Probability distribution

- *Probability distribution P:* attaches a number in (closed) interval $[0, 1]$ to *Boolean expressions*
- *Boolean algebra $\mathcal{B}$* (for two variables *rain* and *happy*):

  $\top$ (true),
  rain, $\neg$*rain*,
  happy, $\neg$*happy*,
  rain $\wedge$ *happy*,..., *rain* $\wedge$ *happy* $\wedge$ $\neg$*happy*,...,
  $\neg$*rain* $\wedge$ *happy*,..., *rain* $\vee$ *happy*,
  $\neg$*rain* $\vee$ *happy*,..., $\bot$ (false)

  such that:

  – $\bot \leq$ *rain*, *rain* $\leq$ (*rain* $\vee$ *happy*), ... (in general $\bot \leq x$ for each Boolean expression $x \in \mathcal{B}$);

  – $x \leq \top$ for each Boolean expression $x \in \mathcal{B}$

Probability distribution

- A **probability distribution** P is defined as a function $P : \mathcal{B} \to [0, 1]$, such that:

  – $P(\bot) = 0$

  – $P(\top) = 1$

  – $P(a \vee b) = P(a) + P(b)$, if $a \wedge b = \bot$ with $a, b \in \mathcal{B}$

- **Examples:**

  – P(*rain* $\vee$ *happy*) $= P$(*rain*) $+ P$(*happy*), as *rain* $\wedge$ *happy* $= \bot$ (why? Because I define it that way)

  – P(*rain* $\wedge$ *happy*) $= P(\bot) = 0$

  – $P(\neg$*rain* $\vee$ *rain*) $= P(\neg$*rain*) $+ P$(*rain*) $= P(\top) = 1$
  $\Rightarrow P(\neg$*rain*) $= 1 - P$(*rain*)

  – $0 \leq P$(*rain*) ≤ 1

Probability distribution

- *Boolean algebras (sets):*

  – $\top \Leftrightarrow \Omega$

  – $\bot \Leftrightarrow \varnothing$

  – $a \Leftrightarrow A$

  – $\neg a \Leftrightarrow \bar{A}$

  – $(a \vee b) \Leftrightarrow (A \cup B)$

  – $(a \wedge b) \Leftrightarrow (A \cap B)$

  – $a \leq (a \vee b) \Leftrightarrow A \subseteq (A \cup B)$

  with $\Leftrightarrow$ 1-1 correspondence, e.g.

  $$P(\overline{\text{Rain}}) = 1 - P(\text{Rain})$$

- *Conditional probability distribution:*

  $$P(\textit{happy} \mid \textit{rain})$$

  probability of *happy* assuming that *rain* is true

Conditional probability

(Example: *flu* and *fever*)

- P(*flu* $\wedge$ *fever*): chance of flu and fever at the same time
- P(*flu* | *fever*): chance of flu knowing that the person already has fever (conditional probability)
- Definition:

  $$P(\textit{flu} \mid \textit{fever}) = \frac{P(\textit{flu} \wedge \textit{fever})}{P(\textit{fever})}$$

  $\nearrow$
  adjust P(*flu* $\wedge$ *fever*), so that uncertainty in 'fever' is removed

- Recall: P(Flu $\cap$ Fever) is different notation, with same meaning as P(*flu* $\wedge$ *fever*)

Reversal of chances

- $P(h \mid e)$ (e.g. $P(\textit{flu} \mid \textit{fever})$) is usually unknown:

$$\underset{\substack{H \\ \text{(hypothesis)}}}{\textit{flu}} \xleftarrow{\;P(\textit{flu} \mid \textit{fever})\;} \underset{\substack{E \\ \text{(evidence)}}}{\textit{fever}}$$

- Known is:

$P(e \mid h)$	$P(\textit{fever} \mid \textit{flu}) = 0.9$
$P(h)$	$P(\textit{flu}) = 0.05$
$P(e)$	$P(\textit{fever}) = 0.09$

$$\underset{P(\textit{flu}) = 0.05}{\textit{flu}} \xrightarrow{\;P(\textit{fever} \mid \textit{flu})\;} \underset{P(\textit{fever}) = 0.09}{\textit{fever}}$$

Bayes' rule (the 'chance reverter'):

$$P(h \mid e) = P(e \mid h)P(h)/P(e)$$

Bayes and marginalisation

- Bayes' rule – reversal of chances:

  $P(\textit{fever} \mid \textit{flu}) = 0.9$
  $P(\textit{flu}) = 0.05$
  $P(\textit{fever}) = 0.09$

$$\begin{aligned} P(\textit{flu} \mid \textit{fever}) &= \frac{P(\textit{fever} \mid \textit{flu})P(\textit{flu})}{P(\textit{fever})} \\ &= 0.9 \cdot 0.05/0.09 = 0.5 \end{aligned}$$

- Marginalisation and conditioning:

$$\begin{aligned} P(e) &= P(e \wedge h) + P(e \wedge \neg h) \\ &= P(e \mid h)P(h) + P(e \mid \neg h)P(\neg h) \end{aligned}$$

  since $P(a \vee b) = P(a) + P(b)$ if $a \wedge b = \bot$,

$$\begin{aligned} P(e) &= P(e \wedge \top) \\ &= P(e \wedge (h \vee \neg h)) \\ &= P((e \wedge h) \vee (e \wedge \neg h)) \end{aligned}$$

  and $P(e \mid h) = P(e \wedge h)/P(h)$