An overview of results on the topological classification of the function spaces $C_p(X)$ by means of absorbing sets

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1. INTRODUCTION

1.1 Introduction

Topology is the study of topological spaces and their properties, especially those properties that remain unchanged after applying homeomorphisms. Together with this, the problem to determine if two spaces are homeomorphic is of major importance in Topology. A method that assist the solution to this problem, is the method of absorbing sets, which is the major topic of this thesis.

To give an intuitive idea of what an absorbing set is, consider the following example from [5]. If we consider the subspace $E = \mathbb{Q} \cup \{\pi\}$ of $\mathbb{R}$, then a straightforward back-and-forth argument yields the existence of a bijection $f : Q \to E$. This mapping can even be extended to an bijection $\overline{f} : \mathbb{R} \to \mathbb{R}$. So $\mathbb{Q}$ and E are homeomorphic, via a homeomorphism that extends to $\mathbb{R}$. In other words, $\mathbb{Q}$ can absorb a point of $\mathbb{R}$. It can be shown that a set $C \subset \mathbb{R}$ can be absorbed by $\mathbb{Q}$ if and only if C is countable. Also, it turns out that absorbers like $\mathbb{Q}$ in $\mathbb{R}$ are unique. If A is a countable subset of $\mathbb{R}$ that can absorb every countable subset of $\mathbb{R}$, then there is a homeomorphism $f : \mathbb{R} \to \mathbb{R}$ such that $f(\mathbb{Q}) = A$.

The aim of this thesis is to present the theory of absorbing sets in a clearer form than is done in the literature. It is primarily addressed to (graduate) students with a basic knowledge of general Topology. The theory has already evolved to a point where it is difficult to access by people starting in this area, especially because there is no standard text-book available related to this subject. A lot of results are well-known to people working in this area and therefore omitted in the literature. This thesis will try to fill in these gaps and moreover to give an overview of the most prominent results made in this area.

This thesis will be structured as follows: Chapter 2 will introduce some definitions and auxiliary result that will be used in the remaining parts of this thesis. The concepts that will be discussed include **A(N)R** theory, filters, the Cantor set and Borel sets and classes. Chapter 3 will define the notion of an absorbing set. We will only concentrate on absorbing sets in $\mathbb{R}^\infty$. Finally in Chapter 4 we will discuss an application. We will look at the space of real-valued functions on a space X with the topology of pointwise convergence. Some of its properties will be derived and an overview of results obtained by the method of absorbing sets will be given.

1.2 Notations and terminology

In this section we fix some notations and conventions for the remaining part of this thesis.

the word "iff" is an abbreviation for "if and only if". The mark "□" indicates the end of a proof.

The set of positive integers, the set of integers, the set of rationals and the set of reals are denoted by $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$ and $\mathbb{R}$, respectively. We let $\overline{\mathbb{R}}$ denote $\mathbb{R} \cup \{-\infty, \infty\}$. As usual, I will denote the unit interval $[0,1]$ and $Q = \prod_{i=1}^{\infty}[-1,1]_i$ the Hilbertcube. Additionally $s = \{x \in Q : \forall i \in \mathbb{N} : -1 < |x_i| < 1\}$ and $B = Q \setminus s$ denotes the pseudo-interior and pseudo-boundary of Q, respectively. Furthermore, $\mathbf{C}$ denotes the Cantor set $\{0,1\}^{\mathbb{N}}$, i.e. the space of all infinite rows of 0's and 1's and $2^{\mathbb{N}}$ denotes the space of all subsets of $\mathbb{N}$.

For any space X we let d denote an admissible metric on X, i.e., a metric that generates the topology. All our metrics are bounded by 1. If $X = \prod_{i=1}^{\infty} X_i$, a countable infinite product, we will use the metric d on X given by

$$d(x,y) = \sum_{i=1}^{\infty} 2^{-i} d_i(x_i, y_i),$$

where d_i for every i is some admissible metric on X_i. Note that if two elements of $\prod_{i=1}^{\infty} X_i$ agree in the first N coordinate directions then their distance is at most 2^{-N}.

We will denote the identity mapping on a space X by 1_X. The projection mapping will be denoted by π. We will use χ to denote the characteristic function $\chi : 2^{\mathbb{N}} \to \mathbf{C}$ (see section 2.2). The distance between two functions $f, g : X \to Y$ is defined by $\hat{d}(f,g) = \sup_{x \in X}\{d(f(x), g(x))\}$ where d is a metric on Y. Furthermore, let $\mathcal{U}$ be an open cover for a space Y and let $f, g : X \to Y$ be two functions. We call f and g $\mathcal{U}$-close if for every $x \in X$ there exists $U \in \mathcal{U}$ such that $\{f(x), g(x)\} \subset U$. For a map $f : X \to Y$ and a subset A of X we will write $f[A]$ for $\{f(a) : a \in A\}$.

For a subset A of a space X, we will denote the product of n copies of A by A^n and the product of a countably infinite set of copies of A by A^{∞} or $A^{\mathbb{N}}$. Furthermore, we denote the closure of A by $\overline{A}$ and the interior of A by A°. When U is an open neighborhood of A we will use the notation U_A. Usually A is just a point, i.e. $A = \{x\}$, and we will write U_x.

2. AUXILIARY RESULTS

2.1 AR's and ANR's

Let X be a space and let $A \subset X$ be closed. We say that A is a *retract* of X provided that there is a continuous function $r : X \to A$ such that r restricted to A is the identity. Such a function r is called a *retraction*. We say that A is a *neighborhood retract* of X provided that there exists a neighborhood U of A in X such that A is a retract of U.

A space is called an *Absolute Retract* (abbreviated **AR**) provided that X is a retract of every space Y containing X as a closed subspace. A space is called an *Absolute Neighborhood Retract* (abbreviated **ANR**) provided that X is a neighborhood retract of every space Y containing X as a closed subspace.

Notice that every **AR** is an **ANR**. Important examples of **AR**'s are the spaces $\mathbb{R}^n$, I^n, Q and $\mathbb{R}^\infty$. The following characterization of **A(N)R**'s is of fundamental importance:

Theorem 2.1.1 *Let X be a separable metric space. The following are equivalent:*

(1) X is an **A(N)R**,

(2) for every separable metric space Y and for every closed subspace A of Y, every continuous function $f : A \to X$ can be extended over Y (over a neighborhood (depending on f) of A in Y).

Proof: See [20], Theorem 1.5.2. □

The following result is well-known.

Theorem 2.1.2 *Let Y be an* **ANR**. *Then for every open cover $\mathcal{U}$ of Y there exists an open refinement $\mathcal{V}$ of $\mathcal{U}$ such that for every space X, any two $\mathcal{V}$-close maps $f, g : X \to Y$ are $\mathcal{U}$-homotopic.*

Proof: See [20], Theorem 5.1.1. □

2.2 The Cantor set

As mentioned in section 1.2, $\mathbf{C}$ denotes the cantor set $\{0,1\}^{\mathbb{N}}$. We shall define a metric d on $\mathbf{C}$. For every $s \in \mathbf{C}$ we define $d(s,s) = 0$. For different $s, t \in \mathbf{C}$, there exists an index $m \in \mathbb{N}$ such that $s_j = t_j$ for $1 \leq j < m$ and $s_m \neq t_m$. We define $d(s,t) = \frac{1}{m}$. Observe, that for a point s in $\mathbf{C}$ the open ball with radius $\frac{1}{n}$ exists of all points t such that $s_j = t_j$ for $1 \leq j \leq n$.

Each subset A of $\mathbb{N}$ can be identified with a point in the Cantor set $\mathbf{C}$, namely with the point χ_A defined by $(\chi_A)_i = 1$ if and only if $i \in A$. We identify $2^{\mathbb{N}}$ with $\mathbf{C}$. It follows that $\mathbf{C}$ and $2^{\mathbb{N}}$ are homeomorphic.

In section 4.3 we will need the following Lemma from [8]:

Lemma 2.2.1 *Let A be a closed and nowhere dense subset of the cantorset* $\mathbf{C}$. *Then for every $n \in \mathbb{N}$, there exists $a \in \{0,1\}^{\mathbb{N}\setminus\{1,2,\dots,n\}}$ such that* $(\{0,1\}^{\{1,2,\dots,n\}} \times \{a\}) \cap A = \varnothing$.

Proof: Take an arbitrary $n \in \mathbb{N}$. Suppose to the contrary that for every $a \in \{0,1\}^{\mathbb{N}\setminus\{1,2,\dots,n\}}$ we have $(\{0,1\}^{\{1,2,\dots,n\}} \times \{a\}) \cap A \neq \varnothing$. Next define for $s \in \{0,1\}^{\{1,2,\dots,n\}}$, $A_s = \{x \in \{0,1\}^{\mathbb{N}\setminus\{1,2,\dots,n\}} : (s,x) \in A\}$. Then by assumption $\bigcup_{s\in\{0,1\}^{\{1,2,\dots,n\}}} A_s = \{0,1\}^{\mathbb{N}\setminus\{1,2,\dots,n\}}$. We will proof that each set A_s is closed in $\{0,1\}^{\mathbb{N}\setminus\{1,2,\dots,n\}}$. Let $x \in \{0,1\}^{\mathbb{N}\setminus\{1,2,\dots,n\}} \setminus A_s$. Then $(s,x) \notin A$. Since A is closed in $\mathbf{C}$ we can find a neighborhood V of x such that $(\{s\}\times V)\cap A = \varnothing$. This gives $V \cap A_s = \varnothing$. Since $\{0,1\}^{\{1,2,\dots,n\}}$ is finite, $\{0,1\}^{\mathbb{N}\setminus\{1,2,\dots,n\}}$ is the union of a finite number of closed subsets. Therefore, there exists $s \in \{0,1\}^{\{1,2,\dots,n\}}$ such that A_s has non-empty interior. Let U be a non-empty open subset of A_s. Then $\{s\} \times U$ is open in $\mathbf{C}$ and $\{s\} \times U \subset A$. This contradicts the fact that A is nowhere dense. □

2.3 Filters

A non empty family F of subsets of a space X is a *filter* if

(1) $\varnothing \notin F$,

(2) If $A, B \in F$, then $A \cap B \in F$,

(3) If $A \in F$ and $A \subset B \subset X$ then $B \in F$.

