CONSTRUCTIVE GELFAND DUALITY FOR C*-ALGEBRAS

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Abstract. We present a constructive proof of Gelfand duality for C*-algebras by reducing the problem to Gelfand duality for real C*-algebras.

1. Introduction

In a sequence of papers starting in a 1980 pre-print and culminating in the references [BM1, BM2, BM3, BM4], Banaschewski and Mulvey explore a constructive version of the Gelfand duality theorem which can be applied internally in an arbitrary topos. Their treatment is not quite constructive: it relies on Barr's Theorem and hence depends on the topos being a Grothendieck topos. We show that a constructive treatment can be obtained by a constructive reduction of Gelfand duality in the complex case to the real case. A constructive presentation of Gelfand duality in the real case has been given in [C]. Our proof uses a concrete presentation of the Gelfand spectrum as a lattice. Such constructive proofs are sometimes more direct [CS] than proofs via an encoding of topology in metric spaces, as is common in Bishop's constructive mathematics [Bis]. Moreover, this construction of the lattice presenting the spectrum as locale is technically advantagous, as it is preserved under inverse images of geometric morphisms. As such it has been applied in [HLS]. An alternative constructive proof announced in [BM4].

2. Preliminaries

We recall here the definition of a commutative C^*-algebra A in a topos. As explained in [BM4], when working in an intuitionistic framework, we cannot assume in general the (semi)norm of an element to be a Dedekind real, but instead it may simply be a *non negative upper real.* We define a non negative upper real to be a nonempty upward closed set of positive rational numbers. We can define the addition and multiplication of non negative upper reals: $U_1 + U_2$ is the set of rationals $r_1 + r_2,\ r_1 \in U_1,\ r_2 \in U_2$ and U_1U_2 is the set of rationals $r_1r_2,\ r_1 \in U_1,\ r_2 \in U_2$. We define also $U_1 \leqslant U_2$ to mean that U_1 is a subset if U_2. Finally we may identify the non negative rational q with the set of rationals r such that $r > q$. The norm $\|a\|$ of a in A is then an upper real. The notation of [BM4] is $a \in N(q)$ for $\|a\| < q$. The conditions for the relation $a \in N(q)$ can then be written as the usual conditions on the seminorm

$$\|0\| = 0,\ \|1\| = 1,\ \|a^*\| = \|a\|,\ \|ab\| \leqslant \|a\|\|b\|$$

$$\|ra\| = |r|\|a\|,\ \|a + b\| \leqslant \|a\| + \|b\|,\ \|aa^*\| = \|a\|^2$$

As in [BM4], we assume finally A to be *complete*: any Cauchy approximation on A has a limit in A.

We let $B = A_{sa}$ be the set of *self-adjoint* elements, i.e. elements a such that $a^* = a$. The algebra B is then a commutative Banach algebra over the rationals. For a in B, we have $\|a^2\| = \|a\|^2$, since $a = a^*$.

Lemma 1. *For a, b in B we have $\|a^2\| \leqslant \|a^2 + b^2\|$.*

Proof. We write $a^2 + b^2 = (a + bi)(a - bi) =: cc^*$. So $\|a^2 + b^2\| = \|cc^*\| = \|c\|^2$. Finally, $2a = c + c^*$, so $\|a\| = \frac{1}{2}\|c + c^*\| \leqslant \frac{1}{2}(\|c\| + \|c^*\|) = \|c\|$ and therefore $\|a^2\| = \|a\|^2 \leqslant \|c\|^2 = \|a^2 + b^2\|$. □

3. Real Banach algebras

In this section, we consider a complete commutative Banach algebra B over the rationals such that $\|a^2\| = \|a\|^2$ and $\|a^2\| \leqslant \|a^2 + b^2\|$. By Lemma 1, this will be the case if we take for B the self-adjoint part of a commutative C^*-algebra.

Lemma 2. *If $\|1 - x\| \leqslant 1$. Then x is a square.*

Proof. We give an explicit proof that the Taylor series for $\sqrt{1-(1-x)}$ converges. We define two sequences: y_n in B and r_n in $\mathbb{Q}$. We take $y_0 = 0$, $r_0 = 0$ and $y_{n+1} = \frac{1}{2}(1 - x + y_n^2)$ and $r_{n+1} = \frac{1}{2}(1 + r_n^2)$.

For all n, $\|y_n\| \leqslant r_n$ by induction. Since we have

$$y_{n+1} - y_n = \frac{1}{2}(y_n + y_{n-1})(y_n - y_{n-1})$$

we get $\|y_{n+1} - y_n\| \leqslant r_{n+1} - r_n$ by induction. Consequently,

$$\|(1 - y_n)^2 - x\| = 2\|y_{n+1} - y_n\| \leqslant 2(r_{n+1} - r_n) \to 0$$

because we have $r_n \to 1$ in a constructive way [C]. □

Lemma 3. *A sum of squares is a square.*

Proof. As in [KeVau]. If both $\|x\|, \|1 - x\| \leqslant 1$, then x and $1 - x$ are squares.

Conversely, if $x = u^2$ and $1 - x = v^2$, then $1 = u^2 + v^2$, so $\|u\|^2, \|v\|^2 \leqslant 1$.

