4

TERM GRAPH REWRITING

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Partially supported by the U.K. ALVEY Project.

Abstract

Graph rewriting (also called reduction) as defined in Wadsworth (1971) was introduced in order to be able to give a more efficient implementation of functional programming languages in the form of lambda calculus or term rewrite systems: identical subterms are shared using pointers.

Several other authors, e.g. Ehrig (1979), Staples (1980a,b,c), Raoult (1984) and van den Broek & van der Hoeven (1986) have given mathematical descriptions of graph rewriting, usually employing concepts from category theory. These papers prove among other things the correctness of graph rewriting in the form of the Church-Rosser property for "well-behaved" (i.e. regular) rewrite systems. However, only Staples has formally studied the soundness and completeness of graph rewriting with respect to term rewriting.

In this paper we give a direct operational description of graph rewriting that avoids the category theoretic notions. We show that if a term t is interpreted as a graph g(t) and is reduced in the graph world, then the result represents an actual reduct of the original term t (*soundness*). For weakly regular term rewrite systems, there is also a *completeness* result: every normal form of a term t can be obtained from the graphical implementation. We also show completeness for all term rewrite systems which possess a so called hypernormalising strategy, and in that case the strategy also gives a normalising strategy for the graphical implementation.

Besides having nice theoretical properties, weakly regular systems offer opportunities for parallelism, since redexes at different places can be executed independently or in parallel, without affecting the final result.

4.1 INTRODUCTION AND BACKGROUND

Graph rewriting is a well-known and standard technique for implementing functional languages based on term rewriting (e.g. Turner (1979a)), but the correctness of this method has received little attention, being simply accepted folklore. For both theory and practice, this makes a poor foundation, especially in the presence of parallelism. Staples (1980a,b,c) provides the only published results we are aware of. (A digested summary of these papers is in Kennaway (1984).) Wadsworth (1971) proves similar results for the related subject of pure lambda calculus.

Our principal result is that the notion of graph rewriting provides a sound and complete representation (in a sense precisely defined below) of *weakly regular* TRS's. A counterexample is given to show that for non-weakly regular TRS's completeness may fail: some term rewriting computations cannot be expressed in the corresponding graph rewrite system. A second result concerns the mapping of evaluation strategies between the term and the graph worlds. A counterexample is exhibited to show that an evaluation strategy which is normalising (i.e. computes normal forms) in the term world may fail to do so when it is transferred to the graph world. We prove that any strategy which satisfies a stronger condition of being *hypernormalising* in the term world is normalising (and indeed hypernormalising) in the graph world. We briefly consider the problem of defining a graph rewriting implementation of non-left linear term rewrite rules.

The general plan of the paper is as follows: Section 4.2 presents basic definitions, and introduces a linear syntax for terms represented as graphs. Section 4.3 introduces a category of term graphs. Section 4.4 defines the notion of graph rewriting, and section 4.5 introduces the notion of *tree rewriting* as a prelude to section 4.6, which develops our theory of how to relate the worlds of term and graph rewriting. Section 4.7 considers the problem of mapping strategies between the two worlds. Finally, section 4.8 gives a summary of the work.

4.2 TERMS AS TREES AND GRAPHS

4.2.1 Definition.

(i) Let **F** be a (finite or infinite) set of objects called *function symbols*. A,B,... range over **F**.

(ii) The set T of *terms* over **F** is defined inductively by:

$A \in \mathbf{F},\ t_1,\ldots,t_n \in T \Rightarrow A(t_1,\ldots,t_n) \in T \qquad (n \geq 0)$

A() is written as just A. n

4.2.2 Example. Let **F**={0,S}. Then T ={0,S(0),0(S,S(0,0)), S(S,S,S)...}. Note that we do not assume that function symbols have fixed arities. This might appear inconvenient if one wished to represent, for example, the Peano integers, with a constant 0 and a successor operator S, since one also obtains extra “unintended” terms such as some of those listed above. When we define rewrite systems in section 4.4, we will see that this does not cause any problems. n

4.2.3 Definition. A *labelled graph* (*over* **F**) is a triple (N,lab,succ) involving a (finite or infinite) set N of *nodes*, a function lab: N→**F**, and a function succ:N→N*. In this case we say that the $n_1,\ldots,n_k$ are the *successors* of n. The ith component of succ(n) is denoted by $\mathrm{succ}(n)_i$. n

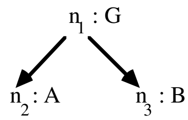
When we draw pictures of graphs, a directed edge will go from each node n to each node in succ(n), with the left-to-right ordering of the sources of the edges corresponding to the ordering of the components of succ(n). The identity of nodes is usually unimportant, and we may omit this information from pictures.

4.2.4 Example. Let $N=\{n_1,n_2,n_3\}$ and define lab and succ on N as follows.

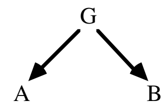
$\mathrm{lab}(n_1)=G, \qquad \mathrm{lab}(n_2)=A, \quad \mathrm{lab}(n_3)=B,$

$succ(n_1)=(n_2,n_3)$, $succ(n_2)=()$, $succ(n_3)=()$.

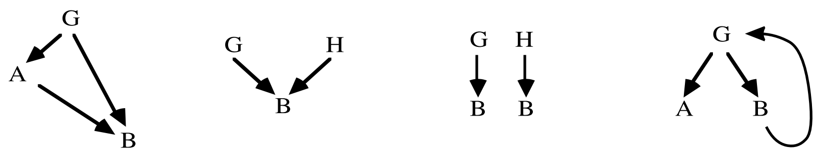
This defines a labelled graph that can be drawn as:



or more simply:



Using this notation, four more examples of graphs are the following.



4.2.5 Definition.

(i) A *path* in a labelled graph (N,lab,succ) is a list $(n_0,i_0,n_1,i_1,\ldots,n_{m-1},i_{m-1},n_m)$ where $m\geq 0$, $n_0,\ldots,n_m \in N$, $i_0,\ldots,i_{m-1} \in \mathbb{N}$ (the natural numbers) and n_{k+1} is the i_k-th successor of n_k. This path is said to be *from* n_0 *to* n_m and m is the *length* of the path.

(ii) A *cycle* is a path of length greater than 0 from a node n to itself. n is called a *cyclic* node.

(iii) A graph is *cyclic* if it contains a cyclic node, otherwise it is *acyclic*. n

4.2.6 Definition.

(i) A *term graph* (often, within this paper, simply a *graph*) is a quadruple (N,lab,succ,r) where (N,lab,succ) is a labelled graph and r is a member of N. The node r is called the *root* of the graph. (We do not require that every node of a term graph is reachable by a path from the root.) For a graph g, the components are often denoted by N_g, lab_g, $succ_g$, and r_g.

(ii) A path in a graph is *rooted* if it begins with the root of the graph. The graph is *root-cyclic* if there is a cycle containing the root.

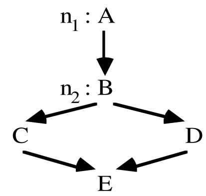
When we draw pictures of term graphs, the topmost node is the root. n

Term graphs are exactly the graphs discussed in the paper Barendregt *et al.* (1987b), which defines a language of generalised graph rewriting of which the rewriting treated in this paper is a special case.

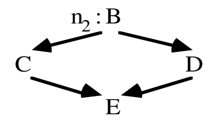
4.2.7 Definition. Let g = (N,lab,succ) be a labelled graph and let $n\in N$. The *subgraph of* g *rooted at* n is the term graph (N',lab',succ',n) where $N'=\{n'\in N \mid \text{there is a path from } n \text{ to } n'\}$ and lab' and succ' are the restrictions of lab and succ to N'. We denote this graph by $g|n$. The definition also applies when g is a term graph. n

4.2.8 Examples.

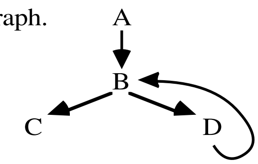
(i)



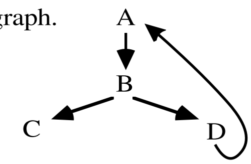
The subgraph rooted at n_2 is:



(ii) A cyclic graph.