Given a filter on $\mathbb{N}$ we denote by $\mathbb{N}_F$ the space $\mathbb{N} \cup \{\infty\}$ topologized by isolating the points of $\mathbb{N}$ and using the family $\{A \cup \{\infty\} : A \in F\}$ to be a neighborhood base at ∞. A filter F is a *free filter* if $\bigcap_{A\in F} A = \varnothing$. The filter consisting of all co-finite subsets of $\mathbb{N}$ is called the *Fréchet filter* and is denoted by F_0. Furthermore, if F is a filter on $\mathbb{N}$, we define

$$c_F = \{(x_n) \in \mathbb{R}^\infty : \forall \epsilon > 0 : \exists A \in F : \forall n \in A : |x_n| < \epsilon\}.$$

Proposition 2.3.1 *Let F be a filter on $\mathbb{N}$. Then the following statements are equivalent:*

(1) F is a free filter on $\mathbb{N}$,

(2) $\mathbb{N}_F$ is a Tychonoff space,

(3) $F_0 \subset F$,

(4) F lies dense in $2^{\mathbb{N}}$.

Proof: We prove (1) $\Rightarrow$ (2)

Because F is a free filter, there exists $A_n \in \mathbb{N}$ for every $n \in \mathbb{N}$ such that $n \notin A_n$. It follows that $\{n\} = \mathbb{N}_F \setminus (\bigcup_{m\in\mathbb{N}, m\neq n}\{m\} \cup (A_n \cup \{\infty\}))$ and that $\{\infty\} = \mathbb{N}_F \setminus \bigcup_{m\in\mathbb{N}}\{m\}$ are closed, proving that $\mathbb{N}_F$ is T_1. Next, take $p \in \mathbb{N}_F$ and $G \subset \mathbb{N}_F$ closed such that $p \notin G$. If $p = \infty$ take the map f such that $f|_G \subset 0$ and $f|_{\mathbb{N}_F\setminus G} \subset 1$. The open neighborhood $\mathbb{N} \setminus G$ of ∞ shows that f

is continuous. If $p \neq \infty$ take the map f such that $f(p) = 0$ and $f(x) = 1$ for every $x \neq p$. The open neighborhood $A_p \cup \{\infty\}$ of ∞ shows that f is continuous.

We prove (2) $\Rightarrow$ (3)

Because every point in $\mathbb{N}_F$ is closed, every finite set in $\mathbb{N}_F$ is closed. Thus every co-finite set is open in $\mathbb{N}_F$. It follows that every co-finite set of $\mathbb{N}$ must be in F.

We prove (3) $\Rightarrow$ (4)

Let $U = \{s_1\} \times \ldots \times \{s_m\} \times \{0,1\}^{\mathbb{N}\setminus\{1,\ldots,m\}}$ be an element of the basis in $\{0,1\}^{\mathbb{N}}$, with $m \in \mathbb{N}$ and $s_j \in \{0,1\}$ for $1 \leq j \leq m$. Next, define the co-finite set $G = \mathbb{N} \setminus \{j : 1 \leq j \leq m, s_j = 0\}$. It follows that $\chi_G \in U$.

We prove (4) $\Rightarrow$ (1)

Fix $n \in \mathbb{N}$. The set $G_n = \{A \subset \mathbb{N} : n \in A\}$ of $2^{\mathbb{N}}$ is closed, because χ_{G_n} is closed. By assumption there exists $A_n \in (2^{\mathbb{N}} \setminus G_n) \cap F$. It follows that $\bigcap_{n\in\mathbb{N}} A_n = \varnothing$. $\square$

2.4 Borel sets and classes

By the family of *Borel sets* in a topological space X we mean the smallest family $\mathcal{S}$ of subsets of X satisfying the following conditions:

(1) The family $\mathcal{S}$ contains all open subsets of X.

(2) If $A \in \mathcal{S}$, then $X \setminus A \in \mathcal{S}$.

(3) If $A_i \in \mathcal{S}$ for $i = 1, 2, \ldots$, then $\bigcup_{i=1}^{\infty} A_i \in \mathcal{S}$.

The most common Borel sets, besides closed and open sets, are countable unions of closed sets called *F_σ-sets* and countable intersections of open sets called *G_δ-sets*. (It is clear that an F_σ-set is the complement of a G_δ-set.) We can continue this indefinite and create $F_{\sigma\delta}$-sets which are countable intersections of F_σ-sets and $G_{\delta\sigma}$-sets which are countable unions of G_δ-sets.

We define for each space X and for each countable ordinal α *the additive Borelian class α, $\mathcal{A}_\alpha(X)$*, and the *multiplicative Borelian class α, $\mathcal{M}_\alpha(X)$*, of subsets of X as follows: $\mathcal{A}_0(X)$ is the collection of all open subsets of X, and $\mathcal{M}_0(X)$ is the collection of all closed subsets op X. Suppose that for $\zeta < \alpha$ the collections $\mathcal{A}_\zeta(X)$ and $\mathcal{M}_\zeta(X)$ have been defined. Then $\mathcal{A}_\alpha(X)$ is the collection of all subsets of X that can be represented as $X_1 \cup X_2 \cup X_3 \cup \cdots$ with each $X_i \in \bigcup_{\zeta<\alpha} \mathcal{M}_\zeta(X)$, and $\mathcal{M}_\alpha(X)$ is the collection of all subsets of X that can be represented as $X_1 \cap X_2 \cap X_3 \cap \cdots$ with each $X_i \in \bigcup_{\zeta<\alpha} \mathcal{A}_\zeta(X)$. For a countable ordinal α we define *the absolute Borelian classes $\mathcal{A}_\alpha$ and $\mathcal{M}_\alpha$*. A space X belongs to $\mathcal{A}_\alpha$ (resp. $\mathcal{M}_\alpha$) if and only if for any embedding $e : X \to Y$ we have $e[X] \in \mathcal{A}_\alpha(Y)$ (resp. $e[X] \in \mathcal{M}_\alpha(Y)$).

Observe that $\mathcal{A}_0 = \varnothing$; $\mathcal{M}_0$ consist of all compacta, $\mathcal{A}_1$ of all σ-compacta; $\mathcal{M}_1$ is the collection of all completely metrizable spaces, and so forth.

Lemma 2.4.1 *Let A be absolute Borel and let B be a G_δ-subset of A. Let $\alpha \geq 2$. If $A \in \mathcal{M}_\alpha$ then $B \in \mathcal{M}_\alpha$ and if $A \in \mathcal{A}_\alpha$ then $B \in \mathcal{A}_\alpha$.*

Proof: Let X be a topological space such that A embeds in X. We may assume that X is perfectly normal. Write $B = \bigcap_{i=1}^{\infty} U_i$ with U_i open in A. For every i there exists an open subset U_i' of X such that $U_i' \cap A = U_i$.

First, assume $A \in \mathcal{M}_2(X)$. Because A is an $F_{\sigma\delta}$ subset of X we can write $A = \bigcap_{i=1}^{\infty} E_i$ with E_i an F_σ-set. Because X is perfectly normal, every open set is an F_σ-set. It follows that $V_i = U_i' \cap E_i$ is an F_σ-set and consequently, $\bigcap_{i=1}^{\infty} V_i = B$ is an $F_{\sigma\delta}$ set.

Next, assume $A \in \mathcal{A}_2(X)$. Because A is an $G_{\delta\sigma}$ subset of X we can write $A = \bigcup_{i=1}^{\infty} F_i$ with F_i a G_δ-set. If follows that $W_i = (\bigcap_{i=1}^{\infty} U_i') \cap F_i$ is a G_δ-set and consequently, $\bigcup_{i=1}^{\infty} W_i = B$ is a $G_{\delta\sigma}$-set.

The rest follows with transfinite induction. $\square$

3. ABSORBING SETS

Basically there are two types of absorbing systems in infinite-dimensional topology. Absorbing systems of the first type are generalizations of the capsets of R.D. Anderson and the skeletoids of Bessaga and Pelczyński (see [2] and references there in). They are of interest if one wishes to prove that pairs of spaces are homeomorphic. Absorbing systems of the second type are the ones of Bestvina and Mogilski (see [3] and references there in). They have proved to be useful for obtaining internal topological characterizations of various interesting spaces.

In this section we will study a variation of absorbing sets in $\mathbb{R}^\infty$. They are of the second type.

3.1 Z-sets

A closed subset A of a topological space X is a *Z-set* if, for every open cover $\mathcal{U}$ of X and every continuous map $f : Q \to X$, there exists a continuous map $g : Q \to X \setminus A$ that is $\mathcal{U}$-close to f. The collection of all Z-sets in X will be denoted by $\mathcal{Z}(X)$. A subset A of X which is a countable union of Z-sets is called a *σZ-set* in X. The collection of all σZ-sets in X will be denoted by $\mathcal{Z}_\sigma(X)$. A space which is a countable union of its own Z-sets is called a Z_σ-space. By a *Z-embedding* we mean an embedding the image of which is a Z-set.

The following properties for Z-sets are elementary and will be used without explicit references:

Proposition 3.1.1 *Let X be a topological space and $A \in \mathcal{Z}(X)$. We have the following properties:*

(1) If $B \subset A$ is closed then $B \in \mathcal{Z}(X)$,

(2) A has empty interior,

(3) If $B \in \mathcal{Z}(X)$ then $A \cup B \in \mathcal{Z}(X)$. Hence, a finite union of Z-sets is a Z-set,

(4) If $h : X \to Y$ is a homeomorphism then $h[A] \in \mathcal{Z}(Y)$,

(5) For any topological space Y we have $A \times Y \in \mathcal{Z}(X \times Y)$.

Proof: The statements are trivial. □

An important example of a σZ-set is the pseudoboundary B of Q. Other examples of σZ-sets in Q are the subspace $\Sigma = \bigcup_{n=1}^\infty \Sigma_n$ with $\Sigma_n = \{x \in Q : (\forall i \in \mathbb{N})(|x_i| \leq 1 - 2^{-n})\}$ (used in [20]) or the space $Z_n = \{x \in Q : |x_i| < 2^{-n}$ for all but finitely many $i\}$ (used in [11]). Examples of Z-sets in Q are the compact subsets of s. Examples of Z_σ-spaces are function spaces which are Borel, which will be studied in chapter 4.

For certain spaces, there are equivalent definitions used throughout the literature. We will state equivalent definitions for metric spaces and **ANR**'s. When X is a metric space we have the following 'metric' translation from [20]:

Lemma 3.1.2 *Let X be a metrizable space and let $A \subset X$ be closed. Then the following statements are equivalent:*

(1) $A \in \mathcal{Z}(X)$,

(2) $\forall \epsilon > 0 : \forall f \in C(Q, X) : \exists g \in C(Q, X \setminus A)$ *such that* $\hat{d}(f, g) < \epsilon$.

