Answers to selected exercises

December 16, 2011

Exercises Logic and Resolution

This set of exercises already contains several examples. We only illustrate unification once more.

Exercise 2.1(iv)

We have to find a substitution θ such that:

$$P(x,z,y)\theta = P(x,z,x)\theta = P(a,x,x)\theta$$

To make the first argument equal, we must replace x by a. This yields:

$$\begin{array}{l} P(x,z,y)\{a/x\} = P(a,z,y) \\ P(x,z,x)\{a/x\} = P(a,z,a) \\ P(a,x,x)\{a/x\} = P(a,a,a) \end{array}$$

To unify the second and third argument, it is clear that z and y also have to be replaced by a. So $\theta = \{a/x, a/y, a/z\}$.

Exercises Description Logics & Frames

Exercise 1

1. Employee $\sqsubseteq$ Human

2. Mother $\equiv$ Female $\sqcap$ $\exists$hasChild.$\top$

3. Parent $\equiv$ Mother $\bigsqcup$ Father

4. Grandmother $\equiv$ Mother $\sqcap$ $\exists$hasChild.Parent

5. $\exists$hasChild.Human $\sqsubseteq$ Human

Exercise 4.a.

Consider the formula in predicate logic:

$$\forall x((\forall y\, r(x,y) \rightarrow A(y) \wedge B(y)) \rightarrow ((\forall y\, r(x,y) \rightarrow A(y)) \wedge (\forall y\, r(x,y) \rightarrow B(y)))$$

Proof: Take an arbitrary x. Suppose that (1) $\forall y\, r(x,y) \rightarrow A(y) \wedge B(y)$. Take an arbitrary y such that $r(x,y)$. Then $A(y)$ follows from (1). So $\forall y\, r(x,y) \rightarrow A(y)$. The same reasoning for $\forall y\, r(x,y) \rightarrow B(y)$.

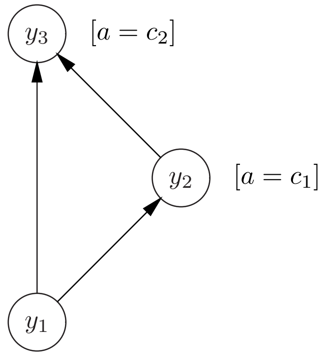


Figure 1: Multiple inheritance with exceptions

Exercise 4.c.

In predicate logic:

$$\forall x((\forall y\, r(x,y) \to A(y) \vee B(y)) \to ((\forall y\, r(x,y) \to A(y)) \wedge (\forall y\, r(x,y) \vee B(y)))$$

Consider the structure with domain $D = \{d_1, d_2, d_3\}$ and interpretation I such that:

$$I(r) = \{(d_1, d_2), (d_1, d_3)\}$$
$$I(A) = \{d_2\}$$
$$I(B) = \{d_3\}$$

Choose $x = d_1$. Then it holds that $\forall y\, r(x,y) \to A(y) \vee B(y)$, but (e.g.) not $\forall y\, r(x,y) \to A(y)$. So the formula does not hold.

Exercise 5

a.

$$\text{car} \sqsubseteq \exists \text{wheels}.\{4\}$$
$$\text{car} \sqsubseteq \exists \text{seats}.\{4\}$$
$$\text{sportscar} \sqsubseteq \text{car}$$
$$\text{sportscar} \sqsubseteq \exists \text{seats}.\{2\}$$

$$\text{Rolls-Royce} : \text{car}$$
$$(\text{Rolls-Royce}, \text{enough}) : \text{max-speed}$$

If a sportscar would have been given, then the set would have become inconsistent. In this case, it is possible that there are no sportscars, so the set is consistent.

b. The problem is in a situation as illustrated in Figure 1. The algorithm is non-deterministic because the order is not specified. If the order is y_3, and then y_2, then the attribute a gets the value c_1 (which happens to be correct). If the order is the other way around then the attribute a gets the value c_2.

