How to Understand What Classical Mathematicians Say

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Incompleteness and Completeness

Formalizing Logic and Analysis in Type Theory

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Constructive Logic

- Constructive logic is usually considered a restriction of classical logic.
  - Law of excluded middle is disallowed.
  - Proof by contradiction is disallowed.

The Classical Fragment

- Better to consider classical logic as a fragment of constructive logic.
- Define the classical (weak) disjunction
  - $A \lor_c B := \neg(\neg A \land \neg B)$
- Define the classical (weak) existential
  - $\exists_c x{:}A.\ B(x) := \neg\forall x{:}A. \neg B(x)$

The Classical Fragment

Excluded Middle

- $A \vee_c \neg A$
- $\neg(\neg A \wedge \neg\neg A)$
- Easy proof

The Classical Fragment

Proof by contradiction

- Call a formula double negation stable (or simply stable) when:
  - $\neg\neg A \Rightarrow A$
- Stability is preserved by all classical connectives:
  - $\Rightarrow$, $\forall$, $\wedge$, $\top$, $\bot$
  - $\neg$, $\vee_c$, $\exists_c$
- All decidable atomic formulas are stable

The Classical Fragment

- Thus all formulas in the “classical fragment” are double negation stable.
- Proof by contradiction is allowed when proving a “classical formula”.

The Classical Fragment

- When proving a formula in the “classical fragment” one can eliminate the classical connectives $\vee_c$, $\exists_c$ the same way that their constructive counterparts are eliminated.
  - $(A \Rightarrow X) \Rightarrow$
    $(B \Rightarrow X) \Rightarrow$
    $(\neg\neg X \Rightarrow X) \Rightarrow$
    $(A \vee_c B) \Rightarrow X$
  - $(\forall x{:}A.B(x) \Rightarrow X) \Rightarrow$
    $(\neg\neg X \Rightarrow X) \Rightarrow$
    $(\exists_c x{:}A.\ B(x)) \Rightarrow X$

The Classical Fragment

- When proving a formula in the “classical fragment” one can eliminate the classical connectives $\vee_c$, $\exists_c$ the same way that their constructive counterparts are eliminated.
- I estimate that 70% of theorems in constructive math are in this classical fragment
- I estimate that 90% of the time when proving theorems, you are in a context when you are proving a formula in the classical fragment
    - These statistics are completely made up.

The Constructive Extension

- Formulas involving constructive existential and constructive disjunction are not necessarily stable.
- Constructive logic extends classical logic by adding
    - Constructive disjunction: ∨
    - Constructive existential: ∃

The Constructive Extension

- Constructive connectives are stronger than their classical counterparts.
- Also choice is provable with the constructive existential
  - if $\forall a{:}A.\ \exists b{:}B.\ R\ a\ b$ then $\exists f{:}A \Rightarrow B.\ \forall a{:}A.\ R\ a\ (f\ a)$

Classical Mathematics

- We know how to interpret classical logic.
- What about classical mathematics?
- Classical mathematics has two other features we need to interpret:
  - Quotients
  - Functions

Quotients

- In type theory we interpret quotients as setoids
  - Setoid is a type paired with and equivalence relation.
- Notice that the constrictive choice theorem does not extend to setoids.
  - If it did we could derive excluded middle.

Functions

- A function in classical mathematics is a binary relation F such that:
  - $\forall a{:}A.\ \exists b{:}B.\ F\ a\ b$
  - $\forall a{:}A.\ \forall b\ b'{:}B.\ F\ a\ b \Rightarrow F\ a\ b' \Rightarrow b = b'$

Functions

- We translate the classical definition of a function to type theory as:
- A classical function is binary relation F on setoids A and B such that:
  - $\forall a{:}A.\ \exists_c b{:}B.\ F\ a\ b$
  - $\forall a\ a'{:}A.\ \forall b\ b'{:}B.\ F\ a\ b \Rightarrow F\ a'\ b' \Rightarrow a \asymp_A a' \Rightarrow b \asymp_B b'$
- Note that a classical function is always a respectful relation.

The Constructive Extension

- Thus constructive type theory also adds:
    - Types
    - Type of constructive functions
- Every type can be considered as a setoid (with equality).
- Every respectful constructive function can be injected into the space of classical functions.
    - Constructive functions on types can be injected to the space of classical functions between types
    - Constructive function on types respect the standard type setoid with equality.

Axiom of Choice

- The classical axiom of choice is translated using:
  - Classical quantifiers
  - Classical function spaces
  - Setoids for domains and codomains
- The full classical axiom is not provable in type theory
- Several other choice axioms available by reinterpreting without some subset of the three translation pieces.

Example

- Translate and prove the classical theorem:
  - If $\forall n.\ \exists m.\ R\ n\ m$ then $\exists f.\ \forall n.\ R\ n\ (f\ n)$ where R is a classical binary predicate over the natural numbers.

Constructive Mathematicans want Classical Math

- Why should constructive mathematicians care about the classical fragment?
- Not all classical theorems can be proved when classical operations are replaced by their constructive counterparts.
  - Note that this can be done for all $\prod_2$ formulas.
- We can prove the translated versions
  - These translated versions (or some partially translated versions) can be useful.

