

An Introduction to Corecursive Algebras

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in collaboration with

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Two example of recursive function definition:

$$\text{qs} : \text{List}(\mathbb{N}) \rightarrow \text{List}(\mathbb{N})$$

$$\text{qs} [] = []$$

$$\text{qs} (x :: l) = (\text{qs } l_{\leq x}) \uparrow\uparrow x :: (\text{qs } l_{> x})$$

where

$$l_{\leq x} = [y \leftarrow l \mid y \leq x]$$

$$l_{> x} = [y \leftarrow l \mid y > x]$$

$$\text{sp} :: \text{Stream}(\mathbb{N}) \rightarrow \text{Stream}(\mathbb{N})$$

$$\text{sp} (x :: s) = x :: (\text{sp } s_{\text{odd}} \oplus \text{sp } s_{\text{even}})$$

where

$$\text{If } s = y_1 :: y_2 :: y_3 :: y_4 :: \dots$$

$$s_{\text{odd}} = y_1 :: y_3 :: \dots$$

$$s_{\text{even}} = y_2 :: y_4 :: \dots$$

$$\text{If } s' = z_1 :: z_2 :: z_3 :: z_4 :: \dots$$

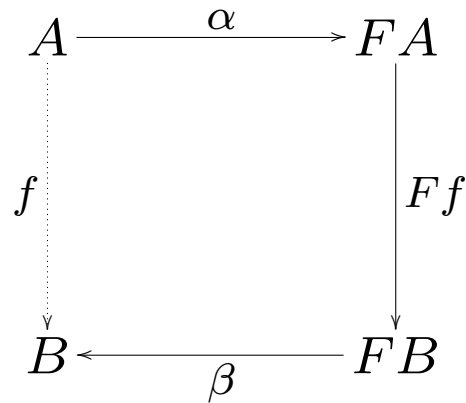
$$s \oplus s' = y_1 \uparrow z_1 :: y_2 \uparrow z_2 :: \dots$$

qs terminates by induction, but is not structurally recursive.

sp is productive, but not guarded.

Both define total functions, for different reasons.

Both recursive function definitions can be written as the solution of a **Recursive Diagram**:



$F : \text{Set} \rightarrow \text{Set}$ functor
 $\alpha : A \rightarrow FA$ coalgebra
 $\beta : FB \rightarrow B$ algebra

A solution of the recursive diagram: $f = \beta \circ Ff \circ \alpha$.

Find conditions on α or β that ensure existence/uniqueness of f .

Quicksort example: $F X = \text{Unit} + \mathbb{N} \times X \times X$, $A = B = \text{List}(\mathbb{N})$.

$$\alpha : \text{List}(\mathbb{N}) \rightarrow \text{Unit} + \mathbb{N} \times \text{List}(\mathbb{N}) \times \text{List}(\mathbb{N})$$

$$\alpha [] = \text{inl } \#$$

$$\alpha (x :: l) = \text{inr } \langle x, l_{\leq x}, l_{> x} \rangle$$

$$\beta : \text{Unit} + \mathbb{N} \times \text{List}(\mathbb{N}) \times \text{List}(\mathbb{N}) \rightarrow \text{List}(\mathbb{N})$$

$$\beta (\text{inl } u) = []$$

$$\beta (\text{inr } \langle x, l_1, l_2 \rangle) = l_1 ++ x :: l_2$$

$$\begin{array}{ccc}
 \text{List}(\mathbb{N}) & \xrightarrow{\alpha} & \text{Unit} + \mathbb{N} \times \text{List}(\mathbb{N}) \times \text{List}(\mathbb{N}) \\
 \text{qs} \downarrow \text{dotted} & & \downarrow \text{id}_{\text{Unit}} + \text{id}_{\mathbb{N}} \times \text{List}(\text{qs}) \times \text{List}(\text{qs}) \\
 \text{List}(\mathbb{N}) & \xleftarrow{\beta} & \text{Unit} + \mathbb{N} \times \text{List}(\mathbb{N}) \times \text{List}(\mathbb{N})
 \end{array}$$

Stream Example: $FX = \mathbb{N} \times X \times X$, $A = \text{Stream}(\mathbb{N})$, $B = \text{Stream}(\mathbb{N})$.

$$\alpha : \text{Stream}(\mathbb{N}) \rightarrow \mathbb{N} \times \text{Stream}(\mathbb{N}) \times \text{Stream}(\mathbb{N})$$

$$\alpha(x :: s) = \langle x, s_{\text{odd}}, s_{\text{even}} \rangle$$

$$\beta : \mathbb{N} \times \text{Stream}(\mathbb{N}) \times \text{Stream}(\mathbb{N}) \rightarrow \text{Stream}(\mathbb{N})$$

$$\beta \langle x, s_1, s_2 \rangle = x :: s_1 \oplus s_2$$

$$\begin{array}{ccc}
 \text{Stream}(\mathbb{N}) & \xrightarrow{\alpha} & \mathbb{N} \times \text{Stream}(\mathbb{N}) \times \text{Stream}(\mathbb{N}) \\
 \text{sp} \downarrow \text{dotted} & & \downarrow \text{id}_{\mathbb{N}} \times \text{sp} \times \text{sp} \\
 \text{Stream}(\mathbb{N}) & \xleftarrow{\beta} & \mathbb{N} \times \text{Stream}(\mathbb{N}) \times \text{Stream}(\mathbb{N})
 \end{array}$$

Existence/Unicity has a different origin in the two examples.
 For **qs** it derives from a property of α .
 For **sp** it derives from a property of β .

