrow \mathbf{OUS}$ of those order unit spaces that are also Banach spaces:*

$$\begin{array}{ccc} \mathbf{BOUS} & \xrightarrow{\cong} & \mathbf{BEMod} \\ \downarrow & & \downarrow \\ \mathbf{OUS} & \xrightarrow{\cong} & \mathbf{AEMod} \\ \downarrow & & \downarrow \\ \mathbf{poVectu} & \xrightarrow[\nu \mapsto ([0, u] \subseteq V)]{\cong} & \mathbf{EMod} \end{array}$$

Proof Like in the proof of Proposition 6 one transforms a Cauchy sequence in $\hat{\mathcal{T}}_o(E)$ into a sequence in $[0, u] \cong E$. \square

Example 3. We review Example 2: both the effect modules $[0, 1]$ and $[0, 1]^X$ are Archimedean, and also Banach effect modules. Norms and distances in $[0, 1]$ are the usual ones, but limits in $[0, 1]^X$ are defined via the supremum (or uniform) norm: for $p \in [0, 1]^X$, we have:

$$\begin{aligned} \|p\| &= \inf \{ r \in [0, 1] \mid p \leq r \cdot u \} \quad \text{where } u \text{ is the constant function } \lambda x. 1 \\ &= \inf \{ r \in [0, 1] \mid \forall x \in X. p(x) \leq r \} \\ &= \sup \{ p(x) \mid x \in X \} \\ &= \|p\|_\infty. \end{aligned}$$

The latter notation $\|p\|_\infty$ is common for this supremum norm. The associated metric on $[0, 1]^X$ is according to (9):

$$\begin{aligned} d(p, q) &= \max \left(\inf \{ r \in (0, 1] \mid \forall x \in X. \frac{1}{2}p(x) \leq \frac{1}{2}q(x) + \frac{r}{2} \}, \right. \\ &\quad \left. \inf \{ r \in (0, 1] \mid \forall x \in X. \frac{1}{2}q(x) \leq \frac{1}{2}p(x) + \frac{r}{2} \} \right) \\ &= \max \left(\sup \{ p(x) - q(x) \mid x \in X \text{ with } p(x) \geq q(x) \}, \right. \\ &\quad \left. \sup \{ q(x) - p(x) \mid x \in X \text{ with } p(x) \leq q(x) \} \right) \\ &= \sup \{ |p(x) - q(x)| \mid x \in X \} \\ &= \|p - q\|_\infty. \end{aligned}$$

Recall that the subset $[X \rightarrow_s [0, 1]] \subseteq [0, 1]^X$ of simple functions contains those $p \in [0, 1]^X$ that take only finitely many values, *i.e.* for which the set $\{p(x) \mid x \in X\}$ is finite. If we write $\{p(x) \mid x \in X\} = \{r_1, \dots, r_n\} \subseteq [0, 1]$, then we obtain n disjoint non-empty sets $X_i = \{x \in X \mid p(x) = r_i\}$ covering X . For a subset $U \subseteq X$, let $\mathbf{1}_U: X \rightarrow [0, 1]$ be the corresponding ‘‘characteristic’’ simple function, with $\mathbf{1}_U(x) = 1$ iff $x \in U$ and $\mathbf{1}_U(x) = 0$ iff $x \notin U$. Hence we can write such a simple function p in a normal form in the effect module $[X \rightarrow_s [0, 1]]$ of simple functions, namely as finite sum of characteristic functions:

$$p = \bigoplus_i r_i \cdot \mathbf{1}_{X_i}. \quad (10)$$

Hence $\|p\| = \max\{r_1, \dots, r_n\}$. These simple functions do not form a Banach effect module, since simple functions are not closed under countable suprema.

Lemma 9. *The inclusion of simple functions on a set X is dense in the Banach effect module of all fuzzy predicates on X :*

$$[X \rightarrow_s [0, 1]] \xrightarrow{\text{dense}} [0, 1]^X$$

Explicitly, each predicate $p \in [0, 1]^X$ can be written as limit $p = \lim_{n \rightarrow \infty} p_n$ of simple functions $p_n \in [0, 1]^X$ with $p_n \leq p$.

Proof Define for instance:

$$p_n(x) = 0.d_1d_2 \cdots d_n \quad \text{where} \quad d_i = \text{the } i\text{-th decimal of } p(x) \in [0, 1].$$

Clearly, p_n is simple, because it can take at most 10^n different values, since $d_i \in \{0, 1, \dots, 9\}$. Also, by construction, $p_n \leq p$. For each $\varepsilon > 0$, take $N \in \mathbb{N}$ such that for all decimals d_i we have:

$$0.\underbrace{00 \cdots 00}_{N \text{ times}} d_1 d_2 d_3 \cdots < \varepsilon.$$

Then for each $n \geq N$ we have $p(x) - p_n(x) < \varepsilon$, for all $x \in X$, and thus $d(p, p_n) \leq \varepsilon$. \square

3.1 Hahn-Banach style extension for effect modules

In this subsection we look at a form of Hahn-Banach theorem for effect modules. We need the following version of the Hahn-Banach extension theorem for partially ordered vector spaces.

Proposition 10. *Let E be a partially ordered vector space and let $F \subseteq E$ be a cofinal subspace (*i.e.* for all $x \in E$, $x \geq 0$ there is $y \in F$ with $x \leq y$). Suppose $f: F \rightarrow \mathbb{R}$ is a monotonic linear function. Then there is a monotonic linear function $g: E \rightarrow \mathbb{R}$ with $g|_F = f$ in:*

$$\begin{array}{ccc} F & \hookrightarrow & E \\ & \searrow f & \downarrow \exists g \\ & & \mathbb{R} \end{array}$$

Proof We define $p: E \rightarrow \mathbb{R}$ by:

$$p(x) = \inf\{f(y) \mid y \in F \text{ and } y \geq x\}.$$

Notice that $p(x)$ is finite since we can find $y, y' \in F$ with $y \leq x \leq y'$ because F is cofinal. We need to check that p is sublinear. So let $x, x' \in E$ and $\varepsilon > 0$ then we can find $y, y' \in F$ with $y \geq x, y' \geq x'$, such that $f(y) < p(x) + \varepsilon > 0$ and $f(y') < p(x') + \varepsilon > 0$. Therefore:

$$p(x+x') \leq f(y+y') = f(y) + f(y') < p(x) + p(x') + 2\varepsilon > 0.$$

Also for $r > 0$ it is obvious that $p(r \cdot x) = r \cdot p(x)$.

Having established that p is sublinear we note that for $y \in F$ we have $p(y) = f(y)$ since f is monotonic. Hence we can apply the standard (dominated) extension version of Hahn-Banach to find a linear function $g: E \rightarrow \mathbb{R}$ with $g < p$ and $g|_F = f$. Since if $x \leq 0$ then $p(x) \leq 0$ because $0 \in F$, hence it follows that g is monotonic. \square

This version translates effortlessly to effect modules

Proposition 11. *Let E be an effect module and $F \subseteq E$ a sub effect module of E . Suppose $f: F \rightarrow [0, 1]$ is an effect module map, then there is an effect module map $g: E \rightarrow [0, 1]$ with $g|_F = f$.*

Proof We translate effect modules to order unit spaces and apply the previous result. Since $u \in \hat{\mathcal{T}}_o(F)$ it's clear that $\hat{\mathcal{T}}_o(F)$ is cofinal in $\hat{\mathcal{T}}_o(E)$. Hence using the previous proposition we can extend $\hat{\mathcal{T}}_o(f)$ to $h: \hat{\mathcal{T}}_o(E) \rightarrow \mathbb{R}$. Hence by restriction to unit intervals $[0, u]$, both in $\hat{\mathcal{T}}_o(E)$ and in \mathbb{R} we get the map $g: E \rightarrow [0, 1]$ that we are looking for. \square

Unfortunately the class of effect module morphisms is too limited to get a full version of the separation theorem. Consider for example $E = [0, 1]^2$ with the two compact convex subsets $C_1 = \{(r, \frac{1}{2} + r) \mid r \in [0, \frac{1}{2}]\}$ and $C_2 = \{(\frac{1}{2} + r, r) \mid r \in [0, \frac{1}{2}]\}$. If $f: E \rightarrow [0, 1]$ is an effect module morphism then the image $f(C_1)$ is the interval $[f(0, \frac{1}{2}), f(\frac{1}{2}, 1)]$, and since $f(\frac{1}{2}, \frac{1}{2}) = \frac{1}{2}$ it follows that this interval has length $\frac{1}{2}$. Analogously the interval $f(C_2)$ is also an interval of length $\frac{1}{2}$ so the two must overlap.

4 The expectation monad

We now apply Lemma 4 to the composable adjunctions in (8) and take a first look at the results. In particular, we investigate different ways of describing the expectation monad \mathcal{E} that arises in this way.

Of the two monads resulting from applying Lemma 4 to the composable adjunctions in Diagram (8), the first one is the well-known distribution monad \mathcal{D} on **Sets**, arising from the adjunction $\mathbf{Sets} \rightleftarrows \mathbf{Alg}(\mathcal{D}) = \mathbf{Conv}$. The second monad on **Sets** arises from the composite adjunction $\mathbf{Sets} \rightleftarrows \mathbf{EMod}^{\text{op}}$ is less familiar (see Section 10 for more information and references). It is what we call the *expectation monad*, written here as \mathcal{E} . Following the description in Lemma 4 this monad is:

$$X \longmapsto \mathbf{EMod}\left(\mathbf{Conv}(\mathcal{D}(X), [0, 1]), [0, 1]\right).$$

Since $\mathcal{D}: \mathbf{Sets} \rightarrow \mathbf{Alg}(\mathcal{D}) = \mathbf{Conv}$ is the free algebra functor, the homset $\mathbf{Conv}(\mathcal{D}(X), [0, 1])$ is isomorphic to the set $[0, 1]^X$ of all maps $X \rightarrow [0, 1]$ in **Sets**. Elements of this set $[0, 1]^X$ can be understood as fuzzy predicates on X . As mentioned, they form a Banach effect module via pointwise operations. Thus we describe the expectation monad $\mathcal{E}: \mathbf{Sets} \rightarrow \mathbf{Sets}$ as:

$$\begin{aligned} \mathcal{E}(X) &= \mathbf{EMod}([0, 1]^X, [0, 1]) \\ \mathcal{E}(X \xrightarrow{f} Y) &= \lambda h \in \mathcal{E}(X). \lambda p \in [0, 1]^Y. h(p \circ f). \end{aligned} \tag{11}$$

The unit $\eta_X: X \rightarrow \mathcal{E}(X)$ is given by:

$$\eta_X(x) = \lambda p \in [0, 1]^X. p(x).$$

And the multiplication $\mu_X: \mathcal{E}^2(X) \rightarrow \mathcal{E}(X)$ is given on $h: [0, 1]^{\mathcal{E}(X)} \rightarrow [0, 1]$ in **EMod** by:

$$\mu_X(h) = \lambda p \in [0, 1]^X. h(\lambda k \in \mathcal{E}(X). k(p)).$$

It is not hard to see that $\eta(x)$ and $\mu(h)$ are homomorphisms of effect modules. We check explicitly that the μ - η laws hold and leave the remaining verifications to the reader. For $h \in \mathcal{E}(X)$,

$$\begin{aligned} (\mu_X \circ \eta_{\mathcal{E}(X)})(h) &= \mu_X(\eta_{\mathcal{E}(X)}(h)) \\ &= \lambda p \in [0, 1]^X. \eta_{\mathcal{E}(X)}(h)(\lambda k \in \mathcal{E}(X). k(p)) \\ &= \lambda p \in [0, 1]^X. (\lambda k \in \mathcal{E}(X). k(p))(h) \\ &= \lambda p \in [0, 1]^X. h(p) \\ &= h \\ (\mu_X \circ \mathcal{E}(\eta_X))(h) &= \mu_X(\mathcal{E}(\eta_X)(h)) \\ &= \lambda p \in [0, 1]^X. \mathcal{E}(\eta_X)(h)(\lambda k \in \mathcal{E}(X). k(p)) \\ &= \lambda p \in [0, 1]^X. h((\lambda k \in \mathcal{E}(X). k(p)) \circ \eta_X) \\ &= \lambda p \in [0, 1]^X. h(\lambda x \in X. \eta_X(x)(p)) \\ &= \lambda p \in [0, 1]^X. h(\lambda x \in X. p(x)) \\ &= \lambda p \in [0, 1]^X. h(p) \\ &= h. \end{aligned}$$

Remark 2. (1) We think of elements $h \in \mathcal{E}(X)$ as measures. Later on, in Theorem 3, it will be proven that $\mathcal{E}(X)$ is isomorphic to the set of finitely additive measures $\mathcal{P}(X) \rightarrow [0, 1]$ on X . The application $h(p)$ of $h \in \mathcal{E}(X)$ to a function $p \in [0, 1]^X$ may then be understood as integration $\int p dh$, giving the expected value of the stochastic variable/predicate p for the measure h .