Exercise 6

a. $\phi = \{\ \forall x(F_1(x) \to F_2(x)),$
$\forall x(F_1(x) \to a(x, c_1)),$
$\forall x(F_1(x) \to a(x, c_2)),$
$\forall x(F_1(x) \to a(x, c_3))\}$

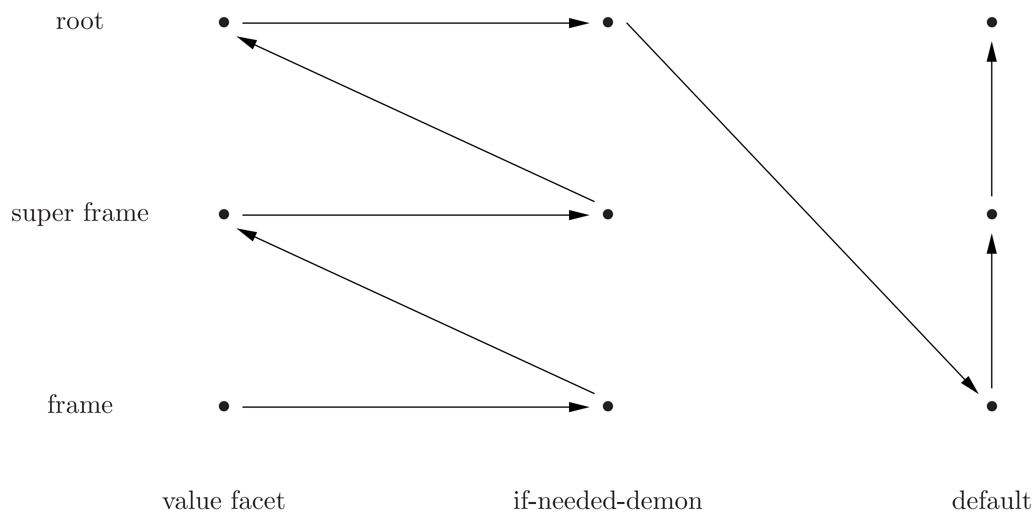


Figure 2: Inheritance relationship

In the function Inherit, replace
attr-value-pairs ← attr-value-pairs ∪ NewAttributes(pairs, attr-value-pairs)
by
attr-value-pairs ← MergeAttributes(pairs, attr-value-pairs)

such that MergeAttributes adds new values for an attribute to the set of values that it has already found. This contrasts NewAttribute, that ignores values of an attribute if it has already found a value for that attribute.

b. Correct values:

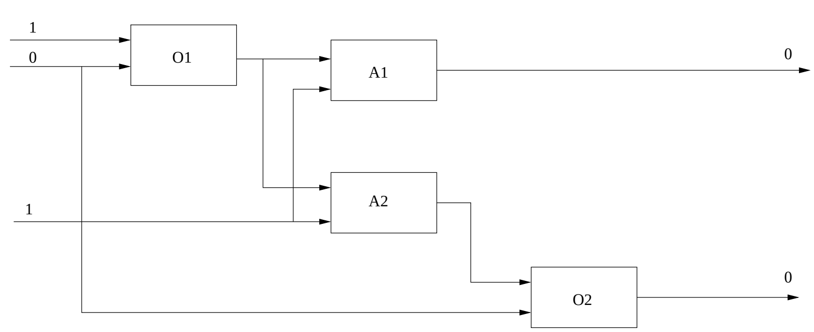
edge = 2	value cube1	not: default cube
base = 4	demon cube	not: default prism
height = 2	demon cube	not: value prism
volume = 8	demon prism	not: default cube1

The inheritance is illustrated in Figure 2.

Exercises Model-based Reasoning

Opgave 4a

The circuit can be visualised as follows:

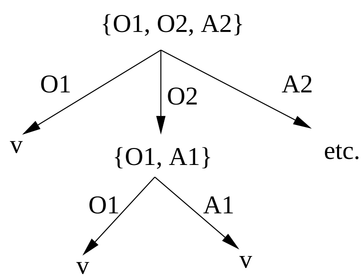


Conflict sets are sets of components that, if we assume they are normal, then we have an inconsistency with the observations. In this cas, we can find 5 conflict sets:

$$\begin{aligned}
\text{CS}_1 &= \{O_1, A_1\}\\
\text{CS}_2 &= \{O_1, O_2, A_2\}\\
\text{CS}_3 &= \{O_1, A_1, A_2\}\\
\text{CS}_4 &= \{O_1, A_1, O_2\}\\
\text{CS}_5 &= \{O_1, O_2, A_1, A_2\}
\end{aligned}$$

The first two conflict sets are (subset) minimal. Every extension of a conflict set is (of course) also a conflict set.

There are many different hitting set trees, depending on the order of 'choosing' a conflict set. A start could look as follows:



The rest is left as an exercise. The diagnoses are the minimal hitting sets, e.g. $\{O_1\}$ and $\{O_2, A_1\}$.

