Matrix Calculations: Vector Spaces and Linear Maps

H. Geuvers (and A. Kissinger)

Institute for Computing and Information Sciences
Radboud University Nijmegen

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Outline

Points in plane

- The set of points in a plane is usually written as

$$\mathbb{R}^2 = \{(x, y) \mid x, y \in \mathbb{R}\} \quad \text{or as} \quad \mathbb{R}^2 = \{\left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right) \mid x, y \in \mathbb{R}\}$$

- Two points can be added, as in:

$$(x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 + y_2)$$

  What is this geometrically?

- Also, points can be multiplied by a number ('scalar'):

$$a \cdot (x, y) = (a \cdot x, a \cdot y)$$

- Several nice properties hold, like:

$$a \cdot \Big((x_1, y_1) + (x_2, y_2)\Big) = a \cdot (x_1, y_1) + a \cdot (x_2, y_2)$$

Points in space

- Points in 3-dimensional space are described as:

$$\mathbb{R}^3 = \{(x, y, z) \mid x, y, z \in \mathbb{R}\} \quad \text{or as} \quad \mathbb{R}^3 = \{\begin{pmatrix} x \\ y \\ z \end{pmatrix} \mid x, y, z \in \mathbb{R}\}$$

- Again such 3-dimensional points can be added and multiplied:

$$\begin{aligned} (x_1, y_1, z_1) + (x_2, y_2, z_2) &= (x_1 + x_2, y_1 + y_2, z_1 + z_2) \\ a \cdot (x, y, z) &= (a \cdot x, a \cdot y, a \cdot z) \end{aligned}$$

  And similar nice properties hold.

- We like to capture such similarities in a general abstract definition
  - sometimes the definition is so abstract one gets lost
  - but then it is good to keep the main examples in mind.

Vector space

Definition

A vector space consists of a set V, whose elements

- are called vectors
- can be added
- can be multiplied with a real number

satisfying precise requirements (to be detailed in later slides).

Example

For each $n \in \mathbb{N}$, n-dimensional space $\mathbb{R}^n$ is a vector space, where

$$\mathbb{R}^n = \{(x_1, x_2, \dots, x_n) \mid x_1, \dots, x_n \in \mathbb{R}\}.$$

This includes the 2-dimensional plane ($n = 2$) and 3-dimensional space ($n = 3$).

Vector space example

Example

The set of solutions of a homogeneous system of equations is a vector space.
Solutions of a homogeneous system of equations

- can be added
- can be multiplied with a real number

to form new solutions.
(This is what we have seen last week.)

- Vector spaces occur at many places in many disguises.
- In general a vector space is a set V with two operations "addition" and "scalar multiplication" that satisfy certain requirements.

Addition for vectors: precise requirements

1. Vector addition is commutative: summands can be swapped:
$$v + w = w + v$$
2. addition is associative: grouping of summands is irrelevant:
$$u + (v + w) = (u + v) + w$$
3. there is a zero vector 0 such that:
$$v + 0 = v, \qquad \text{and hence by (1) also:} \qquad 0 + v = v.$$
4. each vector v has an additive inverse (minus) $-v$ such that:
$$v + (-v) = 0$$

One writes $v - w$ for $v + (-w)$.

Scalar multiplication for vectors: precise requirements

1. $1 \in \mathbb{R}$ is unit for scalar multiplication:

$$1 \cdot v = v$$

2. two scalar multiplications can be done as one:

$$\underbrace{a \cdot (b \cdot v)}_{\text{twice scalar mult.}} \quad = \quad \underbrace{(ab)}_{\text{mult. in } \mathbb{R}} \cdot v$$

3. distributivity

$$\begin{aligned} a \cdot (v + w) &= (a \cdot v) + (a \cdot w) \\ (a + b) \cdot v &= (a \cdot v) + (b \cdot v). \end{aligned}$$

Exercise

Check for yourself that all these properties hold for $\mathbb{R}^n$ and for a set of sulutions of a homogeneous set of equations.

Base in space

- In $\mathbb{R}^3$ we can distinguish three special vectors:

$$(1,0,0) \in \mathbb{R}^3 \qquad (0,1,0) \in \mathbb{R}^3 \qquad (0,0,1) \in \mathbb{R}^3$$

- These vectors form a basis:
  1. each vector (x, y, z) can be expressed in terms of these three special vectors:

$$\begin{array}{rcl} (x,y,z) & = & (x,0,0) + (0,y,0) + (0,0,z) \\ & = & x \cdot (1,0,0) + y \cdot (0,1,0) + z \cdot (0,0,1) \end{array}$$

  2. Moreover, these three special vectors are linearly independent

Remember: Independence

From last week:

Definition

Vectors $v_1, \ldots, v_n$ in a vector space V are called independent if for all scalars $a_1, \ldots, a_n \in \mathbb{R}$ one has:

$$a_1 \cdot v_1 + \cdots + a_n \cdot v_n = 0 \text{ in } V \quad \text{implies} \quad a_1 = a_2 = \cdots = a_n = 0$$