(2) The description $\mathcal{E}(X) = \mathbf{EMod}([0, 1]^X, [0, 1])$ of the expectation monad in (11) bears a certain formal resemblance to the ultrafilter monad \mathcal{UF} from Subsection 2.2. Recall from (3) that:

$$\mathcal{UF}(X) \cong \mathbf{BA}(\{0, 1\}^X, \{0, 1\}).$$

Thus, the expectation monad \mathcal{E} can be seen as a “fuzzy” or “probabilistic” version of the ultrafilter monad \mathcal{UF} , in which the set of Booleans $\{0, 1\}$ is replaced by the set $[0, 1]$ of probabilities. The relation between the two monads is further investigated in Section 5.

(3) Using the equivalence $\mathbf{poVectu} \simeq \mathbf{EMod}$ from Proposition 6 via totalization we may equivalently describe the expectation monad as the homset:

$$\mathcal{E}(X) \cong \mathbf{poVectu}(\mathbb{R}^X, \mathbb{R}).$$

It contains the linear monotone functions $\mathbb{R}^X \rightarrow \mathbb{R}$ that send the unit $\lambda x. 1 \in \mathbb{R}^X$ to $1 \in \mathbb{R}$.

The following result is not a surprise, given the resemblance between the unit and multiplication for the expectation monad and the ones for the continuation monad (see Subsection 2.3).

Lemma 12. *The inclusion maps:*

$$\mathcal{E}(X) = \mathbf{EMod}([0, 1]^X, [0, 1]) \hookrightarrow [0, 1]^{([0, 1]^X)}$$

form a map of monads, from the expectation monad to the continuation monad (with the set $[0, 1]$ as constant). \square

We conclude with an alternative description of the sets $\mathcal{E}(X)$, in terms of finitely additive measures, described as effect algebra homomorphisms. It also occurs as [18, Cor. 4.3].

Theorem 3. *For each set X there is a bijection:*

$$\mathcal{E}(X) = \mathbf{EMod}([0, 1]^X, [0, 1]) \xrightarrow[\cong]{\Phi} \mathbf{EA}(\mathcal{P}(X), [0, 1])$$

given by $\Phi(h)(U) = h(\mathbf{1}_U)$.

Proof We first check that Φ is injective: assume $\Phi(h) = \Phi(h')$, for $h, h' \in \mathcal{E}(X)$. We need to show $h(p) = h'(p)$ for an arbitrary $p \in [0, 1]^X$. We first prove $h(q) = h'(q)$ for a simple function $q \in [0, 1]^X$. Recall that such a simple q can be written as $q = \bigoplus_i r_i \mathbf{1}_{X_i}$, like in (10), where the (disjoint) subsets $X_i \subseteq X$ cover X . Since $h, h' \in \mathcal{E}(X)$ are maps of effect modules we get:

$$h(q) = \sum_i r_i h(\mathbf{1}_{X_i}) = \sum_i r_i \Phi(h)(X_i) = \sum_i r_i \Phi(h')(X_i) = \sum_i r_i h'(\mathbf{1}_{X_i}) = h'(q).$$

For an arbitrary $p \in [0, 1]^X$ we first write $p = \lim_n p_n$ as limit of simple functions p_n like in Lemma 9. Lemma 7 implies that h, h' are continuous, and so we get $h = h'$ from:

$$h(p) = \lim_n h(p_n) = \lim_n h'(p_n) = h'(p).$$

For surjectivity of Φ assume a finitely additive measure $m: \mathcal{P}(X) \rightarrow [0, 1]$. We need to define a function $h \in \mathcal{E}(X)$ with $\Phi(h) = m$. We define such a h first on a simple function $q = \bigoplus_i r_i \mathbf{1}_{X_i}$ as $h(q) = \sum_i r_i m(X_i)$. For an arbitrary $p \in [0, 1]^X$, written as $p = \lim_n p_n$, like in Lemma 9, we define $h(p) = \lim_n h(p_n)$. Then $\Phi(h) = m$, since for $U \subseteq X$ we have:

$$\Phi(h)(U) = h(\mathbf{1}_U) = m(U). \quad \square$$

The inverse $h = \Phi^{-1}(m)$ that is constructed in this proof may be understood as an integral $h(p) = \int p dm$. The precise nature of the bijection Φ remains unclear at this stage since we have not yet identified the (algebraic) structure of the sets $\mathcal{E}(X)$. But via this bijection we can understand mapping a set to its finitely additive measures, *i.e.* $X \mapsto \mathbf{EA}(\mathcal{P}(X), [0, 1])$, as a monad.

Yet another perspective is useful in this context. The characteristic function mapping:

$$[0, 1] \times \mathcal{P}(X) \longrightarrow [0, 1]^X \quad \text{given by} \quad (r, U) \longmapsto r \cdot \mathbf{1}_U$$

is a bihomomorphism of effect modules. Hence it gives rise to a map of effect modules $[0, 1] \otimes \mathcal{P}(X) \rightarrow [0, 1]^X$, where the tensor product $[0, 1] \otimes \mathcal{P}(X)$ forms a more abstract description of the effect module of simple (step) functions $[X \rightarrow_s [0, 1]]$ from Lemma 9 (see also [18, Thm. 5.6]). Lemma 9 says that this map is dense. This gives a quick proof of Theorem 3:

$$\begin{aligned} \mathcal{E}(X) &= \mathbf{EMod}([0, 1]^X, [0, 1]) \\ &\cong \mathbf{EMod}([0, 1] \otimes \mathcal{P}(X), [0, 1]) \quad \text{by denseness} \\ &\cong \mathbf{EA}(\mathcal{P}(X), [0, 1]). \end{aligned}$$

This last isomorphism is standard, because $[0, 1] \otimes \mathcal{P}(X)$ is the free effect module on $\mathcal{P}(X)$.

5 The expectation and ultrafilter monads

In this section we show that the sets $\mathcal{E}(X)$ carry a compact Hausdorff structure and we identify its topology. The unit interval $[0, 1]$ plays an important role. It is a compact Hausdorff space, which means that it carries an algebra of the ultrafilter monad, see Subsection 2.2. We shall write this algebra as $\text{ch} = \text{ch}_{[0,1]}: \mathcal{UF}([0, 1]) \rightarrow [0, 1]$. What this map precisely does is described in Example 1; but mostly we use it abstractly, as an \mathcal{UF} -algebra. The technique we use to define the following map of monads is copied from Lemma 3.

Proposition 13. *There is a map of monads $\tau: \mathcal{UF} \Rightarrow \mathcal{E}$, given on an ultrafilter $\mathcal{F} \in \mathcal{UF}(X)$ and $p \in [0, 1]^X$ by:*

$$\begin{aligned} \tau_X(\mathcal{F})(p) &= \text{ch}(\mathcal{UF}(p)(\mathcal{F})) \\ &= \inf\{s \in [0, 1] \mid [0, s] \in \mathcal{UF}(p)(\mathcal{F})\} && \text{by (5)} \\ &= \inf\{s \in [0, 1] \mid \{x \in X \mid p(x) \leq s\} \in \mathcal{F}\}. \end{aligned}$$

In this description the functor \mathcal{UF} is applied to p , as function $X \rightarrow [0, 1]$, giving $\mathcal{UF}(p): \mathcal{UF}(X) \rightarrow \mathcal{UF}([0, 1])$.

Proof We first have to check that τ is well-defined, i.e. that $\tau_X(\mathcal{F}): [0, 1]^X \rightarrow [0, 1]$ is a morphism of effect modules.

- Preservation $\tau_X(\mathcal{F})(r \cdot p) = r \cdot p \tau_X(\mathcal{F})$ of multiplication with scalar $r \in [0, 1]$. This follows by observing that multiplication $r \cdot (-): [0, 1] \rightarrow [0, 1]$ is a continuous function, and thus a morphism of algebras in the square below.

$$\begin{array}{ccc} \mathcal{UF}([0, 1]) & \xrightarrow{\mathcal{UF}(r \cdot (-))} & \mathcal{UF}([0, 1]) \\ \text{ch} \downarrow & & \downarrow \text{ch} \\ [0, 1] & \xrightarrow{r \cdot (-)} & [0, 1] \end{array}$$

Thus:

$$\begin{aligned} \tau(\mathcal{F})(r \cdot p) &= (\text{ch} \circ \mathcal{UF}(r \cdot (-) \circ p))(\mathcal{F}) \\ &= (r \cdot (-) \circ \text{ch} \circ \mathcal{UF}(p))(\mathcal{F}) = r \cdot \tau(\mathcal{F})(p). \end{aligned}$$

- Preservation of \oplus , is obtained in the same manner, using that addition $+: [0, 1] \times [0, 1] \rightarrow [0, 1]$ is continuous.
- Constant functions $\lambda x. a \in [0, 1]^X$, including 0 and 1, are preserved:

$$\begin{aligned} \tau_X(\mathcal{F})(\lambda x. a) &= \text{ch}(\mathcal{UF}(\lambda x. a)(\mathcal{F})) \\ &= \text{ch}(\{U \in \mathcal{P}([0, 1]) \mid (\lambda x. a)^{-1}(U) \in \mathcal{F}\}) \\ &= \text{ch}(\{U \in \mathcal{P}([0, 1]) \mid \{x \in X \mid a \in U\} \in \mathcal{F}\}) \\ &= \text{ch}(\{U \in \mathcal{P}([0, 1]) \mid a \in U\}) && \text{since } \emptyset \notin \mathcal{F} \\ &= \text{ch}(\eta(a)) \\ &= a. \end{aligned}$$

We leave naturality of τ and commutation with units to the reader and check that τ commutes with multiplications $\mu^{\mathcal{E}}$ and $\mu^{\mathcal{UF}}$ of the expectation and ultrafilter monads. Thus, for $\mathcal{A} \in \mathcal{UF}^2(X)$ and $p \in [0, 1]^X$, we calculate:

$$\begin{aligned}
(\mu^{\mathcal{E}} \circ \tau \circ \mathcal{UF}(\tau))(\mathcal{A})(p) &= \mu\left(\tau(\mathcal{UF}(\tau)(\mathcal{A}))\right)(p) \\
&= \tau(\mathcal{UF}(\tau)(\mathcal{A}))(\lambda k. k(p)) \\
&= \text{ch}\left(\mathcal{UF}(\lambda k. k(p))(\mathcal{UF}(\tau)(\mathcal{A}))\right) \\
&= \text{ch}\left(\mathcal{UF}(\lambda \mathcal{F}. \tau(\mathcal{F})(p))(\mathcal{A})\right) \\
&= \text{ch}\left(\mathcal{UF}(\lambda \mathcal{F}. \text{ch}(\mathcal{UF}(p)(\mathcal{F})))(\mathcal{A})\right) \\
&= \text{ch}\left(\mathcal{UF}(\text{ch} \circ \mathcal{UF}(p))(\mathcal{A})\right) \\
&= (\text{ch} \circ \mathcal{UF}(\text{ch} \circ \mathcal{UF}(p)))(\mathcal{A}) \\
&= (\text{ch} \circ \mathcal{UF}(\text{ch}) \circ \mathcal{UF}^2(p))(\mathcal{A}) \\
&= (\text{ch} \circ \mu^{\mathcal{UF}} \circ \mathcal{UF}^2(p))(\mathcal{A}) \\
&= (\text{ch} \circ \mathcal{UF}(p) \circ \mu^{\mathcal{UF}})(\mathcal{A}) \\
&= \text{ch}(\mathcal{UF}(p)(\mu^{\mathcal{UF}}(\mathcal{A}))) \\
&= (\tau \circ \mu^{\mathcal{UF}})(\mathcal{A})(p). \quad \square
\end{aligned}$$

Corollary 14. *There is a functor $\text{Alg}(\mathcal{E}) \rightarrow \text{Alg}(\mathcal{UF}) = \mathbf{CH}$, by pre-composition: $(\mathcal{E}(X) \xrightarrow{\alpha} X) \mapsto (\mathcal{UF}(X) \xrightarrow{\alpha \circ \tau} X)$. This functor has a left adjoint by Lemma 1.*

In particular, the underlying set X of each \mathcal{E} -algebra $\alpha: \mathcal{E}(X) \rightarrow X$ carries a compact Hausdorff topology, with $U \subseteq X$ closed iff for each $\mathcal{F} \in \mathcal{UF}(X)$ with $U \in \mathcal{F}$ one has $\alpha(\tau(\mathcal{F})) \in U$, as described in Subsection 2.2. \square

With respect to this topology on $\mathcal{E}(X)$, several maps are continuous.

