Marginalisation Explained

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November 13, 2000

Let $P : \mathcal{B} \to [0,1]$ be a probability distribution, where $\mathcal{B}$ is the set of all possible Boolean expressions which can be formed from a finite number of variables, including X and A. If $X = \mathit{true}$ this is denoted by x; if $X = \mathit{false}$ this is denoted by $\neg x$.

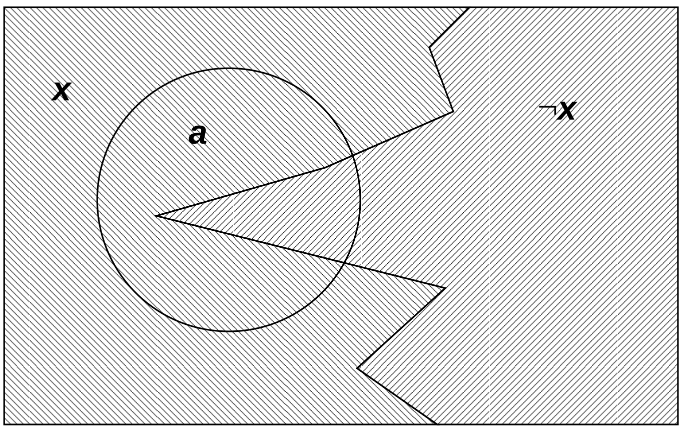
The basic axioms of probability theory are as follows:

1. $P(\top) = 1$;
2. $P(\bot) = 0$;
3. $P(u \vee v) = P(u) + P(v)$ if $u \wedge v = \bot$, with u, v Boolean expressions.

Let $P(X)$ be the probability distribution defined for the stochastic variable X, e.g. $P(x) = 0.3$ and $P(\neg x) = 0.7$. Of course, it holds that

$$P(x) + P(\neg x) = 1$$

since $P(x) + P(\neg x) = P(x \vee \neg x) = P(\top) = 1$ by definition.



Consider the Figure above, which is a Venn diagram; it is used to represent Boolean expressions. We also associate numbers with the various subareas of this diagram. The area of the rectangular, largest Venn diagram is assumed to be 1. The rectangle itself stands for $\top$, i.e. the universal upper bound of all Boolean expressions; all Boolean expressions are contained in this. This area has been subdivided into two subareas: the left indicates x with area equal to $P(x)$, which is smaller than 1; at the right part we have $\neg x$ with area $P(\neg x)$. As together they fill up the entire area ($\top = x \vee \neg x$), it is easily seen that $P(x) + P(\neg x) = 1$. The circular area in the middle stands for $P(a)$, which is also smaller than 1. The expression a intersects both x and $\neg x$. The area of the left part of this intersection, which is $a \wedge x$, is equal to $P(a \wedge x)$. The right intersection, which is $a \wedge \neg x$, has an area equal to $P(a \wedge \neg x)$. From the figure, it can be concludes that

$$P(a) = P(a \wedge x) + P(a \wedge \neg x) = \sum_X P(a \wedge X)$$

which is known as the *marginalisation* rule.