(iii) A root-cyclic graph.



A formal description of a graph requires a complete specification of the quadruple (N,lab,succ,r). When writing down examples of finite graphs, it is convenient to adopt a more concise notation, which abstracts away from details such as the precise choice of the elements of N. We will use a notation based on the definition of terms in definition 4.2.1, but with the addition of node-names, which can express the sharing of common subexpressions. The notation is defined by the following context-free grammar, with the restrictions following it.

4.2.9 Definition (linear notation for graphs).

graph ::= node | node + graph
node ::= A(node,...,node) | x | x : A(node,...,node)

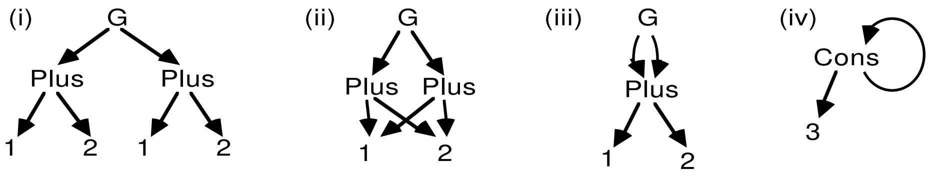
A ranges over **F**. x ranges over a set, disjoint from **F**, of *nodeid's* ('node identifiers'). Any nodeid x which occurs in a graph must occur exactly once in the context x **:** A(node,...,node). Nodeid's are represented by tokens beginning with a lower-case letter. Function symbols will be non-alphabetic, or begin with an upper-case letter. We again abbreviate A() to A. n

This syntax is, with minor differences, the same as the syntax for graphs in the language LEAN (Barendregt *et al.* (1987b)). The five graphs of the examples 4.2.4 are in this notation: G(A,B), G(A(x:B),x), G(x:B) + H(x), G(B) + H(B) and x:G(A,B(x)). Note that multiple uses of the same nodeid express multiple references to the same node.

The definition of terms in 4.2.1 corresponds to a sublanguage of our shorthand notation, consisting of those graphs obtained by using only the first production for graph and the first production for node. *So terms have a natural representation as graphs.*

4.2.10 Examples.

(i) G(Plus(1,2), Plus(1,2)) (ii) G(Plus(n_1: 1, n_2: 2), Plus(n_1, n_2))
(iii) G(n:Plus(1,2), n) (iv) n_1: Cons(3, n_1)



4.2.11 Definition. A *tree* is a graph (N,lab,succ,r) such that there is exactly one path from r to each node in N. n

Thus example (i) above is a tree, and (ii), (iii), and (iv) are not. Trees are always acyclic. Notice that a graph g is a finite tree iff g can be written by the grammar of 4.2.9 without using any nodeid's.

The natural mapping of terms to graphs represents each term as a finite tree. However, some terms can also be represented as proper graphs, by sharing of repeated subterms. For example, the term G(Plus(1,2),Plus(1,2)) can be represented by any of the graphs pictured in example 4.2.10 (i), (ii), or (iii), as well as by the graphs G(Plus(x:1,2),Plus(x,2)) or G(Plus(1,x:2),Plus(1,x)).

4.3 HOMOMORPHISMS OF GRAPHS AND TREES

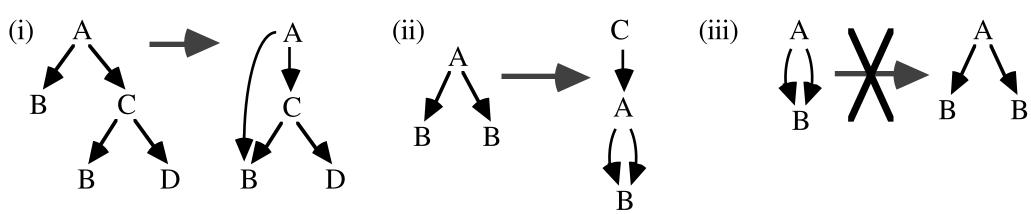
4.3.1 Definition. Given two graphs $g_1 = (N_1,lab_1,succ_1,r_1)$ and $g_2 = (N_2,lab_2,succ_2,r_2)$, a *homomorphism* from g_1 to g_2 is a map $f:N_1 \rightarrow N_2$ such that for all $n \in N_1$,

$$\begin{aligned} lab_2(f(n)) &= lab_1(n) \\ succ_2(f(n)) &= \mathbf{f}(succ_1(n)) \end{aligned}$$

where $\mathbf{f}$ is defined by $\mathbf{f}(n_1,\ldots,n_k) = (f(n_1),\ldots,f(n_k))$. That is, homomorphisms preserve labels, successors, and their order. n

4.3.2 Definition. Graph(**F**) is the category whose objects are graphs over **F** and whose morphisms are homomorphisms. Tree(**F**) is the full subcategory of Graph(**F**) whose objects are the trees over **F**. It is easy to verify that these are categories. n

4.3.3 Examples. We shall write $g_1 \longrightarrow g_2$
when there is a homomorphism from g_1 to g_2. We have the following pictures.



4.3.4 Definition.

(i) A homomorphism $f:g_1 \rightarrow g_2$ is *rooted* if $f(r_1)=r_2$.

(ii) An *isomorphism* is a homomorphism which has an inverse. We write g ~ g' when g and g' are isomorphic.

(iii) Two graphs are *equivalent* when they are isomorphic by a rooted isomorphism. We write g ≈ g' when g and g' are equivalent. n

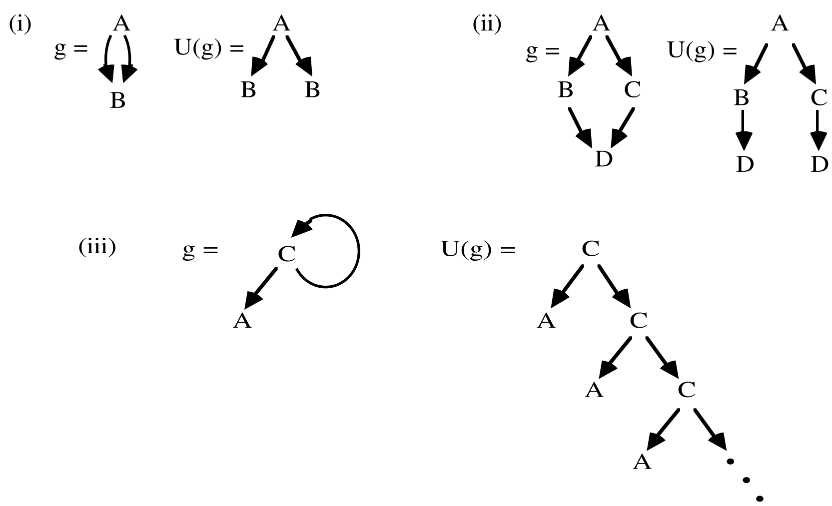
4.3.5 Proposition.

(i) *For any graphs g_1 and g_2 we have $g_1 \approx g_2 \Rightarrow g_1 \sim g_2$.*

(ii) *Every rooted homomorphism from one tree to another is an isomorphism.* n

4.3.6 Example. These two graphs are isomorphic but not equivalent (recall that in diagrammatic representations the root node is the topmost):

4.3.7 Definition. Given any graph g=(N,lab,succ,r) we can define a tree U(g) which results from "unravelling" g from the root. We start with some examples.