α is a **recursive coalgebra** if, for every algebra β , there exists a unique f satisfying the recursive diagram.

(G. Osius, 1974; P. Taylor, 1996; CUV, 2004-6; Adámek, Lüke, Milius, 2007)

β is a **corecursive algebra** if, for every coalgebra α , there exists a unique f satisfying the recursive diagram.

Recursive Coalgebra α

$$\begin{array}{ccc}
 A & \xrightarrow{\alpha} & FA \\
 \exists! f \downarrow \text{dotted} & & \downarrow Ff \\
 B & \xleftarrow{\forall \beta} & FB
 \end{array}$$

CoRecursive Algebra β

$$\begin{array}{ccc}
 A & \xrightarrow{\forall \alpha} & FA \\
 \exists! f \downarrow \text{dotted} & & \downarrow Ff \\
 B & \xleftarrow{\beta} & FB
 \end{array}$$

In Set, \mathcal{P} powerset functor. $\mathcal{P}(X) = \text{subsets of } X$.

\mathcal{P} doesn't have any fixed points.

No initial algebra. No terminal coalgebra.

But \mathcal{P} has recursive coalgebras.

\mathcal{P} -coalgebras are relations:

$$\begin{array}{lll} \alpha : A \rightarrow \mathcal{P}A & \iff & R_\alpha \subseteq A \times A \\ x_1 \in \alpha(x_2) & \iff & x_1 R_\alpha x_2 \\ \alpha \text{ recursive} & \iff & R_\alpha \text{ well-founded} \end{array}$$

Proof: Direct or

Special case of a more general result (Taylor, 1996).

Generalize the notion of well-founded relation to any functor and coalgebra.

Let F be a functor and $\alpha : A \rightarrow FA$ a coalgebra.

Idea: Generate the *accessible* part of A
by iterating the *next-time operator* nt (Jacobs):

Given a subobject $U \xrightarrow{i} A$, define $\text{nt}(U)$ as the result of the pullback

$$\begin{array}{ccc} \text{nt}(U) & \xrightarrow{\alpha[i]} & FU \\ \text{nt}(i) \downarrow & & \downarrow Fi \\ A & \xrightarrow{\alpha} & FA \end{array}$$

Intuitively: $\text{nt}(U) = \{x \in A \mid \alpha(x) \in FU\}$.

Induction principle for a well-founded relation R :

If U is a subset of A such that:

for every $x \in A$, if $\forall y \in A, y R x \rightarrow y \in U$, then $x \in U$;

then $U = A$.

Definition (Taylor):

α is a **well-founded** coalgebra if:

For every subobject $i : U \hookrightarrow A$ such that:

$\text{nt}(i)$ factors through i ;

then $U \cong A$.

$$\begin{array}{ccc} U & \xleftarrow{\text{nt}(U)} & \text{nt}(U) \\ \downarrow i & \swarrow \text{nt}(i) & \\ A & & \end{array}$$

We can define the **accessible part** of A (w.r.t. α) by a fixed point of nt obtained by iterating nt starting with the empty subobject.

$$\begin{aligned} A_0 &= \emptyset & A_1 &= \text{nt}(A_0) & \cdots & A_{i+1} &= \text{nt}(A_i) \\ A_\omega &= \varinjlim_{i < \omega} A_i & & (= \cup_{i < \omega} A_i) \\ A_\gamma &= \varinjlim_{\delta < \gamma} A_i \end{aligned}$$

Accessible part of A (Bove/C., 2001):

$A_\zeta = \text{nt}(A_\zeta)$, if it exists.

α is a **inductive coalgebra** if

There is an ordinal ζ such that $A_\zeta = A$.

Theorem (Taylor)

Under some conditions on the category and the functor F ;

The three conditions:

- α is a recursive coalgebra,
- α is a well-founded coalgebra,
- α is a inductive coalgebra,

are equivalent.

Does the previous characterization hold
also for corecursive algebras?

NO

Dual of the next-time operator.

The dual of a subset is a quotient, $Q = (B / \equiv)$.
 Categorically, an epimorphism $q : B \twoheadrightarrow Q$.

The next-time of Q , $\text{nt}(Q)$ is given by a push-out:

$$\begin{array}{ccc}
 FB & \xrightarrow{\beta} & B \\
 \downarrow Fq & & \downarrow \text{nt}(q) \\
 FQ & \xrightarrow{\beta[q]} & \text{nt}(Q)
 \end{array}$$

Intuitively: $\text{nt}(Q) = (B / \equiv')$ where
 \equiv' is the reflexive/transitive closure of:
 $\forall y_1, y_2 \in FB, y_1 \equiv^F y_2 \rightarrow \beta(y_1) \equiv' \beta(y_2)$.

