Reasoning with Uncertainty

Exercise 1

a. A reasoning system employs the certainty-factor calculus for reasoning with uncertain information. Consider the following five uncertain implications (rules):

$$\{R_1 : a \vee b \to f_{0.5},$$
$$R_2 : b \to e_{0.5},$$
$$R_3 : c \to e_{1.0},$$
$$R_4 : c \wedge d \to f_{0.8},$$
$$R_5 : e \to f_{0.9}\}$$

We wish to derive how certain we are about proposition f. Furthermore, it is known that $a_{0.7}$, $b_{0.8}$, $c_{0.5}$ and $d_{0.7}$ are true with the associated uncertainty, which are shorthand notations for $\mathrm{CF(a, e')} = 0.7$, etc. Determine $\mathrm{CF}(f, e')$.

b. The certainty-factor calculus employs the following combination functions to compute the conjunction and disjunction, respectively, of uncertain propositions:

$$\mathrm{CF}(e_1 \wedge e_2, e') = \min\{\mathrm{CF}(e_1, e'), \mathrm{CF}(e_2, e')\}$$

$$\mathrm{CF}(e_1 \vee e_2, e') = \max\{\mathrm{CF}(e_1, e'), \mathrm{CF}(e_2, e')\}$$

Show by means of a counter-example that these functions are incorrect from the point of view of probability theory.

c. (This exercise is similar to exercise a). A reasoning system employs the certainty-factor calculus for reasoning with uncertain information. Consider the following four uncertain implications (rules):

$$\{R_1\text{: } a \vee b \vee c \to f_{1.0},$$
$$R_2\text{: } c \wedge d \to f_{0.5},$$
$$R_3\text{: } f \to g_{0.2},$$
$$R_4\text{: } e \to f_{0.6}\}$$

The user supplies the following certainty factors 0.8, 0.4, 0.7, 0.6 and 1.0 for the propositions a, b, c, d and e. Compute the certainty factor $\mathrm{CF}(g, e')$ for proposition g using these facts and rules.

d. Show that the combination function for co-concluding rules $f_{\mathbf{co}}(x, y) = x + y(1 - x)$ is not idempotent. (Idempotency of an operator o means that if you apply the operator again, you get the same result, i.e., $o(z) = o(o(z))$.) Do you think that idempotency would be a good property of $f_{\mathbf{co}}$? Provide motivation for your solution.

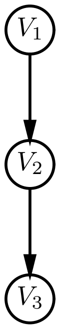
Exercise 2

Let $\mathcal{B} = (G, P)$ be Bayesian network with acyclic directed graph $G = (V(G), A(G))$ and associated joint probability distribution P, as shown in Figure 1.

a. Which probabilistic information must be locally available for vertex V_2 to compute locally, i.e., in V_2, the (marginal) probability distribution $P(V_2)$?

b. Compute $P(V_3)$ by marginalization and compute the number of operations involved in the computation.

c. Show how distributing sums during marginalization reduces the number of operations involved.

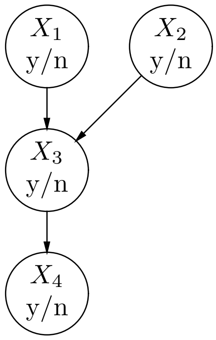


$$P(v_1) = 0.4, P(\neg v_1) = 0.6$$

$$P(v_2|v_1) = 0.3, P(\neg v_2|v_1) = 0.7$$
$$P(v_2|\neg v_1) = 0.9, P(\neg v_2|\neg v_1) = 0.1$$

$$P(v_3|v_2) = 0.7, P(\neg v_3|v_2) = 0.3$$
$$P(v_3|\neg v_2) = 0.1, P(\neg v_3|\neg v_2) = 0.9$$

Figure 1: Bayesian network; the notation $\neg v$ stands for $V = \textit{false}$ and v for $V = \textit{true}$.



$$P(X_4 = y \mid X_3 = y) = 0.4$$
$$P(X_4 = y \mid X_3 = n) = 0.1$$
$$P(X_3 = y \mid X_1 = y, X_2 = y) = 0.3$$
$$P(X_3 = y \mid X_1 = n, X_2 = y) = 0.5$$
$$P(X_3 = y \mid X_1 = y, X_2 = n) = 0.7$$
$$P(X_3 = y \mid X_1 = n, X_2 = n) = 0.9$$
$$P(X_1 = y) = 0.6$$
$$P(X_2 = y) = 0.2$$

Figure 2: Bayesian network defined on four random variables

d. Suppose that $V_1 = \textit{true}$ has been observed. Determine now $P(V_3 \mid V_1 = \textit{true})$.

e. Now, suppose that in addition to $V_1 = \textit{true}$ also the value *false* has been observed for variable V_3. Determine now the probability distributions $P(V_i \mid V_1 = \textit{true}, V_3 = \textit{false})$ for $i \in \{1, 2, 3\}$.

Exercise 4

Parts of the certainty-factor calculus can be translated to Bayesian networks. This offers a lot of insight into the nature of this calculus.

a. Give the translation of the combination function f_{prop} from the certainty-factor calculus, and give the conditions that must be satisfied for the resulting probability distribution P.

b. Show how we might redefine f_{prop} such that is comes closer to probability theory in general by dropping one of the conditions. How would an uncertain implication look like in this case?

c. The combination function f_{co} can be interpreted in probability theory as a noisy-OR Bayesian network. Show that the combination function would look like if the noisy-OR definition is replaced by the noisy-AND. (In the noisy-AND a logical AND is used to determine the probability distributions $P(E \mid C_1, \ldots, C_n)$.)

Exercise 4

Consider the Bayesian network shown in Figure 2 above.

a. Use variable elimination to compute $P(X_3 = y \mid X_2 = y)$

b. Show how we can arrive at the answer more easily by making use of two conditional probabilities and two prior probabilities

Exercise 5

A physician observes a symptom S which is indicative of a disease D. The prevalence of the disease is 30% of the admitted population. The symptom will be observed 70% or 10% of the time, depending on whether the disease is present or absent, respectively. The physician can give treatment T. The utility of this treatment is given by $u_1(d,t) = 100$, $u_1(d,\neg t) = -100$, $u_1(\neg d,t) = -10$, $u_1(\neg d,\neg t) = 0$. Furthermore, there is a cost to treatment: $u_2(t) = -20$, $u_2(\neg t) = 0$. Assume the utilities are additive. Determine the optimal policy for T and compute the expected utility given this policy.