Now we give the formal definition. U(g) has as nodes the rooted paths of g. The root of U(g) is the path (r). For a path $p=(n_0,i_0,...,n_{m-1},i_{m-1},n_m)$, $lab_{U(g)}(p) = lab_g(n_m)$ and $succ_{U(g)}(p) = (p_1,...,p_k)$ where p_i is the result of appending $(i,succ_g(n_m)_i)$ to p. Clearly this is a tree. n

4.3.8 Proposition. *For every graph g there is a rooted homomorphism u_g: $U(g) \rightarrow g$ defined by:* $u_g(n_0,i_0,...,n_m) = n_m$. n

4.3.9 Proposition. *A graph g is a tree iff $g \approx U(g)$.* n

4.3.10 Definition. Two graphs g and g' are *tree-equivalent*, notated $g \approx_t g'$, if $U(g) \approx U(g')$ n

For example, the graphs of example 4.2.10 (i), (ii) and (iii) are all tree-equivalent. So are these two graphs:

4.4 GRAPH REWRITING

We now turn to rewriting. First we recall the familiar definitions of terms with free variables and term rewriting. We then explain informally how we represent terms with free variables as 'open' graphs, and define our notion of graph rewriting. Our definition is quite similar to the one in Staples (1980a).

4.4.1 Definition (term rewriting).

(i) Let **V** be a fixed set of function symbols, disjoint from **F**. The members of **V** are called *variables*, and are denoted by lower-case letters. An *open term* over a set of function symbols **F** is a term over **F**∪**V** in which every node labelled with a variable has no successors. An open term containing no variables (that is, what we have been calling simply a term) is a *closed* term.

(ii) A *term rewrite rule* is a pair of terms t_L and t_R (written $t_L \rightarrow t_R$) such that every variable occurring in t_R occurs in t_L. t_L and t_R are, respectively, the *left-* and *right-hand* sides of the *term rewrite rule* $t_L \rightarrow t_R$.

(iii) A term rewrite rule is *left-linear* if no variable occurs more than once in its left-hand side. n

The usual definition of a term rewrite rule requires that t_L be not just a variable. However, our results are not affected by the presence of such rules, so we do not bother to exclude them.

In order to introduce graph rewriting, first we need some preparatory definitions.

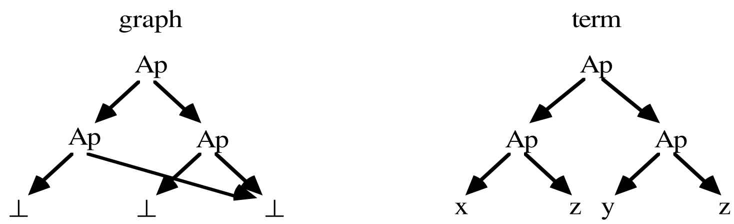
4.4.2 Definition.

(i) An *open labelled graph* is an object (N,lab,succ) like a labelled graph, except that lab and succ are only required to be partial functions on N, with the same domain. A node on which lab and succ are undefined is said to be *empty*. The definition of an *open (term) graph* bears the same relation to that of a (term) graph. When we write open graphs, we will use the symbol ⊥ to denote empty nodes. As with terms, we talk of closed (labelled or term) graphs and closed trees as being those containing no empty nodes.

(ii) A *homomorphism* from one open graph g_1 to another g_2 is defined as for graphs, except that the "structure preserving" conditions are only required to hold at nonempty nodes of g_1. n

Open term graphs are intended to represent terms with variables. Instead of using the set **V** of variables, we find it more convenient, for technical reasons, to follow Staples (1980a) by using empty nodes. The precise translation from open graphs to open terms is as follows. Given an open graph over **F**, we first replace each empty node in it by a different variable symbol from **V**, and then unravel the resulting closed graph over **F**∪**V**, obtaining an open term over **F**. Thus where a graph has multiple edges pointing to the same empty node, the term will have multiple occurrences of the same nodeid.

For example, the graph Ap(Ap(⊥,w:⊥),Ap(⊥,w)) translates to the term Ap(Ap(x,z),Ap(y,z)):



We could obtain any term which only differs from this one by changes of variables. We shall treat such terms as the same.

We now turn to the graph representation of term rewrite rules. We only deal with left-linear rules in this paper. In 4.6.13 we discuss briefly the problems in graphically describing non-left-linear rules.

4.4.3 Definition.

(i) A *graph rewrite rule* is a triple (g,n,n'), where g is an open labelled graph and n and n' are nodes of g, called respectively the *left root* and the *right root* of the rule.

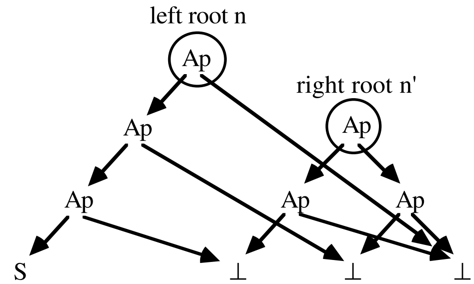
(ii) A *redex* in a graph g_0 is a pair Δ = (R,f), where R is a graph rewrite rule (g,n,n') and f is a homomorphism from g|n to g_0. The homomorphism f is called an *occurrence* of R. n

Rather than introduce our formal definition of graph rewriting immediately, we begin with some examples. The formal definition is given in section 4.4.6.

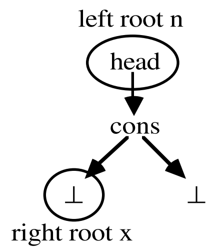
4.4.4 Translation of term rules to graph rules.

Let $t_L \rightarrow t_R$ be a left-linear term rewrite rule. We construct a corresponding graph rewrite rule (g,n,n'), where g is a labelled graph and n and n' are nodes of g. First take the graphs representing t_L and t_R. Form the union of these, sharing those empty nodes which represent the same variables in t_L and t_R. This graph is g. Take n and n' to be the respective roots of t_L and t_R. Here are two examples which should make the correspondence between term and graph rewrite rules clear.

(i) Term rule: Ap(Ap(Ap(S,x),y),z) → Ap(Ap(x,z),Ap(y,z))
Graph rule: (n:Ap(Ap(Ap(S,x:⊥),y:⊥),z:⊥) + n':Ap(Ap(x,z),Ap(y,z)), n, n')

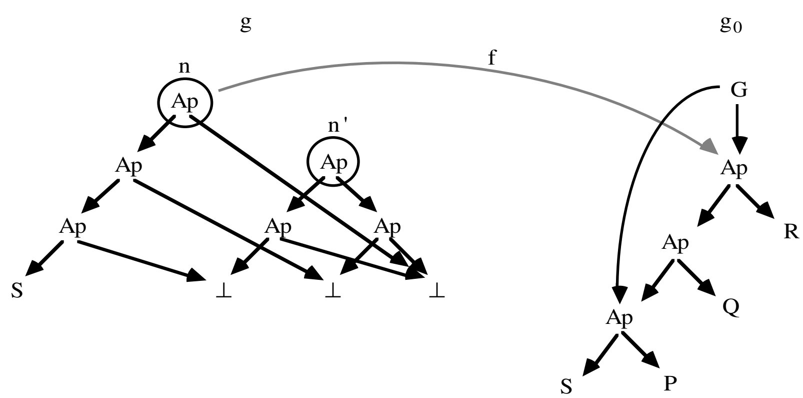


(ii) Term rule: head(cons(x,y)) → x
Graph rule: (n:head(cons(x:⊥,⊥)), n, x)

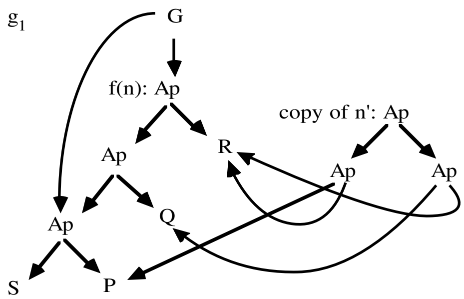


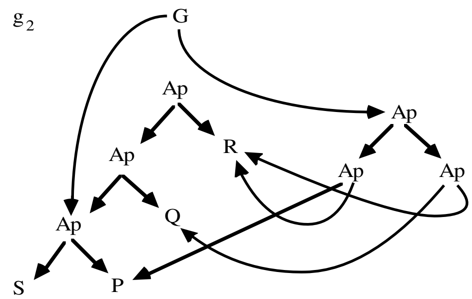
4.4.5 Informal definition of graph rewriting.