Dichotomy for Real Inequality

- It is well known that $\forall x\ y{:}\mathbb{R}.\ x \leq y \lor y \leq x$ is not provable constructively.
- The translated version $\forall x\ y{:}\mathbb{R}_c.\ x \leq y \lor_c y \leq x$ is provable.
- The partially translated version:
  $\forall x\ y{:}\mathbb{R}.\ x \leq y \lor_c y \leq x$
  is provable and useful.
  - Same also holds for trichotomy.

Dichotomy for Real Inequality

- In Feb. 2007, Robert Kam asked on the C-CoRN mailing list how to combine his two proofs of:
    - $\forall x{:}\mathbb{R}.\ x \le \pi/2 \Rightarrow P(x)$
    - $\forall x{:}\mathbb{R}.\ \pi/2 \le x \Rightarrow P(x)$
- I told him he would have to extend one of the theorems so that the intervals overlap
    - This was bad advice

Dichotomy for Real Inequality

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  - $\forall x{:}\mathbb{R}.\ x \le \pi/2 \Rightarrow P(x)$
  - $\forall x{:}\mathbb{R}.\ \pi/2 \le x \Rightarrow P(x)$
- I should have said that his $P(x)$ is almost certainly in the classical fragment, and if so then is it possible to do case analysis on $x \le \pi/2 \lor_c \pi/2 \le x$ to combine the two theorems into:
  - $\forall x{:}\mathbb{R}.\ P(x)$

Metric Space of Finite Sets

$$H_\varepsilon\ S\ T := \forall x \in S.\ \exists_c x \in T.\ B_\varepsilon\ x\ y$$

$$B_\varepsilon\ S\ T := H_\varepsilon\ S\ T \wedge H_\varepsilon\ T\ S$$

- The classical exists is essential for proving the closedness property.
  - (closed) $\forall \varepsilon{:}\mathbb{Q}^+.\ \forall x\ y : X.\ (\forall \delta{:}\mathbb{Q}^+.\ B_{\delta+\varepsilon}\ x\ y) \Rightarrow B_\varepsilon\ x\ y$
- Requires a classical proof of the infinite pigeon hole principle:
  - If $\forall n.\ \exists_c a \in_c S.\ \wedge\ F\ n\ a$ then $\exists_c a \in_c S.\ \forall n.\ \exists_c m \geq n.\ F\ m\ a$

Computer Scientists want Constructive Math

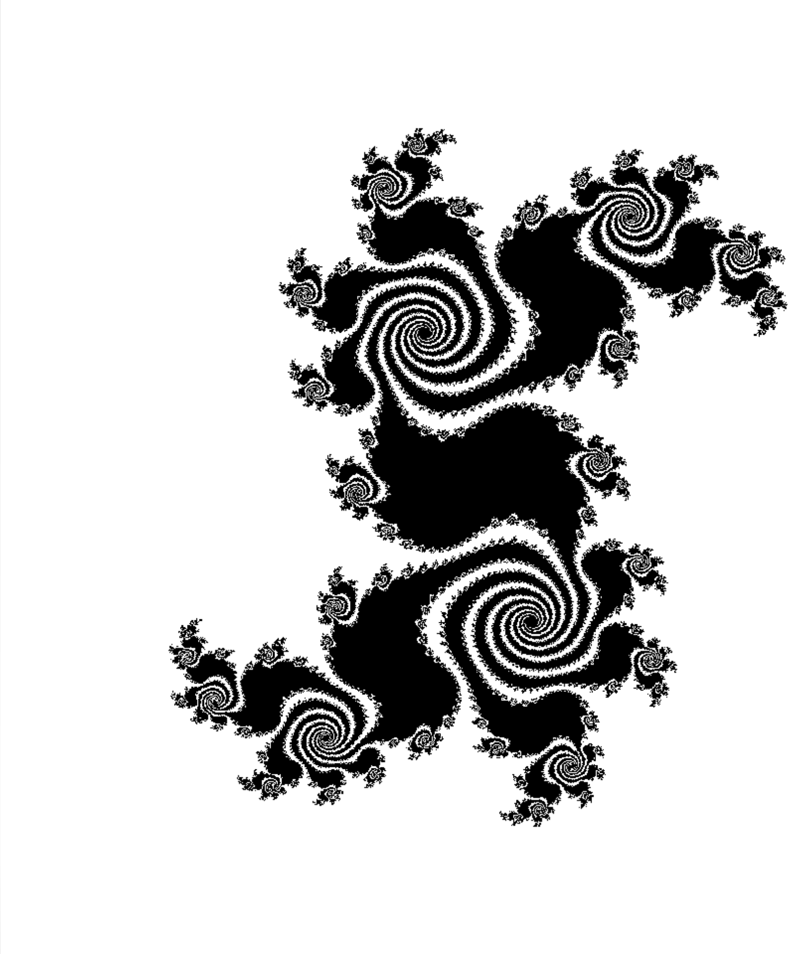
- Is there a program that
  - outputs 0 if the Riemann Hypothesis is True
  - outputs 1 if the Riemann Hypothesis if False