Lemma 15. *The following maps are continuous functions.*

$$\mathcal{UF}(X) \xrightarrow{\tau_X} \mathcal{E}(X) \quad \mathcal{E}(X) \xrightarrow[\text{algebra}]{\alpha} X \quad \mathcal{E}(X) \xrightarrow{\mathcal{E}(f)} \mathcal{E}(Y) \quad \mathcal{E}(X) \xrightarrow[\lambda h. h(p)]{\text{ev}_p} [0, 1].$$

Proof One shows that these maps are morphisms of \mathcal{UF} -algebras. For instance, τ_X is continuous because it is a map of monads: commutation with multiplications, as required in (2), precisely says that it is a map of algebras, in the square on the left below.

$$\begin{array}{ccc}
\mathcal{UF}^2(X) & \xrightarrow{\mathcal{UF}(\tau_X)} & \mathcal{UF}(\mathcal{E}(X)) & & \mathcal{UF}(\mathcal{E}(X)) & \xrightarrow{\mathcal{UF}(\alpha)} & \mathcal{UF}(X) \\
\mu_X \downarrow & & \downarrow \mu_X \circ \tau_{\mathcal{E}(X)} & & \mu_X \circ \tau_{\mathcal{E}(X)} \downarrow & & \downarrow \alpha \circ \tau_X \\
\mathcal{UF}(X) & \xrightarrow{\tau_X} & \mathcal{E}(X) & & \mathcal{E}(X) & \xrightarrow{\alpha} & X
\end{array}$$

The rectangle on the right expresses that an Eilenberg-Moore algebra $\alpha: \mathcal{E}(X) \rightarrow X$ is a continuous function. It commutes by naturality of τ :

$$\alpha \circ \tau_X \circ \mathcal{UF}(\alpha) = \alpha \circ \mathcal{E}(\alpha) \circ \tau_{\mathcal{E}(X)} = \alpha \circ \mu_X \circ \tau_{\mathcal{E}(X)}.$$

For $f: X \rightarrow Y$, continuity of $\mathcal{E}(f): \mathcal{E}(X) \rightarrow \mathcal{E}(Y)$ follows directly from naturality of τ . Finally, for $p \in [0, 1]^X$ the map $\text{ev}_p = \lambda h. h(p): \mathcal{E}(X) \rightarrow [0, 1]$ is continuous because for $\mathcal{F} \in \mathcal{UF}(\mathcal{E}(X))$,

$$\begin{aligned}
(\text{ev}_p \circ \mu_X \circ \tau_{\mathcal{E}(X)})(\mathcal{F}) &= \mu_X(\tau_{\mathcal{E}(X)}(\mathcal{F}))(p) \\
&= \tau_{\mathcal{E}(X)}(\mathcal{F})(\lambda k. k(p)) \\
&= \tau_{\mathcal{E}(X)}(\mathcal{F})(\text{ev}_p) \\
&= \text{ch}(\mathcal{UF}(\text{ev}_p)(\mathcal{F})) \\
&= (\text{ch} \circ \mathcal{UF}(\text{ev}_p))(\mathcal{F}). \quad \square
\end{aligned}$$

The next step is to give a concrete description of this compact Hausdorff topology on sets $\mathcal{E}(X)$, as induced by the algebra $\mathcal{UF}(\mathcal{E}(X)) \rightarrow \mathcal{E}(X)$.

Proposition 16. *Fix a set X . For a predicate $p \in [0, 1]^X$ and a rational number $s \in [0, 1] \cap \mathbb{Q}$ write:*

$$\square_s(p) = \{h \in \mathcal{E}(X) \mid h(p) > s\}.$$

These sets $\square_s(p) \subseteq \mathcal{E}(X)$ form a subbasis for the topology on $\mathcal{E}(X)$.

Proof We reason as follows. The subsets $\square_s(p)$ are open in the compact Hausdorff topology induced on $\mathcal{E}(X)$ by the algebra structure $\mathcal{UF}(\mathcal{E}(X)) \rightarrow \mathcal{E}(X)$. They form a subbasis for a Hausdorff topology on $\mathcal{E}(X)$. Hence by Lemma 2 this topology is the induced one. We now elaborate these steps.

The Eilenberg-Moore algebra $\mathcal{UF}(\mathcal{E}(X)) \rightarrow \mathcal{E}(X)$ is given by $\mu_X \circ \tau_{\mathcal{E}(X)}$. Hence the associated closed sets $U \subseteq \mathcal{E}(X)$ are those satisfying $U \in \mathcal{F} \Rightarrow \mu_X(\tau_{\mathcal{E}(X)}(\mathcal{F})) \in U$, for each $\mathcal{F} \in \mathcal{UF}(\mathcal{E}(X))$, see Subsection 2.2. We wish to show that $\neg \square_s(p) = \{h \mid h(p) \leq s\} \subseteq \mathcal{E}(X)$ is closed. We reason backwards, starting with the required conclusion.

$$\begin{aligned}
\mu(\tau(\mathcal{F})) \in \neg \square_s(p) &\iff \mu(\tau(\mathcal{F}))(p) \leq s \\
&\iff \text{ch}(\mathcal{UF}(\lambda k. k(p))(\mathcal{F})) \in [0, s] \\
&\quad \text{since } \mu(\tau(\mathcal{F}))(p) = \tau(\mathcal{F})(\lambda k. k(p)) = \text{ch}(\mathcal{UF}(\lambda k. k(p))(\mathcal{F})) \\
&\iff [0, s] \in \mathcal{UF}(\lambda k. k(p))(\mathcal{F}) \\
&\quad \text{since } [0, s] \subseteq [0, 1] \text{ is closed} \\
&\iff (\lambda k. k(p))^{-1}([0, s]) \in \mathcal{F} \\
&\iff \{h \in \mathcal{E}(X) \mid h(p) \in [0, s]\} = \neg \square_s(p) \in \mathcal{F}.
\end{aligned}$$

Hence $\neg \square_s(p) \subseteq \mathcal{E}(X)$ is closed, making $\square_s(p)$ open.

Next we need to show that these $\square_s(p)$'s give rise to a Hausdorff topology. So assume $h \neq h' \in \mathcal{E}(X)$. Then there must be a $p \in [0, 1]^X$ with $h(p) \neq h'(p)$. Without loss of generality we assume $h(p) < h'(p)$. Find an $s \in [0, 1] \cap \mathbb{Q}$ with $h(p) < s < h'(p)$. Then $h' \in \square_s(p)$. Also:

$$h(p^\perp) = 1 - h(p) > 1 - s > 1 - h'(p) = h'(p^\perp).$$

Hence $h \in \square_{1-s}(p^\perp)$. These sets $\square_s(p)$ and $\square_{1-s}(p^\perp)$ are disjoint, since: $k \in \square_s(p) \cap \square_{1-s}(p^\perp)$ iff both $k(p) > s$ and $1 - k(p) > 1 - s$, which is impossible. \square

As is well-known, ultrafilters on a set X can also be understood as finitely additive measures $\mathcal{P}(X) \rightarrow \{0, 1\}$. Using Theorem 3 we can express more precisely how the expectation monad \mathcal{E} is a probabilistic version of the ultrafilter monad \mathcal{UF} , namely via the descriptions:

$$\mathcal{E}(X) \cong \mathbf{EA}(\mathcal{P}(X), [0, 1]) \quad \text{and} \quad \mathcal{UF}(X) \cong \mathbf{EA}(\mathcal{P}(X), \{0, 1\}).$$

We have $\mathbf{EA}(\mathcal{P}(X), \{0, 1\}) = \mathbf{BA}(\mathcal{P}(X), \{0, 1\})$ because in general, for Boolean algebras B, B' a homomorphism of Boolean algebras $B \rightarrow B'$ is the same as an effect algebra homomorphism $B \rightarrow B'$.

Lemma 17. *The components $\tau_X: \mathcal{UF}(X) \rightarrow \mathcal{E}(X)$ are injections.*

Proof Because there are isomorphisms:

$$\begin{array}{ccc} \mathcal{UF}(X) & \xrightarrow{\tau_X} & \mathcal{E}(X) \\ \wr \parallel & & \wr \parallel \\ \mathbf{EA}(\mathcal{P}(X), \{0, 1\}) & \xrightarrow{\quad} & \mathbf{EA}(\mathcal{P}(X), [0, 1]) \end{array} \quad \square$$

6 The expectation and distribution monads

We continue with the implications of Lemma 4 in the current situation, especially with the natural transformation (6). This leads to convex structure on sets $\mathcal{E}(X)$.

Lemma 18. *There is a map of monads:*

$$\sigma: \mathcal{D} \Longrightarrow \mathcal{E} \quad \text{given by} \quad \sigma_X(\varphi) = \lambda p \in [0, 1]^X. \sum_x \varphi(x) \cdot p(x), \quad (12)$$

where the dot \cdot describes multiplication in $[0, 1]$.

All components $\sigma_X: \mathcal{D}(X) \rightarrow \mathcal{E}(X)$ are injections. And for finite sets X the component at X is an isomorphism $\mathcal{D}(X) \xrightarrow{\cong} \mathcal{E}(X)$.

With this result we have completed the positioning of the expectation monad in Diagram (1), in between the distribution and ultrafilter monad on the hand, and the continuation monad on the other.

Proof By construction via (6) the natural transformation $\sigma: \mathcal{D} \Rightarrow \mathcal{E}$ is a map of monads. Next, assume X is finite, say $X = \{x_1, \dots, x_n\}$. Each $p \in [0, 1]^X$ is determined by the values $p(x_i) \in [0, 1]$. Using the effect module structure of $[0, 1]^X$, this p can be written as sum of scalar multiplications:

$$p = p(x_1) \cdot \mathbf{1}_{x_1} \otimes \cdots \otimes p(x_n) \cdot \mathbf{1}_{x_n},$$

where $\mathbf{1}_{x_i}: X \rightarrow [0, 1]$ is the characteristic function of the singleton $\{x_i\} \subseteq X$. A map of effect modules $h \in \mathcal{E}(X) = \mathbf{EMod}([0, 1]^X, [0, 1])$ will thus send such a predicate p to:

$$\begin{aligned} h(p) &= h(p(x_1) \cdot \mathbf{1}_{x_1} \otimes \cdots \otimes p(x_n) \cdot \mathbf{1}_{x_n}) \\ &= p(x_1) \cdot h(\mathbf{1}_{x_1}) + \cdots + p(x_n) \cdot h(\mathbf{1}_{x_n}), \end{aligned}$$

since \otimes is $+$ in $[0, 1]$. Hence h is completely determined by these values $h(\mathbf{1}_{x_i}) \in [0, 1]$. But since $\sum_i \mathbf{1}_{x_i} = \mathbf{1}$ in $[0, 1]^X$ we also have $\sum_i h(\mathbf{1}_{x_i}) = 1$. Hence h can be described by the convex sum $\varphi \in \mathcal{D}(X)$

given by $\varphi(x) = h(\mathbf{1}_x)$. Thus we have a bijection $\mathcal{E}(X) \cong \mathcal{D}(X)$. In fact, σ_X describes (the inverse of) this bijection, since:

$$\begin{aligned} \sigma_X(\varphi)(p) &= \sum_i \varphi(x_i) \cdot p(x_i) \\ &= \sum_i p(x_i) \cdot h(\mathbf{1}_{x_i}) \\ &= h(\bigoplus_i p(x_i) \cdot \mathbf{1}_{x_i}) \\ &= h(p). \end{aligned} \quad \square$$

Corollary 19. *There is a functor $\text{Alg}(\mathcal{E}) \rightarrow \text{Alg}(\mathcal{D}) = \mathbf{Conv}$, by pre-composition: $(\mathcal{E}(X) \xrightarrow{\alpha} X) \mapsto (\mathcal{D}(X) \xrightarrow{\alpha \circ \sigma} X)$. It has a left adjoint by Lemma 1.* \square

Explicitly, for each \mathcal{E} -algebra $\alpha: \mathcal{E}(X) \rightarrow X$, the set X is a convex set, with sum of a formal convex combination $\sum_i r_i x_i$ given by the element:

$$\alpha(\sigma_X(\sum_i r_i x_i)) = \alpha(\lambda p \in [0, 1]^X \cdot \sum_i r_i \cdot p(x_i)) \in X.$$

Lemma 18 implies that if the carrier X is finite, the algebra structure α corresponds precisely to such convex structure on X . If X is non-finite we still have to find out what α involves.

Here is another (easy) consequence of Lemma 18.

Corollary 20. *On the first few finite sets: empty 0, singleton 1, and two-element 2 one has:*

$$\mathcal{E}(0) \cong 0 \quad \mathcal{E}(1) \cong 1 \quad \mathcal{E}(2) \cong [0, 1].$$

The isomorphism in the middle says that \mathcal{E} is an affine functor.

Proof The isomorphisms follow easily from $\mathcal{E}(X) \cong \mathcal{D}(X)$ for finite X . \square

Remark 3. (1) The natural transformation $\sigma: \mathcal{D} \Rightarrow \mathcal{E}$ from (12) implicitly uses that the unit interval $[0, 1]$ is convex. This can be made explicit in the following way. Describe this convexity via an algebra $\text{cv}: \mathcal{D}([0, 1]) \rightarrow [0, 1]$. Then we can equivalently describe σ as:

$$\sigma_X(\varphi)(p) = \text{cv}(\mathcal{D}(p)(\varphi)).$$

This alternative description is similar to the construction in Proposition 13, for a natural transformation $\mathcal{UF} \Rightarrow \mathcal{E}$ (see also Lemma 3).

(2) From Corollaries 14 and 19 we know that the sets $\mathcal{E}(X)$ are both compact Hausdorff and convex. This means that we can take free extensions of the maps $\tau: \mathcal{UF}(X) \rightarrow \mathcal{E}(X)$ and $\sigma: \mathcal{D}(X) \rightarrow \mathcal{E}(X)$, giving maps $\mathcal{D}(\mathcal{UF}(X)) \rightarrow \mathcal{E}(X)$ and $\mathcal{UF}(\mathcal{D}(X)) \rightarrow \mathcal{E}(X)$, etc. The latter map is the composite:

$$\mathcal{UF}(\mathcal{D}(X)) \xrightarrow{\mathcal{UF}(\sigma)} \mathcal{UF}(\mathcal{E}(X)) \xrightarrow{\tau} \mathcal{E}^2(X) \xrightarrow{\mu} \mathcal{E}(X).$$

Using Example 1, it can be described more concretely on $\mathcal{F} \in \mathcal{UF}(\mathcal{D}(X))$ and $p \in [0, 1]^X$ as:

$$\inf\{s \in [0, 1] \mid \{\varphi \in \mathcal{D}(X) \mid \sum_x \varphi(x) \cdot p(x) \leq s\} \in \mathcal{F}\}.$$

The next result is the affine analogue of Lemma 15.