A redex ((g,n,n'), f: g|n → g_0) in a graph g_0 is reduced in three steps. We shall use the following redex as an example: (g,n,n') is the S-rule above, g_0 = G(a,Ap(Ap(a:Ap(S,P),Q),R)) and f operates on n as indicated in the picture (which completely determines how f behaves on the rest of g|n).

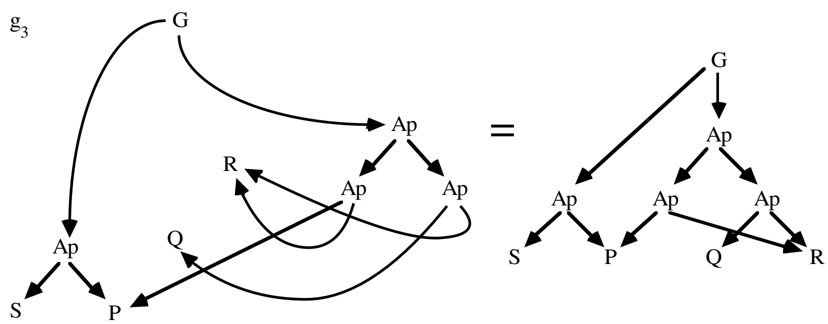


First (the *build* phase) an isomorphic copy of that part of g|n' not contained in g|n is added to g_0, with lab, succ, and root defined in the natural way. Call this graph g_1. Then (the *redirection* phase) all edges of g_1 pointing to f(n) are replaced by edges pointing to the copy of n', giving a graph g_2. The root of g_2 is the root of g_1, if that node is not equal to f(n). Otherwise, the root of g_2 is the copy of n'.





Lastly (the *garbage collection* phase), all nodes not accessible from the root of g_2 are removed, giving g_3, which is the result of the rewrite.



Note that the bottommost Ap node of the redex graph and the S node remain after garbage collection, since they are still accessible from the root of the graph. The other two Ap nodes of the redex vanish.

4.4.6 Formal definition of graph rewriting.

We now give a formal definition of the general construction. Let $((g,n,n'),\ f: g|n \to g_0)$ be a redex in a graph g_0. The graphs g_1 (the build phase), g_2 (the redirection phase) and g_3 (the garbage collection phase) are defined as follows.

(i) The node-set N of g_1 is the disjoint union of N_{g0} and $N_{g|n'} - N_{g|n}$. The root is r_{g0}. The functions lab_{g1} and $succ_{g1}$ are given by:

$$
\begin{array}{lll}
lab_{g1}(m) & = lab_{g0}(m) & (m \in N_{g0}) \\
 & = lab_g(m) & (m \in N_{g|n'} - N_{g|n}) \\
succ_{g1}(m)_i & = succ_{g0}(m)_i & (m \in N_{g0}) \\
 & = succ_g(m)_i & (m,\ succ_g(m)_i \in N_{g|n'} - N_{g|n}) \\
 & = f(succ_g(m)_i) & (m \in N_{g|n'} - N_{g|n},\ succ_g(m)_i \in N_{g|n})
\end{array}
$$

We write $g_1 = g_0 +_f (g,n,n')$.

(ii) The next step is to replace in g_1 all references to f(n) by references to n'. We can define a substitution operation in general for any term graph h and any two nodes a and b of h.

h[a:=b] is a term graph (N_h,lab,succ,r), where lab, succ, and r are given as follows.

lab(c) = lab_h(c) for each node c of N_h
if $succ_h(c)_i$ = a then $succ(c)_i$ = b, otherwise $succ(c)_i = succ_h(c)_i$
if r_h = a then r = b, otherwise r = r_h

With this definition, g_2 is g_1[f(n):=n'].

(iii) Finally, we take the part of g_2 which is accessible from its root, by defining $g_3 = g_2|r_{g2}$. We give this operation a name: for any term graph h, we denote $h|r_h$ by GC(h) (*G*arbage *C*ollection).
We denote the result of reducing a redex Δ in a graph g by RED(Δ,g). Collecting the notations we have introduced, we have
RED(((g,n,n'),f),g_0) = GC(($g_0 +_f$ (g,n,n'))[f(n):=n']). ∎

Our definition of graph rewriting is a special case of a more general notion, defined in Glauert *et al.* (1987) by a category-theoretic construction. Those familiar with category theory may recognise the build phase of a rewrite as a pushout, and redirection and garbage collection can be given definitions in the same style (though the categories involved are not those defined in this paper). For the purpose of this paper - describing graph rewritings which correspond to conventional term rewritings - the direct “operational” definition we have given is simpler.

4.4.7 Definition.

(i) If g reduces to g' by reduction of a redex Δ, we write $g \rightarrow^{\Delta} g'$, or g → g' if we do not wish to indicate the identity of the redex. The reflexive and transitive closure of the relation → is $\rightarrow^*$.

(ii) A *graph rewriting system* (GRS) over **F** consists of a pair (G,**R**) where **R** is a set of rewrite rules and G is a set of graphs over **F** closed under rewriting by the members of **R**.

(iii) We write $g \rightarrow_{\mathbf{R}} g'$ if g → g' by reduction of a redex using one of the rules in **R**. The reflexive and transitive closure of $\rightarrow_{\mathbf{R}}$ is $\rightarrow^*_{\mathbf{R}}$. If clear from the context, we omit the subscript **R**.

(iv) A graph g such that for no g' does one have $g \rightarrow_{\mathbf{R}} g'$ is said to be an **R**-normal form (or to be in **R**-normal form). If $g \rightarrow^*_{\mathbf{R}} g'$ and g' is in **R**-normal form, we say that g' is an **R**-normal form of g, and that g has an **R**-normal form. Again, we often omit the **R**. ∎

Note that a GRS is not required to include all the graphs which can be formed from the given set of function symbols **F**. Any subset closed under rewriting will do. This allows our definition to automatically handle such things as, for example, sorted rewrite systems, where there are constraints over what function symbols can be applied to what arguments, or arities, where each function symbol may only be applied to a specified number of arguments. From our point of view, this amounts to simply restricting the set of graphs to those satisfying these constraints. So long as rewriting always yields allowed graphs from allowed graphs, we do not need to develop any special formalism for handling restricted rewrite systems, nor do we need to prove new versions of our results.

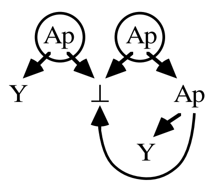
Our definition of a graph rewrite rule allows any conventional term rewrite rule to be interpreted as a graph rewrite rule, provided that the term rewrite rule is left-linear, that is, if no variable occurs twice or more on its left-hand side. As some of the following examples show, however, some new phenomena arise with graph rewrite rules.

