Bicategories in Univalent Foundations

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This Talk in Under A Minute

- We studied bicategories in UF
- We defined the notion of **univalent bicategory**
- We defined the notion of **displayed bicategory**
- We used **displayed bicategories to construct univalent bicategories**
- This way, we constructed univalent bicategories of pseudofunctors, CwFs, and algebraic structures
- We formalized this in UniMath

Getting higher and higher, but why?

- Categories are useful for reasoning about structures like sets, groups, rings, ...
- But less so for categories (category of categories).
- Bicategories come into play!

Briefly: bicategories are a tool to study structures like categories.

Even the types!?

We work in univalent foundations:

- Equality is proof relevant (*goodbye UIP*)
- Interpretation in simplicial sets (*types are spaces, terms are points, equalities are paths, equalities between equalities are homotopies, ...*)
- Univalence Axiom (*equality of types is isomorphism*)

Let's put them together!

We are interested in numerous examples:

- 1-types and groupoids (to study HITs)
- comprehension categories and CwFs (for the semantics of TT)

These have the structure of **bicategories**.

For these two applications: we need a formalization of bicategories.

The Basics: Categories

Definition

A **category** $\mathcal{C}$ consists of

- A type $\mathcal{C}_0$ of *objects*
- For $X, Y : \mathcal{C}_0$ a set $\mathcal{C}_1(X, Y)$ of *morphisms*
- *Identities* $\mathrm{id}_X : \mathcal{C}_1(X, X)$
- A *composition function* $\circ : \mathcal{C}_1(Y, Z) \times \mathcal{C}_1(X, Y) \to \mathcal{C}_1(X, Y)$
- An equality $\mathrm{id}_Y \circ f = f$ (*left unitality*)
- An equality $f \circ \mathrm{id}_X = f$ (*right unitality*)
- An equality $h \circ (g \circ f) = (h \circ g) \circ f$ (*associativity*)

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What's that set doing there?

In UF: a **set** is a type for which all proofs of equality are equal.
Recall: we work in a setting with proof relevant equality.
Examples of sets: $\mathbb{N}$, unit type.
These are not: the type of types, functors between categories.
We need it, because equality on arrows give higher structure.

From Categories to Bicategories

Definition

A **category** $\mathcal{C}$ consists of

- A type $\mathcal{C}_0$ of objects
- For $X, Y : \mathcal{C}_0$ a **set** $\mathcal{C}_1(X, Y)$ of morphisms
- Identities $\mathrm{id}_X : \mathcal{C}_1(X, X)$
- A composition **function** $\circ : \mathcal{C}_1(Y, Z) \times \mathcal{C}_1(X, Y) \to \mathcal{C}_1(X, Y)$
- An **equality** $\mathrm{id}_Y \circ f = f$ (left unitality)
- An **equality** $f \circ \mathrm{id}_X = f$ (right unitality)
- An **equality** $h \circ (g \circ f) = (h \circ g) \circ f$ (associativity)

From Categories to Bicategories

Definition

A **bicategory** $\mathcal{C}$ consists of

- A type $\mathcal{C}_0$ of objects
- For $X, Y : \mathcal{C}_0$ a **category** $\mathcal{C}_1(X, Y)$ of morphisms
- Identities $\mathrm{id}_X : \mathcal{C}_1(X, X)$
- A composition **functor** $\circ : \mathcal{C}_1(Y, Z) \times \mathcal{C}_1(X, Y) \to \mathcal{C}_1(X, Y)$
- A **natural iso** $\mathrm{id}_Y \circ f \Rightarrow f$ (left unitalit**or**)
- A **natural iso** $f \circ \mathrm{id}_X \Rightarrow f$ (right unitalit**or**)
- A **natural iso** $h \circ (g \circ f) \Rightarrow (h \circ g) \circ f$ (associat**or**)

such that (big scary coherencies)

Now let's implement it

Note: for the implementation, use an unfolded definition. This separates data and properties, so

Definition

A bicategory $\mathcal{C}$ consists of

- A type $\mathcal{C}_0$ of 0-cells
- For $X, Y : \mathcal{C}_0$ a type $\mathcal{C}_1(X, Y)$ of 1-cells
- For $f, g : \mathcal{C}_1(X, Y)$, a **set** $\mathcal{C}_2(f, g)$ of 2-cells
- Identities $\mathrm{id}_2(f) : \mathcal{C}_2(f, f)$ for $f : \mathcal{C}_1(X, Y)$
- Compositions $\beta \circ \alpha$ for $\beta : \mathcal{C}_2(g, h)$ and $\alpha : \mathcal{C}_2(f, g)$
- (and more)

Our Favorite Bicategories

Some examples of bicategories:

- Categories
- Groupoids
- 1-types
- Comprehension Categories
- Categories with Families (CwFs)

All seems good, **but...**

This might look good, **but**

- UF has a model in simplicial sets
- We want "our bicategories" to be interpreted as actual set-theoretic bicategories in this interpretation
- But for that, we need an extra condition

They need to be univalent!

Univalent category: equality of objects = isomorphisms
Now for bicategories:

- Note: equivalence for 0-cells is *adjoint equivalence*
- Note: equivalence for 1-cells is *invertible 2-cells*

A bicategory is **univalent** if

- equality on 0-cells = adjoint equivalence
- equality on 1-cells = invertible 2-cells

And they are

Examples of univalent bicategories

- univalent categories
- 1-types
- univalent groupoids

But what about CwFs?

And they are

Examples of univalent bicategories

- univalent categories
- 1-types
- univalent groupoids

But what about CwFs?
Difficult to show directly!

Build Them with Displayed Bicategories!

Build bicategories from small simple parts

We need new machinery

- Tool: **displayed bicategories** (see: displayed categories)
- Then: define bicategories by adding layers of structure

Main point: each displayed bicategory $\mathcal{D}$ over $\mathcal{B}$ gives rise to a total one.

This is $\mathcal{B}$ with the additional structure described by $\mathcal{D}$.

Let's put them into action: categories with an operation

- Our wish: categories with a binary operation
- Build it from the bicategory of categories

Define a displayed bicategory D over categories

- Displayed 0-cells over C: functors $m_C : C \times C \to C$
- Displayed 1-cells $F : C \to D$ from m_C to m_D: natural transformations

$$F \circ m_C \Rightarrow m_D \circ (F, F)$$

- Displayed 2-cells: (longer formula)

This gives a total bicategory E.
It is the bicategory of category with an operation.

But we also wanted commutativity

- But now I want (C, m_C) with a natural transformation

$$m_C \Rightarrow m_C \circ \sigma$$

(σ switches the arguments)

- Build them from categories with an operation!

Define a displayed bicategory D' over E (from last slide)

- Displayed 0-cells over (C, m_C): natural transformations $m_C \Rightarrow m_C \circ \sigma$
- Displayed 1-cells and 2-cells have more complicated formulae.

The total bicategory: categories with a commutative operation.
Note: we reused previous definitions.

More Complicated Examples

We can use it to make complicated bicategories:

- Pseudofunctors between bicategories
- Algebraic structures
- CwFs

And now: univalence with displayed bicategories

Theorem

Let B be a bicategory and let D be a displayed bicategory on B. If both B and D are univalent, then so is the total bicategory of D.

We use this to prove univalence of

- Pseudofunctors between bicategories (if target is univalent)
- Algebraic structures
- CwFs

Reason about small edible pieces rather than huge chunks

Take-Home Message

With displayed bicategories one can conveniently construct complicated bicategories layerwise and prove their univalence

Note: all is formalized in UniMath.