Computational Intelligence 2008–2009
Tutorial II

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Solutions

Exercise 1

Let V be a set of statistical variables. Let P be a joint probability distribution on V and let $\perp\!\!\!\perp_P$ be its independence relation. Show that $\perp\!\!\!\perp_P$ satisfies the properties

a. $X \perp\!\!\!\perp_P Y \mid Z \Rightarrow Y \perp\!\!\!\perp_P X \mid Z$

b. $X \perp\!\!\!\perp_P Y \cup W \mid Z \Rightarrow X \perp\!\!\!\perp_P Y \mid Z \wedge X \perp\!\!\!\perp_P W \mid Z$ (note the difference with the decomposition axiom on the slides)

c. $X \perp\!\!\!\perp_P Y \cup W \mid Z \Rightarrow X \perp\!\!\!\perp_P Y \mid Z \cup W$ (note the difference with the weak-union axiom on the slides)

d. $X \perp\!\!\!\perp_P Y \mid W \wedge X \perp\!\!\!\perp_P Z \mid W \cup Y \Rightarrow X \perp\!\!\!\perp_P Y \cup Z \mid W$ (note again the difference with the slides)

for all mutually disjoint sets of variables $X, Y, Z, W \subseteq V$.

You need to prove these properties by translating them to statements concerning the probability distribution P. For example $X \perp\!\!\!\perp_P Y \mid Z$ is first translated into $P(X \mid Y, Z) = P(X \mid Z)$.

a. Similar to symmetry property proof on lecture notes.

b. It is given that $X \perp\!\!\!\perp_P Y \cup W \mid Z$ holds. Therefore, by definition, we have that

$$P(X \mid Y, W, Z) = P(X \mid Z)$$

Given $V = W/Y$ (i. e., the set W excluding the elements in Y), $P(X \mid Z, Y)$ can be defined as

$$\begin{aligned} P(X \mid Z, Y) &= \sum_V P(X, V \mid Z, Y) \\ &= \sum_V P(X \mid Z, Y, V) P(V \mid Z, Y) \end{aligned}$$

Knowing that $P(X \mid Y, W, Z) = P(X \mid Z)$ we can then obtain that

$$\begin{aligned} P(X \mid Z, Y) &= \sum_V P(X \mid Z, Y, V) P(V \mid Z, Y) \\ &= \sum_V P(X \mid Z) P(V \mid Z, Y) \\ &= P(X \mid Z) \sum_V P(V \mid Z, Y) \\ &= P(X \mid Z) \end{aligned}$$

The last term holds because $\sum_V P(V \mid Z, Y) = 1$, as the sum of the probabilities over all possible values of a random variable is equal to 1.
Thus, $P(X \mid Z, Y) = P(X \mid Z)$ and, hence $X \perp\!\!\!\perp_P Y \mid Z$.
Similarly proof is obtained for $X \perp\!\!\!\perp_P W \mid Z$.

c. We show that the independence relation $\perp\!\!\!\perp_P$ satisfies the property

$$X \perp\!\!\!\perp_P Y \cup W \mid Z \Rightarrow X \perp\!\!\!\perp_P Y \mid Z \cup W$$

for all mutually disjoint sets of variables $X, Y, Z, W \subseteq V$.
We assume that $X \perp\!\!\!\perp_P Y \cup W \mid Z$. From this observation, we have

$$P(X \mid Z \wedge Y \wedge W) = P(X \mid Z)$$

From our assumption $X \perp\!\!\!\perp_P Y \cup W \mid Z$, we further have $X \perp\!\!\!\perp_P W \mid Z$ by the second property stated in the exercise. By definition, we therefore have that

$$P(X \mid Z \wedge W) = P(X \mid Z)$$

Now consider the conditional probability $P(X \mid Z \wedge W \wedge Y)$. We find that

$$\begin{aligned} P(X \mid Z \wedge W \wedge Y) &= P(X \mid Z) \\ &= P(X \mid Z \wedge W) \end{aligned}$$