Lemma 21. *The following maps are affine functions.*

$$\mathcal{D}(X) \xrightarrow{\sigma_X} \mathcal{E}(X) \quad \mathcal{E}(X) \xrightarrow[\text{algebra}]{\alpha} X \quad \mathcal{E}(X) \xrightarrow{\mathcal{E}(f)} \mathcal{E}(Y) \quad \mathcal{E}(X) \xrightarrow[\lambda h \cdot h(p)]{\text{cv}_p} [0, 1].$$

Proof Verifications are done like in the proof of Lemma 15. We only do the last one. We need to prove that the following diagram commutes,

$$\begin{array}{ccc} \mathcal{D}(\mathcal{E}(X)) & \xrightarrow{\mathcal{D}(\text{ev}_p)} & \mathcal{D}([0, 1]) \\ \mu_X \circ \sigma_X \downarrow & & \downarrow \text{cv} \\ \mathcal{E}(X) & \xrightarrow{\text{ev}_p} & [0, 1] \end{array}$$

where the algebra cv interprets formal convex combinations as actual combinations. For a distribution $\Phi = \sum_i r_i h_i \in \mathcal{D}(\mathcal{E}(X))$ we have:

$$\begin{aligned} (\text{ev}_p \circ \mu \circ \sigma)(\Phi) &= \mu(\sigma(\Phi))(p) \\ &= \sigma(\Phi)(\text{ev}_p) \\ &= \sum_i r_i \cdot \text{ev}_p(h_i) \\ &= \text{cv}(\sum_i r_i \text{ev}_p(h_i)) \\ &= \text{cv}(\mathcal{D}(\text{ev}_p)(\sum_i r_i h_i)) \\ &= (\text{cv} \circ \mathcal{D}(\text{ev}_p))(\Phi). \end{aligned} \quad \square$$

The \mathcal{D} -algebras obtained from \mathcal{E} -algebras turn out to be continuous functions. This connects the convex and topological structures in such algebras.

Lemma 22. *The maps $\sigma_X: \mathcal{D}(X) \rightarrow \mathcal{E}(X)$ are (trivially) continuous when we provide $\mathcal{D}(X)$ with the subspace topology with basic opens $\square_s(p) \subseteq \mathcal{D}(X)$ given by restriction: $\square_s(p) = \{\varphi \in \mathcal{D}(X) \mid \sum_x \varphi(x) \cdot p(x) > s\}$, for $p \in [0, 1]^X$ and $s \in [0, 1] \cap \mathbb{Q}$.*

For each \mathcal{E} -algebra $\alpha: \mathcal{E}(X) \rightarrow X$ the associated \mathcal{D} -algebra $\alpha \circ \sigma: \mathcal{D}(X) \rightarrow X$ is then also continuous.

Proof Lemma 15 states that \mathcal{E} -algebras $\alpha: \mathcal{E}(X) \rightarrow X$ are continuous. Hence $\alpha \circ \sigma: \mathcal{D}(X) \rightarrow X$, as composition of continuous maps, is also continuous. \square

The following property of the map of monads $\mathcal{D} \Rightarrow \mathcal{E}$ will play a crucial role.

Proposition 23. *The inclusions $\sigma_X: \mathcal{D}(X) \rightarrow \mathcal{E}(X)$ are dense: the topological closure of $\mathcal{D}(X)$ is the whole of $\mathcal{E}(X)$.*

Proof We need to show that for each non-empty open $U \subseteq \mathcal{E}(X)$ there is a distribution $\varphi \in \mathcal{D}(X)$ with $\sigma(\varphi) \in U$. By Proposition 16 we may assume U is of the form $U = \square_{s_1}(p_1) \cap \dots \cap \square_{s_m}(p_m)$, for certain $s_i \in [0, 1] \cap \mathbb{Q}$ and $p_i \in [0, 1]^X$. For convenience we do the proof for $m = 2$. Since U is non-empty there is some inhabitant $h \in \square_{s_1}(p_1) \cap \square_{s_2}(p_2)$. Thus $h(p_i) > s_i$. We claim there are simple functions $q_i \leq p_i$ with $h(q_i) > s_i$.

In general, this works as follows. If $h(p) > s$, write $p = \lim_n p_n$ for simple functions $p_n \leq p$, like in Lemma 9. Then $h(p) = \lim_n h(p_n) > s$. Hence $h(p_n) > s$ for some simple $p_n \leq p$.

In a next step we write the simple functions as weighted sum of characteristic functions, like in (10). Thus, let

$$q_1 = \bigoplus_j r_j \mathbf{1}_{U_j} \quad \text{and} \quad q_2 = \bigoplus_k t_k \mathbf{1}_{V_k},$$

where these $U_j \subseteq X$ and $V_k \subseteq X$ form non-empty partitions, each covering X . We form a new, refined partition $(W_\ell \subseteq X)_{\ell \in L}$ consisting of the non-empty intersections $U_j \cap V_k$, and choose $x_\ell \in W_\ell$. Then:

- $\sum_{\ell} h(\mathbf{1}_{W_{\ell}}) = h(\bigoplus_{\ell} \mathbf{1}_{W_{\ell}}) = h(\mathbf{1}_X) = 1.$
- There are subsets $L_j \subseteq L$ so that each $U_j \subseteq X$ can be written as disjoint union $U_j = \bigcup_{\ell \in L_j} W_{\ell}.$
- Similarly, $V_k = \bigcup_{\ell \in L_k} W_{\ell}$ for subsets $L_k \subseteq L.$

We take as distribution $\varphi = \sum_{\ell \in L} h(\mathbf{1}_{W_{\ell}}) x_{\ell} \in \mathcal{D}(X).$ Then $\sigma(\varphi) \in \square_{s_i}(p_i).$ We do the proof for $i = 1.$

$$\begin{aligned}
\sigma(\varphi)(p_1) &= \sum_{\ell \in L} \varphi(x_{\ell}) \cdot p_1(x_{\ell}) \\
&\geq \sum_{\ell \in L} h(\mathbf{1}_{W_{\ell}}) \cdot q_1(x_{\ell}) \\
&= \sum_j \sum_{\ell \in L_j} h(\mathbf{1}_{W_{\ell}}) \cdot q_1(x_{\ell}) \\
&= \sum_j \sum_{\ell \in L_j} h(\mathbf{1}_{W_{\ell}}) \cdot r_j \\
&= \sum_j h(\bigoplus_{\ell \in L_j} \mathbf{1}_{W_{\ell}}) \cdot r_j \\
&= \sum_j h(\mathbf{1}_{U_j}) \cdot r_j \\
&= h(\bigoplus_j r_j \cdot \mathbf{1}_{U_j}) \\
&= h(p_1) \\
&> s_1. \quad \square
\end{aligned}$$

Corollary 24. *Each map $\mathcal{UF}(\mathcal{D}(X)) \rightarrow \mathcal{E}(X),$ described in Example 3.(3), is onto (surjective).*

Proof Since $\mathcal{D}(X) \rightarrow \mathcal{E}(X)$ is dense, each $h \in \mathcal{E}(X)$ is a limit of elements in $\mathcal{D}(X).$ Such limits can be described for instance via nets or via ultrafilters. In the present context we choose the latter approach. Thus there is an ultrafilter $\mathcal{F} \in \mathcal{UF}(\mathcal{D}(X))$ such that h is the limit of this ultrafilter $\mathcal{UF}(\sigma)(\mathcal{F}) \in \mathcal{UF}(\mathcal{E}(X)),$ when mapped to $\mathcal{E}(X).$ The limit is expressed via the ultrafilter algebra $\mu \circ \tau: \mathcal{UF}(\mathcal{E}(X)) \rightarrow \mathcal{E}(X).$ This means that $(\mu \circ \tau \circ \mathcal{UF}(\sigma))(\mathcal{F}) = h.$ \square

7 Algebras of the expectation monad

This section describes algebras of the expectation monad via barycenters of measures. It leads to an equivalence of categories between ‘observable’ algebras and ‘observable’ convex compact Hausdorff spaces. We shall write **CCH** for the category of these convex compact Hausdorff spaces, with affine continuous maps between them.

We start with the unit interval $[0, 1].$ It is both compact Hausdorff and convex. Hence it carries algebras $\mathcal{UF}([0, 1]) \rightarrow [0, 1]$ and $\mathcal{D}([0, 1]) \rightarrow [0, 1].$ This interval also carries an algebra structure for the expectation monad.

Lemma 25. *The unit interval $[0, 1]$ carries an \mathcal{E} -algebra structure:*

$$\mathcal{E}([0, 1]) \xrightarrow{\text{ev}_{\text{id}}} [0, 1] \quad \text{by} \quad h \longmapsto h(\text{id}_{[0,1]}).$$

More generally, for an arbitrary set A the set of (all) functions $[0, 1]^A$ carries an \mathcal{E} -algebra structure:

$$\mathcal{E}([0, 1]^A) \longrightarrow [0, 1]^A \quad \text{namely} \quad h \longmapsto \lambda a \in A. h(\lambda f \in [0, 1]^A. f(a)).$$

Proof It is easy to see that the evaluation-at-identity map $\text{ev}_{\text{id}}: \mathcal{E}([0,1]) \rightarrow [0,1]$ is an algebra. We explicitly check the details:

$$\begin{aligned}
 (\text{ev}_{\text{id}} \circ \eta)(x) &= \text{ev}_{\text{id}}(\eta(x)) \\
 &= \eta(x)(\text{id}) \\
 &= \text{id}(x) \\
 &= x \\
 (\text{ev}_{\text{id}} \circ \mathcal{E}(\text{ev}_{\text{id}}))(H) &= \text{ev}_{\text{id}}(\mathcal{E}(\text{ev}_{\text{id}})(H)) \\
 &= \mathcal{E}(\text{ev}_{\text{id}})(H)(\text{id}) \\
 &= H(\text{id} \circ \text{ev}_{\text{id}}) \\
 &= H(\lambda k \in \mathcal{E}([0,1]).k(\text{id})) \\
 &= \mu(H)(\text{id}) \\
 &= \text{ev}_{\text{id}}(\mu(H)) \\
 &= (\text{ev}_{\text{id}} \circ \mu)(H).
 \end{aligned}$$

Since Eilenberg-Moore algebras are closed under products, there is also an \mathcal{E} -algebra on $[0,1]^A$. \square

From Corollaries 14 and 19 we know that the underlying set X of an algebra $\mathcal{E}(X) \rightarrow X$ is both compact Hausdorff and convex. Additionally, Lemma 22 says that the algebra $\mathcal{D}(X) \rightarrow X$ is continuous.

We first characterize algebra maps.

Lemma 26. *Consider Eilenberg-Moore algebras $(\mathcal{E}(X) \xrightarrow{\alpha} X)$ and $(\mathcal{E}(Y) \xrightarrow{\beta} Y)$. A function $f: X \rightarrow Y$ is an algebra homomorphism if and only if it is both continuous and affine, that is, iff the following two diagrams commute.*

$$\begin{array}{ccc}
 \mathcal{U}\mathcal{F}(X) & \xrightarrow{\mathcal{U}\mathcal{F}(f)} & \mathcal{U}\mathcal{F}(Y) \\
 \alpha \circ \tau \downarrow & & \downarrow \beta \circ \tau \\
 X & \xrightarrow{f} & Y
 \end{array}
 \qquad
 \begin{array}{ccc}
 \mathcal{D}(X) & \xrightarrow{\mathcal{D}(f)} & \mathcal{D}(Y) \\
 \alpha \circ \sigma \downarrow & & \downarrow \beta \circ \sigma \\
 X & \xrightarrow{f} & Y
 \end{array}$$

Thus, the functor $\text{Alg}(\mathcal{E}) \rightarrow \mathbf{CCH}$ is full and faithful.

Proof If f is an algebra homomorphism, then $f \circ \alpha = \beta \circ \mathcal{E}(f)$. Hence the two rectangles above commute by naturality of τ and σ .

For the (if) part we use the property from Proposition 23 that the maps $\sigma_X: \mathcal{D}(X) \rightarrow \mathcal{E}(X)$ are dense monos. This means that for each map $g: \mathcal{D}(X) \rightarrow Z$ into a Hausdorff space Z there is at most one continuous $h: \mathcal{E}(X) \rightarrow Z$ with $h \circ \sigma = g$. We use this property as follows.

$$\begin{array}{ccc}
 \mathcal{D}(X) & \xrightarrow[\text{dense}]{\sigma} & \mathcal{E}(X) \\
 & \searrow & \downarrow f \circ \alpha \\
 & & Y \\
 & & \downarrow \beta \circ \mathcal{E}(f)
 \end{array}$$

The triangle commutes for both maps since f is affine:

$$f \circ \alpha \circ \sigma = \beta \circ \sigma \circ \mathcal{D}(f) = \beta \circ \mathcal{E}(f) \circ \sigma.$$

Also, both vertical maps are continuous, by Lemma 15. Hence $f \circ \alpha = \beta \circ \mathcal{E}(f)$, so that f is an algebra homomorphism. \square

For convex compact Hausdorff spaces $X, Y \in \mathbf{CCH}$ one (standardly) writes $\mathcal{A}(X, Y) = \mathbf{CCH}(X, Y)$ for the homset of affine continuous functions $X \rightarrow Y$. In light of the previous result, we shall also use this notation $\mathcal{A}(X, Y)$ when X, Y are carriers of \mathcal{E} -algebras, in case the algebra structure is clear from the context.