4.4.8 Examples.

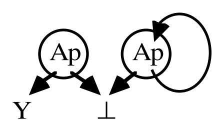
(i) Term rule: $A(x) \rightarrow B(x)$; Graph rule: ($n{:}A(x{:}\perp) + n'{:}B(x)$, n, n')
Graph: $x{:}A(x)$; Result of rewriting: $x{:}B(x)$

(ii) Term rule: $I(x) \rightarrow x$; Graph rule:($n{:}I(n'{:}\perp)$, n, n')
Graph: $I(3)$; Result of rewriting: 3

(iii) The fixed point combinator Y has the term rewrite rule $Ap(Y,x) \rightarrow Ap(x,Ap(Y,x))$. This can be transformed into the graph rewrite rule $(n{:}Ap(Y,x{:}\perp) + n'{:}Ap(x,Ap(Y,x)),n,n')$. However, it can also be given the graph rewrite rule: $(n{:}Ap(Y,x{:}\perp) + n'{:}Ap(x,n'),n,n')$:

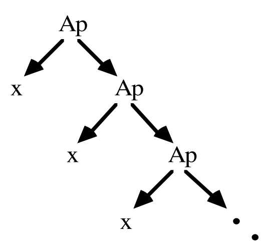


$(n{:}Ap(Y,x{:}\perp) + n'{:}Ap(x,Ap(Y,x)),n,n')$



$(n{:}Ap(Y,x{:}\perp\ ) + n'{:}Ap(x,n'),n,n')$

This captures the fact that the Böhm tree (Barendregt (1984)) of the term Ap(Y,x) is:



The graph rule can do all the ‘unravelling’ in one step, which in the term rewrite world requires an infinite sequence of rewritings.

(iv) Here is a more subtle example of the same phenomenon illustrated by (iii). Consider the term rewrite rule

$F(Cons(x,y)) \rightarrow G(Cons(x,y))$

Our standard representation of this as a graph rewrite rule is:

(n:F(Cons(x:⊥,y:⊥)) + n':G(Cons(x,y)), n, n')

Note that each application of this rule will create a new node of the form Cons(...,...), which will have the same successors as an existing node Cons (...,...). In a practical implementation, there is no need to do this. One might as well use that existing node, instead of making a copy of it. The following graph rewrite rule does this:

(n:F(z:Cons(⊥,⊥)) + n':G(z), n, n')

Both the languages Standard ML and Hope, which are languages of term rewriting, allow an enhanced form of term rewrite rules such as (using our syntax):

F(z:Cons(x,y)) → G(z)

with precisely this effect. Of course, given referential transparency (which ML lacks) there is no reason for an implementation not to make this optimisation wherever there is an opportunity, even if the programmer does not. But providing this feature to the programmer may make his programs more readable.

(v)	Term rule: I(x) → x;	Graph rule: (n:I(n':⊥), n, n')
	Graph: x:I(x);	Result of rewriting: x:I(x)

Example 4.4.8(v) is deliberately pathological. Consider the GRS for combinatory logic, whose rules are those for the S, K, and I combinators. The graph can be interpreted as “the least fixed point of I” (cf. the example of the Y combinator above), and in the usual denotational semantics in terms of reflexive domains should have the bottom, “undefined” value. As the graph reduces to itself (and to nothing else), it has no normal form. Thus our operational semantics of graph rewriting agrees with the denotational semantics. This is not true for some other attempts we have seen at formalising graphical term rewriting.

We now study some properties of graph rewrite systems. We establish a version of the theorem of finite developments for term rewriting, and the confluence of weakly regular systems. For reasons of space, the longer proofs are omitted from this paper. They appear in Barendregt *et al.* (1986a).

4.4.9 Proposition. *Garbage collection can be postponed. That is, given* $g \rightarrow^{\Delta 1} g_1 \rightarrow^{\Delta 2} g_2$, $\Delta_i = (R_i,f_i)$, $R_i = (g_i,n_i,n'_i)$ *(i = 1,2) and* $g'_1 = (g +_{f1} R_1)[f(n) = n']$, *then* Δ_2 *is also a redex of* g'_1, *and* $g'_1 \rightarrow^{\Delta 2} g_2$. ∎

4.4.10 Definition. Two redexes $\Delta_1 = ((g_1,n_1,n'_1),f_1)$ and $\Delta_2 = ((g_2,n_2,n'_2),f_2)$ of a graph g are *disjoint* if:

(i) $f_2(n_2)$ is not equal to $f_1(n)$ for any nonempty node n of $g_1|n_1$, and
(ii) $f_1(n_1)$ is not equal to $f_2(n)$ for any nonempty node n of $g_2|n_2$.

Δ_1 and Δ_2 are *weakly disjoint* if either they are disjoint, or the only violation of conditions (i) and (ii) is that $f_1(n_1) = f_2(n_2)$, and the results of reducing Δ_1 or Δ_2 are identical.

A GRS is *regular* (resp. *weakly regular*) if for every graph g of the GRS, every two distinct redexes in g are disjoint (resp. weakly disjoint). n

4.4.11 Proposition. *Let $\Delta_1 = ((g_1,n_1,n'_1), f_1)$ and $\Delta_2 = ((g_2,n_2,n'_2), f_2)$ be two disjoint redexes of a graph g. Let $g \rightarrow^{\Delta 1} g'$. Then either $f_2(n_2)$ is not a node of g', or there is a redex $((g_2,n_2,n'_2),f)$ of g' such that $f(n_2) = f_2(n_2)$.* n

4.4.12 Definition.

(i) With the notations of the preceding proposition, if $f_2(n_2)$ is not a node of g' then Δ_2/Δ_1 is the empty reduction sequence from g' to g'; otherwise, Δ_2/Δ_1 is the one-step reduction sequence consisting of the reduction of $((g_2,n_2,n'_2),f)$. This redex is the *residual* of Δ_2 by Δ_1 and is denoted by $\Delta_2//\Delta_1$. For weakly disjoint Δ_1 and Δ_2, Δ_2/Δ_1 is the empty reduction sequence from g' to g'. Δ_2/Δ_1 is not defined when Δ_1 and Δ_2 are not weakly disjoint, and $\Delta_2//\Delta_1$ is not defined when Δ_1 and Δ_2 are not disjoint.

(ii) Given a reduction sequence $g \rightarrow^{\Delta 1} \rightarrow^{\Delta 2} \ldots \rightarrow^{\Delta i} g'$ and a redex Δ of g, the *residual* of Δ by the sequence $\Delta_1 \ldots \Delta_i$, denoted $\Delta//(\Delta_1 \ldots \Delta_i)$ is $(\Delta//(\Delta_1 \ldots \Delta_{i-1}))//\Delta_i$ (provided that $(\Delta//(\Delta_1 \ldots \Delta_{i-1}))$ exists and is disjoint from Δ_i). n

4.4.13 Proposition. *Let Δ_1 and Δ_2 be weakly disjoint redexes of g, and let $g \rightarrow^{\Delta i} g_i$ (i = 1,2). Then there is a graph h such that $g_1 \rightarrow^{\Delta 2/\Delta 1} h$ and $g_2 \rightarrow^{\Delta 1/\Delta 2} h$. That is, weakly disjoint redexes are subcommutative.* n

4.4.14 Corollary. *Every weakly regular GRS is confluent. That is, if $g \rightarrow^* g_i$ (i = 1,2), then there is an h such that $g_i \rightarrow^* h$ (i = 1,2).* n

4.4.15 Definition. Let g be a graph and F be a set of disjoint redexes of g. A *development* of F is a reduction sequence in which the redex reduced at each step is a residual, by the preceding steps of the sequence, of a member of F. A *complete development* of F is a development of F, at the end of which there remain no residuals of members of F. n

4.4.16 Proposition. *Every complete development of a finite set of pairwise disjoint redexes F is finite. In fact, its length is bounded by the number of redexes in F.* n

4.4.17 Proposition. *Let F be a set of redexes of a graph g. Every finite complete development of F ends with the same graph (up to isomorphism). This graph is:*

$$GC(\ (g +_{f1} R_1 +_{f2} \ldots +_{fi} R_i)[f_1(n_1) := n'_1] \ldots [f_i(n_i) := n'_i]\ )$$

where the redexes whose residuals are reduced in the complete development are $\Delta_1 = (f_1,R_1),\ldots,\Delta_i = (f_i,R_i)$. n

Note that since we allow infinite graphs, a set of redexes F as in the last two propositions may be infinite. Nevertheless, it may have a finite complete development, if rewriting of some members of F causes all but finitely many members of F to be erased.