Exercises Uncertainty Reasoning

Exercise 1

a. $CF(a, e') = 0.7; CF(b, e') = 0.8; CF(c, e') = 0.5; CF(d, e') = 0.7$

$$CF(a\,\textbf{or}\,b, e') = \max\{CF(a, e'), CF(b, e')\} = \max\{0.7, 0.8\} = 0.8(e_1)$$
$$CF(f, e_1') = CF(f, e_1) \cdot \max\{0, CF(e_1, e')\} = 0.5 \cdot 0.8 = 0.4$$

$$CF(c\,\textbf{and}\,d, e') = \min\{CF(c, e'), CF(d, e')\} = \min\{0.5, 0.7\} = 0.5(e_2)$$
$$CF(f, e_2') = CF(f, e_2) \cdot \max\{0, CF(e_2, e')\} = 0.8 \cdot 0.5 = 0.4$$

$$CF(e, b') = CF(e, b) \cdot \max\{0, CF(b, e')\} = 0.5 \cdot 0.8 = 0.4(e_3)$$
$$CF(e, c') = CF(e, c) \cdot \max\{0, CF(c, e')\} = 1.0 \cdot 0.5 = 0.5(e_4)$$
$$CF(e, e_3'\,\textbf{co}\,e_4') = 0.4 + 0.5(1 - 0.4) = 0.7(e_5)$$
$$CF(f, e_5') = CF(f, e_5) \cdot \max\{0, CF(e_5, e')\} = 0.9 \cdot 0.7 = 0.63$$

$$CF(f, e_1'\,\textbf{co}\,e_2') = 0.5 + 0.8(1 - 0.5) = 0.9$$
$$CF(f, (e_1'\,\textbf{co}\,e_2')\,\textbf{co}\,e_5') = 0.9 + 0.63(1 - 0.9) \approx 0.963$$

b. Suppose $P(a \mid b, c) = x$; $P(a \mid \neg b, c) = y$, $P(b \mid c) = z$, $P(\neg b \mid c) = (1 - z)$;
Then $P(a \wedge b \mid c) = P(a \mid b, c)P(b \mid c) = x \cdot z$
The certainty factor interpretation gives $P(a \wedge b \mid c) = \min\{P(a \mid c), P(b \mid c)\}$ with $P(a \mid c) = P(a \mid b, c)P(b \mid c) + P(a \mid \neg b, c)P(\neg b \mid c) = x \cdot z + y \cdot (1 - z)$
However, it does not hold in general that $xz = \min\{xz + y(1 - z), z\}$.

Also, $P(a \vee b \mid c) = P(a \mid c) + P(b \mid c) - P(a \wedge b \mid c) = xz + y(1 - z) + z - xz = z + y(1 - z)$

The certainty factor interpretation gives $P(a \vee b \mid c) = \max\{P(a \mid c), P(b \mid c)\}$
However, it does not hold in general that $z + y(1-z) = \max\{xz + y(1-z), z\}$.

c. $CF(a, e') = 0.8; CF(b, e') = 0.4; CF(c, e') = 0.7; CF(d, e') = 0.6; CF(e, e') = 1.0$

$CF(a \,\mathbf{or}\, b \,\mathbf{or}\, c, e') = \max\{\max\{CF(a, e'), CF(b, e')\}, CF(c, e')\} = \max\{0.8, 0.7\} = 0.8(e_1)$
$CF(f, e_1') = CF(f, e_1) \cdot \max\{0, CF(e_1, e')\} = 1 \cdot 0.8 = 0.8$

$CF(c \,\mathbf{and}\, d, e') = \min\{CF(c, e'), CF(d, e')\} = \min\{0.7, 0.6\} = 0.6(e_2)$
$CF(f, e_2') = CF(f, e_2) \cdot \max\{0, CF(e_2, e')\} = 0.5 \cdot 0.6 = 0.3$

$CF(e, e') = 1.0(e_3)$
$CF(f, e_3') = CF(f, e_3) \cdot \max\{0, CF(e_3, e')\} = 0.6 \cdot 1 = 0.6$

$CF(f, e_1' \,\mathbf{co}\, e_2') = 0.8 + 0.3(1 - 0.8) = 0.86$
$CF(f, (e_1' \,\mathbf{co}\, e_2') \,\mathbf{co}\, e_3') = 0.86 + 0.6(1 - 0.86) = 0.94(e_4)$

$CF(g, e_4') = CF(g, e_4) \cdot \max\{0, CF(e_4, e')\} = 0.2 \cdot 0.94 = 0.188$

d. To see what it means that such a rule is idempotent, take a value for y, for example c; then $f_{co}(x, c)$ is an operator on the argument x (we could call that $o(x)$). So idempotence then means that $f_{co}(x, c) = f_{co}(f_{co}(x, c), c)$. This is not the case for the rule mentioned. For example, take $x = 0.5$ and $c = 0.4$. Then $f_{co}(x, c) = 0.5 + 0.4(1 - 0.5) = 0.7$ and $f_{co}(f_{co}(x, c), c) = 0.7 + 0.4(1 - 0.7) = 0.82$.