Suppose that x, y are squares. We can assume $\|x\|, \|y\| \leqslant 1$. Then $1-x$ and $1-y$ are squares and so $\|1-x\|, \|1-y\| \leqslant 1$. Hence $\|1-\frac{(x+y)}{2}\| \leqslant \frac{1}{2}(\|1-x\|+\|1-y\|) \leqslant 1$, and so $(x + y)/2$ and hence $x + y$ are squares. □

Let P be the set of all squares. Then P is a *cone*: it contains the squares and is closed under multiplication and addition, and it defines an ordering on the algebra B. As in [C] we define $r \ll a$ to mean $a - s \in P$ for some $s > r$. By Lemma 2 we have $r - a$ in P if $\|a\| \leqslant r$ and hence B has a strong unit for this ordering. Consequently, all the results of the first part of [C] are available.

We define $\mathsf{MFn}(A)$ to be the locale generated by symbols $D(a)$ and relations

(1) $D(1) = 1$
(2) $D(-a^2) = 0$
(3) $D(a + b) \leqslant D(a) \vee D(b)$
(4) $D(a) \wedge D(-a) = 0$
(5) $D(ab) = (D(a) \wedge D(b)) \vee (D(-a) \wedge D(-b))$
(6) $D(a) = \vee_{r>0} D(a - r)$

Lemma 4. *If $0 \ll ac$ and $0 \leqslant c$ then $0 \ll a$.*

Proof. See [Kri] Théorème 12. We give a sketch of the argument. Since the ring is Archimedean, we have N in $\mathbb{N}$ such that $-N \leqslant a \leqslant N$. Since $0 \leqslant c$ and $1 \leqslant ac$ we have $1 \leqslant Nc$ and thus $1/N \leqslant c$. We have also L in $\mathbb{N}$ such that $c \leqslant L$ and we get $1/N \leqslant c \leqslant L$. If we write $b = 1 - c/L$ we have $0 \leqslant b \leqslant 1 - 1/NL$ and $1/L \leqslant a(1-b)$. By multiplying by $1 + \cdots + b^{n-1}$ we get $1/L \leqslant a(1-b^n)$ and so $(1/L) + ab^n \leqslant a$. For n big enough we have $b^n \leqslant 1/2NL$; hence $1/2L \leqslant a$. ☐

One of the main results of [C] is a constructive proof of the following result.

Proposition 1. *We have* $D(a) = 1$ *in* $\mathsf{MFn}(A)$ *iff* $0 \ll a$ *in* B*.*

The proof is a combination of Lemma 4 and a cut-elimination argument [CC, CLR], which is an important technique in proof theory.

We shall now see that this result is a way to state Gelfand duality in the real case.

For this, we define first the upper real $\|a\|_0$ by: $\|a\|_0 < r$ iff $0 \ll r - a$ and $0 \ll r + a$. Another result from [C] is that $\|a\|_0$ defines a seminorm on B which satisfies $\|a^2\|_0 = \|a\|_0^2$.

Each element a defines a map of locales $\hat{a} : \mathsf{MFn}(A) \to \mathbb{R}$ by taking $\hat{a}^{-1}(r,s)$ to be the open $D(a-r) \wedge D(s-a)$. It is natural to define $\|\hat{a}\|$ as the upper real such that $\|\hat{a}\| < r$ iff $1 = D(r-a) \wedge D(a+r)$. By Proposition 1, this is equivalent to $0 \ll a - r$ and $0 \ll a + r$. Hence we get the following result.

Proposition 2. $\|\hat{a}\| = \|a\|_0$.

Corollary 1. $\|a\|_0^2 = \|a^2\|_0$.

Proof. This follows from $\|\hat{a^2}\| = \|\hat{a}\|^2$ and Proposition 2.

Since Proposition 1 is a combination of Lemma 4 and cut-elimination, we can also expect a direct proof from Lemma 4. Here is such a direct argumennt. If $0 \leqslant r$ and $0 \ll r^2 - a^2$ then we have $0 \ll uv$ where $u = r - a,\ v = r + a$. Hence $0 \ll u(u+v)$ and $0 \ll v(u+v)$. Since $0 \leqslant 2r = u + v$ we can apply Lemma 4 and deduce $0 \ll r + a$ and $0 \ll r - a$. ☐

To get Gelfand duality in the real case, we need to establish that $\|a\|_0$ and $\|a\|$ coincide. The usual Stone-Weierstrass Theorem, which has a constructive proof [C, BM3], then establishes the surjectivity of the map $a \longmapsto \hat{a}$.

Lemma 5. $\|a^2\| \leqslant \|a^2\|_0$.

Proof. Suppose that $\|a^2\|_0 < r$, then $r - a^2$ is a square, b^2. So $\|a^2\| \leqslant \|a^2 + b^2\| = r$. ☐

Theorem 1. $\|a\|_0 = \|a\| = \|\hat{a}\|$.