The next result gives a better understanding of \mathcal{E} -algebras: it shows that such algebras send measures to barycenters (like for instance in [24]).

Proposition 27. *Assume an \mathcal{E} -algebra $\mathcal{E}(X) \xrightarrow{\alpha} X$. For each (algebra) map $q \in \mathcal{A}(X, [0, 1])$ the following diagram commutes.*

$$\begin{array}{ccc} \mathcal{E}(X) & & \\ \alpha \downarrow & \searrow \text{ev}_q = \lambda h. h(q) & \\ X & \xrightarrow{q} & [0, 1] \end{array}$$

This says that $x = \alpha(h) \in X$ is a barycenter for $h \in \mathcal{E}(X)$, in the sense that $q(x) = h(q)$ for all affine continuous $q: X \rightarrow [0, 1]$.

Proof Since $\text{ev}_q = \text{ev}_{\text{id}} \circ \mathcal{E}(q)$ the above triangle can be morphed into a rectangle expressing that q is a map of algebras:

$$\begin{array}{ccc} \mathcal{E}(X) & \xrightarrow{\mathcal{E}(q)} & \mathcal{E}([0, 1]) \\ \alpha \downarrow & \searrow \text{ev}_q & \downarrow \text{ev}_{\text{id}} \\ X & \xrightarrow{q} & [0, 1] \end{array}$$

where ev_{id} is the \mathcal{E} -algebra on $[0, 1]$ from Lemma 25.

Now that we have a reasonable grasp of \mathcal{E} -algebras, namely as convex compact Hausdorff spaces with a barycentric operation, we wish to comprehend how such algebras arise. We first observe that measures in $\mathcal{E}(X)$ in the images of $\mathcal{D}(X) \rightarrow \mathcal{E}(X)$ and $\mathcal{UF}(X) \rightarrow \mathcal{E}(X)$ have barycenters, if X carries appropriate structure.

Lemma 28. *Assume X is a convex compact Hausdorff space, described via \mathcal{D} - and \mathcal{UF} -algebra structures $\text{cv}: \mathcal{D}(X) \rightarrow X$ and $\text{ch}: \mathcal{UF}(X) \rightarrow X$. Then:*

1. $\text{cv}(\varphi) \in X$ is a barycenter of $\sigma(\varphi) \in \mathcal{E}(X)$, for $\varphi \in \mathcal{D}(X)$;
2. $\text{ch}(\mathcal{F}) \in X$ is a barycenter of $\tau(\mathcal{F}) \in \mathcal{E}(X)$, for $\mathcal{F} \in \mathcal{UF}(X)$.

Proof We write $\text{cv}_{[0,1]}: \mathcal{D}([0,1]) \rightarrow [0,1]$ and $\text{ch}_{[0,1]}: \mathcal{UF}([0,1]) \rightarrow [0,1]$ for the convex and compact Hausdorff structure on the unit interval. Then for $q \in \mathcal{A}(X, [0,1])$,

$$\begin{aligned} q(\text{cv}(\varphi)) &= \text{cv}_{[0,1]}(\mathcal{D}(q)(\varphi)) && \text{since } q \text{ is affine} \\ &= \text{cv}_{[0,1]}(\sum_i r_i q(x_i)) && \text{if } \varphi = \sum_i r_i x_i \\ &= \sum_i r_i \cdot q(x_i) \\ &= \sigma(\varphi)(q) \\ q(\text{ch}(\mathcal{F})) &= \text{ch}_{[0,1]}(\mathcal{UF}(q)(\mathcal{F})) && \text{since } q \text{ is continuous} \\ &= \tau(\mathcal{F})(q). \end{aligned}$$

□

We call a convex compact Hausdorff space X *observable* if the collection of affine continuous maps $X \rightarrow [0, 1]$ is jointly monic. This means that $x = x'$ holds if $q(x) = q(x')$ for all $q \in \mathcal{A}(X, [0, 1])$. In a similar manner we call an \mathcal{E} -algebra observable if its underlying convex compact Hausdorff space is observable. This yields full subcategories $\mathbf{CCH}_{\text{obs}} \hookrightarrow \mathbf{CCH}$ and $\mathbf{Alg}_{\text{obs}}(\mathcal{E}) \hookrightarrow \mathbf{Alg}(\mathcal{E})$. By definition, $[0, 1]$ is a cogenerator in these categories $\mathbf{CCH}_{\text{obs}}$ and $\mathbf{Alg}_{\text{obs}}(\mathcal{E})$.

Proposition 29. *Assume X is a convex compact Hausdorff space, described via \mathcal{D} - and \mathcal{UF} -algebra structures $\text{cv}: \mathcal{D}(X) \rightarrow X$ and $\text{ch}: \mathcal{UF}(X) \rightarrow X$.*

1. *Via the Axiom of Choice one obtains a function $\alpha: \mathcal{E}(X) \rightarrow X$ such that $\alpha(h) \in X$ is a barycenter for $h \in \mathcal{E}(X)$; that is, $q(\alpha(h)) = h(q)$ for each $q \in \mathcal{A}(X, [0, 1])$.*
2. *If X is observable, there is precisely one such $\alpha: \mathcal{E}(X) \rightarrow X$; moreover, it is an \mathcal{E} -algebra; and its induced convex and topological structures are the original ones on X , as expressed via the commuting triangles:*

$$\begin{array}{ccc} \mathcal{D}(X) & \xrightarrow{\sigma} & \mathcal{E}(X) & \xleftarrow{\tau} & \mathcal{UF}(X) \\ & \searrow \text{cv} & \downarrow \alpha & & \swarrow \text{ch} \\ & & X & & \end{array}$$

This yields a functor $\mathbf{CCH}_{\text{obs}} \rightarrow \mathbf{Alg}_{\text{obs}}(\mathcal{E})$.

Proof Recall from Corollary 24 that the function $\mu \circ \tau \circ \mathcal{UF}(\sigma): \mathcal{UF}(\mathcal{D}(X)) \rightarrow \mathcal{E}(X)$ is surjective. Using the Axiom of Choice we choose a section $s: \mathcal{E}(X) \rightarrow \mathcal{UF}(\mathcal{D}(X))$ with $\mu \circ \tau \circ \mathcal{UF}(\sigma) \circ s = \text{id}_{\mathcal{E}(X)}$. We now obtain, via the choice of s , a map $\alpha: \mathcal{E}(X) \rightarrow X$ in:

$$\begin{array}{ccc} & & \mathcal{E}(X) \\ & \xleftarrow{s} & \downarrow \alpha = \text{ch} \circ \mathcal{UF}(\text{cv}) \circ s \\ \mathcal{UF}(\mathcal{D}(X)) & \xrightarrow{\mu \circ \tau \circ \mathcal{UF}(\sigma)} & X \\ & \searrow \text{ch} \circ \mathcal{UF}(\text{cv}) & \end{array}$$

We show that $\alpha(h) \in X$ is a barycenter for the measure $h \in \mathcal{E}(X)$. For each $q \in \mathcal{A}(X, [0, 1])$ one has:

$$\begin{aligned} h(q) &= (\mu \circ \tau \circ \mathcal{UF}(\sigma) \circ s)(h)(q) \\ &= \mu((\tau \circ \mathcal{UF}(\sigma) \circ s)(h))(q) \\ &= (\tau \circ \mathcal{UF}(\sigma) \circ s)(h)(\text{ev}_q) \\ &= (\text{ch}_{[0,1]} \circ \mathcal{UF}(\text{ev}_q) \circ \mathcal{UF}(\sigma) \circ s)(h) \\ &= (\text{ch}_{[0,1]} \circ \mathcal{UF}(\lambda \varphi. \text{ev}_q(\sigma(\varphi))) \circ s)(h) \\ &= (\text{ch}_{[0,1]} \circ \mathcal{UF}(\lambda \varphi. \text{cv}_{[0,1]}(\mathcal{D}(q)(\varphi))) \circ s)(h) && \text{see Remark 3.(1)} \\ &= (\text{ch}_{[0,1]} \circ \mathcal{UF}(\text{cv}_{[0,1]} \circ \mathcal{D}(q)) \circ s)(h) \\ &= (\text{ch}_{[0,1]} \circ \mathcal{UF}(q \circ \text{cv}) \circ s)(h) && \text{since } q \text{ is affine} \\ &= (q \circ \text{ch} \circ \mathcal{UF}(\text{cv}) \circ s)(h) && \text{since } q \text{ is continuous} \\ &= (q \circ \alpha)(h) \\ &= q(\alpha(h)). \end{aligned}$$

For the second point, assume the collection of maps $q \in \mathcal{A}(X, [0, 1])$ is jointly monic. Barycenters are then unique, since if both $x, x' \in X$ satisfy $q(x) = h(q) = q(x')$ for all $q \in \mathcal{A}(X, [0, 1])$, then $x = x'$.

Hence the function $\alpha: \mathcal{E}(X) \rightarrow X$ picks barycenters, in a unique manner. We need to prove the algebra equations (see the beginning of Section 2). They are obtained via the barycentric property $q(\alpha(h)) = h(q)$ and observability. First, the equation $\alpha \circ \eta = \text{id}$ holds, since for each $x \in X$ and $q \in \mathcal{A}(X, [0, 1])$,

$$q((\alpha \circ \eta)(x)) = q(\alpha(\eta(x))) = \eta(x)(q) = q(x) = q(\text{id}(x)).$$

In the same way we obtain the equation $\alpha \circ \mu = \alpha \circ \mathcal{E}(\alpha)$. For $H \in \mathcal{E}^2(X)$ we have:

$$\begin{aligned} (q \circ \alpha \circ \mu)(H) &= q(\alpha(\mu(H))) \\ &= \mu(H)(q) \\ &= H(\lambda k \in \mathcal{E}(X).k(q)) \\ &= H(\lambda k \in \mathcal{E}(X).q(\alpha(k))) \\ &= H(q \circ \alpha) \\ &= \mathcal{E}(\alpha)(H)(q) \\ &= q(\alpha(\mathcal{E}(\alpha)(H))) \\ &= (q \circ \alpha \circ \mathcal{E}(\alpha))(H). \end{aligned}$$

We need to show that α induces the original convexity and topological structures. Since barycenters are unique, the equations $\alpha(\sigma(\varphi)) = \text{cv}(\varphi)$ and $\alpha(\tau(\mathcal{F})) = \text{ch}(\mathcal{F})$ follow directly from Lemma 26.

Finally, we need to check functoriality. So assume $f: X \rightarrow Y$ is a map in $\mathbf{CCH}_{\text{obs}}$, and let $\alpha: \mathcal{E}(X) \rightarrow X$ and $\beta: \mathcal{E}(Y) \rightarrow Y$ be the induced algebras obtained by picking barycenters. We need to prove $\beta \circ \mathcal{E}(f) = f \circ \alpha$. Of course we use that Y is observable. For $h \in \mathcal{E}(X)$, one has for all $q \in \mathcal{A}(Y, [0, 1])$,

$$\begin{aligned} q(\beta(\mathcal{E}(f)(h))) &= \mathcal{E}(f)(h)(q) \\ &= h(q \circ f) \\ &= (q \circ f)(\alpha(h)) \\ &= q(f(\alpha(h))). \end{aligned} \quad \square$$

In the approach followed above barycenters are obtained via the Axiom of Choice. Alternatively, they can be obtained via the Hahn-Banach theorem, see for instance [3, Prop. I.2.1].

Theorem 4. *There is an isomorphism $\text{Alg}_{\text{obs}}(\mathcal{E}) \cong \mathbf{CCH}_{\text{obs}}$ between the categories of observable \mathcal{E} -algebras and observable convex compact Hausdorff spaces in a situation:*

$$\begin{array}{ccc} \text{Alg}_{\text{obs}}(\mathcal{E}) & \xrightleftharpoons{\cong} & \mathbf{CCH}_{\text{obs}} \\ \downarrow & & \downarrow \\ \text{Alg}(\mathcal{E}) & \xrightarrow{\text{full \& faithful}} & \mathbf{CCH} \end{array}$$

Proof Obviously the full and faithful functor $\text{Alg}(\mathcal{E}) \rightarrow \mathbf{CCH}$ from Lemma 26 restricts to $\text{Alg}_{\text{obs}}(\mathcal{E}) \rightarrow \mathbf{CCH}_{\text{obs}}$. We need to show that it is an inverse to the functor $\mathbf{CCH}_{\text{obs}} \rightarrow \text{Alg}_{\text{obs}}(\mathcal{E})$ from Proposition 29.(2).

- Starting from an algebra $\alpha: \mathcal{E}(X) \rightarrow X$, we know by Proposition 27 that $\alpha(h)$ is a barycenter for $h \in \mathcal{E}(X)$. The underlying set X is an observable convex compact Hausdorff space. This structure gives by Proposition 29.(2) rise to an algebra $\alpha': \mathcal{E}(X) \rightarrow X$ such that $\alpha'(h)$ is barycenter for h . Since X is observable, barycenters are unique, and so $\alpha'(h) = \alpha(h)$.