Advantage of idempotence: two or more identical production rule only change the CF once. Disadvantage of idempotence: of different rules that result into an equal CF, only 1 of them contributes to the final CF.

Exercise 2

a. $P(V_1)$ and $P(V_2 \mid V_1)$

b.

$$\begin{aligned}
P(v_3) &= \sum_{y \in \mathrm{dom}(V_2)} P(v_3 \mid y) \sum_{x \in \mathrm{dom}(V_1)} P(y \mid x) P(x) \\
&= \sum_{y \in \mathrm{dom}(V_2)} P(v_3 \mid y)(P(y \mid v_1)P(v_1) + P(y \mid \neg v_1)P(\neg v_1)) \\
&= \sum_{y \in \mathrm{dom}(V_2)} P(v_3 \mid y) f(y) \\
&= P(v_3 \mid v_2) f(v_2) + P(v_3 \mid \neg v_2) f(\neg v_2)
\end{aligned}$$

6 operations (two multiplications and one summation, repeated twice)

c.

$$\begin{aligned}
P(v_3 \mid v_1) &= \frac{P(v_1, v_3)}{P(v_1)} = \frac{\sum_{x \in \mathrm{dom}(V_2)} P(v_1, x, v_3)}{P(v_1)} = \frac{\sum_{x \in \mathrm{dom}(V_2)} P(v_3 \mid x) P(x \mid v_1) P(v_1)}{P(v_1)} \\
&= \sum_{x \in \mathrm{dom}(V_2)} P(v_3 \mid x) p(x \mid v_1) = P(v_3 \mid v_2) P(v_2 \mid v_1) + P(v_3 \mid \neg v_2) P(\neg v_2 \mid v_1) \\
&= 0.7 \cdot 0.3 + 0.1 \cdot 0.7 = 0.28
\end{aligned}$$

d. $P(v_1 \mid V_1 = true, V_3 = false) = P(\neg v_3 \mid V_1 = true, V_3 = false) = 1$

$$\begin{aligned}
P(v_2 \mid V_1 = true, V_3 = false) &= \frac{P(v_1, v_2, \neg v_3)}{P(v_1, \neg v_3)} \\
&= \frac{P(\neg v_3 \mid v_2)P(v_2 \mid v_1)}{P(\neg v_3 \mid v_2)P(v_2 \mid v_1) + P(\neg v_3 \mid \neg v_2)P(\neg v_2 \mid v_1)} \\
&= \frac{1}{1 + P(\neg v_3 \mid \neg v_2)P(\neg v_2 \mid v_1)} \\
&= \frac{1}{1 + 0.9 \cdot 0.7} = 0.61
\end{aligned}$$

Exercise 4

a. In the certainty factor model, CFs are propagated using the following rule: $CF(h, e') = CF(h, e) \cdot \max\{0, CF(e, e')\}$ which we could interpret as the probabilistic statement: $P(h|e') = P(h|e) \cdot \max\{0, P(e|e')\} = P(h|e)P(e|e')$.

According to the interpretation of CF rules, we can model the distribution using a Bayesian network $E' \to E \to H$. Then it holds that: $P(h|e') = P(h|e)P(e|e') + P(h|\neg e)P(\neg e|e')$. This is close to the CF model, but we still need to make sure that $P(h|\neg e)P(\neg e|e') = 0$. If $P(\neg e|e') = 0$ then $P(e|e') = 1$, which is (usually) inconsistent with the CF model, so this is a bad solution. So apparently we also need to require that $P(h|\neg e) = 0$.

b. We can make the CF factor closer to the probabilistic model by including (besides $CF(h, e)$) a statement $CF(h, \neg e)$ to a rule. Different definitions are possible, for example: $CF(h, e') = CF(h, e) \cdot \max\{0, CF(e, e') + CF(h, \neg e) \cdot \max\{0, -CF(e, e')\}\}$

c. In the noisy-AND model it holds that:

$$P(e \mid C_1, C_2) = \sum_{C_1 \wedge C_2 = e} P(e \mid I_1, I_2) \prod_{k=1}^{2} P(I_k \mid C_k) = P(i_1 \mid C_1)P(i_2 \mid C_2)$$