4.5 TREE REWRITING

In order to study the relationship between term rewriting and graph rewriting, we define the notion of *tree rewriting*. This is a formalisation of conventional term rewriting within the framework of our definitions of graph rewriting.

4.5.1 Definition.

(i) A *tree rewrite rule* is a graph rewrite rule (g,n,n') such that g|n is a tree. For a set of tree rewrite rules **R**, the relation $\rightarrow_{t\mathbf{R}}$ of *tree rewriting* with respect to **R** is defined by:

$$t_1 \rightarrow_{t\mathbf{R}} t_2 \Leftrightarrow \text{for some graph } g,\ t_1 \rightarrow_{\mathbf{R}} g \text{ and } U(g) = t_2$$

(ii) A *tree rewrite system* (TreeRS) over **F** is a pair (T,R) where R is a set of tree rewrite rules and T is a set of trees over **F** closed under $\rightarrow_{t\mathbf{R}}$. A *term rewrite system* (TRS) is a TreeRS, all of whose trees are finite.

When t_1 reduces to t_2 by tree rewriting of a redex Δ, we write $t_1 \rightarrow_t^{\Delta} t_2$, or $t_1 \rightarrow_t t_2$ when we do not wish to indicate the identity of the redex. ∎

Tree rewrite systems differ from conventional term rewrite systems in two ways. Firstly, infinite trees are allowed. We need to handle infinite trees, since they are produced by the unravelling of cyclic graphs. We need to handle cyclic graphs because some implementors of graph rewriting use them, and we do not want to limit the scope of this paper unnecessarily. Secondly, the set of trees of a TreeRS may be any set of trees over the given function symbols which is closed under tree rewriting. This is for the same reason as was explained above for GRS's.

If for each rule (g,n,n') in the rule-set, g is finite and acyclic, the set of all finite trees generated by the function symbols will be closed under tree rewriting. This is true for those rules resulting from term rewrite rules by our standard representation. Thus the conventional notion of a TRS is included in ours.

4.5.2 Definition. Let t, t_1,... t_i be trees, and $n_1,\ldots,n_i$ be distinct nodes of t. We define $t[n_1{:=}t_1,\ldots,n_i{:=}t_i]$ to be the tree whose nodes are

(i) all paths of t which do not include any of $n_1,\ldots,n_i$, and

(ii) every path obtained by taking a path p of t, which ends at n_j ($1 \le j \le i$) and contains no other occurrence of $n_1 \ldots n_i$, and replacing the last node of p by any path of t_j.

For any of these paths p, the label of p is the label of the last node in p, in whichever of t, t_1,... t_i that came from. The successors function is defined similarly. ∎

The results concerning disjointness, regularity, and confluence which we proved for graph rewriting all have versions for tree rewriting. Again we omit proofs. We also have the following:

4.5.3 Proposition. *Unravelling can be postponed. That is, if $t_1 \to_t t_2 \to_t t_3$, then there are graphs g and g' such that*

(1) $t_2 = U(g)$ *and* $t_1 \to g$ *(by graph rewriting)*
(2) $g \to g'$ *and* $t_3 \to^*_t U(g')$. ∎

4.6 RELATIONS BETWEEN TREE AND GRAPH REWRITING

In this section we prove our principal result: for weakly regular rule-systems, graph rewriting is a sound and complete implementation of term rewriting.

4.6.1 Definition. Let (T,**R**) be a TreeRS.

(i) L(T,**R**), the *lifting* of this system, is the GRS whose set of graphs is L(T) = {g | U(g)∈ T}, and whose rule set is **R** (but now interpreted as graph rewrite rules). It is trivial to verify that L(T) is closed under $\to_{\mathbf{R}}$.

(ii) A *graphical term rewrite system* (GTRS) is a GRS of the form L(T,**R**), where (T,**R**) is a term rewrite system.

(iii) A GRS (G,**R**) is *acyclic* if every member of G is acyclic. ∎

When (T,**R**) is a term rewrite system, L(T,**R**) represents its graphical implementation. There are two fundamental properties it must have to be a correct implementation, which we now define.

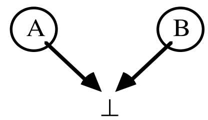
4.6.2 Definition.

(i) A TreeRS (T,**R**) is called *graph-reducible* if for every graph g in L(T), if t is a normal form of U(g) in (T,**R**), then there is a normal form g' of g in L(T,**R**) such that U(g') = t, and if U(g) has no normal form in (T,**R**), then g has no normal form in L(T,**R**).

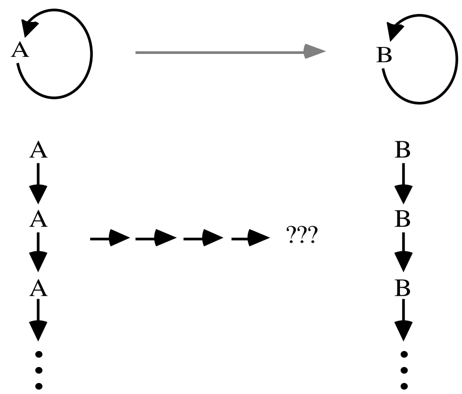
(ii) A GRS (G,**R**) is *tree-reducible* if there is a TreeRS (T,**R**) such that (G,**R**) = L(T,**R**), and such that if g' is a normal form of g in (G,**R**), then U(g') is a normal form of U(g) in (T,R), and if g has no normal form in (G,**R**), then U(g) has no normal form in (T,**R**). ∎

L(T,**R**) is the graphical implementation of (T,**R**). Tree-reducibility of L(T,**R**) expresses soundness: every result which is obtainable by graph rewriting in L(T,**R**) is also obtainable by tree rewriting in (T,**R**). Graph-reducibility of (T,**R**) expresses completeness: every result which is obtainable by tree rewriting is also obtainable by graph rewriting. We shall see that every GTRS is tree-reducible, and every weakly regular TRS is graph-reducible. Not all GRS's, even those of the form L(T,**R**), are tree-reducible, nor is every TreeRS graph-reducible, as the following examples show.

4.6.3 Example. Tree reducibility can fail when there are cyclic graphs. Consider the term rewrite rule A(x) → B(x), represented graphically by:



A cyclic graph may contain a single redex with respect to this rule, while its unravelling contains infinitely many:

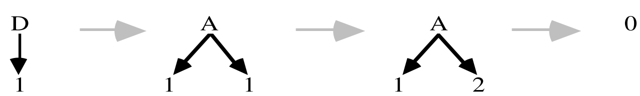


4.6.4 **Example.** The following TreeRS is not graph-reducible:

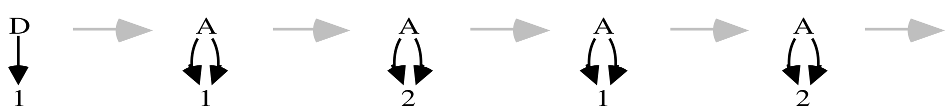
T: trees over {A,D,0,1,2}, with the following arities: A is binary, D is unary, and 0, 1, and 2 are nullary.

