An Introduction to Corecursive Algebras

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Two example of recursive function definition:

\[qs : \text{List}(\mathbb{N}) \rightarrow \text{List}(\mathbb{N})\]

\[qs [] = []\]

\[qs (x :: l) = (qs l_{\leq x}) \mathbin{\uplus} x :: (qs l_{> x})\]

where

\[l_{\leq x} = [y \leftarrow l \mid y \leq x]\]

\[l_{> x} = [y \leftarrow l \mid y > x]\]

\[sp : \text{Stream}(\mathbb{N}) \rightarrow \text{Stream}(\mathbb{N})\]

\[sp (x :: s) = x :: (sp s_{\text{odd}} \oplus sp s_{\text{even}})\]

where

If \(s = y_1 :: y_2 :: y_3 :: y_4 :: \cdots\)

\[s_{\text{odd}} = y_1 :: y_3 :: \cdots\]

\[s_{\text{even}} = y_2 :: y_4 :: \cdots\]

If \(s' = z_1 :: z_2 :: z_3 :: z_4 :: \cdots\)

\[s \oplus s' = y_1 + z_1 :: y_2 + z_2 :: \cdots\]

qs terminates by induction, but is not structurally recursive.
sp is productive, but not guarded.

Both define total functions, for different reasons.
Both recursive function definitions can be written as the solution of a Recursive Diagram:

\[
\begin{array}{ccc}
A & \xrightarrow{\alpha} & FA \\
\downarrow & & \downarrow Ff \\
B & \xleftarrow{\beta} & FB
\end{array}
\]

\[
F : \text{Set} \to \text{Set} \quad \text{functor}
\]
\[
\alpha : A \to FA \quad \text{coalgebra}
\]
\[
\beta : FB \to B \quad \text{algebra}
\]

A solution of the recursive diagram: \( f = \beta \circ Ff \circ \alpha \).

Find conditions on \( \alpha \) or \( \beta \) that ensure existence/uniqueness of \( f \).
Quicksort example: \( FX = \text{Unit} + \mathbb{N} \times X \times X, \quad A = B = \text{List}(\mathbb{N}) \).

\[ \begin{align*}
\alpha &: \text{List}(\mathbb{N}) \to \text{Unit} + \mathbb{N} \times \text{List}(\mathbb{N}) \times \text{List}(\mathbb{N}) \\
\alpha &([]) = \text{inl} \ \text{tt} \\
\alpha (x :: l) = \text{inr} \ \langle x, l_{\leq x}, l_{> x} \rangle \\
\beta &: \text{Unit} + \mathbb{N} \times \text{List}(\mathbb{N}) \times \text{List}(\mathbb{N}) \to \text{List}(\mathbb{N}) \\
\beta (\text{inl} \ u) = [] \\
\beta (\text{inr} \ \langle x, l_1, l_2 \rangle) = l_1 + + x :: l_2
\end{align*} \]
Stream Example: \( F X = \mathbb{N} \times X \times X, \ A = \text{Stream}(\mathbb{N}), \ B = \text{Stream}(\mathbb{N}) \).

\[ \alpha : \text{Stream}(\mathbb{N}) \rightarrow \mathbb{N} \times \text{Stream}(\mathbb{N}) \times \text{Stream}(\mathbb{N}) \]
\[ \alpha (x :: s) = \langle x, s_{\text{odd}}, s_{\text{even}} \rangle \]

\[ \beta : \mathbb{N} \times \text{Stream}(\mathbb{N}) \times \text{Stream}(\mathbb{N}) \rightarrow \text{Stream}(\mathbb{N}) \]
\[ \beta \langle x, s_1, s_2 \rangle = x :: s_1 \oplus s_2 \]
Existence/Unicity has a different origin in the two examples. For $qs$ it derives from a property of $\alpha$. For $sp$ it derives from a property of $\beta$.

$\alpha$ is a **recursive coalgebra** if, for every algebra $\beta$, there exists a unique $f$ satisfying the recursive diagram.