Proof: We only have to prove (2) $\Rightarrow$ (1)

Let $\mathcal{U}$ be an open cover of X and let $f : Q \to X$ be continuous. By compactness of $f[Q]$, there exists $\delta > 0$ with the property that every $A \subset X$ with $\operatorname{diam}(A) < \delta$ and which moreover intersects $f[Q]$, is contained in an element $U \in \mathcal{U}$. (The number δ is the *Lebesque number* for $\mathcal{U}$, see [20] or [7].) By assumption there is a function $g \in C(Q, X \setminus A)$ with $\hat{d}(f, g) < \delta$. We claim that f and g are $\mathcal{U}$-close. Pick an arbitrary element $x \in Q$. Then $d(f(x), g(x)) < \delta$ and since $f(x) \in f[Q]$, $\{f(x), g(x)\}$ is contained in an element $U \in \mathcal{U}$. $\square$

There is also an equivalent notion for **AR**'s and **ANR**'s:

Theorem 3.1.3 *Let X be an* **ANR** *and let $A \subset X$ be closed. The following statements are equivalent:*

(1) $A \in \mathcal{Z}(X)$,

(2) for every open cover $\mathcal{U}$ of X there exists a continuous function $f : X \to X \setminus A$ such that f and 1_X are $\mathcal{U}$-close,

(3) for every open cover $\mathcal{U}$ of X and every continuous function $f : K \to X$ of a compactum K there exists $g : K \to X \setminus A$ such that f and g are $\mathcal{U}$-close.

Proof: See [20] Theorem 7.2.5 and Corollary 7.2.6. $\square$

We finish this section with a Proposition that relates some Z-sets in Q with some Z-sets in its pseudointerior s.

Proposition 3.1.4 *Let D be a (relatively) closed subset of s and let E be its closure in Q. Then D is a Z-set in s if and only if E is a Z-set in Q.*

Proof: First assume D is a Z-set in s. Let $\epsilon > 0$ and let $f \in C(s, s)$ such that $f[s] \cap D = \varnothing$ and $\hat{d}(f, 1_s) < \frac{1}{2}\epsilon$. There exists $g \in C(Q, Q)$ with $g[Q] \cap B = \varnothing$ and $\hat{d}(g, 1_Q) < \frac{1}{2}\epsilon$ (see [20]). Then $f \circ g \in C(Q, Q)$ and $(f \circ g)[Q] \cap E = \varnothing$. Moreover, $\hat{d}(f \circ g, 1_Q) \leq \hat{d}(f \circ g, g) + \hat{d}(g, 1_Q) \leq \hat{d}(f, 1_s) + \hat{d}(g, 1_Q) < \epsilon$.

Next assume E is a Z-set in Q. Let $\epsilon > 0$ and let $f \in C(Q, Q)$ such that $f[Q] \cap (B \cup E) = \varnothing$ and $\hat{d}(f, 1_Q) < \epsilon$. Let $g = f|_s$, then g is as desired. $\square$

3.2 Locally homotopy negligible sets

A subset A of a space X is said to be *locally homotopy negligible* if, given $x \in X$, $k < \infty$ and a neighborhood U_x of x, there is a neighborhood $V_x \subset U_x$ of x such that for each $f : (B^k, S^{k-1}) \to (V_x, V_x \setminus A)$ there is a homotopy $H_t : (B^k, S^{k-1}) \to (U_x, U_x \setminus A)$ with $H_0 = f$ and $H_1[B^k] \subset U_x \setminus A$.

This concept was first defined by Toruńczyk [18]. Several equivalent definitions are used in the literature. The following is taken from [8].

A subset A of a space X is *locally homotopy negligible* in X if for every map $f : M \to X$, where M is any ANR, and for every open cover $\mathcal{U}$ of X there exists a homotopy $H : M \times I \to X$ such that $\{H[\{x\} \times I]\}_{x \in M}$ refines $\mathcal{U}$, $H_0 = f$ and $H[M \times (0,1]] \subset X \setminus A$.

By a result of Toruńczyk [18], if X is an ANR, for a set $A \subset X$ to be locally homotopy negligible it suffices to consider maps $f : M \to X$, where $M \in \{\{pt\}, I, I^2, \ldots, I^n, \ldots\}$.

3.3 Absorbing sets

A class $\mathcal{C}$ of spaces is said to be *topological* if it contains every homeomorph of an element of $\mathcal{C}$. It is *additive* if, given an element $C \in \mathcal{C}$ such that $C = C_1 \cup C_2$ with C_1 and C_2 closed in C, it follows that C_1 and C_2 belong to $\mathcal{C}$. Finally, it is called *hereditary with respect to closed subsets* if, for every $C \in \mathcal{C}$ and every closed $D \subset C$, D belongs to $\mathcal{C}$.

Let $\mathcal{C}$ be a class of spaces. A space X is strongly *$\mathcal{C}$-universal* if for every map $f : C \to X$ from a space $C \in \mathcal{C}$, for every closed subset $D \subset C$ such that $f|_D : D \to X$ is a Z-embedding, and for every open cover $\mathcal{U}$ of X, there exists a Z-embedding $g : C \to X$ such that $g|_D = f|_D$ and f and g are $\mathcal{U}$-close.

Let $\mathcal{C}$ be a class of (separable, metric) spaces which is topological, additive and hereditary with respect to closed subsets. A subset X of $\mathbb{R}^\infty$ is called a *$\mathcal{C}$-absorbing set* if it satisfies the following conditions:

(1) $\mathbb{R}^\infty \setminus X$ is locally homotopy negligible in $\mathbb{R}^\infty$,

(2) $X = \bigcup_{n=1}^\infty Z_n$, where each Z_n is a Z-set in X and belongs to $\mathcal{C}$,

(3) X is strongly $\mathcal{C}$-universal.

It follows from [18] (Theorem 3.1) that every $\mathcal{C}$-absorbing set is an **ANR**. The usefullness of absorbing sets is stated in the following Theorem:

Theorem 3.3.1 *Any two $\mathcal{C}$-absorbing sets in $\mathbb{R}^\infty$ are homeomorphic.*

Proof: This a particular case of Theorem 3.1 in [3]. □

The method of absorbing sets now uses Theorem 3.3.1 to reduce the problem of homeomorphy of the spaces under consideration to the problem of recognizing wether they are absorbing sets for the same class.

Some examples of absorbing set in $\mathbb{R}^\infty$ are the following:

(1) Every countable dense subset of $\mathbb{R}^\infty$ is an absorbing set for the class of all finite sets (see [2]),

(2) The space $\sigma = \{(x_i) \in \mathbb{R}^\infty : x_i = 0 \text{ for all but finitely many } i\}$ is an absorbing set for the class of all finite-dimensional compacta (see [3]),

(3) For every countable ordinal $\alpha \geq 1$ there exists a $\mathcal{A}_\alpha$-absorbing set Λ_α (see [3]),

(4) For every countable ordinal $\alpha \geq 2$ there exists a $\mathcal{M}_\alpha$-absorbing set Ω_α (see [3]).

For a more comprehensive list of absorbing sets in $\mathbb{R}^\infty$ see [6].

4. FUNCTION SPACES

4.1 The function space $C_p(X)$

Let X and Y be topological spaces and $C(X,Y)$ the set of all continuous mappings from X into Y. By $C^*(X,Y)$ we denote the set of all *bounded* mappings in $C(X,Y)$. We will abbreviate $C(X,\mathbb{R})$ to $C(X)$.

For any points $x_1,\ldots,x_n$ in X and open sets $V_1,\ldots,V_n$ in $\mathbb{R}$, put

$$U(x_1,\ldots,x_n;V_1,\ldots,V_n)=\{f\in C(X): f(x_i)\in V_i \text{ for } i=1,\ldots,n\}.$$

Then the family

$$\{U(x_1,\ldots,x_n;V_1,\ldots,V_n): x_1,\ldots,x_n\in X; V_1,\ldots,V_n \text{ open in } \mathbb{R}, n\in\mathbb{N}\},$$

forms a base and generates the *topology of pointwise convergence* on $C(X)$ and will be denoted by $C_p(X)$. From this, it is easily seen that the collection of all sets of the form

$$N(x_1,\ldots,x_n,s_1,\ldots,s_n,\epsilon)=\{f\in\mathbb{R}^X:(\forall i)(|f(x_i)-s_i|<\epsilon)\},$$

where $x_1,\ldots,x_n\in X$, $s_1,\ldots,s_n\in\mathbb{R}$ and $\epsilon>0$, forms a base.

When $f\in C_p(X)$, we can identify f with the point $y\in\mathbb{R}^X$ such that $\pi_x(y)=f(x)$. It follows that the space $C_p(X)$ can be identified with a subset of $\mathbb{R}^X$, hence $C_p(X)$ is a Tychonoff space. If X is discrete then $C_p(X)=\mathbb{R}^X$, because every function $f:X\to\mathbb{R}$ is continuous. If moreover, X is countable then $C_p(X)=\mathbb{R}^k$ or $C_p(X)=\mathbb{R}^\infty$. If X is not discrete, but at least functionally Hausdorff then $C_p(X)$ is a dense subspace of $\mathbb{R}^X$. This is a standard result, that is often used in the literature. Because the proof is almost always omitted, we present a constructive proof.

Lemma 4.1.1 *Let X be a topological space. Then the following statements are equivalent:*

(1) $C_p(X)$ is a dense subspace of $\mathbb{R}^X$,

(2) $(\forall x,y\in X)(x\neq y\Rightarrow \exists f\in C_p(X): f(x)\neq f(y))$, i.e. X is functionally Hausdorff.

Proof: We prove (1) $\Rightarrow$ (2)

Take $x_1,x_2\in X$ with $x_1\neq x_2$ and $s_1,s_2\in\mathbb{R}$ with $s_1\neq s_2$. Define $\epsilon=\frac{1}{4}\cdot|s_1-s_2|$. Because $N(x_1,x_2,s_1,s_2,\epsilon)\cap C_p(X)$ is not empty, the result follows.

We prove (2) $\Rightarrow$ (1)

We will prove with induction on n that $C_p(X)$ meets all non-empty base-elements $N(x_1,\ldots,x_n,s_1,\ldots,s_n,\epsilon)$.

