Pictures of complete positivity in arbitrary dimension

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Background

Categorical quantum mechanics:

- study quantum mechanics from *minimal assumptions*
- category with tensor, dagger, and *compact structure*

Get rid of compactness:

- construction turning arrows into *completely positive* ones
- *axiomatize* resulting categories
Graphical language for dagger tensor categories

Trade algebraic reasoning with morphisms for graphical calculus:

\[ g \circ f = g \otimes f \]

\[ f \otimes g = f \otimes g \]

\[ \text{id}_X = \]

\[ \text{swap}_{X,Y} = \]
Compactness

\[ I \to X^* \otimes X \]

\[ \quad \Rightarrow \quad \]

\[ X \otimes X^* \to I \]

\[ \Rightarrow \quad \text{such that} \quad \]

\[ X \]

\[ = \]

\[ X \]

\[ \Rightarrow \quad \text{four orientations of morphisms:} \quad \]

\[ f^* \]

\[ = \]

\[ f \]

\[ : Y^* \to X^* \]

\[ f_\ast = (f^*)^\dagger : X^* \to Y^* \]

\[ \Rightarrow \quad \text{Example: category of finite-dimensional Hilbert spaces} \]
Selinger’s CPM-construction

- dagger compact category $\mapsto$ new dagger compact category
- $\text{fdHilb} \mapsto$ fin-dim $\ast$-algebras and completely positive maps

Definition of category $\text{CPM}(\mathcal{C})$:

- objects: those of $\mathcal{C}$

- arrows $X \rightarrow Y$:
  \[
  \begin{aligned}
  \begin{array}{c}
  f_\ast \\
  f
  \end{array}
  \end{aligned}
  \end{array}
  \quad
  \begin{array}{c}
  f \\
  \in \mathcal{C}(X, Z \otimes Y)
  \end{array}
  \)

- identities: those of $\mathcal{C}$
Selinger’s CPM-construction

\[
\text{CPM}(\mathbf{C}) \text{ is a dagger compact category:}
\]

\[
\begin{pmatrix}
 f_* & f \\
\end{pmatrix} \otimes \begin{pmatrix}
 g_* & g \\
\end{pmatrix} = \begin{pmatrix}
 g_* & f_* \\
\end{pmatrix} \otimes \begin{pmatrix}
 f & g \\
\end{pmatrix}
\]

\[
\begin{pmatrix}
 f_* & f \\
\end{pmatrix}^\dagger = \begin{pmatrix}
 f^* & f^\dagger \\
\end{pmatrix}
\]
General CP-construction: idea

- $\text{CPM}(\text{fdHilb}) = \text{fin-dim } \ast$-algebras, completely positive maps
- But completely positive maps make sense for all C*-algebras

- Question: $\text{CP}(\text{Hilb}) = \text{C}^*$-algebras, completely positive maps
- Main obstruction: $\text{Hilb}$ noncompact dagger tensor category

- Idea: ‘unbend’ $f_\ast$ into $f^\dagger$
General CP-construction

Definition of category $\text{CP}(\mathbf{C})$:

- **objects**: those of $\mathbf{C}$

- **arrows** $X \to Y$: 
  \[
  \begin{align*}
    \left\{ \begin{array}{l}
      f^\dagger f \\
      f 
    \end{array} \right\} 
    \quad \in \mathbf{C}(X, Z \otimes Y)
  \end{align*}
  \]

- **composition**:
  \[
  \begin{align*}
    \left( \begin{array}{c}
      g^\dagger \\
      g
    \end{array} \right) 
    \circ 
    \left( \begin{array}{c}
      f^\dagger \\
      f
    \end{array} \right) 
    = 
    \left( \begin{array}{c}
      f^\dagger \\
      g^\dagger \\
      g \\
      f
    \end{array} \right)
  \end{align*}
  \]

- **identity**:
  \[
  \text{id}_X = \text{swap}_{X,X}
  \]
General CP-construction makes sense: compact case

\[ \text{CP}(C) \text{ is a tensor category:} \]

\[
\begin{pmatrix}
  f^\dagger \\
  f
\end{pmatrix} \otimes
\begin{pmatrix}
  g^\dagger \\
  g
\end{pmatrix}
= \begin{pmatrix}
  g^\dagger \\
  f^\dagger \\
  g \\
  f
\end{pmatrix}
\]

But not compact or dagger in general

**Proposition:** If \( C \) is compact, then \( \text{CPM}(C) \) and \( \text{CP}(C) \) are isomorphic as (dagger compact) tensor categories
A linear map $\varphi : A \rightarrow B$ between C*-algebras is

- **positive** when $\varphi(a^*a) \geq 0$
- **completely positive** when $\varphi \otimes \text{id} : A \otimes M_n \rightarrow B \otimes M_n$ positive
- ***-homomorphism** when $\varphi(ab) = \varphi(a)\varphi(b)$ and $\varphi(a^*) = \varphi(a)^*$
- **normal** when it is ultraweakly continuous (for von Neumann algebras: when $\varphi(\bigvee p_i) = \bigvee_i \varphi(p_i)$ for increasing projections)
- a **quantum operation** when normal and completely positive

