

Probability distribution

- **Probability distribution P** : attaches a number in (closed) interval $[0, 1]$ to *Boolean expressions*
- **Boolean algebra \mathcal{B}** (for two variables *rain* and *happy*):
 - \top (true),
 - rain*, \neg *rain*,
 - happy*, \neg *happy*,
 - rain* \wedge *happy*, ..., *rain* \wedge *happy* \wedge \neg *happy*, ...,
 - \neg *rain* \wedge *happy*, ..., *rain* \vee *happy*,
 - \neg *rain* \vee *happy*, ..., \perp (false)

such that:

- $\perp \leq \text{rain}$, $\text{rain} \leq (\text{rain} \vee \text{happy})$, ... (in general $\perp \leq x$ for each Boolean expression $x \in \mathcal{B}$);
- $x \leq \top$ for each Boolean expression $x \in \mathcal{B}$

Probability distribution

- A **probability distribution P** is defined as a function $P : \mathcal{B} \rightarrow [0, 1]$, such that:
 - $P(\perp) = 0$
 - $P(\top) = 1$
 - $P(a \vee b) = P(a) + P(b)$, if $a \wedge b = \perp$ with $a, b \in \mathcal{B}$
- **Examples:**
 - $P(\text{rain} \vee \text{happy}) = P(\text{rain}) + P(\text{happy})$, as $\text{rain} \wedge \text{happy} = \perp$ (why? Because I define it that way)
 - $P(\text{rain} \wedge \text{happy}) = P(\perp) = 0$
 - $P(\neg \text{rain} \vee \text{rain}) = P(\neg \text{rain}) + P(\text{rain}) = P(\top) = 1$
 $\Rightarrow P(\neg \text{rain}) = 1 - P(\text{rain})$
 - $0 \leq P(\text{rain}) \leq 1$

Probability distribution

- **Boolean algebras (sets):**

- $\top \Leftrightarrow \Omega$
- $\perp \Leftrightarrow \emptyset$
- $a \Leftrightarrow A$
- $\neg a \Leftrightarrow \bar{A}$
- $(a \vee b) \Leftrightarrow (A \cup B)$
- $(a \wedge b) \Leftrightarrow (A \cap B)$
- $a \leq (a \vee b) \Leftrightarrow A \subseteq (A \cup B)$

with \Leftrightarrow 1-1 correspondence, e.g.

$$P(\overline{\text{Rain}}) = 1 - P(\text{Rain})$$

- **Conditional probability distribution:**

$$P(\text{happy} \mid \text{rain})$$

probability of *happy* assuming that *rain* is true

Conditional probability

(Example: *flu* and *fever*)

- $P(\text{flu} \wedge \text{fever})$: chance of flu and fever at the same time
- $P(\text{flu} \mid \text{fever})$: chance of flu knowing that the person already has fever (conditional probability)
- Definition:

$$P(\text{flu} \mid \text{fever}) = \frac{P(\text{flu} \wedge \text{fever})}{P(\text{fever})}$$

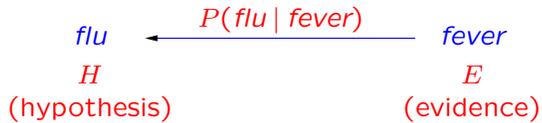


adjust $P(\text{flu} \wedge \text{fever})$, so that uncertainty in 'fever' is removed

- Recall: $P(\text{Flu} \cap \text{Fever})$ is different notation, with same meaning as $P(\text{flu} \wedge \text{fever})$

Reversal of chances

- $P(h | e)$ (e.g. $P(\text{flu} | \text{fever})$) is usually unknown:



- Known is:

$$P(e | h) \quad P(\text{fever} | \text{flu}) = 0.9$$

$$P(h) \quad P(\text{flu}) = 0.05$$

$$P(e) \quad P(\text{fever}) = 0.09$$



$$P(\text{flu}) = 0.05$$

$$P(\text{fever}) = 0.09$$

Bayes' rule (the 'chance reverter'):

$$P(h | e) = P(e | h)P(h)/P(e)$$

Bayes and marginalisation

- Bayes' rule – reversal of chances:

$$P(\text{fever} | \text{flu}) = 0.9$$

$$P(\text{flu}) = 0.05$$

$$P(\text{fever}) = 0.09$$

$$\begin{aligned}
 P(\text{flu} | \text{fever}) &= \frac{P(\text{fever} | \text{flu})P(\text{flu})}{P(\text{fever})} \\
 &= 0.9 \cdot 0.05 / 0.09 = 0.5
 \end{aligned}$$

- Marginalisation and conditioning:

$$\begin{aligned}
 P(e) &= P(e \wedge h) + P(e \wedge \neg h) \\
 &= P(e | h)P(h) + P(e | \neg h)P(\neg h)
 \end{aligned}$$

since $P(a \vee b) = P(a) + P(b)$ if $a \wedge b = \perp$,

$$\begin{aligned}
 P(e) &= P(e \wedge \top) \\
 &= P(e \wedge (h \vee \neg h)) \\
 &= P((e \wedge h) \vee (e \wedge \neg h))
 \end{aligned}$$

$$\text{and } P(e | h) = P(e \wedge h) / P(h)$$