For $n=1$ we can take the constant function, so assume $n=2$. By assumption there exists a function $f\in C_p(X)$ such that $f(x_1)\neq f(x_2)$. Let $\xi:\mathbb{R}\to\mathbb{R}$ be a continuous map such that $\xi(f(x_i))=s_i$ for $i=1,2$. Then $\xi\circ f\in C_p(X)\cap N(x_1,x_2,s_1,s_2,\epsilon)$.

Next, assume $n > 2$ and that for $1 \le m < n$, $C_p(X)$ meets all base-elements $N(x_1, \dots, x_m, s_1, \dots, s_m, \epsilon)$. By our induction hypothesis we may assume the existence of functions $f_1, \dots, f_n \in C_p(X)$ such that $f_i(x_j) = n^j$ whenever $i \neq j$. If there exists i such that $f_i(x_1), \dots, f_i(x_n)$ are all distinct, we can take a continuous map $\xi : \mathbb{R} \to \mathbb{R}$ such that $\xi(f_i(x_j)) = s_j$ for $j = 1, \dots, n$. So assume that $f_i(x_i)$ is an element of $\{n^j : 1 \le j \le n, j \neq i\}$ for $i = 1, \dots, n$. Now define the continuous map $g = f_1 + \dots + f_n$ and observe that $g(x_i) \neq g(x_j)$ whenever $i \neq j$. $\square$

This result has also been mentioned in [9], but they used the term *completely Hausdorff* for condition (2) in Lemma 4.1.1. This might be confusing, because in the literature the term completely Hausdorff is also used for Urysohn spaces, but these definitions are not equivalent. Remember that we call a space X a *Urysohn space* if for every distinct points x and y in X there exists an open neighborhood U_x and an open neighborhood U_y such that $\overline{U_x} \cap \overline{U_y} = \varnothing$. For the sake of completeness we shall show that functionally Hausdorff is more restrictive than Urysohn space. The following Proposition shows that every space that is functionally Hausdorff is a Urysohn space.

Proposition 4.1.2 *Let X be a topological space. If X is functionally Hausdorff, then X is an Urysohn space.*

Proof: Let x and y be different points in X. By assumption there exists $f \in C_p(X)$ with $f(x) \neq f(y)$. Choose open neighborhoods $U_{f(x)}$ and $U_{f(y)}$ such that $\overline{U_{f(x)}} \cap \overline{U_{f(y)}} = \varnothing$. Then $U_x = f^{-1}[U_{f(x)}]$ and $U_y = f^{-1}[U_{f(y)}]$ show that X is a Urysohn space. $\square$

To show that functionally Hausdorff is more restrictive than a Urysohn space, we shall present an example of a regular T_1-space that is not functionally Hausdorff. The example is taken from [17].

Lemma 4.1.3 *There exists a regular T_1-space that is not functionally Hausdorff.*

Proof: First we shall construct a number of subsets of $\mathbb{R}^2$. For even $n \in \mathbb{Z}$ define $L_n = \{(n, y) : 0 \le y < \frac{1}{2}\}$. For odd $n \in \mathbb{Z}$ and $k \in \mathbb{N}$ with $k \ge 2$, define the point $p_{n,k} = (n, 1 - \frac{1}{k})$ and the set $T_{n,k} = \{(n \pm t, 1 - t - \frac{1}{k}) : 0 < t \le 1 - \frac{1}{k}\}$. Let S_1 be the union of all L_n, S_2 the union of all $p_{n,k}$ and S_3 the union of all $T_{n,k}$. The underlying set for our topological space X is the union of S_1, S_2 and S_3, plus two additional points p_- and p_+ (see figure 4.1).

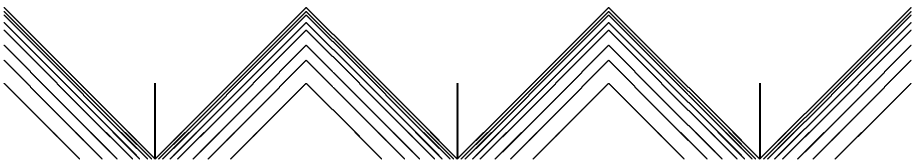

Figure 4.1: A regular space, not functionally Hausdorff.

Next we shall define a topology $\mathcal{T}$ on X. The topology is discrete for every point in S_3. A neighborhood of $p_{n,k}$ must contain all but finitely many points of $T_{n,k}$. A neighborhood of a point $(n, y) \in L_n$ contains all but finitely many points in $\{(n \pm t, y) : 0 \leq t < 1\}$. A neighborhood of p_- contains all points with first coordinate less than some real number c, and a neighborhood of p_+ contains all points with first coordinate greater than some real number c. Finally $\mathcal{T}$ contains all sets U such that for each $x \in U$, U is a neighborhood for x as the ones we just specified. It is easily verified that $\mathcal{T}$ is a topology on X.

CLAIM 1: X is a T_1-space.

Let $p \in X$. We shall show that $\{p\}$ is closed. If $p \in S_3$, then there exists n, k such that $p \in T_{n,k}$. Because $X \setminus \{p\}$ contains all but finitely many points of $T_{n,k}$ it is an open neighborhood of $p_{n,k}$. Similarly, if $p \in S_1$, $p \in S_2$, $p = p_-$ or $p = p_+$, $X \setminus \{p\}$ is an open neighborhood for some $p_{n,k}$.

CLAIM 2: X is regular.

It is easily seen that for every point $p \in X$ and open neighborhood O_p, there exists an open neighborhood U_p such that $p \in U_p \subset \overline{U}_p \subset O_p$, proving that X is regular.

CLAIM 3: X is not functionally Hausdorff.

We shall prove that every continuous real-valued function on X takes the same value at p_- and p_+. Let $x \in X$ and let $f \in C_p(X)$. Define for $n \in \mathbb{N}$, $N_n = \{y \in X : |f(x) - f(y)| < \frac{1}{n}\}$, then $\{y \in X : f(y) = f(x)\} = \bigcap_{n=1}^{\infty} N_n$. Now take $x = p_{n,k}$. A neighborhood of $p_{n,k}$ contains all but finitely many points of $T_{n,k}$. Hence, the set $S_{n,k} = \{y \in T_{n,k} : f(y) \neq f(p_{n,k})\}$ is countable. Let $S_n = \bigcup_k S_{n,k}$. Pick a point p in L_{n-1} or L_{n+1} such that there exists no point in S_n with the same second coordinate as p. Let $U_{f(p)}$ be an arbitrary open neighborhood of $f(p)$ and let $U_p \subset f^{-1}[U_{f(p)}]$ be an open neighborhood of p that misses every point of S_n. This neighborhood must intersect all but finitely many $T_{n,k}$. Thus, there exists $N \in \mathbb{N}$ such that for every $m \geq N$ we have $U_p \cap T_{n,m} \neq \varnothing$. For each $x \in U_p \cap T_{n,m}$ we have $f(x) = f(p_{n,m})$, hence, for all $m \geq N$ we have $f(p_{n,m}) \in U_{f(p)}$. Consequently,

$$f(p) = \lim_{k \to \infty} f(p_{n,k}) = c_n.$$

Analogous, $f(p) = \lim_{k \to \infty} f(p_{n-2,k}) = c_{n-2}$. If follows that $c_n = c$ for every n in $\mathbb{N}$ and some c in $\mathbb{R}$. Consequently, $f(p_-) = f(p_+) = c$, since f takes the value c in every neighborhood of both points. $\square$

Let us note that there even exists regular T_1-spaces on which every continuous real-valued function is constant. Most of these examples are, however, fairly complicated. For more information see references in [17].

As we already mentioned, the function space $C_p(X)$ can be identified with a subset of the Cartesian product $\mathbb{R}^X$. We will finish this section with a result that shows that the topology of pointwise convergence coincides with the topology of a subspace of the Cartesian product. The result is taken from [7] (Proposition 2.6.3).

Proposition 4.1.4 *The topology of pointwise convergence on $C_p(X)$ coincides with the topology of a subspace of the Cartesian product $\mathbb{R}^X$.*

Proof: Every open set in $C_p(X)$ with the topology of a subspace of the Cartesian product $\mathbb{R}^X$ is a union of sets of the form

$$C_p(X) \cap \pi_{x_1}^{-1}[V_1] \cap \pi_{x_2}^{-1}[V_2] \cap \ldots \cap \pi_{x_n}^{-1}[V_n],$$

where $x_i \in X$ and V_i is open in $\mathbb{R}$ for $i = 1, \ldots, n$. However,

$$C_p(X) \cap \pi_x^{-1}[V] = U(x; V),$$

so that all sets of the form in the upper equation, and all sets that are open with respect to the topology of a subspace of the Cartesian product, are open with respect to the topology of pointwise convergence.

Conversely, from the lower equation it follows that for $x_1, \ldots, x_n \in X$ and V open in $\mathbb{R}$, we have

$$U(x_1, \ldots, x_n; V) = C_p(X) \cap \pi_{x_1}^{-1}[V] \cap \pi_{x_2}^{-1}[V] \cap \ldots \cap \pi_{x_n}^{-1}[V],$$

so that all sets that are open with respect to the topology of pointwise convergence are open with respect to the topology of a subspace of the Cartesian product. □

4.2 Complement of $C_p(X)$

In this section we will show that $\mathbb{R}^\infty \setminus C_p(X)$ is locally homotopy negligible when X is a countable Tychonoff space. We will show that this is a consequence of the result proven in [15], that every function space $C_p(X)$ is strongly $\mathcal{F}_0(C_p(X))$-universal, where $\mathcal{F}_0(C_p(X))$ is the class of all spaces homeomorphic to a closed subset of $C_p(X)$ and X is a countable Tychonoff space. First, we need the following result.

Proposition 4.2.1 *Let $E \subset \mathbb{R}^\infty$ be a dense linear subspace. Then for all $\epsilon > 0$ there exists a continuous function $f : \mathbb{R}^\infty \to E$ such that $\hat{d}(f, 1_E) < \epsilon$.*

Proof: We identify $\mathbb{R}^\infty$ with s. Let $N \in \mathbb{N}$ such that $\sum_{n=N}^{\infty} 2^{-n} < \frac{1}{2}\epsilon$. Let $\pi : s \to (0,1)^N$ be the projection. We may identify $(0,1)^N$ with the subspace $(0,1)^N \times (0,0,\ldots)$ in s. Clearly we have $\hat{d}(\pi, 1_s) < \frac{1}{2}\epsilon$. For $i = 1, \ldots, n$ let e_i denote the vector having all entries zero except for a $\frac{1}{2}$ in the i^{th} entry. Because E is a dense subspace of s there exist for $i = 1, \ldots, n$ vectors s_i in E such that $d(e_i, s_i) < \frac{1}{4N}\epsilon$. For $x \in (0,1)^N$ such that $x = \sum_{n=1}^{N} x_n \cdot e_n$, define $g : (0,1)^N \to E$ by

$$g(\sum_{n=1}^{N} x_n \cdot e_n) = \sum_{n=1}^{N} x_n \cdot s_n.$$

It is easily verified that $d(x, g(x)) < \frac{1}{2}\epsilon$ for every $x \in (0,1)^N$. Hence, $f = g \circ \pi$ is as desired. □

Next, we shall show that, under certain conditions, locally homotopy negligible is a consequence of strong universality.