R: A(1,2) → 0; 1 → 2; 2 → 1; D(x) → A(x,x).

For a counterexample, consider the following tree rewriting sequence:



In the graph rewriting system we have:



In this example, the sharing of (tree) subterms in the graph world has excluded from the graph world certain rewrite sequences of the tree world. Distinct subterms of A(1,1) correspond to the same subgraph of A(x:1,x), forcing synchronized rewriting of siblings, which makes the normal form inaccessible.

4.6.5 **Definition.**

(i) Redexes Δ = ((g,n,n'),f) and Δ' = ((g',m,m'),f') in a graph h are *siblings* if $h|f(n) \approx_t h|f'(m)$.

(ii) For a redex $\Delta = ((U(g),n,n'),f)$ of a tree $U(g)$ we define $u_g(\Delta)$ to be the redex $((g,n,n'),u_g \cdot f)$ of g.

(iii) For a redex Δ of a graph g, the set of redexes Δ' of $U(g)$ such that $u_g(\Delta') = \Delta$ is denoted by $u_g^{-1}(\Delta)$. For a set of redexes $\mathcal{F}$ of a graph g, $u_g^{-1}(\mathcal{F})$ denotes $\bigcup\{\ u_g^{-1}(\Delta') \mid \Delta' \in \mathcal{F}\ \}$.

(iv) A redex Δ of a graph G is *acyclic* if $u_g^{-1}(\Delta)$ is finite. n

4.6.6 Proposition. *Let $g \to g'$ by rewriting of an acyclic redex Δ. Then $U(g) \to^*_t U(g')$ by complete development of $u_g^{-1}(\Delta)$. For any redex Δ' of g, weakly disjoint from Δ, $u_g^{-1}(\Delta'/\!/\Delta) = u_g^{-1}(\Delta')/\!/u_g^{-1}(\Delta)$.* n

4.6.7 Proposition. *Let $g \to^* g'$ by a complete development of a set $\mathcal{F}$ of disjoint acyclic redexes of g whose associated rewrite rules are acyclic. Then $U(g) \to^*_t U(g')$ by a complete development of $u_g^{-1}(\mathcal{F})$.* n

4.6.8 Definition.

(i) In a weakly regular GRS, the relation of *Gross-Knuth* reduction, notation $\to^{GK}$, is defined as follows

$g \to^{GK} g' \Leftrightarrow g \to^* g'$ by complete development of the set of all redexes of g.

(ii) In a weakly regular TreeRS we define *Gross-Knuth* reduction by

$t \to^{GK}_t t' \Leftrightarrow t \to^*_t t'$ by complete development of the set of all redexes of t. n

4.6.9 Proposition. *Let $(T,\mathbf{R})$ be a weakly regular TRS. Then $L(T,\mathbf{R})$ is weakly regular. Let g and g' be graphs in L(T) such that $g \to^{GK} g'$. Then $U(g) \to^{GK}_t U(g')$.* n

4.6.10 Proposition. *If every graph in L(T) is acyclic, then $L(T,\mathbf{R})$ is tree-reducible. In particular, a graphical term rewrite system is tree-reducible.* n

4.6.11 Proposition. *For any TreeRS $(T,\mathbf{R})$ and any graph g in L(T), g is a normal form of $L(T,\mathbf{R})$ iff U(g) is a normal form of $(T,\mathbf{R})$.* n

Thus in a graphical term rewrite system $L(T,\mathbf{R})$, everything which can happen can also happen in the term rewrite system, and all the normal forms are the same. Graph-reducibility may fail, however, since it may be that for some graph g, U(g) has a normal form but g does not.

4.6.12 Theorem. *Every weakly regular TRS is graph-reducible.*

Proof. Let $(T,\mathbf{R})$ be a weakly regular TRS. Let g be a graph of $L(T,\mathbf{R})$ such that U(g) has a normal form. Proposition 4.6.7 relates the Gross-Knuth reduction sequences for g and U(g) in the following way.

$$g \xrightarrow{GK} g_1 \xrightarrow{GK} g_2 \xrightarrow{GK} \cdots$$

$$U(g) \xrightarrow[t]{GK} U(g_1) \xrightarrow[t]{GK} U(g_2) \xrightarrow[t]{GK} \cdots$$

It is a standard result that for regular TRS's, Gross-Knuth reduction is normalising (Klop (1980)), and the proof carries over immediately to weakly regular TreeRS's. Therefore the bottom line of the diagram terminates with some tree U(g') in normal form such that g reduces to g' in L(T,**R**). Therefore g' is a normal form of g, and (T,**R**) is graph-reducible. ∎

4.6.13 Non-left-linearity.

We shall now discuss non-left-linearity, and indicate why we excluded non-left-linear TRS's from consideration. In term rewriting theory, for a term to match a non-linear left-hand side, the subterms corresponding to all the occurrences of a repeated variable must be identical.

Our method of using empty nodes to represent the variables of term rewrite rules suggests a very different semantics for non-left-linear rules. Our representation of a term rule A(x,x) → B would be (n:A(x:⊥,x), n, x). This will only match a subgraph of the form a:A(b: ..., b). That is, the subgraphs matched by the repeated variable must be not merely textually equal, but identical - the very same nodes. If one is implementing graph rewriting as a computational mechanism in its own right, rather than considering it merely as an optimisation of term rewriting, then this form of non-left-linearity may be useful. However, it is not the same as non-left-linearity for term rules.

To introduce a concept more akin to the non-left-linearity of term rules, we could use variables in graphs, just as for terms, instead of empty nodes. A meaning must then be chosen for the matching of a graph A(Var1,Var1) where Var1 is a variable symbol, occurring at two different nodes. Two possibilities naturally suggest themselves. The subgraphs rooted at nodes matched by the same variable may be required to be equivalent, or they may only be required to be tree-equivalent. The latter definition is closer to the term rewriting concept.

When a variable occurs twice or more on the left-hand side of a rule, there is also a problem of deciding which of the subgraphs matched by it is referred to by its occurrences on the right-hand side. One method would be to cause those subgraphs to be first coalesced, replacing the equivalence or tree-equivalence which the matching detected by pointer equality. This technique may be useful in implementing logic programming languages, where non-linearity is much more commonly used than in functional term rewriting. Further investigation of the matter is outside the scope of the present paper.

Lastly, we note that although some term rewriting languages, such as SASL (Turner (1979b)) and Miranda (Turner (1986)), allow non-left-linear rules, they generally interpret the implied equality test neither as textual equality, nor as pointer equality, but as the equality operator of the language (although pointer equality may be used as an optimisation). In these languages, any program containing non-left-linear rules can be transformed to one which does not.

4.7 NORMALISING STRATEGIES

In this section we define the notion of an evaluation strategy in a general setting which includes term and graph rewrite systems. We then study the relationships between strategies for term rewrite systems and for the corresponding graph systems.

4.7.1 Definition.

(i) An *abstract reduction system* (*ARS*) is a pair $(O,\to)$, where O is a set of *objects* and $\to$ is a binary relation on O. This notion abstracts from term and graph rewrite systems. The transitive reflexive closure of $\to$ is denoted by $\to^*$.

(ii) An element x of an ARS is a *normal form* (nf) if for no y does one have $x\to y$.

(iii) An element x *has a normal form* if $x\to^* y$ and y is a normal form.

(iv) A *reduction sequence* of an ARS is a sequence $x_0\to x_1\to\ldots\to x_n$. The *length* of this sequence is n. A sequence of length 0 is *empty*. n

4.7.2 Definition.

(i) Given an ARS $(O,\to)$, a *strategy* for this system is a function S which takes each $x\in O$ to a set S(x) of nonempty finite reduction sequences, each beginning with x. Note that S can be empty.