- Starting from an observable convex compact Hausdorff space X , we obtain an algebra $\alpha: \mathcal{E}(X) \rightarrow X$ by Proposition 29.(2), whose induced convex and topological structure is the original one. \square

Thus we have characterized *observable* \mathcal{E} -algebras. The characterization of arbitrary \mathcal{E} -algebras remains open. Possibly the functor $\text{Alg}(\mathcal{E}) \rightarrow \mathbf{CCH}$ is (also) an isomorphism. For the duality in the next section the characterization of observable algebras is sufficient.

We conclude this section with some further results on observability. We show that *observable* convex compact Hausdorff spaces can be considered as part of an enveloping locally convex topological vector space. This is the more common way of describing such structures, see e.g. [3, 4].

Lemma 30. *Let X be a convex compact Hausdorff space; write $A = \mathcal{A}(X, [0, 1])$. If X is observable, there is (by definition) an injection:*

$$X \xrightarrow{x \mapsto \text{ev}_x} [0, 1]^A \quad \text{where} \quad \text{ev}_x = \lambda q \in A. q(x).$$

1. *This map is both affine and continuous—where $[0, 1]^A$ carries the product topology.*
2. *Hence if X is the carrier of an \mathcal{E} -algebra, this map is a homomorphism of algebras—where $[0, 1]^A$ carries the \mathcal{E} -algebra structure from Lemma 25.*

Proof The second point follows from the first one via Lemma 26, so we only do point 1. Obviously, $x \mapsto \text{ev}_x$ is affine. In order to see that it is also continuous, assume we have a basic open set $U \subseteq [0, 1]^A$. The product topology says that U is of the form $U = \prod_{q \in A} U_q$, with $U_q \subseteq [0, 1]$ open and $U_q \neq [0, 1]$ for only finitely many q 's, say q_1, \dots, q_n . Thus:

$$\text{ev}^{-1}(U) = \{x \mid q_1(x) \in U_{q_1} \wedge \dots \wedge q_n(x) \in U_{q_n}\} = \bigcap_i q_i^{-1}(U_{q_i}).$$

This intersection of opens is clearly an open set of X . \square

Proposition 31. *Each observable convex compact Hausdorff space $X \in \mathbf{CCH}_{\text{obs}}$ occurs as subspace of a locally convex topological vector space, namely via:*

$$X \xrightarrow{\quad} [0, 1]^A \hookrightarrow \mathbb{R}^A$$

where $A = \mathcal{A}(X, [0, 1])$ like in the previous lemma.

Proof It is standard that the vector space \mathbb{R}^A with product topology is locally convex. We write $\mathcal{O}(X)$ for the original compact Hausdorff topology on X and $\mathcal{O}_i(X)$ for the topology induced by the injection $X \hookrightarrow \mathbb{R}^A$. The latter contains basic opens of the form $q_1^{-1}(U_1) \cap \dots \cap q_n^{-1}(U_n)$ for $q_i \in A = \mathcal{A}(X, [0, 1])$ and $U_i \subseteq \mathbb{R}$ open. Thus $\mathcal{O}_i(X) \subseteq \mathcal{O}(X)$. We wish to use Lemma 2 to prove the equality $\mathcal{O}_i(X) = \mathcal{O}(X)$. Since $\mathcal{O}(X)$ is compact we only need to show that the induced topology $\mathcal{O}_i(X)$ is Hausdorff. This is easy since X is observable: if $x \neq x'$ for $x, x' \in X$, then there is a $q \in \mathcal{A}(X, [0, 1])$ with $q(x) \neq q(x')$ in $[0, 1] \subseteq \mathbb{R}$. Hence there are disjoint opens $U, U' \subseteq \mathbb{R}$ with $q(x) \in U$ and $q(x') \in U'$. Thus $q^{-1}(U), q^{-1}(U') \in \mathcal{O}_i(X)$ are disjoint (induced) opens containing x, x' . \square

8 Algebras of the expectation monad and effect modules

In this section we relate algebras of the expectation monad to effect modules via a (dual) adjunction. By suitable restriction this adjunction gives rise to an equivalence (duality) between observable \mathcal{E} -algebras

and Banach effect modules. Via a combination with Theorem 4 we then get our main duality result (see Theorem 5 below).

We first return to Section 2.4. When we apply Lemma 4 to the adjunctions involving convex sets and effect modules in Diagram (8), the (upper) comparison functor in (7) says that each effect module $M \in \mathbf{EMod}$ gives rise to a \mathcal{E} -algebra on the homset $\mathbf{EMod}(M, [0, 1])$, namely:

$$\begin{array}{ccc} \mathcal{E}(\mathbf{EMod}(M, [0, 1])) & \xrightarrow{\alpha_M} & \mathbf{EMod}(M, [0, 1]) \\ h \vdash & \longrightarrow & \lambda y \in M. h(\lambda k \in \mathbf{EMod}(M, [0, 1]).k(y)) \end{array} \quad (13)$$

In order to simplify notation we write:

$$\begin{array}{ll} S_M & = \mathbf{EMod}(M, [0, 1]) & \text{for the set of "states" of } M \\ \alpha_M(h)(y) & = h(\text{ev}_y) & \text{where } \text{ev}_y = \lambda k. k(y). \end{array}$$

Thus, Diagram (7) becomes:

$$\begin{array}{ccc} \mathbf{EMod}^{\text{op}} & \xrightarrow{S_{(-)}} & \text{Alg}(\mathcal{E}) \\ & \searrow^{S_{(-)}} & \swarrow^{(-) \circ \sigma} \\ & & \text{Alg}(\mathcal{D}) = \mathbf{Conv} \\ & & \downarrow \\ & & \mathbf{Sets} \end{array} \quad (14)$$

Proposition 32. Consider for an effect module M , the \mathcal{E} -algebra structure (13) on the homset of states $S_M = \mathbf{EMod}(M, [0, 1])$.

1. The induced topology is like the weak-* topology, with subbasic opens

$$\square_s(y) = \{g \in S_M \mid g(y) < s\},$$

where $y \in M$ and $s \in [0, 1] \cap \mathbb{Q}$. It thus generalizes the topology on $\mathcal{E}(X) = \mathbf{EMod}([0, 1]^X, [0, 1])$ in Proposition 16.

2. This convex compact Hausdorff space $\mathbf{EMod}(M, [0, 1])$ is observable.

Hence the states functor $S_{(-)}$ at the top of (14) restricts to $\mathbf{EMod}^{\text{op}} \rightarrow \text{Alg}_{\text{obs}}(\mathcal{E})$.

Proof The proof of Proposition 16 generalizes directly from an effect module of the form $[0, 1]^X$ to an arbitrary effect module M .

Next suppose $f, g \in S_M = \mathbf{EMod}(M, [0, 1])$ satisfy $q(f) = q(g)$ for each affine continuous $q: S_M \rightarrow [0, 1]$. This applies especially to the functions $\text{ev}_y = \lambda k. k(y)$, which are both continuous and affine. Hence $f(y) = \text{ev}_y(f) = \text{ev}_y(g) = g(y)$ for each $y \in M$, and thus $f = g$. \square

We can also form a functor $\text{Alg}(\mathcal{E}) \rightarrow \mathbf{EMod}^{\text{op}}$, in the reverse direction in (14), by “homming” into the unit interval $[0, 1]$. Recall that this interval carries an \mathcal{E} -algebra, identified in Lemma 25 as evaluation-at-identity ev_{id} . For an algebra $\alpha: \mathcal{E}(X) \rightarrow X$ we know that the algebra homomorphisms $X \rightarrow [0, 1]$ are precisely the affine continuous maps $X \rightarrow [0, 1]$, by Lemma 26. We shall be a bit sloppy in our notation

and write the homset $\mathit{Alg}(\mathcal{E})(\alpha, \mathit{ev}_{\text{id}}) = \{q: X \rightarrow [0, 1] \mid q \circ \alpha = \mathit{ev}_{\text{id}} \circ \mathcal{E}(q)\}$ of algebras map in various ways, namely as:

$$\left\{ \begin{array}{ll} \mathit{Alg}(\mathcal{E})(\alpha, [0, 1]) & \text{leaving the algebra structure } \mathit{ev}_{\text{id}} \text{ on } [0, 1] \text{ implicit,} \\ \mathit{Alg}(\mathcal{E})(X, [0, 1]) & \text{also leaving the algebra structure } \alpha \text{ on } X \text{ implicit,} \\ \mathcal{A}(X, [0, 1]) & \text{as set of affine continuous functions, via Lemma 26.} \end{array} \right.$$

Proposition 33. *The states functor $S_{(-)} = \mathbf{EMod}(-, [0, 1]): \mathbf{EMod}^{\text{op}} \rightarrow \mathit{Alg}(\mathcal{E})$ from (14) has a left adjoint, also given by “homming into $[0, 1]$ ”:*

$$\begin{array}{ccc} & \xrightarrow{S_{(-)} = \mathbf{EMod}(-, [0, 1])} & \\ \mathbf{EMod}^{\text{op}} & \xleftrightarrow{\quad \top \quad} & \mathit{Alg}(\mathcal{E}) \\ & \xleftarrow{\mathit{Alg}(\mathcal{E})(-, [0, 1])} & \end{array}$$

Proof Assume an \mathcal{E} -algebra $\alpha: \mathcal{E}(X) \rightarrow X$. We should first check that the set of affine continuous functions is a sub-effect module: $\mathit{Alg}(\mathcal{E})(\alpha, [0, 1]) = \mathcal{A}(X, [0, 1]) \hookrightarrow [0, 1]^X$. The top and bottom elements $1 = \lambda y. 1$ and $0 = \lambda y. 0$ are clearly in $\mathcal{A}(X, [0, 1])$. Also, $\mathcal{A}(X, [0, 1])$ is closed under (partial) sums \oplus and scalar multiplication with $r \in [0, 1]$. Next, if we have a map of algebras $g: X \rightarrow Y$, from $\mathcal{E}(X) \xrightarrow{\alpha} X$ to $\mathcal{E}(Y) \xrightarrow{\beta} Y$. Then we get a map of effect modules $g^* = (-) \circ g: \mathcal{A}(X, [0, 1]) \rightarrow \mathcal{A}(Y, [0, 1])$. This is easy because g is itself affine and continuous, by Lemma 26.

We come to the adjunction $\mathit{Alg}(\mathcal{E})(-, [0, 1]) \dashv \mathbf{EMod}(-, [0, 1])$. For $M \in \mathbf{EMod}$ and $(\mathcal{E}(X) \xrightarrow{\alpha} X) \in \mathit{Alg}(\mathcal{E})$ there is a bijective correspondence:

$$\begin{array}{ccc} \mathcal{E}(X) & \xrightarrow{\mathcal{E}(f)} & \mathcal{E}(S_M) \\ \alpha \downarrow & & \downarrow \alpha_M \\ X & \xrightarrow{f} & S_M \\ \hline M & \xrightarrow{g} & \mathit{Alg}(\mathcal{E})(\alpha, [0, 1]) = \mathcal{A}(X, [0, 1]) \end{array}$$

We proceed as follows.

- Given an algebra map $f: X \rightarrow S_M = \mathbf{EMod}(M, [0, 1])$ as indicated, define $\bar{f}: M \rightarrow \mathcal{A}(X, [0, 1])$ by $\bar{f}(y)(x) = f(x)(y)$. We leave it to the reader to check that \bar{f} is a map of effect modules, but we do verify that $\bar{f}(y)$ is an algebra map $X \rightarrow [0, 1]$; so for $h \in \mathcal{E}(X)$,

$$\begin{aligned} (\mathit{ev}_{\text{id}} \circ \mathcal{E}(\bar{f}(y)))(h) &= \mathcal{E}(\bar{f}(y))(h)(\text{id}) \\ &= h(\text{id} \circ \bar{f}(y)) \\ &= h(\bar{f}(y)) \\ &= h(\lambda x. f(x)(y)) \\ &= h((\lambda k. k(y)) \circ f) \\ &= \mathcal{E}(f)(h)(\lambda k. k(y)) \\ &= \alpha_M(\mathcal{E}(f)(h))(y) \\ &= f(\alpha(h))(y) \quad \text{since } f \text{ is an algebra map} \\ &= (\bar{f}(y) \circ \alpha)(h). \end{aligned}$$

- Now assume we have a map of effect modules $g: M \rightarrow \mathbf{Alg}(\mathcal{E})(\alpha, [0, 1]) = \mathcal{A}(X, [0, 1])$. We turn it into a map of algebras $\bar{g}: X \rightarrow \mathcal{S}_M$, again by twisting arguments: $\bar{g}(x)(y) = g(y)(x)$. Via calculations as above one checks that \bar{g} is a map of algebras.