Lemma 4.2.2 *Let X be a dense linear subspace of $\mathbb{R}^\infty$ and let $\mathcal{C}$ be a class of spaces that contains the collection of all compact spaces. Suppose X is strongly $\mathcal{C}$-universal. Then $\mathbb{R}^\infty \setminus X$ is locally homotopy negligible.*

Proof: Observe that $A \setminus (\mathbb{R}^\infty \setminus X) = X \cap A$ for every subset A in $\mathbb{R}^\infty$ Let $\mathcal{U}$ be an arbitrary open cover for $\mathbb{R}^\infty$. Pick a point $x \in \mathbb{R}^\infty$ and a neighborhood $U_x \in \mathcal{U}$. Let $V_x \subset U_x$ and $f : (B^k, S^{k-1}) \to (V_x, V_x \cap X)$ be arbitrary. Because X is a dense linear subspace of $\mathbb{R}^\infty$ we can approximate f arbitrarily closely by a map $g : B^k \to (X \cap V_x)$ (Proposition 4.2.1). Because we assumed that X was strongly $\mathcal{C}$-universal, we can approximate g arbitrarily closely by a Z-embedding $h : B^k \to X$ such that $h|_{S^{k-1}} = g|_{S^{k-1}}$. Moreover, because $d(g[B^k], X \setminus U_x) > 0$ we may assume that $h(B^k) \subset U_x$. Because $\mathbb{R}^\infty$ is an **ANR** we may assume that f, g and h are homotopic (Theorem 2.1.2). $\square$

Finally, we come to the main result of this section.

Corollary 4.2.3 *Let X be a countable Tychonoff space. Then $\mathbb{R}^\infty \setminus C_p(X)$ is locally homotopy negligible.*

Proof: By Lemma 4.1.1, $C_p(X)$ is a dense linear subspace of $\mathbb{R}^\infty$. Because $C_p(X)$ is strongly $\mathcal{F}_0(C_p(X))$-universal, the result follows immediately from Lemma 4.2.2. $\square$

4.3 Z_σ-property of function spaces

To apply the method of absorbing sets to the topological classification of function spaces, we need to restrict ourselves to function spaces that are Z_σ-spaces. In this section we will show that for a countable, non-discrete Tychonoff space X the class of function spaces $C_p(X)$ that are *analytic*, i.e. are a continuous image of the irrationals, satisfy this property.

First, we shall derive a well-known fact that allows us to loosen the restriction of Z_σ-spaces a bit.

Proposition 4.3.1 *Let X and Y be dense linear subspaces of $\mathbb{R}^\infty$ such that $X \subset Y$ and Y is a Z_σ-space. Then X is a Z_σ-space.*

Proof: By assumption we can write $Y = \bigcup_{n=1}^\infty Y_n$ with Y_n a Z-set in Y. Let $X_n = X \cap Y_n$, then clearly $X = \bigcup_{n=1}^\infty X_n$ with X_n relatively closed in X. Let K be a compactum and $f : K \to X$ a continuous function. Pick $n \in \mathbb{N}$ and let $\epsilon > 0$. Because Y_n is a Z-set in Y, there is a continuous function $\xi : K \to Y$ such that $\hat{d}(\xi, f) < \frac{1}{2}\epsilon$ and $\xi[K] \cap Y_n = \varnothing$. Consequently, we have $\xi[K] \cap X_n = \varnothing$. Because K is compact, $\xi[K]$ is compact. Hence, $\xi[K]$ is closed and consequently $\delta = d(\xi[K], X_n) > 0$. It follows from Proposition 4.2.1 that there exists a continuous function $\eta : Y \to X$ such that $\hat{d}(\eta, 1_X) < \min\{\delta, \frac{1}{2}\epsilon\}$. Now define the continuous function $g = \eta \circ \xi$, then $\hat{d}(f, g) < \epsilon$ and $g[K] \cap X_n = \varnothing$. This shows that X_n is a Z-set in X. $\square$

This already allows us to derive the following result from [16].

Lemma 4.3.2 *Let X be a countable non-discrete Tychonoff space. Then $C_p^*(X)$ is a Z_σ-space.*

Proof: Let $\epsilon > 0$ and $N \in \mathbb{N}$ such that $\sum_{n=N}^{\infty} 2^{-n} < \epsilon$. Let $f \in C_p(X)$ be arbitrary. Define $g \in \mathbb{R}^X$ by $g = (f(x_1), \ldots, f(x_N), 0, 0, \ldots)$. Then $\hat{d}(f, g) < \epsilon$. Hence, every continuous function can be approximated arbitrarily closely by a bounded mapping. It follows from Lemma 4.1.1 that $C_p^*(X)$ is dense in $\mathbb{R}^\infty$. Because the subspace $C_p^*(X)$ is contained in the subspace $\mathbb{R}^\infty_{bd} = \{(x_i) \in \mathbb{R}^\infty : \sup |x_i| < \infty\}$, which is a Z_σ-space, it follows from Proposition 4.3.1 that $C_p^*(X)$ is a Z_σ-space. $\square$

Before we can derive the result for $C_p(X)$, we need some results on the spaces c_F and $C_p(\mathbb{N}_F)$ where F is a filter on $\mathbb{N}$. The following result is from [8]:

Lemma 4.3.3 *Let F be filter on $\mathbb{N}$ such that F is a first category subset of $2^{\mathbb{N}}$. Then c_F is contained in a σZ-set of $\overline{\mathbb{R}}^\infty$.*

Proof: Let $\overline{T} = \overline{\mathbb{R}} \setminus ((-3, -1) \cup (1, 3))$. Define $g : \overline{T} \to \overline{\mathbb{R}}$ by

$$g(x) = \begin{cases} x & x \in [-1, 1], \\ x - 2 & x \in [3, \infty], \\ x + 2 & x \in [-\infty, -3]. \end{cases}$$

Let $g^\infty : \overline{T}^\infty \to \overline{\mathbb{R}}^\infty$ be defined by $g^\infty((x_i)_i) = (g(x_i))_i$. Then g^∞ is obviously continuous. Because F is a first category subset there are nowhere dense closed subsets A_n ($n \in \mathbb{N}$) of $2^{\mathbb{N}}$ such that $F \subset \bigcup_{i=1}^\infty A_n$. For $n \in \mathbb{N}$ let $B_n = \{x \in \overline{T}^\infty : \{i : |x_i| < 2\} \in A_n\}$ and let $\chi_{A_n} = \bigcup_{A \in A_n} \chi_A$.

CLAIM 1: B_n is closed in $\overline{T}^\infty$.

Let $x \in \overline{T}^\infty \setminus B_n$. Then $K = \{i : |x_i| < 2\} \notin A_n$ and consequently, $\chi_K \notin \chi_{A_n}$. Because, χ_{A_n} is closed there exists $m \in \mathbb{N}$ and $U_i \subset \{0, 1\}$ for $1 \leq i \leq m$ such that if

$$U = U_1 \times \ldots \times U_m \times \{0, 1\} \times \{0, 1\} \times \ldots$$

then $\chi_K \in U$ and $U \cap \chi_{A_n} = \varnothing$. For $1 \leq i \leq m$ such that $|x_i| \leq 1$ let $V_i = [-1, 1]$ and for $1 \leq i \leq m$ such that $|x_i| \geq 3$ let $V_i = [-\infty, -3] \cup [3, \infty]$. Put $V = V_1 \times \ldots \times V_m \times \overline{T}^\infty$. Then V is open in $\overline{T}^\infty$, $x \in V$ and $V \cap B_n = \varnothing$, because for $y \in V$ we have $\{i : |y_i| < 2\} = \{i : |x_i| < 2\} \notin A_n$ and for $y \in B_n$ we have $\{i : |y_i| < 2\} \in A_n$.

CLAIM 2: $g^\infty[B_n]$ is a Z-set in $\overline{\mathbb{R}}^\infty$.

Let $\epsilon > 0$. Find $i \in \mathbb{N}$ such that $\sum_{j=i}^\infty 2^{-j} < \frac{1}{2}\epsilon$. It follows from Lemma 2.2.1 that there exists $a = (a_{i+1}, a_{i+2}, \ldots) \in \{0, 1\}^{\mathbb{N} \setminus \{1, 2, \ldots, i\}}$ such that $(\{0, 1\}^{\{1, 2, \ldots, i\}} \times \{a\}) \cap \chi_{A_n} = \varnothing$. Define $f : \overline{\mathbb{R}} \to \overline{\mathbb{R}}$ by

$$f(x)_j = \begin{cases} x_j & j \leq i, \\ 0 & a_j = 1, \ j > i, \\ 10 & a_j = 0, \ j > i. \end{cases}$$

Then clearly f is a continuous function and $\hat{d}(f, 1_{\overline{\mathbb{R}}}) < \epsilon$. We claim that $f[\overline{\mathbb{R}}] \cap g^\infty[B_n] = \varnothing$. Suppose to the contrary that there exists $x \in B_n$, $y \in \overline{\mathbb{R}}$ such that $f(y) = g^\infty(x)$. By definition of B_n we have $D = \{i : |x_i| < 2\} \in A_n$, so consequently $\chi_D \notin \{0,1\}^{\{1,2,\ldots,i\}} \times \{a\}$. Thus there exists $j > i$ such that $a_j \neq (\chi_D)_j$. If $|x_j| \leq 1$, $j \in D$, thus $(\chi_D)_j = 1$ and $a_j = 0$. If $|x_j| \geq 3$, $j \notin D$, thus $(\chi_D)_j = 0$ and $a_j = 1$. If $a_j = 0$ and $|x_j| \leq 1$, then $|g(x_j)| \leq 1$, hence $|f(y)_j| \leq 1$. This gives $|f(y)_j| = 0$, so $a_j = 1$ which is a contradiction. Similarly we arrive at a contradiction when $a_j = 1$ and $|x_j| \geq 3$.