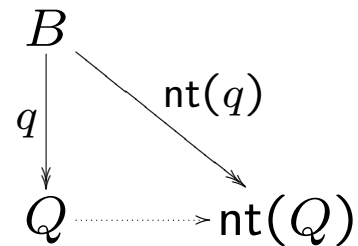
Definition:

β is a **discriminating** algebra if:

For every quotient $q : B \twoheadrightarrow Q$ such that:

$\text{nt}(q)$ factors through q ;

then $Q \cong B$.



Sequence of iterations of nt:

$$Q_0 = \text{Unit} \equiv (B / \equiv_0) \text{ with } \equiv_0 \text{ the total relation}$$

$$Q_{i+1} = \text{nt}(Q_i)$$

$$Q_{\delta < \gamma} = \varprojlim_{\delta < \gamma} Q_\delta$$

Definition:

β is a **focusing algebra** if

There is an ordinal ζ such that $Q_\zeta = Q$.

Problem:

Transfinite iterations of nt are not necessarily quotients of B .

focusing $\xrightarrow{\neq}$ discriminating

focusing $\xrightarrow{\neq}$ corecursive

Counterexample 1

Consider the following algebra for the functor $FX = X \times X$:

$$\beta : 3 \times 3 \rightarrow 3$$

$$\beta(1, 2) = 2$$

$$\beta(n, m) = 0 \text{ otherwise}$$

β is corecursive

β is neither discriminating nor focusing:

$$Q_0 = \text{Unit} \quad Q_1 = \{1, \perp\} \quad \text{where } \perp = \{0, 2\}$$

Q_1 is already a fixed point.

Counterexample 2

Consider the following algebra for the functor $FX = X \times X$:

$$\begin{aligned}\beta &: \mathbb{N} \times \mathbb{N} \rightarrow \mathbb{N} \\ \beta(1, m) &= m + 2 \\ \beta(n, m) &= 0 \quad \text{otherwise}\end{aligned}$$

β is corecursive and discriminating

β is not focusing:

$$\begin{aligned}Q_0 &= \{\perp\} & Q_3 &= \{0, 1, 2, 3, 5, \perp\} & Q_\omega &= \mathbb{N} \cup \{\perp\} \\ Q_1 &= \{1, \perp\} & Q_4 &= \{0, 1, 2, 3, 4, 5, 7, \perp\} \\ Q_2 &= \{0, 1, 3, \perp\} & Q_i &= \{0, \dots, 2i - 3, 2i - 1, \perp\}\end{aligned}$$

Coda: A recursive diagram that has a unique solution but such that α is not recursive and β is not corecursive.

$\text{down_runs} : \text{Stream}(\mathbb{N}) \rightarrow \text{Stream}(\mathbb{N})$

$\text{down_runs } s = \text{drf } \langle 1, s \rangle$

$\text{drf} : \mathbb{N} \times \text{Stream}(\mathbb{N}) \rightarrow \text{Stream}(\mathbb{N})$

$\text{drf } \langle n, (x_1 :: x_2 :: s) \rangle = \begin{cases} \text{drf } \langle (n + 1), (x_2 :: s) \rangle & \text{if } x_1 > x_2 \\ n :: \text{drf } \langle 1, (x_2 :: s) \rangle & \text{if } x_1 \leq x_2 \end{cases}$

Diagram: $FX = X + \mathbb{N} \times X$, $A = \mathbb{N} \times \text{Stream}(\mathbb{N})$, $B = \text{Stream}(\mathbb{N})$.

$$\alpha : \mathbb{N} \times \text{Stream}(\mathbb{N}) \rightarrow \mathbb{N} \times \text{Stream}(\mathbb{N}) + \mathbb{N} \times \mathbb{N} \times \text{Stream}(\mathbb{N})$$

$$\alpha \langle n, x_1 :: x_2 :: s \rangle = \begin{cases} \text{inl} \langle (n + 1), (x_2 :: s) \rangle & \text{if } x_1 > x_2 \\ \text{inr} \langle n, 1, (x_2 :: s) \rangle & \text{if } x_1 \leq x_2 \end{cases}$$

$$\beta : \text{Stream}(\mathbb{N}) + \mathbb{N} \times \text{Stream}(\mathbb{N}) \rightarrow \text{Stream}(\mathbb{N})$$

$$\beta (\text{inl } u) = u$$

$$\beta (\text{inr} \langle n, u \rangle) = n :: u$$

$$\begin{array}{ccc} \mathbb{N} \times \mathbb{S} & \xrightarrow{\alpha} & \mathbb{N} \times \mathbb{S} + \mathbb{N} \times \mathbb{N} \times \mathbb{S} \\ \text{drf} \downarrow \text{dotted} & & \downarrow \text{drf} + \text{id}_{\mathbb{N}} \times \text{drf} \\ \mathbb{S} & \xleftarrow{\beta} & \mathbb{S} + \mathbb{N} \times \mathbb{S} \end{array}$$