A corresponding CF definition could look as follows:

$$CF(h, e'_1 \,\mathbf{co}\, e'_2) = CF(h, e'_1)CF(h, e'_2)$$

Exercise 5

a.

$$\begin{aligned}
P(x_3 \mid X_2 = y) &= \sum_{x_1}\sum_{x_4} P(x_4 \mid x_3)P(x_3 \mid x_1, X_2 = y)P(x_1)P(X_2 = y) \\
&= \sum_{x_1}\sum_{x_4} f_1(x_3, x_4)f_2(x_1, x_3)f_3(x_1)f_4(X_2 = y) \\
&\propto \sum_{x_1}\sum_{x_4} f_1(x_3, x_4)f_2(x_1, x_3)f_3(x_1) \\
&= \sum_{x_4} f_1(x_3, x_4) \sum_{x_1} f_2(x_1, x_3)f_3(x_1) \\
&= \sum_{x_4} f_1(x_3, x_4) \sum_{x_1} f_5(x_1, x_3) \\
&= \sum_{x_4} f_1(x_3, x_4)f_6(x_3) \\
&= \sum_{x_4} f_7(x_3, x_4) \\
&= f_8(x_3)
\end{aligned}$$

with

x_1	x_3	f_2
y	y	0.3
y	n	0.7
n	y	0.5
n	n	0.5

x_1	x_3	f_5
y	y	$0.3 \cdot 0.6 = 0.18$
y	n	$0.7 \cdot 0.6 = 0.42$
n	y	$0.5 \cdot 0.4 = 0.2$
n	n	$0.5 \cdot 0.4 = 0.2$

x_3	f_6
y	$0.18 + 0.2 = 0.38$
n	$0.42 + 0.2 = 0.62$

x_3	x_4	f_7
y	y	$0.4 \cdot 0.38 = 0.152$
y	n	$0.6 \cdot 0.38 = 0.228$
n	y	$0.1 \cdot 0.62 = 0.062$
n	n	$0.9 \cdot 0.62 = 0.558$

x_3	f_8
y	$0.152 + 0.228 = 0.38$
n	$0.062 + 0.558 = 0.62$

$$P(X_3 = y \mid X_2 = y) = 0.38/(0.38 + 0.62) = 0.38$$

b.

$$P(X_3 = y \mid X_2 = y) = \sum_{x_1} P(X_3 = y \mid x_1, X_2 = y)P(x_1) = 0.3 \cdot 0.6 + 0.5 \cdot 0.4 = 0.38$$

Exercise 6

$P(D) = 0.3$, $P(S \mid D) = 0.7$, $P(S \mid \neg D) = 0.1$, $u_1(d,t) = 100$, $u_1(d,\neg t) = -100$, $u_1(\neg d, t) = -10$, $u_1(\neg d, \neg t) = 0$, $u_2(t) = -20$, $u_2(\neg t) = 0$. Define $u(D,T) = u_1(D,T) + u_2(T)$. That is, $u(d,t) = 80$, $u(d,\neg t) = -100$, $u(\neg d, t) = -30$, $u(\neg d, \neg t) = 0$.

$$\begin{aligned} u^* &= \sum_{S,D} \max_T f_1(D) f_2(S,D) u(D,T) \\ &= \sum_S \max_T f_6(S,T) \end{aligned}$$

where (summing out D)

$$\begin{aligned} f_6(s,t) &= 0.3 \cdot 0.7 \cdot 80 + 0.7 \cdot 0.1 \cdot -30 = 14.7 \\ f_6(s,\neg t) &= 0.3 \cdot 0.7 \cdot -100 + 0.7 \cdot 0.1 \cdot 0 = -21 \\ f_6(\neg s,t) &= 0.3 \cdot 0.3 \cdot 80 + 0.7 \cdot 0.9 \cdot -30 = -11.7 \\ f_6(\neg s,\neg t) &= 0.3 \cdot 0.3 \cdot -100 + 0.7 \cdot 0.9 \cdot 0 = -9 \end{aligned}$$

Optimal policy: given a symptom, treat; given no symptom, do not treat: $f_7(s) = 14.7$, $f_7(\neg s) = -9$; $u^* = 14.7 - 9 = 5.7$.