$\beta$ is a **corecursive algebra** if, for every coalgebra $\alpha$, there exists a unique $f$ satisfying the recursive diagram.
In Set, \( \mathcal{P} \) powerset functor. \( \mathcal{P}(X) = \text{subsets of } X \).
\( \mathcal{P} \) doesn’t have any fixed points.
No initial algebra. No terminal coalgebra.

But \( \mathcal{P} \) has recursive coalgebras.
\( \mathcal{P} \)-coalgebras are relations:
\[ \alpha : A \to \mathcal{P}A \iff R_\alpha \subseteq A \times A \]
\[ x_1 \in \alpha(x_2) \iff x_1 R_\alpha x_2 \]
\[ \alpha \text{ recursive } \iff R_\alpha \text{ well-founded} \]

Proof: Direct or
Special case of a more general result (Taylor, 1996).
Generalize the notion of well-founded relation to any functor and coalgebra.
Let $F$ be a functor and $\alpha : A \to FA$ a coalgebra.

Idea: Generate the \textit{accessible} part of $A$
by iterating the \textit{next-time operator} $nt$ (Jacobs):
Given a subobject $U \xrightarrow{i} A$, define $nt(U)$ as the result of the pullback

\[
\begin{array}{ccc}
nt(U) & \xrightarrow{\alpha[i]} & FU \\
\downarrow \quad nt(i) & & \downarrow \quad Fi \\
A & \xrightarrow{\alpha} & FA
\end{array}
\]

Intuitively: $nt(U) = \{x \in A \mid \alpha(x) \in FU\}$. 
Induction principle for a well-founded relation $R$:
If $U$ is a subset of $A$ such that:
for every $x \in A$, if $\forall y \in A, y R x \rightarrow y \in U$, then $x \in U$;
then $U = A$.

Definition (Taylor):
$\alpha$ is a well-founded coalgebra if:
For every subobject $i : U \hookrightarrow A$ such that:
$\text{nt}(i)$ factors through $i$;
then $U \cong A$.
We can define the accessible part of $A$ (w.r.t. $\alpha$) by a fixed point of $nt$ obtained by iterating $nt$ starting with the empty subobject.

$$A_0 = \emptyset \quad A_1 = nt(A_0) \quad \cdots \quad A_{i+1} = nt(A_i)$$

$$A_\omega = \lim_{i<\omega} A_i \quad (= \cup_{i<\omega} A_i)$$

$$A_\gamma = \lim_{\delta<\gamma} A_i$$

Accessible part of $A$ (Bove/C., 2001):

$A_\zeta = nt(A_\zeta)$, if it exists.

$\alpha$ is a inductive coalgebra if

There is an ordinal $\zeta$ such that $A_\zeta = A$. 
**Theorem** (Taylor)

Under some conditions on the category and the functor $F$;

The three conditions:

- $\alpha$ is a recursive coalgebra,

- $\alpha$ is a well-founded coalgebra,

- $\alpha$ is a inductive coalgebra,

are equivalent.
Does the previous characterization hold also for corecursive algebras?

NO
Dual of the next-time operator.

The dual of a subset is a quotient, \( Q = (B/ \equiv) \). Categorically, an epimorphisms \( q : B \to Q \).

The next-time of \( Q \), \( \text{nt}(Q) \) is given by a push-out:

\[
\begin{array}{ccc}
FB & \xrightarrow{\beta} & B \\
\downarrow Fq & & \downarrow \beta \\
FQ & \xrightarrow{\beta[q]} & \text{nt}(Q)
\end{array}
\]

Intuitively: \( \text{nt}(Q) = (B/ \equiv') \) where \( \equiv' \) is the reflexive/transitive closure of:
\[
\forall y_1, y_2 \in FB, y_1 \equiv^F y_2 \to \beta(y_1) \equiv' \beta(y_2).
\]
Definition:

$\beta$ is a **discriminating** algebra if:

For every quotient $q : B \to Q$ such that:

$nt(q)$ factors through $q$;

then $Q \simeq B$. 

```
\begin{tikzcd}
B \arrow{dr}{nt(q)} \arrow[shift left=3]{d}{q} \\
Q \arrow{d}{nt(Q)} \\
&
\end{tikzcd}
```
Sequence of iterations of $\text{nt}$:

$$Q_0 = \text{Unit} \equiv (B / \equiv_0) \text{ with } \equiv_0 \text{ the total relation}$$
$$Q_{i+1} = \text{nt}(Q_i)$$
$$Q_{\delta < \gamma} = \lim_{\delta < \gamma} Q_{\delta}$$

Definition:
$I$ is a focusing algebra if
There is an ordinal $\zeta$ such that $Q_\zeta = Q$.

Problem:
Transfinite iterations of $\text{nt}$ are not necessarily quotients of $B$. 
focusing $\leftrightarrow$ discriminating $\leftrightarrow$ corecursive

Counterexample 1
Consider the following algebra for the functor $FX = X \times X$:

$$\beta : 3 \times 3 \rightarrow 3$$
$$\beta(1, 2) = 2$$
$$\beta(n, m) = 0 \text{ otherwise}$$

$\beta$ is corecursive

$\beta$ is neither discriminating nor focusing:

$$Q_0 = \text{Unit} \quad Q_1 = \{1, \bot\} \quad \text{where } \bot = \{0, 2\}$$
$Q_1$ is already a fixed point.
Counterexample 2
Consider the following algebra for the functor $FX = X \times X$:

$$\beta : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$$
$$\beta(1, m) = m + 2$$
$$\beta(n, m) = 0 \text{ otherwise}$$

$\beta$ is corecursive and discriminating

$\beta$ is not focusing:

$$Q_0 = \{ \bot \} \quad Q_3 = \{0, 1, 2, 3, 5, \bot\} \quad Q_\omega = \mathbb{N} \cup \{ \bot \}$$
$$Q_1 = \{1, \bot\} \quad Q_4 = \{0, 1, 2, 3, 4, 5, 7, \bot\}$$
$$Q_2 = \{0, 1, 3, \bot\} \quad Q_i = \{0, \ldots, 2i - 3, 2i - 1, \bot\}$$
Coda: A recursive diagram that has a unique solution but such that $\alpha$ is not recursive and $\beta$ is not corecursive.

\[
down\_runs : \text{Stream}(\mathbb{N}) \rightarrow \text{Stream}(\mathbb{N})
\]

\[
down\_runs s = \text{drf} \langle 1, s \rangle
\]

\[
drf : \mathbb{N} \times \text{Stream}(\mathbb{N}) \rightarrow \text{Stream}(\mathbb{N})
\]

\[
drf \langle n, (x_1 :: x_2 :: s) \rangle = \begin{cases} 
  \text{drf} \langle (n + 1), (x_2 :: s) \rangle & \text{if } x_1 > x_2 \\
  n :: \text{drf} \langle 1, (x_2 :: s) \rangle & \text{if } x_1 \leq x_2
\end{cases}
\]
Diagram: $FX = X + \mathbb{N} \times X$, $A = \mathbb{N} \times \text{Stream}(\mathbb{N})$, $B = \text{Stream}(\mathbb{N})$.

$\alpha : \mathbb{N} \times \text{Stream}(\mathbb{N}) \rightarrow \mathbb{N} \times \text{Stream}(\mathbb{N}) + \mathbb{N} \times \mathbb{N} \times \text{Stream}(\mathbb{N})$

$\alpha \langle n, x_1 :: x_2 :: s \rangle = \begin{cases} 
\text{inl} \langle (n + 1), (x_2 :: s) \rangle & \text{if } x_1 > x_2 \\
\text{inr} \langle n, 1, (x_2 :: s) \rangle & \text{if } x_1 \leq x_2 
\end{cases}$

$\beta : \text{Stream}(\mathbb{N}) + \mathbb{N} \times \text{Stream}(\mathbb{N}) \rightarrow \text{Stream}(\mathbb{N})$

$\beta (\text{inl} \ u) = u$

$\beta (\text{inr} \langle n, u \rangle) = n :: u$