Clearly $\bar{\bar{f}} = f$ and $\bar{\bar{g}} = g$. \square

Let's write the unit and counit of this adjunction as η^\dagger and ε^\dagger , in order make a distinction with the unit η of the monad \mathcal{E} , see below. The unit and counit are maps:

$$\begin{array}{ccc} X & \xrightarrow{\eta^\dagger} & \mathbf{EMod}\left(\mathbf{Alg}(\mathcal{E})(\alpha, [0, 1]), [0, 1]\right) & \text{in } \mathbf{Alg}(\mathcal{E}) \\ \mathbf{Alg}(\mathcal{E})\left(\mathbf{EMod}(M, [0, 1]), [0, 1]\right) & \xrightarrow{\varepsilon^\dagger} & M & \text{in } \mathbf{EMod}^{\text{op}} \end{array}$$

both given by point evaluation:

$$\eta^\dagger(x) = \lambda f \in \mathcal{A}(X, [0, 1]). f(x) \quad \varepsilon^\dagger(y) = \lambda g \in \mathbf{EMod}(M, [0, 1]). g(y) \quad (15)$$

The unit of this adjunction is related to the unit of the monad \mathcal{E} , written explicitly as $\eta^\mathcal{E}$ in the following way.

$$\begin{array}{ccc} & & \mathbf{EMod}(\mathcal{A}(X, [0, 1]), [0, 1]) \\ & \nearrow \eta^\dagger & \uparrow \\ X & & \mathcal{E}(X) = \mathbf{EMod}([0, 1]^X, [0, 1]) \\ & \searrow \eta^\mathcal{E} & \end{array}$$

Lemma 34. Consider the unit η^\dagger in (15) of the adjunction from Proposition 33, at an algebra $\mathcal{E}(X) \xrightarrow{\alpha} X$.

1. This unit is injective if and only if X is observable.
2. In fact, it is an isomorphism if and only if X is observable.

Hence the adjunction $\mathbf{Alg}(\mathcal{E}) \rightleftarrows \mathbf{EMod}^{\text{op}}$ from Proposition 33 restricts to a coreflection $\mathbf{Alg}_{\text{obs}}(\mathcal{E}) \rightleftarrows \mathbf{EMod}^{\text{op}}$.

Proof The first statement holds by definition of ‘observable’. So for the second statement it suffices to assume that X is observable and show that $\eta^\dagger: X \rightarrow \mathbf{EMod}(\mathcal{A}(X, [0, 1]), [0, 1])$ is surjective. This unit is by construction affine and continuous. Hence its image in the space $\mathbf{EMod}(\mathcal{A}(X, [0, 1]), [0, 1])$ is compact, and thus closed.

We are done if η^\dagger is dense. Thus we assume a non-empty open set $U \subseteq \mathbf{EMod}(\mathcal{A}(X, [0, 1]), [0, 1])$ and need to prove that there is an $y \in X$ with $\eta^\dagger(y) \in U$. By Proposition 32 we may assume $U = \square_{s_1}(q_1) \cap \dots \cap \square_{s_n}(q_n)$, for $q_i \in \mathcal{A}(X, [0, 1])$ and $s_i \in [0, 1] \cap \mathbb{Q}$. Thus we may assume a map of effect modules $h: \mathcal{A}(X, [0, 1]) \rightarrow [0, 1]$ inhabiting all these \square 's. Hence $h(q_i) < s_i$.

Since $\iota: \mathcal{A}(X, [0, 1]) \hookrightarrow [0, 1]^X$ is a sub-effect module, by extension, see Proposition 11, we get a map of effect modules $h': [0, 1]^X \rightarrow [0, 1]$ with $h' \circ \iota = h$. Thus, we can take the inverse image of the open set U along the continuous map:

$$\mathcal{E}(X) = \mathbf{EMod}([0, 1]^X, [0, 1]) \xrightarrow{(-) \circ \iota} \mathbf{EMod}(\mathcal{A}(X, [0, 1]), [0, 1])$$

The resulting open set is:

$$\begin{aligned} V &\stackrel{\text{def}}{=} ((-) \circ \iota)^{-1}(U) = \{k \in \mathcal{E}(X) \mid k \circ \iota \in U\} \\ &= \{k \in \mathcal{E}(X) \mid \forall i. k(\iota(q_i)) < s_i\}. \end{aligned}$$

This subset $V \subseteq \mathcal{E}(X)$ contains h' and is thus non-empty. Since $\sigma: \mathcal{D}(X) \rightarrow \mathcal{E}(X)$ is dense, by Proposition 23, there is a distribution $\varphi = (\sum_j r_j x_j) \in \mathcal{D}(X)$ with $\sigma(\varphi) \in V$. We take $x = \sum_j r_j x_j \in X$ to be the interpretation of φ in X , using that X is convex. We claim $\eta^{-1}(x) \in U$. Indeed, $\eta^{-1}(x) \in \square_s(q_i)$, for each i , since:

$$\begin{aligned} \eta^{-1}(x)(q_i) &= q_i(x) = q_i(\sum_j r_j x_j) \\ &= \sum_j r_j \cdot q_i(x_j) && \text{since } q_i: X \rightarrow [0, 1] \text{ is affine} \\ &= \sigma(\varphi)(\iota(q_i)) \\ &< s_i && \text{since } \sigma(\varphi) \in V. \end{aligned} \quad \square$$

We turn to the counit (15) of the adjunction in Proposition 33.

Lemma 35. *For an effect module M consider the counit $\varepsilon^{-1}: M \rightarrow \mathcal{A}(S_M, [0, 1])$, where $S_M = \mathbf{EMod}(M, [0, 1])$ is the convex compact Hausdorff space of states.*

1. *The effect module $\mathcal{A}(S_M, [0, 1])$ is “Banach”, i.e. complete.*
2. *The counit map ε^{-1} is a dense embedding of M into this Banach effect module $\mathcal{A}(S_M, [0, 1])$.*
3. *Hence it is an isomorphism if and only if M is a Banach effect module.*

Proof Completeness of $\mathcal{A}(S_M, [0, 1])$ is inherited from $[0, 1]$, since its norm is the supremum norm, like in Example 3.

For the second point we use the corresponding result for order unit spaces, via the equivalence $\hat{\mathcal{T}}_o: \mathbf{AEMod} \xrightarrow{\cong} \mathbf{OUS}$ from Proposition 6. If (V, u) is an order unit space then it is well known (see [4]) that the evaluation map $\theta: V \rightarrow \mathcal{A}(S, \mathbb{R})$ is a dense embedding. Here $S = \mathbf{OUS}(V, \mathbb{R})$ is the state space of V . However if we take V to be the totalization $\hat{\mathcal{T}}_o(M)$ of M , then θ is precisely $\hat{\mathcal{T}}_o(\varepsilon^{-1})$, since:

$$\begin{aligned} \hat{\mathcal{T}}_o(M) = V &\xrightarrow[\text{dense}]{\theta} \mathcal{A}(S, \mathbb{R}) &= \mathcal{A}(\mathbf{OUS}(V, \mathbb{R}), \mathbb{R}) \\ &&\cong \mathcal{A}(\mathbf{EMod}(M, [0, 1]), \mathbb{R}) \\ &&\cong \hat{\mathcal{T}}_o\left(\mathcal{A}(\mathbf{EMod}(M, [0, 1]), [0, 1])\right). \end{aligned}$$

and both θ and $\hat{\mathcal{T}}_o(\varepsilon^{-1})$ are the evaluation map.

For the third point, one direction is easy: if the counit is an isomorphism, then M is isomorphic to the complete effect module $\mathcal{A}(S_M, [0, 1])$, and thus complete itself. In the other direction, denseness of $M \rightarrow \mathcal{A}(S_M, [0, 1])$ means that each $h \in \mathcal{A}(S_M, [0, 1])$ can be expressed as limit $h = \lim_n \varepsilon^{-1}(x_n)$ of elements $x_n \in M$. But if M is complete, there is already a limit $x = \lim_n x_n \in M$. Hence $\varepsilon^{-1}(x) = h$, making ε^{-1} an isomorphism. \square

Combining lemmas 35 and 34 gives us the main result of this paper.

Theorem 5. *The adjunction $\mathbf{Alg}(\mathcal{E}) \rightleftarrows \mathbf{EMod}^{\text{op}}$ from Proposition 33 restricts to a duality $\mathbf{Alg}_{\text{obs}}(\mathcal{E}) \simeq \mathbf{BEMod}^{\text{op}}$ between observable \mathcal{E} -algebras and Banach effect modules. In combination with Theorem 4 we obtain:*

$$\mathbf{CCH}_{\text{obs}} \cong \mathbf{Alg}_{\text{obs}}(\mathcal{E}) \simeq \mathbf{BEMod}^{\text{op}}. \quad \square$$

This result can be seen as a probabilistic version of fundamental results of Manes (Theorem 1) and Gelfand (Theorem 2).

9 A new formulation of Gleason's theorem

Gleason's theorem in quantum mechanics says that every state on a Hilbert space of dimension three or greater corresponds to a density matrix [17]. In this section we introduce a reformulation of Gleason's theorem, and prove the equivalence via Banach effect modules (esp. Lemma 35). This reformulation says that effects are the free effect module on projections. In formulas: $\text{Ef}(\mathcal{H}) \cong [0, 1] \otimes \text{Pr}(\mathcal{H})$, for a Hilbert space \mathcal{H} .

Gleason's theorem is not easy to prove (see *e.g.* [14]). Even proofs using elementary methods are quite involved [12]. A state on a Hilbert space \mathcal{H} is a certain probability distribution on the projections $\text{Pr}(\mathcal{H})$ of \mathcal{H} . These projections $\text{Pr}(\mathcal{H})$ form an orthomodular lattice, and thus an effect algebra [15, 21]. In our current context these are exactly the effect algebra maps $\text{Pr}(\mathcal{H}) \rightarrow [0, 1]$. So Gleason's (original) theorem states:

$$\mathbf{EA}(\text{Pr}(\mathcal{H}), [0, 1]) \cong \text{DM}(\mathcal{H}). \quad (16)$$

This isomorphism, from right to left, sends a density matrix M to the map $p \mapsto \text{tr}(Mp)$ —where tr is the trace map acting on operators.

Recall that $\text{Ef}(\mathcal{H})$ is the set of positive operators on \mathcal{H} below the identity. It is a Banach effect module. One can also consider the effect module maps $\text{Ef}(\mathcal{H}) \rightarrow [0, 1]$. For these maps there is a “lightweight” version of Gleason's theorem:

$$\mathbf{EMod}(\text{Ef}(\mathcal{H}), [0, 1]) \cong \text{DM}(\mathcal{H}). \quad (17)$$

This isomorphism involves the same trace computation as (16). This statement is significantly easier to prove than Gleason's theorem itself, see [11].

Because Gleason's original theorem (16) is so much harder to prove than the lightweight version (17) one could wonder what Gleason's theorem states that Gleason light doesn't. In Theorem 6 we will show that the difference amounts exactly to the statement:

$$[0, 1] \otimes \text{Pr}(\mathcal{H}) \cong \text{Ef}(\mathcal{H}), \quad (18)$$

where \otimes is the tensor of effect algebras (see [21]). A general result, see [29, VII,§4], says that the tensor product $[0, 1] \otimes \text{Pr}(\mathcal{H})$ is the free effect module on $\text{Pr}(\mathcal{H})$.

The following table gives an overview of the various formulations of Gleason's theorem.

Description	Formulation	Label
original Gleason, for projections	$\mathbf{EA}(\text{Pr}(\mathcal{H}), [0, 1]) \cong \text{DM}(\mathcal{H})$	(16)
lightweight version, for effects	$\mathbf{EMod}(\text{Ef}(\mathcal{H}), [0, 1]) \cong \text{DM}(\mathcal{H})$	(17)
effects as free module on projections	$[0, 1] \otimes \text{Pr}(\mathcal{H}) \cong \text{Ef}(\mathcal{H})$	(18)

In this section we shall prove (16) \iff (18), in presence of (17), see Theorem 6. Since (16) is true, for dimension ≥ 3 , the same then holds for (18).

We first prove a general result based on the duality from the previous section. There we used the shorthand S_M for the algebra of states $\mathbf{EMod}(M, [0, 1])$. We now extend this notation to effect algebras

and write $S_D = \mathbf{EA}(D, [0, 1])$, where D is an effect algebra. We recall from Section 3 that S_D is a convex set. We will topologize it via the weakest topology that makes all point evaluations continuous.

Since the tensor product $[0, 1] \otimes D$ of effect algebras is the free effect module on D it follows that there is an isomorphism:

$$\begin{array}{ccc} \mathbf{EA}(D, [0, 1]) & \xrightarrow[\cong]{\widehat{(-)}} & \mathbf{EMod}([0, 1] \otimes D, [0, 1]) & \text{with } \widehat{f}(s \otimes x) = s \cdot f(x) \\ \parallel & & \parallel & \\ S_D & & S_{[0,1] \otimes D} & \end{array} \quad (19)$$

Lemma 36. *The mapping $\widehat{(-)}: S_D \xrightarrow{\cong} S_{[0,1] \otimes D}$ in (19) is an affine homeomorphism.*

Proof We only show that $\widehat{(-)}$ is a homeomorphism. For an arbitrary element $\bigoplus_i r_i \otimes x_i \in [0, 1] \otimes D$ we have in $[0, 1]$,

$$\widehat{f}(\bigoplus_i r_i \otimes x_i) = \sum_i r_i \cdot f(x_i).$$

Since the maps $f \mapsto f(x_i)$ are continuous by definition of the topology on S_D , and since addition and multiplication on $[0, 1]$ are continuous, it follows that $f \mapsto \widehat{f}(\bigoplus_i r_i \otimes x_i)$ is continuous. Hence by definition of the topology on $S_{[0,1] \otimes D}$ we see that the mapping $\widehat{(-)}$ is continuous.