Computer Scientists want Constructive Math

- We do not know yet how to constructively prove
  - $\mathrm{RH} \vee \neg\mathrm{RH}$

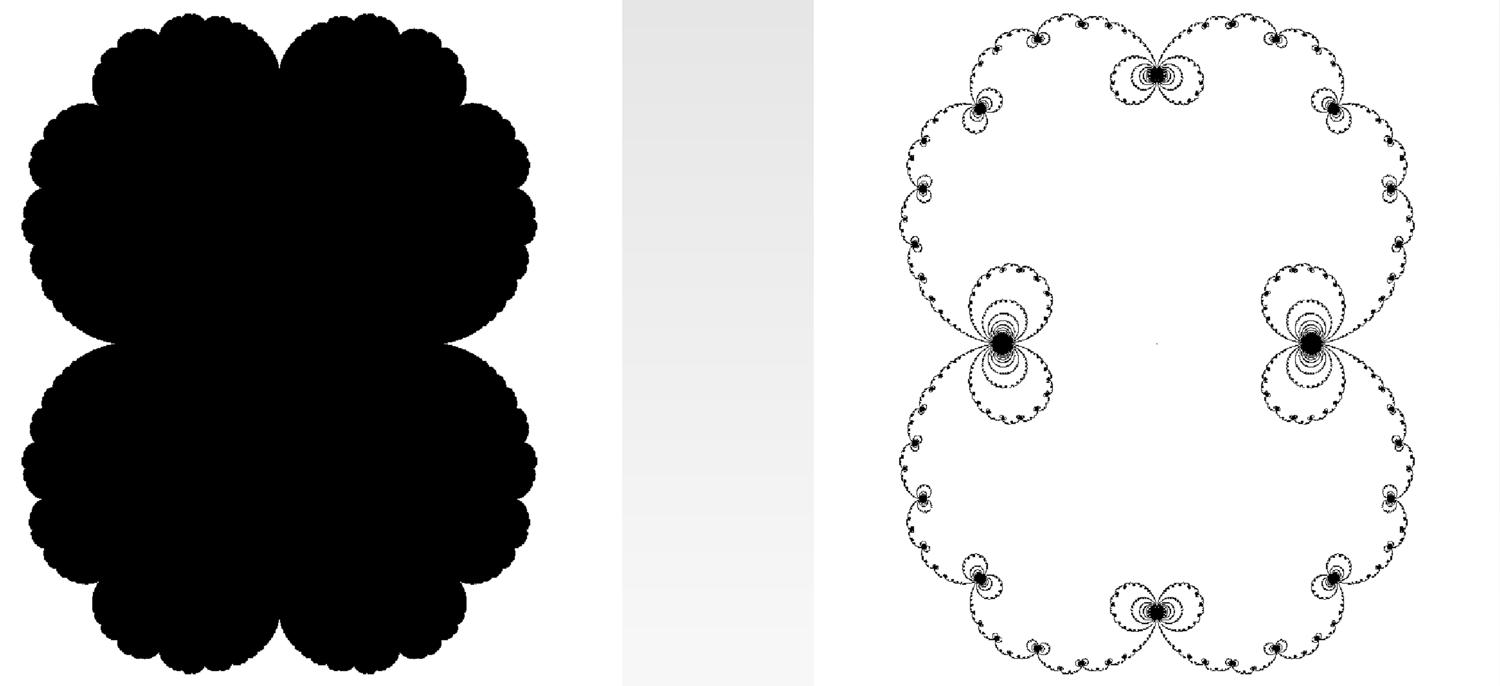
Filled Julia Sets

- Is every quadratic filled Julia set computable?



Filled Julia Sets

- Yes, but there exists quadratic filled Julia sets that we do not know how to draw.



Filled Julia Sets

- If Braverman had used constructive logic he would have only claimed:
- A quadratic filled julia set is compact when
    - The polynomial has an attracting orbit or
    - The polynomial has a parabolic orbit or
    - The polynomial has a Siegel orbit or
    - The polynomial has a Cremer orbit or
    - All the orbits of the polynomial are repelling

Classical Mathematicians want Constructive Math

- “The familiar statement that a countable union of countable sets is countable requires the axiom of countable choice.
- “However, if we do have explicit enumerations of the sets A_n then the proof yields for us an explicit enumeration of their union.

—gowers

Constructive Math is Useful for Formalization

- The way that classical mathematicians (think HOL-light minus indefinite description) formalize constructive math is by:
  - Skolemizing all the constructive existentials
  - Writing programs for the constructive witnesses
  - Prove the remainder of formula with the programs inserted.
    - This remainder is in the classical fragment of constructive logic.

Constructive Math is Useful for Formalization

- Why force this separation between programming and proving?
- Keeping constructive program data along side its classical invariants is a useful way to formalize.
- Anytime you are in the classical fragment, you can use classical reasoning.
  - Constructive theorem provers should provide more support for reasoning in the classical fragment.