CLAIM 3: $c_F \subset \bigcup_{n=1}^\infty g^\infty[B_n]$.

Let $x \in c_F \subset \mathbb{R}^\infty$. Define the point $y = (y_i)_{i \in \mathbb{N}}$ by

$$y_i = \begin{cases} x_i & x_i \in [-1, 1], \\ x_i + 2 & x_i \in (1, \infty), \\ x_i - 2 & x_i \in (-\infty, -1). \end{cases}$$

Then $y \in \overline{T}^\infty$ and $g^\infty(y) = x$. Since $x \in c_F$, the set $\{i : |x_i| < 1\} \in F$. Because $\{i : |x_i| < 1\} \subset \{y_i : |y_i| < 2\}$ and since F is a filter, it follows that $\{y_i : |y_i| < 2\} \in F \subset \bigcup_{i=1}^\infty A_n$. We conclude that $y \in \bigcup_{n=1}^\infty B_n$. □

In [16] it was proven that c_F is a Z_σ-space when F is a free Borelian filter on $\mathbb{N}$. The following result from [11] allows us to relate the spaces c_F and $C_p(\mathbb{N}_F)$ with each other:

Lemma 4.3.4 *Let F be a filter on $\mathbb{N}$ and let $\overline{\mathbb{N}} = \mathbb{N} \cup \{\infty\}$. There exists a homeomorphism $\Phi : \overline{\mathbb{R}}^{\mathbb{N}} \times \overline{\mathbb{R}} \to \overline{\mathbb{R}}^{\overline{\mathbb{N}}}$ such that $\Phi[c_F \times \mathbb{R}] = C_p(\mathbb{N}_F)$.*

Proof: In $\overline{\mathbb{R}}$ and $\overline{\mathbb{N}}$ we use the following arithmetic: $1/0 = \infty$ and $\infty + a = \infty$, if a is finite. Define the continuous function $\Psi : \overline{\mathbb{R}} \to \overline{\mathbb{R}}^{\overline{\mathbb{N}}}$ by

$$\Psi(r)(n) = \operatorname{sign}(r) \min\{|r|, n\}.$$

Note that $\Psi(r)(n)$ is finite if $n \neq \infty$ and $\lim_{n\to\infty} \Psi(r)(n) = \Psi(r)(\infty) = r$. Consequently, $\Psi[\mathbb{R}]$ is a subset of $C_p(\mathbb{N}_F)$. For $f \in \overline{\mathbb{R}}^{\mathbb{N}}$ let $\overline{f}$ be the extension of f over $\overline{\mathbb{N}}$ that assigns 0 to ∞. Define a map $\Phi : \overline{\mathbb{R}}^{\mathbb{N}} \times \overline{\mathbb{R}} \to \overline{\mathbb{R}}^{\overline{\mathbb{N}}}$ by $\Phi(f, r) = \overline{f} + \Psi(r)$. Then Φ is a well-defined map from $\overline{\mathbb{R}}^{\mathbb{N}} \times \overline{\mathbb{R}}$ to $\overline{\mathbb{R}}^{\overline{\mathbb{N}}}$. For $h \in \overline{\mathbb{R}}^{\overline{\mathbb{N}}}$ we have $\Phi^{-1}(h) = \{(f, r) \in \overline{\mathbb{R}}^{\mathbb{N}} \times \mathbb{R} : \forall n \in \mathbb{N} : \overline{f}(n) + \Psi(r)(n) = h(n)\}$. If $(f, r) \in \Psi^{-1}(h)$ we must have $h(\infty) = \overline{f}(\infty) + \Psi(r)(\infty) = 0 + r = r$, and for all $n \in \mathbb{N}$ we must have $h(n) = \overline{f}(n) + \Psi(h(\infty))(n) = f(n) + \Psi(h(\infty))(n)$, so consequently $f(n) = h(n) - \Psi(h(\infty))(n)$. From this, it follows that $\Phi^{-1}(h) = ((h - \Psi(h(\infty)))|_{\mathbb{N}}, h(\infty))$ which shows that Φ is a homeomorphism. It is easily verified that $\Phi[c_F \times \mathbb{R}] = C_p(\mathbb{N}_F)$. □

Corollary 4.3.5 *Let F be a filter on $\mathbb{N}$ such that F is a first category subset of $2^{\mathbb{N}}$. Then $C_p(\mathbb{N}_F)$ is contained in a σZ-set of $\mathbb{R}^\infty$.*

Proof: This follows immediately from Lemma 4.3.3, Lemma 4.3.4 and Proposition 3.1.4. □

Hence, using the remark after Lemma 4.3.3 we see that when F is a free Borelian filter on $\mathbb{N}$, $C_p(\mathbb{N}_F)$ is a Z_σ-space. We still need some technical results. (Lemma 4.3.7 is taken from [16].)

Proposition 4.3.6 *Let S be the σ-algebra generated by the open subsets and first category subsets of a topological space X. Then every element $F \in S$ can be written as $F = (U \setminus A) \cup B$ where U is an open subset and both A and B are first category subsets of X.*

Proof: Clearly every open subset or first category subset can be written as stated. It therefore suffices to prove that all sets F such as in the Proposition form a σ-algebra.

Assume we have for $i \in \mathbb{N}$ the sets $F_i = (U_i \setminus A_i) \cup B_i$ where U_i is an open subset and both A_i and B_i are first category subsets of X. Then $U = \bigcup_{i=1}^{\infty} U_i$ is an open subset and both $A = \bigcup_{i=1}^{\infty} A_i$ and $B = \bigcup_{i=1}^{\infty} B_i$ are first category subsets of X. Hence $\bigcup_{i=1}^{\infty} F_i = (U \setminus A) \cup B$ satisfies the stated property.

Next assume we have $F = (U \setminus A) \cup B$. Then it can easily be verified that $X \setminus F = ((X \setminus \overline{U}) \setminus B) \cup ((\overline{U} \setminus U) \cap (X \setminus B)) \cup (A \cap (X \setminus B))$. Because a subset of a first category subset is a first category subset and $\overline{U} \setminus U$ is a first category subset $X \setminus F$ satisfies the stated property. $\square$

Lemma 4.3.7 *Let F be a free filter on $\mathbb{N}$. Then the following conditions are equivalent:*

(1) F is an element of the σ-algebra generated by the open subsets and first category subsets of $2^{\mathbb{N}}$.

(2) F is a first category subset of $2^{\mathbb{N}}$

Proof: We only have to prove (1) $\Rightarrow$ (2).

By Proposition 4.3.6 we can write $F = (U \setminus A) \cup B$, where U is an open subset and both A and B are first category subsets of $2^{\mathbb{N}}$. Suppose to the contrary that F is not a first category subset, then $U \neq \varnothing$. From this, it follows that there exists $m \in \mathbb{N}$ and $S \subset \{1, \ldots, m\}$ such that $\{C \subset 2^{\mathbb{N}} : C \cap \{1, \ldots, m\} = S\} \subset U$. Write $N' = \mathbb{N} \setminus \{1, \ldots, m\}$ and let $F' = \{A \cap N' : A \in F\}$. Since F is dense in $2^{\mathbb{N}}$ by Proposition 2.3.1, F' is dense in $2^{N'}$. Moreover, since F contains a dense G_δ-subset of U, F' contains a dense G_δ-subset of $2^{N'}$. Let $\xi : 2^{N'} \to 2^{N'}$ be the homeomorphism assigning to each C the set $N' \setminus C$. If $A \in (\xi[F'] \cap F')$ we must have $N' \setminus A \in F'$ and $A \in F'$. Because F' is a filter this would lead to $(N' \setminus A) \cap A = \varnothing \in F'$, thus $\xi[F'] \cap F' = \varnothing$. Consequently, $2^{N'}$ contains 2 disjoint dense G_δ-subsets which contradicts the Baire category theorem. $\square$

Finally, we come to the main result of this section. The result is from [16].

Lemma 4.3.8 *Let X be a countable non-discrete Tychonoff space such that the space $C_p(X)$ is analytic. Then $C_p(X)$ is a Z_σ-space.*

Proof: Let $a \in X$ be an accumulation point. Let $Y = X \setminus \{a\}$ and define the filter $F = \{A \subset Y : a \text{ is an interior point of } A \cup \{a\}\}$. Note that F is a free filter, because for every $A \in F$ and $b \in Y$ we have

$A \setminus \{b\} \in F$. Next, define $G = \{B \subset X : B \text{ clopen in X and a } \in \text{B}\}$ and $G' = \{(A,B) \in 2^Y \times 2^X : B \in G \text{ and } B \subset A \cup \{a\}\}$.

CLAIM 1: G is analytic.

Define the set $H = \{f \in C_p(X) : f[X] \subset \{0,1\} \text{ and } f(a) = 0\}$. We claim that H is closed in $C_p(X)$. For this, let $X = \{x_1, x_2, \ldots\}$ and define $A_n = \{f \in C_p(X) : f(x_n) \subset \{0,1\}\}$. If $f \notin A_n$ then $\epsilon = \min\{|f(x_n)|, |f(x_n) - 1|\} > 0$. The set $U(x_n, B(f(x_n), \frac{1}{2}\epsilon))$ is open, contains f and is disjunct from A_n. Consequently A_n is closed. From this it easily follows that $H = \bigcap_{i=1}^{\infty} A_n \cap \{f \in C_p(X) : f(a) = 0\}$ is closed. By assumption there exists a continuous surjection $f : \mathbb{P} \to C_p(X)$. Because H is closed in $C_p(X)$, $f^{-1}[H]$ is closed in $\mathbb{P}$. By Theorem XIII.1 in [13], there exists a continuous surjection $g : \mathbb{P} \to f^{-1}[H]$. Consequently, $f \circ g$ is a continuous surjection from $\mathbb{P}$ onto H. This shows that H is analytic. Because H can be identified with G it follows that G is analytic.

CLAIM 2: G' is analytic.

From the previous claim it is clear that it is enough to prove that G' is a closed subspace of the analytical space $2^Y \times G$. Pick an element $(A,B) \in (2^Y \times G) \setminus G'$. This means that $B \not\subset A \cup \{a\}$. Thus there exists $b \in B \setminus A$ with $b \neq a$. Now put $U = \{D \subset Y : \pi_b^Y[D] = 0\}$ and $V = \{E \subset X : \pi_b^X[E] = 1\}$. Then U is clopen in 2^Y, V is clopen in 2^X and $(A,B) \in U \times V$. Moreover, $U \times V$ misses G', because for every $(D,E) \in U \times V$ we have $\pi_b^Y[D] = 0$ and $\pi_b^X[E] = 1$. Thus $b \in E \setminus D$ and consequently $F \not\subset E \cup \{a\}$. We conclude that G' is closed in $2^Y \times G$.