Proof. We have $\|a\|_0 \leqslant \|a\|$ since $r - a$ is a square if $r \geqslant \|a\|$ by Lemma 2. On the other hand, we have $\|a\|^2 = \|a^2\| \leqslant \|a^2\|_0 = \|a\|_0^2$ by Lemmas 1 and 5. Hence the result. ☐

4. Constructive Gelfand duality

We now have all the pieces for constructive proof of Gelfand duality, also in the complex case. Let A be a commutative C^*-algebra and $B = A_{sa}$ its self-adjoint part. The locale $\mathsf{MFn}(B)$ defined in [BM1] is isomorphic to the locale $\mathsf{MFn}(A)$

defined above by interpreting the element $a_1 + ia_2 \in (r_1 + ir_2, s_1 + is_2)$ in $\mathsf{MFn}(B)$ by the element

$$D(a_1 - r_1) \wedge D(s_1 - a_1) \wedge D(a_2 - r_2) \wedge D(s_2 - a_2)$$

in $\mathsf{MFn}(A)$.

Each element b of B defines a map of locales $\hat{b} : \mathsf{MFn}(B) \to \mathbb{C}$ by taking $\hat{b}^{-1}(r, s)$ to be $b \in (r, s)$.

Theorem 2. $\|b\| = \|\hat{b}\|$.

Proof. This follows from Theorem 1. $\square$

5. Some Simple Applications

We give some instances of simple properties of C^*-algebra, that are proved in [BM4] by using Barr's Theorem. All these cases are direct consequences of Lemma 3.

Proposition 3. *If* $\|a\| \leqslant 1$*, then* $\|1 - a^*a\| \leqslant 1$.

Proof. Suppose that $\|a\| \leqslant 1$. Then $\|a^*a\| \leqslant 1$. Write $a = b + ci$, where b, c are the real and complex part. Then $a^*a = b^2 + c^2$. Since $b^2 + c^2$ is a square it suffices to prove: If $\|d^2\| \leqslant 1$, then $\|1 - d^2\| \leqslant 1$. Suppose that $\|d^2\| \leqslant 1$. Then $1 - d^2 = e^2$, so $1 = d^2 + e^2$ and hence $\|1 - d^2\| = \|e^2\| \leqslant 1$. $\square$

Lemma 6. *The absolute value* $(\sqrt{a^*a})$ *exists.*

Proof. We can assume $\|a\| \leqslant 1$. Then $\|a^*a\| = \|a\|^2 \leqslant 1$, so $\|1 - a^*a\| \leqslant 1$. The result now follows from Lemma 2: the Taylor series $\sqrt{a^*a}$ converges. $\square$

Lemma 7. $1 + a^*a$ *is invertible.*

Proof. As in [Joh]. Let $b^2 = a^*a$. Choose $n \geqslant 1 + b^2$. Define $c = (1 - \frac{1}{n}) - \frac{b^2}{n}$. By Proposition 3, $\|c\| \leqslant 1 - \frac{1}{n}$. It follows that $(1 - c)^{-1} = 1 + c + c^2 + \ldots$ exists and $n(1 - c)^{-1}$ is the inverse of $1 + b^2$. $\square$

References

[Bis] E. Bishop. *Foundations of constructive analysis.* McGraw-Hill Book Co., New York-Toronto, Ont.-London 1967.

[BM1] B. Banaschewski and C. Mulvey. The spectral theory of commutative C^*-algebras: the constructive spectrum. Quaest.Math. 23 (2000), no. 4, 425–464.

[BM2] B. Banaschewski and C. Mulvey. The spectral theory of commutative C^*-algebras: the constructive Gelfand-Mazur theorem. Quaest. Math. 23 (2000), no. 4, 465–488.

[BM3] B. Banaschewski and C. Mulvey. A constructive proof of the Stone-Weierstrass theorem. J. Pure Appl. Algebra 116 (1997), no. 1-3, 25–40.

[BM4] B. Banaschewski and C. Mulvey. A globalisation of the Gelfand duality theorem. Ann. Pure Appl. Logic 137 (2006), no. 1-3, 62–103.

[CC] J. Cederquist and Th. Coquand. Entailment Relations and Distributive Lattices. *Proceeding of Logic Colloquium 1998.*

[C] Th. Coquand. About Stone's notion of spectrum. J. Pure Appl. Algebra 197 (2005) 141–158.

[CS] Th. Coquand and B. Spitters. Formal Topology and Constructive Mathematics: the Gelfand and Stone-Yosida Representation Theorems. J. UCS 11(12) (2005) 1932–1944.

[CLR] M. Coste and H. Lombardi and M.F. Roy Dynamical method in algebra: effective Nullstellenstze. Ann. Pure Appl. Logic 111 (2001), no. 3, 203–256.

[Joh] P. T. Johnstone. *Stone Spaces.* Cambridge studies in advanced mathematics 3, 1982.

[HLS] Chris Heunen, Klaas Landsman and Bas Spitters. A topos presentation of algebraic quantum theory. to appear

[KeVau] J.L. Kelley and R.L. Vaught. The positive cone in Banach algebras. Trans. Amer. Math. Soc. 74, (1953). 44–55.

[Kri] J.-L. Krivine. (1964) Anneaux préordonnés. J. Analyse Math. 12, p. 307-326.