**Theorem (Stinespring):** If $\varphi : A \rightarrow B(H)$ is completely positive and $A$ is a von Neumann algebra, then

$$\varphi(a) = v^\dagger \pi(a) v$$

for ***-homomorphism** $\pi : A \rightarrow B(K)$ and bounded linear $v : H \rightarrow K$
General CP-construction makes sense: Hilbert space

A linear map \( \varphi: A \to B \) between C*-algebras is

- **positive** when \( \varphi(a^*a) \geq 0 \)
- **completely positive** when \( \varphi \otimes \text{id}: A \otimes M_n \to B \otimes M_n \) positive
- **\(*\)-homomorphism** when \( \varphi(ab) = \varphi(a)\varphi(b) \) and \( \varphi(a^*) = \varphi(a)^* \)
- **normal** when it is ultraweakly continuous (for von Neumann algebras: when \( \varphi(\bigvee p_i) = \bigvee_i \varphi(p_i) \) for increasing projections)
- a ***quantum operation*** when normal and completely positive

**Theorem (Dixmier):** If \(*\)-homomorphism \( \pi: B(H) \to B(K) \) is normal, then \( \pi = \left( B(H) \xrightarrow{\pi_1} B(H \otimes H') \xrightarrow{\pi_2} B(K') \xrightarrow{\pi_3} B(K) \right) \) for

- \( \pi_1(f) = f \otimes \text{id}_{H'} \)
- \( \pi_2(f) = pf \) for projection \( p \) onto subspace \( K' \subseteq H \otimes H' \)
- \( \pi_3(f) = u^\dagger fu \) for unitary \( u: K \to K' \)
General CP-construction makes sense: Hilbert space

**Corollary**: A linear map $\varphi: B(H) \to B(K)$ is a quantum operation iff $\varphi(f) = g^\dagger(f \otimes \text{id}_{H'})g$ for bounded linear $g: K \to H \otimes H'$.

**Theorem**: $\text{CP(Hilb)}$ is isomorphic as a tensor category to Hilbert spaces and quantum operations

\[
\begin{pmatrix}
H & K \\
\downarrow & \downarrow \\
H' & H'
\end{pmatrix} \quad \mapsto \quad \left(g^\dagger(- \otimes \text{id}_{H'})g\right)
\]
Environment structures

Question: when is a given category of the form $\text{CP}(\mathbf{C})$?

Answer: an *environment structure* for a dagger tensor category $\mathbf{C}$ is a dagger tensor supercategory $\hat{\mathbf{C}}$ with the same objects, in which each object $X$ has a morphism $\uparrow_X$ such that:

- $\uparrow_X Y = X \otimes Y$ in $\hat{\mathbf{C}}$
- $f^\dagger f^\dagger = g^\dagger g^\dagger$ in $\mathbf{C}$ $\iff$ $f = g$ in $\hat{\mathbf{C}}$
- $\forall \hat{f} \in \hat{\mathbf{C}}(X, Y) \exists f \in \mathbf{C}(X, Z \otimes Y): \hat{f} = f$ in $\hat{\mathbf{C}}$.
Environment structures: use

**Theorem:** An environment structure on a tensor category $\mathbf{C}$ induces an isomorphism $\xi: \text{CP}(\mathbf{C}) \to \hat{\mathbf{C}}$ of tensor categories.
Axiomatizing the general CP-construction: idea

There is a canonical functor $\mathbf{C} \rightarrow \text{CP}(\mathbf{C})$

Let

- $\mathbf{D}$ be the image of this functor ('double' of $\mathbf{C}$)
- $\hat{\mathbf{D}} = \text{CP}(\mathbf{C})$

When is this an environment structure?
The doubling axiom

A tensor category satisfies the *doubling axiom* when

\[ f f = g g \iff f = g \]

for all parallel morphisms \( f \) and \( g \).

**Proposition**: If a dagger compact category \( \mathbf{C} \) obeys the doubling axiom, then \( \text{CPM}(\mathbf{C}) \) satisfies *preparation-state agreement*:

\[ f f^\dagger = g g^\dagger \iff f = g \]
Axiomatizing the general CP-construction

Canonical functor $\mathbf{C} \rightarrow \text{CP}(\mathbf{C})$ is not faithful, so cannot recover $\mathbf{C}$ from abstract category $\text{CP}(\mathbf{C})$ alone. Best we can get: when $\text{CP}(\mathbf{C})$ arises from environment structure.

**Theorem** A tensor category $\hat{\mathcal{D}}$ is of the form $\text{CP}(\mathbf{C})$ if and only if it is part of an environment structure and satisfies the doubling axiom.
Axiomatizing the general CP-construction

Theorem A tensor category $\hat{D}$ is of the form $\text{CP}(C)$ iff it is part of an environment structure and satisfies the doubling axiom.

(1) $\iff$ (2) always
(2) $\iff$ (3) means doubling axiom for $\text{CP}(C)$
(1) $\iff$ (3) means $\frac{\bigoplus}{X} = \left( \begin{array}{c} X \\ X \end{array} \right)$ is environment structure