CLAIM 3: F is analytic.

We shall prove that F is an image of G' by the projection onto the first axis. Denote this projection by π_1. First pick a point $A \in \pi_1[G']$. Then there exists $B \in G$ with $B \subset A \cup \{a\}$. Because B is clopen and contains a, a is an interior point of $A \cup \{a\}$. Thus $A \in F$ and consequently $\pi_1[G'] \subset F$.

Next pick a point $f \in F$. Then a is an interior point of $f \cup \{a\}$, thus $a \in (f \cup \{a\})^\circ$. Because X is zero-dimensional (see [7], Corollary 6.2.8 and Theorem 6.2.6), it has a base consisting of clopen sets. Hence, there exists a clopen B in X such that $a \in B \subset (f \cup \{a\})^\circ$. We conclude that $(f, B) \in G'$ and consequently $F \subset \pi_1[G']$.

This shows that $F = \pi_1[G']$, thus F is analytic.

CLAIM 4: $C_p(X)$ is a Z_σ-space.

Now, by [12] and Lemma 4.3.7, F is a first category subset of 2^Y. Observe, that $C_p(X) \subset E$, where

$$E = \{f \in \mathbb{R}^X : (\forall \epsilon > 0)(\exists A \in F)(\forall x \in A)(|f(a) - f(x)| < \epsilon)\}.$$

The space E can be identified with $C_p(\mathbb{N}_F)$. Hence, by Corollary 4.3.5, the space E is contained in a σZ-set of $\mathbb{R}^\infty$. By Proposition 4.3.1 $C_p(X)$ is a Z_σ-space $\square$

In [13] (Chapter XIII, Theorem 6) it is proven that every separable, metrizable space that is absolutely Borel is analytic. Consequently, this last result allows us to apply the method of absorbing sets to function spaces $C_p(X)$ that are Borel, which will be studied in the following section.

4.4 Function spaces $C_p(X)$ which are Borel

In this section we will look at function spaces $C_p(X)$ that are Borel. In [4] it has been shown that $C_p(X)$ can have arbitrary high Borel complexity. First we will look at some general results that restrict the Borel complexity of $C_p(X)$.

The following results are from [9]. Remember that $C_p(X)$ is dense in $\mathbb{R}^X$ if and only if X is functionally Hausdorff (Lemma 4.1.1).

Theorem 4.4.1 *If $C_p(X)$ contains a dense G_δ-set in $\mathbb{R}^X$, then X is discrete.*

Proof: Suppose to the contrary that X is not discrete. Then there is a point $p \in X$ with $p \in \overline{X \setminus \{p\}}$. Define $g : X \to \mathbb{R}$ by $g(p) = 1$ and $g(y) = 0$ for every $y \in X \setminus \{p\}$. Then g is not continuous. Define a homeomorphism $\psi : \mathbb{R}^X \to \mathbb{R}^X$ by $\psi(f) = f + g$ for every $f \in \mathbb{R}^X$. Because $C_p(X)$ contains a dense G_δ-set, it follows that $\psi[C_p(X)] \subset \mathbb{R}^X \setminus C_p(X)$ also contains a dense G_δ-set. Then $\mathbb{R}^X$ contains 2 disjoint dense G_δ-set, which is a contradiction because $\mathbb{R}^X$ is a Baire space. $\square$

Theorem 4.4.2 *If $C_p(X)$ is a dense F_σ-subset in $\mathbb{R}^X$ then X is discrete.*

Proof: We claim that it is enough to prove the Theorem for every Tychonoff space X. Suppose the Theorem is proven for all Tychonoff spaces and suppose that $\mathcal{T}$ is a topology on X such that $C_p(X, \mathcal{T})$ is a dense F_σ subset of $\mathbb{R}^X$. Then $(X, \mathcal{T})$ is functionally Hausdorff by Lemma 4.1.1. Let $\mathcal{S}$ be the topology on X having the family $\{f^{-1}[U] : f \in C_p(X, \mathcal{T})$ and U is open in $\mathbb{R}\}$ as a subbase. Then $\mathcal{S} \subset \mathcal{T}$ and $(X, \mathcal{S})$ is a Tychonoff space. Furthermore $C_p(X, \mathcal{T})$ and $C_p(X, \mathcal{S})$ are identical subsets of $\mathbb{R}^X$ so that $C_p(X, \mathcal{S})$ is a dense F_σ-set subset of $\mathbb{R}^X$. But then $(X, \mathcal{S})$ is a discrete space. Because $\mathcal{S} \subset \mathcal{T}$, so is $(X, \mathcal{T})$. Therefore it will be enough to prove the Theorem for an arbitrary Tychonoff space X.

Let $C_p(X) = \bigcup_{n=0}^{\infty} F_n$ with each F_n a closed subset of $\mathbb{R}^X$ and $F_0 = \varnothing$. Suppose to the contrary that X is not discrete. Then there is a nonisolated point p in X. We shall construct inductively a sequence $f_0, f_1, \ldots$ of continuous functions in I^X and a sequence $U_0, U_1, \ldots$ of open neighborhoods of p with the properties:

(1) $f_0 \leq f_1 \leq f_2 \ldots$,
(2) $U_0 \supset \overline{U_1} \supset U_1 \supset \overline{U_2} \supset U_2 \supset \ldots$,
(3) $f_n(p) = 1$,
(4) $f_n[U_n \setminus \{p\}] = \{1 - 2^{-n}\}$,
(5) $f_n|_{X \setminus \{p\}}$ is continuous,
(6) $f_{n+1}|_{X \setminus U_n} = f_n|_{X \setminus U_n}$,
(7) If $f \in \mathbb{R}^X$ coincides with f_n on $(X \setminus U_n) \cup \{p\}$, then $f \notin F_n$.

As basis step we put $f_0(p) = 1$, $f_0(x) = 0$ if $x \neq p$, and $U_0 = X$. Suppose that for each i with $0 \leq i \leq n$, f_i and U_i are well defined. Because X is

Tychonoff there exists a continuous function ϕ from X into $[0, 2^{-n}]$ such that $\phi(X \setminus U_n) \subset \{0\}$ and $\phi(p) = 2^{-n}$. Define $f_{n+1} : X \to [0, 1]$ by $f_{n+1}(p) = 1$ and for $x \neq p$, $f_{n+1}(x) = f_n(x) + \min\{2^{-n-1}, \phi(x)\}$.

Note that $V = \phi^{-1}[(2^{-n-1}, 2^{-n}]]$ is an open neighborhood of p contained in U_n and $f_{n+1}[V \setminus \{p\}] = \{1 - 2^{-n-1}\}$. Since p is not isolated, this means that f_{n+1} is not continuous and hence not an element of F_{n+1}. Since $\mathbb{R}^X$ carries the product topology, it follows from Proposition 4.1.4, that there exists a finite set $A \subset X$ such that for every $f \in \mathbb{R}^X$ that coincides with f_{n+1} on A is not in F_{n+1}. Put $U_{n+1} = (V \setminus A) \cup \{p\}$. Then it is easily verified that f_{n+1} and U_{n+1} satisfy the induction hypothesis.

Now let f be the pointwise limit of the sequence $(f_n)_{n=0}^{\infty}$. Then we have $f|_{X \setminus U_n} = f_n|_{X \setminus U_n}$ for every n and $f[\bigcap_{n=0}^{\infty} U_n] = \{1\}$. This means, according to the induction hypothesis that $f \notin \bigcup_{n=1}^{\infty} F_n$. Hence, f is not continuous.

On the other hand, we shall see that f is continuous, leading to a contradiction. Because, for any point $x \in X$, if $x \notin \bigcap_{n=0}^{\infty} U_n$, then for some n, $x \notin \overline{U_n}$ and $f|_{X_n \setminus \overline{U_n}} = f_n|_{X_n \setminus \overline{U_n}}$. By (5), f is continuous at x. If $x \in \bigcap_{n=0}^{\infty} U_n$, then U_n is a neighborhood of x such that $f_n[U_n] \subset \{1 - 2^{-n}, 1\}$ by (3) and (4). Consequently, by (1) $f_m[U_n] \subset [1 - 2^{-n}, 1]$ for every $m \geq n$ and hence $f[U_n] \subset [1 - 2^{-n}, 1]$. This yields the continuity of f at x. As a consequence f is continuous, which is a contradiction. $\square$

Theorem 4.4.3 *If $C_p(X)$ is a dense $G_{\delta\sigma}$-subset in $\mathbb{R}^X$ then X is discrete.*

Proof: The proof is in contrast to the G_δ and F_σ case quite technical and therefore omitted. For a proof see [9]. $\square$

Let us note that the Theorems 4.4.1 and 4.4.2 can also be found in [14], but they restrict the underlying space X to the class of Tychonoff space. This allows them to omit the first part of the proof of Theorem 4.4.2.

In contrast to the previous Theorems, which stated that for non-discrete X, $C_p(X)$ cannot be an G_δ, F_σ and $G_{\delta\sigma}$-subset of $\mathbb{R}^X$, it is easily seen that $C_p(X)$ can be an $F_{\sigma\delta}$-subset of $\mathbb{R}^X$. For example, let X be any countable metric space, then

$$C_p(X) = \bigcap_{x \in X} \bigcap_{n=1}^{\infty} \bigcup_{m=1}^{\infty} \{g \in \mathbb{R}^X : g(B(x, \frac{1}{m})) \subset [g(x) - \frac{1}{n}, g(x) + \frac{1}{n}]\},$$

which is an $F_{\sigma\delta}$-subset of $\mathbb{R}^X$ (see [9]).

When we restrict the space X to the class of countable infinite regular spaces, if $C_p(X)$ is Borel then $C_p(X)$ can only be of an *exact multiplicative class*, i.e. $C_p(X) \in \mathcal{M}_\alpha \setminus \mathcal{A}_\alpha$ for some $\alpha \geq 1$. This was proven in [15] and is mainly based on a Lemma that relates the Borel class of $C_p(X)$ with the Borel class of $C_p(X, \mathbb{P})$, with $\mathbb{P}$ denoting the set of irrationals $\mathbb{R} \setminus \mathbb{Q}$. For the sake of completeness we present this Lemma.