Similarly, the inverse, say written as $\widetilde{(-)}: S_{[0,1] \otimes D} \rightarrow S_D$, is continuous. It is given by $\widetilde{k}(x) = k(1 \otimes x)$. Continuity again follows from the definition of the topology on $S_{[0,1] \otimes D}$. \square

Lemma 37. *Suppose $f: D \rightarrow E$ is an effect algebra map between an effect algebra D and a Banach effect module E such that the following hold.*

- *The induced map $\widehat{f}: [0, 1] \otimes D \rightarrow E$ is surjective—obtained like in (19) as $\widehat{f}(s \otimes x) = s \bullet f(x)$.*
- *The “precompose with f ” map $- \circ f: S_E \rightarrow S_D$ is a homeomorphism.*

The map \widehat{f} is then an isomorphism between $[0, 1] \otimes D$ and E .

Proof Using Lemma 35, there are for the Banach effect module E and for the (free) effect module $[0, 1] \otimes D$, maps ε_E and ϕ_D in:

$$\begin{array}{ccc} & [0, 1] \otimes D & \xrightarrow[\text{dense}]{\varepsilon_{[0,1] \otimes D}} \mathcal{A}(S_{[0,1] \otimes D}, [0, 1]) \\ E \xrightarrow[\cong]{\varepsilon_E} \mathcal{A}(S_E, [0, 1]) & & \searrow \phi_D \quad \cong \downarrow h \mapsto h(\widehat{(-)}) \\ & & \mathcal{A}(S_D, [0, 1]) \end{array}$$

The operation $\widehat{(-)}$ on the right is as in (19). We claim that the following diagram commutes.

$$\begin{array}{ccc} [0, 1] \otimes D & \xrightarrow[\text{dense}]{\phi_D} & \mathcal{A}(S_D, [0, 1]) \\ \widehat{f} \downarrow & & \cong \downarrow k \mapsto k(- \circ f) \\ E & \xrightarrow[\varepsilon_E]{\cong} & \mathcal{A}(S_E, [0, 1]) \end{array}$$

If this is indeed true the map \widehat{f} is an embedding followed by two isomorphism and therefore injective (and thus an isomorphism). To prove the claim, we assume $\bigoplus_i r_i \otimes x_i \in [0, 1] \otimes D$ and $g \in S_E$ and compute

first the east-south direction:

$$\begin{aligned}
((\lambda k. k(- \circ f)) \circ \phi_D)(\bigotimes_i r_i \otimes x_i)(g) &= \phi_D(\bigotimes_i r_i \otimes x_i)(g \circ f) \\
&= \varepsilon_{[0,1] \otimes D}(\bigotimes_i r_i \otimes x_i)(\widehat{(g \circ f)}) \\
&= \widehat{(g \circ f)}(\bigotimes_i r_i \otimes x_i) \\
&= \sum_i r_i \cdot g(f(x_i)) \\
&= g(\bigotimes_i r_i \bullet f(x_i)) \quad \text{since } g \text{ is affine} \\
&= g(\widehat{f}(\bigotimes_i r_i \otimes x_i)) \\
&= \varepsilon_E(\widehat{f}(\bigotimes_i r_i \otimes x_i))(g) \\
&= (\varepsilon_E \circ \widehat{f})(\bigotimes_i r_i \otimes x_i)(g). \quad \square
\end{aligned}$$

As a consequence we obtain the isomorphism (18). We will show next that it is equivalent to Gleason's (original) theorem.

Theorem 6. (16) \iff (18), in presence of (17).

That is, using Gleason light (17) the following statements are equivalent.

(16): $\mathbf{EA}(\text{Pr}(\mathcal{H}), [0, 1]) \cong \text{DM}(\mathcal{H})$, i.e. Gleason's original theorem;

(18): The canonical map $[0, 1] \otimes \text{Pr}(\mathcal{H}) \rightarrow \text{Ef}(\mathcal{H})$ is an isomorphism.

Proof Assuming $[0, 1] \otimes \text{Pr}(\mathcal{H}) \xrightarrow{\cong} \text{Ef}(\mathcal{H})$ we get Gleason's theorem:

$$\begin{aligned}
\mathbf{EA}(\text{Pr}(\mathcal{H}), [0, 1]) &\cong \mathbf{EMod}([0, 1] \otimes \text{Pr}(\mathcal{H}), [0, 1]) && \text{by freeness} \\
&\cong \mathbf{EMod}(\text{Ef}(\mathcal{H}), [0, 1]) && \text{by assumption} \\
&\cong \text{DM}(\mathcal{H}) && \text{by Gleason light (17)}.
\end{aligned}$$

In the other direction assume $S_{\text{Pr}(\mathcal{H})} = \mathbf{EA}(\text{Pr}(\mathcal{H}), [0, 1]) \cong \text{DM}(\mathcal{H})$. We apply the previous lemma to the inclusion $f: \text{Pr}(\mathcal{H}) \hookrightarrow \text{Ef}(\mathcal{H})$. Then indeed:

- the induced map $\widehat{f}: [0, 1] \otimes \text{Pr}(\mathcal{H}) \rightarrow \text{Ef}(\mathcal{H})$ is surjective: each effect $A \in \text{Ef}(\mathcal{H})$ can be written as convex combination of projections $A = \sum_i r_i P_i$, via the spectral theorem.
- the precomposition $- \circ f: S_{\text{Ef}(\mathcal{H})} \rightarrow S_{\text{Pr}(\mathcal{H})}$ is an isomorphism since:

$$S_{\text{Ef}(\mathcal{H})} \stackrel{(17)}{\cong} \text{DM}(\mathcal{H}) \stackrel{(16)}{\cong} S_{\text{Pr}(\mathcal{H})}.$$

Since both these isomorphisms involve the same trace computation, this isomorphism is in fact the map induced by the inclusion $f: \text{Pr}(\mathcal{H}) \hookrightarrow \text{Ef}(\mathcal{H})$.

Thus the conditions of Lemma 37 are met and so $[0, 1] \otimes \text{Pr}(\mathcal{H}) \cong \text{Ef}(\mathcal{H})$. \square

10 The expectation monad for program semantics

This paper uses the expectation monad $\mathcal{E}(X) = \mathbf{EMod}([0, 1]^X, [0, 1])$ in characterization and duality results for convex compact Hausdorff spaces. Elements of $\mathcal{E}(X)$ are characterized as (finitely additive) measures (see esp. Theorem 3). The way the monad \mathcal{E} is defined, via the adjunction $\mathbf{Sets} \rightleftarrows \mathbf{EMod}^{\text{op}}$, is new. This approach deals effectively with the rather subtle preservation properties for maps $h \in \mathcal{E}(X) =$

$\mathbf{EMod}([0, 1]^X, [0, 1])$, namely preservation of the structure of effect modules (with non-expansiveness, and thus continuity, as consequence, see Lemma 7).

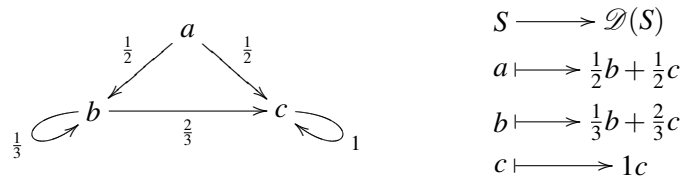
Measures have been captured via monads before, first by Giry [16] following ideas of Lawvere. Such a description in terms of monads is useful to provide semantics for probabilistic programs [26, 23, 30, 31]. The term ‘expectation monad’ seems to occur first in [33], where it is formalized in Haskell. Such a formalization in a functional language is only partial, because the relevant equations and restrictions are omitted, so that there is not really a difference with the continuation monad $X \mapsto [0, 1]^{([0, 1]^X)}$. A formalization of what is also called ‘expectation monad’ in the theorem prover Coq occurs in [5] and is more informative. It involves maps $h: [0, 1]^X \rightarrow [0, 1]$ which are required to be monotone, continuous, linear (preserving partial sum \odot and scalar multiplication) and compatible with inverses—meaning $h(1 - p) \leq 1 - h(p)$. This comes very close to the notion of homomorphism of effect module that is used here, but effect modules themselves are not mentioned in [5]. This Coq formalization is used for instance in the semantics of game-based programs for the certification of cryptographic proofs in [9] (see [35] for an overview of this line of work). Finally, in [25] a monad is used of maps $h: [0, 1]^X \rightarrow [0, 1]$ that are (Scott) continuous and sublinear—*i.e.* $h(p \odot q) \leq h(p) \odot h(q)$, and $h(r \cdot p) = r \cdot h(p)$.

The definition $\mathcal{E}(X) = \mathbf{EMod}([0, 1]^X, [0, 1])$ of the expectation monad that is used here has good credentials to be the right definition, because:

- The monad \mathcal{E} arises in a systematic (not *ad hoc*) manner, namely via the composable adjunctions (8).
- The sets $\mathcal{E}(X)$ as defined here form a stable collection, in the sense that its elements can be characterized in several other ways, namely as finitely additive measures (Theorem 3) or as maps of partially ordered vector spaces with strong unit (via Proposition 5, see Remark 2 (3)).
- Its (observable) algebras correspond to well-behaved mathematical structures (convex compact Hausdorff spaces), via the isomorphism $\mathbf{Alg}_{\text{obs}}(\mathcal{E}) \cong \mathbf{CCH}_{\text{obs}}$ in Theorem 4.
- There is a dual equivalence $\mathbf{Alg}_{\text{obs}}(\mathcal{E}) \simeq \mathbf{BEMod}^{\text{op}}$ that can be exploited for program logics, see [13].

It is thus worthwhile to systematically develop a program semantics and logic based on the expectation monad and its duality. This is a project on its own. We conclude by sketching some ingredients, focusing on the program constructs that can be used.

First we include a small example. Suppose we have a set of states $S = \{a, b, c\}$ with probabilistic transitions between them as described on the left below.



On the right is the same system described as a function, namely as coalgebra of the distribution monad \mathcal{D} . It maps each state to the corresponding discrete probability distribution. We can also describe the same system as coalgebra $S \rightarrow \mathcal{E}(S)$ of the expectation monad, via the map $\mathcal{D} \mapsto \mathcal{E}$. Then it looks as

follows:

$$\begin{aligned}
S &\longrightarrow \mathcal{E}(S) \\
a &\longmapsto \lambda q \in [0, 1]^S. \frac{1}{2}q(b) + \frac{1}{2}q(c) \\
b &\longmapsto \lambda q \in [0, 1]^S. \frac{1}{3}q(b) + \frac{2}{3}q(c) \\
c &\longmapsto \lambda q \in [0, 1]^S. q(c)
\end{aligned}$$

Thus, via the \mathcal{E} -monad we obtain a probabilistic continuation style semantics.

Let's consider this from a more general perspective. Assume we now have an arbitrary, unspecified set of states S , for which we consider programs as functions $S \rightarrow \mathcal{E}(S)$, *i.e.* as Kleisli endomaps or as \mathcal{E} -coalgebras. In a standard way the monad structure provides a monoid structure on these maps $S \rightarrow \mathcal{E}(S)$ for sequential composition, with the unit $S \rightarrow \mathcal{E}(S)$ as neutral element 'skip'. We briefly sketch some other algebraic structure on such programs (coalgebras), see also [31].

Programs $S \rightarrow \mathcal{E}(S)$ are closed under convex combinations: if we have programs $P_1, \dots, P_n: S \rightarrow \mathcal{E}(S)$ and probabilities $r_i \in [0, 1]$ with $\sum_i r_i = 1$, then we can form a new program $P = \sum_i r_i P_i: S \rightarrow \mathcal{E}(S)$. For $q \in [0, 1]^S$,

$$P(s)(q) = \sum_i r_i \cdot P_i(s)(q).$$

Since the sets $S \rightarrow \mathcal{E}(S)$ carries a pointwise order with suprema of ω -chains we can also give meaning to iteration constructs like 'while' and 'for ... do'.

Further we can also do "probabilistic assignment", written for instance as $n := \varphi$, where n is a variable, say of integer type int , and φ is a distribution of type $\mathcal{D}(\text{int})$. The intended meaning of such an assignment $n := \varphi$ is that afterwards the variable n has value $m: \text{int}$ with probability $\varphi(m) \in [0, 1]$. In order to model this we assume an update function $\text{upd}_n: S \times \text{int} \rightarrow S$, which we leave unspecified (similar functions exist for other variables). The interpretation $\llbracket n := \varphi \rrbracket$ of the probabilistic assignment is a function $S \rightarrow \mathcal{E}(S)$, defined as follows.

$$\begin{aligned}
\llbracket n := \varphi \rrbracket(s) &= \mathcal{E}(\text{upd}_n(s, -))(\sigma(\varphi)) \\
&= \lambda q \in [0, 1]^S. \sum_i r_i \cdot q(\text{upd}_n(m_i)), \quad \text{if } \varphi = \sum_i r_i m_i.
\end{aligned}$$

It applies the functor \mathcal{E} to the function $\text{upd}_n(s, -): \text{int} \rightarrow S$ and uses the natural transformation $\sigma: \mathcal{D} \Rightarrow \mathcal{E}$ from (6).

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