(ii) S is *deterministic* if, for all x, S(x) contains at most one element.

(iii) S is a *one-step strategy* (or *1-strategy*) if for every x in O, every member of S(x) has length 1.

(iv) Write $x\to_S y$ if S(x) contains a reduction sequence ending with y. By abuse of notation, we may write $x\to_S y$ to denote some particular but unspecified member of S(x).

(v) An S-*sequence* is a reduction sequence of the form $x_0\to_S x_1 \to_S x_2 \to_S \ldots\ldots$

(vi) S is *normalising* if for all x having a normal form any sequence $x_0\to_S x_1 \to_S x_2 \to_S \ldots\ldots$ must eventually terminate with a normal form. n

4.7.3 Definition.

(i) Let S be a strategy of an ARS $(O,\to)$. *Quasi*-S is the strategy defined by:

$$\text{quasi-S}(x) = \{x \to^* x' \to_S y \mid x' \text{ in } O\}.$$

Thus a quasi-S path is an S-path diluted with arbitrary reduction steps.

(ii) A strategy S is *hypernormalising* if quasi-S is normalising. n

A 1-strategy for a TreeRS or GRS can be specified as a function which takes the objects of the system to some subset of its redexes. This will be done from now on.

4.7.4 Definition. Let S be a 1-strategy for a TreeRS (T,**R**). The strategy S_L for the lifted graph rewrite system L(T,**R**) is defined by $S_L(g) = u_g(S(U(g)))$. ■

For 1-strategies on TreeRS's, this is a natural definition of lifting. For multi-step strategies, it is less clear how to define a lifting, and we do not do so in this paper.

Although a 1-strategy for a TreeRS may be normalising, its lifting may not be. This may be because the lifting of the TreeRS does not preserve normal forms (e.g. as in example 4.6.4), or for more subtle reasons, such as in the following example.

4.7.5 Example. Consider the following TreeRS:
Function symbols: A (binary), B, 1, 2 (nullary).
Rules: $1 \rightarrow 2$, $2 \rightarrow 1$, $A(x,y) \rightarrow B$.

By stipulating that A is binary and B, 1, and 2 are nullary, we mean, as discussed following definitions 4.4.7 and 4.5.1, that trees not conforming to these arities are not included in the system. Define a strategy S as follows (where the redexes chosen by S are boldfaced):

A(**1**,1) → A(2,1)   A(2,**2**) → A(2,1)
A(x,y) → B, if neither of the preceding cases applies

S takes the tree A(1,1) to normal form B in two steps. S_L takes the graph A(x:1,x) to A(x:2,x) and back again in an infinite loop.

The next theorem shows that if a 1-strategy S for a TreeRS is hypernormalising, then S_L is hypernormalising for the corresponding GRS.

4.7.6 Theorem. *Let (T,**R**) be a TreeRS and let S be a 1-strategy for it. Let (G,**R**') be the lifting of (T,**R**). If S is hypernormalising then S_L is hypernormalising.*

Proof. Assume S is hypernormalising. Let g be a graph in G having a normal form, and consider a quasi-S_L reduction sequence starting from g.

By proposition 4.6.7 and the definition of S_L, we can construct the following diagram, where the top line is the quasi-S_L reduction sequence:

$$\begin{tikzcd}
g \arrow[rr, "S_L"] & & g' \arrow[r, twoheadrightarrow] & g_1 \arrow[rr, "S_L"] & & g_1' \arrow[r, twoheadrightarrow] & g_2 \arrow[rr, "S_L"] & & g_2' \arrow[r, twoheadrightarrow] & \cdots \\
U(g) \arrow[u, "u_g"] \arrow[r, "S"] & t \arrow[r, twoheadrightarrow] & U(g') \arrow[u] \arrow[r, twoheadrightarrow] & U(g_1) \arrow[u, "u_{g_1}"] \arrow[r, "S"] & t_1 \arrow[r, twoheadrightarrow] & U(g_1') \arrow[u] \arrow[r, twoheadrightarrow] & U(g_2) \arrow[u, "u_{g_2}"] \arrow[r, "S"] & t_2 \arrow[r, twoheadrightarrow] & U(g_2') \arrow[u] \arrow[r, twoheadrightarrow] & \cdots
\end{tikzcd}$$

Since g has a normal form, so does U(g), so since quasi-S is normalising, the bottom line must stop at some point, with a normal form of U(g). Therefore the top line also stops, and must do so with a graph which unravels to the normal form in the bottom line. ∎

4.7.7 Example. The converse does not hold. If S_L is hypernormalising, S need not be. Consider the following TRS.

Function symbols: A (binary), B (nullary)
Rules: $A(x,y) \to B \quad A(x,y) \to A(x,x)$

Every non-normal form of this system has the form $A(\alpha,\beta)$ for some terms α and β. Let S be the strategy:

$$A(\alpha,\beta) \to_S A(\alpha,\alpha) \quad (\text{if } \alpha \neq \beta)$$

$$A(\alpha,\alpha) \to_S B$$

The first S_L-step in any quasi-S_L-sequence will produce either a graph of the form $A(x{:}\alpha,x)$ or the normal form B. In the former case, whatever extra steps are then inserted, the result can only be either another term of the same form or B. In the former case, the next S_L-step will reach B. Therefore S_L is hypernormalising. However, S is not hypernormalising. A counterexample is provided by the term A(A(B,B),B). An infinite quasi-S sequence beginning with this term is:

$$A(A(B,B),B) \to_S A(A(B,B),A(B,B)) \to A(A(B,B),B) \to_S A(A(B,B),A(B,B)) \to \ldots$$

4.7.8 Corollary. *If a TreeRS (T,**R**) has a hypernormalising 1-strategy then it is graph reducible.*

Proof. By theorem 4.7.5 the lifting (G,**R**) of the TreeRS has a normalising strategy. Now assume U(g) = t. Suppose g has no nf. Then the S_L path of g is infinite. This gives, by the construction of 4.7.6, an infinite quasi-S-path of t, hence t has no normal form. ∎

An application of this result is that strongly sequential TRS's (in the sense of Huet & Lévy (1979)) are graph reducible. This follows from their theorem that the 1-strategy which chooses any needed redex is hypernormalising.

The condition that a strategy be hypernormalising is unnecessarily strong. Inspection of the proofs of the preceding theorem and corollary shows that the following weaker concept suffices.

4.7.9 Definition.

(i) Let S be a 1-strategy of a TreeRS (T,**R**). Then *sib*-S is the strategy defined by:
sib-S(x) = $\{ x \to_S y \to^* z \mid$ the sequence $y \to^* z$ consists of siblings of $S(x)\}$.
That is, a sib-S path is an S-path diluted with arbitrary sib-steps from the reduction relation.

(ii) A strategy S is *sib-normalising* if sib-S is normalising. ∎

4.7.10 Theorem. *Let (T,**R**) be a TreeRS and let S be a 1-strategy for it. Let (G,**R**') be the lifting of (T,**R**). If S is sib-normalising then S_L is sib-normalising and (T,**R**) is graph-reducible.*

Proof. Immediate from the proofs of theorem 4.7.6 and corollary 4.7.8. n

4.8 CONCLUSION

Graph rewriting is an efficient way to perform term rewriting. We have shown:

1. Soundness: for all TRS's, graph rewriting cannot give incorrect results.
2. Completeness: for weakly regular TRS's, graph rewriting gives all results.
3. Many normalising strategies (the hypernormalising, or even the sib-normalising ones) on terms can be lifted to graphs to yield normalising strategies there. In particular, for strongly sequential term rewrite systems, the strategy of contracting needed redexes can be lifted to graphs.

We have also given counterexamples illustrating incompleteness for non-weakly regular TRS's and for liftings of non-sib-normalising strategies.