Lemma 4.4.4 *Let $C_p(X,\mathbb{P})$ be the subspace of $C_p(X)$ consisting of functions that take values in $\mathbb{P}$. For every countable infinite regular space X, $C_p(X)$ is Borel if and only if $C_p(X,\mathbb{P})$ is Borel. Moreover, the exact Borel classes of $C_p(X)$ and $C_p(X,\mathbb{P})$ coincide.*

Proof: For discrete X, $C_p(X) = \mathbb{R}^X$ and $C_p(X,\mathbb{P}) = \mathbb{P}^X$ and both belong to $\mathcal{M}_1 \setminus \mathcal{A}_1$. Suppose X is not discrete. Let $X = \{x_1, x_2, \ldots\}$ and $\mathbb{Q} = \{q_1, q_2, \ldots\}$. Define for $n, m \in \mathbb{N}$

$$A(n,m) = \{f \in C_p(X) : f(x_m) = q_n\}.$$

We claim that $A(n,m)$ is closed in $C_p(X)$. Let $f \notin A(n,m)$ be a continuous map. Thus, $f(x_m) \neq q_n$ and $\epsilon = |f(x_m), q_n| > 0$. Then $U(x_m; B(f(x_m), \frac{1}{2}\epsilon))$ is a base element that contains f and is disjunct from $A(n,m)$. Hence, $A(n,m)$ is closed. This shows that $C_p(X,\mathbb{P}) = C_p(X) \setminus \bigcup_{n,m\in\mathbb{N}} A(n,m)$ is a G_δ subset of $C_p(X)$. Consequently $C_p(X,\mathbb{P})$ is Borel whenever $C_p(X)$ is Borel. Using Lemma 2.4.1 and that $C_p(X) \notin \mathcal{A}_1$ by Theorem 4.4.1, we conclude that $C_p(X,\mathbb{P}) \in \mathcal{M}_\alpha$ (resp. $\mathcal{A}_\alpha$) provided $C_p(X) \in \mathcal{M}_\alpha$ (resp. $\mathcal{A}_\alpha$).

Conversely, suppose $C_p(X,\mathbb{P})$ is Borel. Put for $n, m, k \in \mathbb{N}$

$$B(n,m,k) = \{(f,t) \in \mathbb{R}^X \times \mathbb{R} : |t| \leq k, f(x_m) + t = q_n\}.$$

Let $(f,t) \notin B(n,m,k)$, then $|t| > k$ or $f(x_n) \neq q_n - t$. It is easily seen that (f,t) has an open neighborhood disjunct from $B(n,m,k)$, hence $B(n,m,k)$ is closed. Now put

$$S = \{(f,t) \in \mathbb{R}^X \times \mathbb{R} : (\forall x \in X)(f(x) + t \in \mathbb{P})\}.$$

Because $S = (\mathbb{R}^X \times \mathbb{R}) \setminus \bigcup_{n,m,k} B(n,m,k)$, S is a G_δ-subset of the complete metrizable space $\mathbb{R}^X \times \mathbb{R}$. Consequently, we can choose a complete metric d on S (see [7], Theorem 4.3.23). Define $\phi : S \to \mathbb{P}^X$ by letting

$$\phi(f,t)(x) = f(x) + t.$$

Then ϕ is continuous, because if $(f_n, t_n) \to (f,t)$ then $\phi(f_n,t_n)(x) = f_n(x) + t_n \to f(x) + t = \phi(f,t)$. Also ϕ satisfies

$$\phi^{-1}[C_p(X,\mathbb{P})] = \{(f,t) : f \in C_p(X)\} = T,$$

because $(f,t) \in \phi^{-1}[C_p(X,\mathbb{P})]$ iff $\phi(f,t) \in C_p(X,\mathbb{P})$ iff $\forall x : x \mapsto f(x) + t$ is continuous and $f(x) + t \in \mathbb{P}$ iff $f \in C_p(X)$ and $f \in S$.

Observe that the projection $\pi : C_p(X) \times \mathbb{R} \to C_p(X)$ restricted to T is a continuous surjection. We claim that $\pi|_T$ is open. For this, define

$$S_f = \{t \in \mathbb{R} : (f,t) \in S\},$$

for $f \in \mathbb{R}^X$. Then $S_f = \{t \in \mathbb{R} : (\forall x \in X)(f(x) + t \in \mathbb{P})\} = \bigcap_{x\in X}(\mathbb{P} - f(x))$. Consequently, S_f is G_δ-subset of a Baire space and therefore dense. Now observe that $\pi^{-1}(f) = \{f\} \times S_f \approx S_f$ for every $f \in C_p(X)$, from which the claim follows. Since $\{f\} \times S_f$ is closed in S, $(\pi^{-1}(f), d)$ is complete. It follows from [10] (Theorem 5.9.16) that if $T \in \mathcal{M}_\alpha$, then $\pi[T] = C_p(X) \in \mathcal{M}_\alpha$ (resp. $\mathcal{A}_\alpha$). The Lemma follows. $\square$

Theorem 4.4.5 *Let X be a countable infinite regular space such that $C_p(X)$ is Borel. Then there exists a countable ordinal $\alpha \geq 1$ such that $C_p(X) \in \mathcal{M}_\alpha \setminus \mathcal{A}_\alpha$.*

Proof: Observe that

$$C_p(X, \mathbb{P}) \approx C_p(X, \mathbb{P}^\infty) = (C_p(X, \mathbb{P}))^\infty.$$

Now, the Theorem follows from Lemma 4.4.4 and Lemma 8.3 in [15], which states that a countable product of a Borel set is of an exact multiplicative class.
□

4.5 Overview of results

In this section we will give an overview of results on the topological classification of the function spaces $C_p(X)$ that were made by using a variation of absorbing sets.

The function spaces $C_p(X)$ have been studied intensively in the last decade, primarily in the Soviet Union. See [1] for general results on the function spaces $C_p(X)$.

In 1987 van Mill showed that for a non-locally compact countable metric space X, $C_p^*(X)$ is homeomorphic to σ_ω (see [19]), where

$$\sigma_\omega = (\ell_f^2)^\infty \text{ and } l_f^2 = \{x \in \ell^2 : x_i = 0 \text{ for all but finitely many } i\}.$$

The same result was later obtained for $C_p(X)$ by Baars, de Groot, van Mill and Pelant in 1989. A few years later, it was shown by Dobrowolski, Gul'ko and Mogilski in 1990 and independently by Cauty in 1991, that if X is any countable metrizable non-discrete space, then $C_p^*(X)$ and $C_p(X)$ are homeomorphic to σ_ω. This result was generalized even further by Dobrowolski, Marciszewski and Mogilski in 1991 as follows: if X is a non-discrete countable Tychonoff space and if $C_p(X)$ is an $F_{\sigma\delta}$-subset of $\mathbb{R}^X$ then $C_p(X)$ is homeomorphic to σ_ω (see [16]). Because, if X is a non-discrete Tychonoff space, then $C_p(X)$ is not a G_δ, an F_σ or an $G_{\delta\sigma}$-subset of $\mathbb{R}^X$ (see Theorem 4.4.1, 4.4.2 and 4.4.3), their result provides a satisfactory classification result for function spaces of the lowest possible Borel complexity.

So far, we have only considered results about spaces that are homeomorphic to $C_p(X)$, but not homeomorphisms of pairs of spaces. Because for countable X we have $C_p(X) \subset \mathbb{R}^X \approx \mathbb{R}^\infty \approx s \subset Q$, we can think of $C_p(X)$ being a subspace of the Hilbert cube Q. Therefore it is natural to try to find homeomorphisms of pairs $(Q, C_p(X)) \approx (Q, C_p(Y))$. By this we mean the existence of a homeomorphism $h : Q \to Q$ such that $h[C_p(X)] = C_p(Y)$.

To deal with this problem, Dijkstra, van Mill and Mogilski introduced in 1992 the notion of absorbing systems in the Hilbert cube (see [11]). They showed that for a countable, metric, non-discrete space X, $C_p(X)$ is an $F_{\sigma\delta}$-absorber in Q. In this case, there exists a homeomorphism op pairs $(Q, C_p(X)) \approx (Q^\infty, B^\infty)$. Their result was generalized by Baars, Gladdiness and van Mill who proved that for countable non-discrete X, if $C_p(X)$ is an $F_{\sigma\delta}$-subset of $\mathbb{R}^X$ then it is an $F_{\sigma\delta}$-absorber in Q. Consequently, there exists a homeomorphism of pairs $(Q, C_p(X)) \approx (Q^\infty, B^\infty)$. Their report also contains a good overview of absorbing systems, which are generalized to arbitrary infinite dimensional spaces.

Similar results were obtained by Cauty who introduced the concept of k-tuples (see [15]). These results contain the equivalence of triples of the form $(Q, s, C_p(X))$ and quadruples of the form $(Q, s, C_p(X), C_p^*(X))$. This completely solves the classification problem for function spaces of the lowest possible Borel complexity.

The question arises if something can be said about function spaces of higher Borel complexity. So far, no proof has been given whatsoever. The following was conjectured by Dobrowolski, Marciszewski and Mogilski in 1991 (see [16])

Conjecture 4.5.1 *If X is countable regular and non-discrete, if $C_p(X)$ is an absolute Borel subset of $\mathbb{R}^X$ then $C_p(X)$ is an absorber for the Borel class to which it belongs. In other words, we conjecture that the topological type of $C_p(X)$ is for absolutely Borel $C_p(X)$ determined by its Borel type.*

In [15] it was proven by Cauty, Dobrowolski and Marciszewski that for every countable regular non-compact space X, $C_p(X)$ is strongly $\mathcal{F}_0(C_p(X))$-universal. From this, they derived that for countable regular spaces X and Y, such that $C_p(X)$ and $C_p(Y)$ are Z_σ-spaces, we have $C_p(X) \approx C_p(Y)$ if and only if $\mathcal{F}_0(C_p(X)) = \mathcal{F}_0(C_p(Y))$. Because Borelian function spaces are Z_σ-spaces (Lemma 4.3.8), and for every infinite countable regular space X, $C_p(X)$ is of an exact multiplicative class (Theorem 4.4.5) the conjecture can be solved if we can answer the following question.

Question 4.5.2 *Let X be a countable regular space such that $C_p(X) \in \mathcal{M}_\alpha \backslash \mathcal{A}_\alpha$ for some $\alpha \geq 2$. Is $\mathcal{F}_0(C_p(X))$ equal to $\mathcal{M}_\alpha$?*

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