From $P(X \mid Z \wedge W \wedge Y) = P(X \mid Z \wedge W)$, we have by definition that $X \perp\!\!\!\perp_P Y \mid Z \cup W$. We conclude that $X \perp\!\!\!\perp_P Y \cup W \mid Z \Rightarrow X \perp\!\!\!\perp_P Y \mid Z \cup W$.

d. From $X \perp\!\!\!\perp_P Z \mid W \cup Y$ we have - by defintion - that $P(X \mid W, Y, Z) = P(X \mid W, Y)$.
From $X \perp\!\!\!\perp_P Y \mid W$ we have that $P(X \mid W, Y) = P(X \mid W)$.
From both sentences above, we can conclude that

$$P(X \mid W, Y, Z) = P(X \mid W, Y) = P(X \mid W)$$

Therefore, it also holds that

$$X \perp\!\!\!\perp_P Y \cup Z \mid W$$

Exercise 2

Let V be a set of statistical variables and let $\perp\!\!\!\perp$ be a semi-graphoid independence relation on V. Show that

$$X \perp\!\!\!\perp Y \cup W \mid Z \wedge Y \perp\!\!\!\perp W \mid Z \Rightarrow X \cup W \perp\!\!\!\perp Y \mid Z$$

for all mutually disjoint sets of variables $X, Y, Z, W \subseteq V$.

We begin our proof by observing that, since $\perp\!\!\!\perp$ is a semi-graphoid independence relation, it obeys the first four axioms of the independence relation $\perp\!\!\!\perp$. Now, we assume that $X \perp\!\!\!\perp Y \cup W \mid Z$ and $Y \perp\!\!\!\perp W \mid Z$. We have that

$$\begin{aligned} X \perp\!\!\!\perp Y \cup W \mid Z \quad & \Rightarrow X \perp\!\!\!\perp Y \mid Z \cup W \\ & \Rightarrow Y \perp\!\!\!\perp X \mid Z \cup W \end{aligned}$$

by the weak union and symmetry axioms; in conjunction with our assumption $Y \perp\!\!\!\perp W \mid Z$, we find

$$\begin{aligned} Y \perp\!\!\!\perp X \mid Z \cup W \wedge Y \perp\!\!\!\perp W \mid Z \quad & \Rightarrow Y \perp\!\!\!\perp W \cup X \mid Z \Rightarrow \\ & \Rightarrow X \cup W \perp\!\!\!\perp Y \mid Z \end{aligned}$$

by the contraction and symmetry axioms.

Exercise 3

Let V be a set of statistical variables and let $\perp\!\!\!\perp$ be a semi-graphoid independence relation on V. Show that

$$X \perp\!\!\!\perp U \cup W \mid Y \cup Z \wedge Y \perp\!\!\!\perp X \mid Z \cup U \Rightarrow X \perp\!\!\!\perp Y \cup W \mid Z \cup U$$

for all mutually disjoint sets of variables $X, Y, Z, U, W \subseteq V$.

Like in the previous exercise,we begin our proof by observing that, since $\perp\!\!\!\perp$ is a semi-graphoid independence relation. Therefore, it obeys the first four axioms of the independence relation $\perp\!\!\!\perp$. We then assume that $X \perp\!\!\!\perp U \cup W \mid Y \cup Z$ and $Y \perp\!\!\!\perp X \mid Z \cup U$. We have that

$$X \perp\!\!\!\perp U \cup W \mid Y \cup Z \Rightarrow X \perp\!\!\!\perp W \mid Y \cup Z \cup U$$

by weak union. Furthermore,

$$Y \perp\!\!\!\perp X \mid Z \cup U \Rightarrow X \perp\!\!\!\perp Y \mid Z \cup U$$

by symmetry.
From contraction on

$$X \perp\!\!\!\perp W \mid Y \cup Z \cup U \wedge X \perp\!\!\!\perp Y \mid Z \cup U$$

we then obtain that

$$X \perp\!\!\!\perp Y \cup W \mid Z \cup U$$

Exercise 4

Let $V = \{X_1, X_2, X_3, X_4\}$ be a set of statistical variables. Furthermore, let $\perp\!\!\!\perp$ be a (semi-graphoid) independence relation on V, containing, amongst others, the following elements:

$$
\begin{array}{rc}
\{X_1\} \perp\!\!\!\perp \{X_4\} \mid \varnothing & \{X_4\} \perp\!\!\!\perp \{X_2\} \mid \{X_1\} \\
\{X_2\} \perp\!\!\!\perp \{X_4\} \mid \varnothing & \{X_4\} \perp\!\!\!\perp \{X_3\} \mid \{X_1\} \\
\{X_3\} \perp\!\!\!\perp \{X_4\} \mid \varnothing & \{X_4\} \perp\!\!\!\perp \{X_2, X_3\} \mid \{X_1\} \\
\{X_4\} \perp\!\!\!\perp \{X_1\} \mid \varnothing & \{X_1\} \perp\!\!\!\perp \{X_4\} \mid \{X_2\} \\
\{X_4\} \perp\!\!\!\perp \{X_2\} \mid \varnothing & \{X_3\} \perp\!\!\!\perp \{X_4\} \mid \{X_2\} \\
\{X_4\} \perp\!\!\!\perp \{X_3\} \mid \varnothing & \{X_1, X_3\} \perp\!\!\!\perp \{X_4\} \mid \{X_2\} \\
\{X_1, X_2\} \perp\!\!\!\perp \{X_4\} \mid \varnothing & \{X_4\} \perp\!\!\!\perp \{X_1\} \mid \{X_2\} \\
\{X_1, X_3\} \perp\!\!\!\perp \{X_4\} \mid \varnothing & \{X_4\} \perp\!\!\!\perp \{X_3\} \mid \{X_2\} \\
\{X_4\} \perp\!\!\!\perp \{X_1, X_3\} \mid \varnothing & \{X_2\} \perp\!\!\!\perp \{X_4\} \mid \{X_3\} \\
\{X_4\} \perp\!\!\!\perp \{X_1, X_3\} \mid \varnothing & \{X_2\} \perp\!\!\!\perp \{X_4\} \mid \{X_3\} \\
\{X_1, X_2, X_3\} \perp\!\!\!\perp \{X_4\} \mid \varnothing & \{X_1\} \perp\!\!\!\perp \{X_2\} \mid \{X_4\} \\
\{X_1\} \perp\!\!\!\perp \{X_2\} \mid \varnothing & \{X_3\} \perp\!\!\!\perp \{X_4\} \mid \{X_1, X_2\} \\
\{X_1, X_4\} \perp\!\!\!\perp \{X_2\} \mid \varnothing & \{X_2\} \perp\!\!\!\perp \{X_4\} \mid \{X_1, X_3\} \\
\{X_2, X_4\} \perp\!\!\!\perp \{X_1\} \mid \varnothing & \{X_4\} \perp\!\!\!\perp \{X_2\} \mid \{X_1, X_3\}
\end{array}
$$

Show that each statement $X \perp\!\!\!\perp Y \mid Z$, $X, Y, Z \subseteq V$, of the independence relation $\perp\!\!\!\perp$ can be derived from the statements $\{X_1, X_2, X_3\} \perp\!\!\!\perp \{X_4\} \mid \varnothing$ and $\{X_1\} \perp\!\!\!\perp \{X_2\} \mid \varnothing$, by the four independence axioms.

- From $\{X_1, X_2, X_3\} \perp\!\!\!\perp \{X_4\} \mid \varnothing$, by symmetry, we have that $\{X_4\} \perp\!\!\!\perp \{X_1, X_2, X_3\} \mid \varnothing$
- From this result, $\{X_4\} \perp\!\!\!\perp \{X_1\} \cup \{X_2, X_3\} \mid \varnothing$, by the decomposition property we obtain
  - $\{X_4\} \perp\!\!\!\perp \{X_1\} \mid \varnothing$
  - $\{X_4\} \perp\!\!\!\perp \{X_2, X_3\} \mid \varnothing$
- From $\{X_4\} \perp\!\!\!\perp \{X_1\} \mid \varnothing$, by symmetry, we have that $\{X_1\} \perp\!\!\!\perp \{X_4\} \mid \varnothing$

and so on and so forth.