Remember: (in)dependence can be proved via equation solving

$$\begin{pmatrix}1\\2\\3\end{pmatrix}, \begin{pmatrix}2\\-1\\4\end{pmatrix}, \text{ and } \begin{pmatrix}0\\5\\2\end{pmatrix} \text{ are dependent}$$

if there are non-zero $a_1, a_2, a_3 \in \mathbb{R}$ with:

$$a_1 \begin{pmatrix}1\\2\\3\end{pmatrix} + a_2 \begin{pmatrix}2\\-1\\4\end{pmatrix} + a_3 \begin{pmatrix}0\\5\\2\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix}$$

Dependence (or non-independence)

- In the plane two vectors $v, w \in \mathbb{R}^2$ are dependent if and only if:
  - they are on the same line
  - that is: $v = a \cdot w$, for some scalar a
- **Example**: for $v = (1, 2)$ and $w = (-2, -4)$ we have:
  - $v = -\frac{1}{2}w$, so they are on the same line
  - $a_1 \cdot v + a_2 \cdot w = 0$, e.g. for $a_1 = 2 \neq 0$ and $a_2 = 1 \neq 0$.
- In space, three vectors $u, v, w \in \mathbb{R}^3$ are dependent if they are in the same plane (or even line)
- One can prove: $v_1, \ldots, v_n \in V$ are dependent, if and only if some v_i can be expressed as a linear combination of the others (the v_j with $j \neq i$).

Basis

Definition

Vectors $v_1, \ldots, v_n \in V$ form a basis for a vector space V if these $v_1, \ldots, v_n$

- are independent, and
- span V in the sense that each $w \in V$ can be written as linear combination of these $v_1, \ldots, v_n$, namely as:

$$w = a_1 v_1 + \cdots + a_n v_n \quad \text{for certain} \quad a_1, \ldots, a_n \in \mathbb{R}$$

- These scalars a_i are uniquely determined by $w \in V$ (see below)
- A space V may have several bases, but the number of elements of a basis for V is always the same; it is called the dimension of V, usually written as $\dim(V) \in \mathbb{N}$.

The standard basis for $\mathbb{R}^n$

For the space $\mathbb{R}^n = \{(x_1, \ldots, x_n) \mid x_i \in \mathbb{R}\}$ there is a standard choice of base vectors:

$$(1, 0, 0 \ldots, 0), \quad (0, 1, 0, \ldots, 0), \quad \cdots \quad (0, \ldots, 0, 1)$$

We have already seen that they are independent; it is easy to see that they span $\mathbb{R}^n$

This enables us to state precisely that $\mathbb{R}^n$ has n dimensions.

An alternative basis for $\mathbb{R}^2$

- The standard basis for $\mathbb{R}^2$ is $(1,0)$, $(0,1)$.
- But many other choices are possible, eg. $(1,1)$, $(1,-1)$
  - **independence**: if $a \cdot (1,1) + b \cdot (1,-1) = (0,0)$, then:

$$\begin{cases} a + b = 0 \\ a - b = 0 \end{cases} \quad \text{and thus} \quad \begin{cases} a = 0 \\ b = 0 \end{cases}$$

  - **spanning**: each point (x,y) can written in terms of $(1,1),(1,-1)$, namely:

$$(x,y) = \tfrac{x+y}{2}(1,1) + \tfrac{x-y}{2}(1,-1)$$

The space of solutions to a set of equations I

- The set of solutions to a set of homogeneous equations forms a vector space.
- How do we compute its basis?

Example:

$$\begin{aligned} x_1 + 2x_2 - 3x_3 &= 0 \\ 2x_1 + 3x_2 + x_3 &= 0 \\ 3x_1 + 4x_2 + 5x_3 &= 0 \\ -2x_1 - 4x_2 + 6x_3 &= 0 \end{aligned}$$

with associated coefficient matrix

$$\begin{pmatrix} 1 & 2 & -3 \\ 2 & 3 & 1 \\ 3 & 4 & 5 \\ -2 & -4 & 6 \end{pmatrix}$$

The space of solutions to a set of equations II

We transform the coefficient matrix to Echelon form:

$$\begin{pmatrix} 1 & 2 & -3 \\ 2 & 3 & 1 \\ 3 & 4 & 5 \\ -2 & -4 & 6 \end{pmatrix} \mapsto \begin{pmatrix} 1 & 2 & -3 \\ 0 & -1 & 7 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

There are 3 variables and 2 pivots, so there is one basic solution (and the $(0, 0, 0)$ solution).
Example of a basic solution: $x_1 = -11, x_2 = 7, x_3 = 1$.

- A basis for the solution space is $(-11, 7, 1)$, but also $(-22, 14, 2)$ forms a basis
- The dimension of the solution space (of this set of eqns) is 1.

Uniqueness of representations

Theorem

- *Suppose V is a vector space, with basis $v_1, \dots, v_n$*
- *assume $x \in V$ can be represented in two ways:*

$$x = a_1 v_1 + \cdots + a_n v_n \qquad \textit{and also} \qquad x = b_1 v_1 + \cdots + b_n v_n$$

Then: $a_1 = b_1$ and $\dots$ and $a_n = b_n$.

Proof: This follows from independence of $v_1, \dots, v_n$ since:

$$\begin{aligned} 0 \;=\; x - x \;&=\; \big(a_1 v_1 + \cdots + a_n v_n\big) - \big(b_1 v_1 + \cdots + b_n v_n\big) \\ &=\; (a_1 - b_1)v_1 + \cdots + (a_n - b_n)v_n. \end{aligned}$$

Hence $a_i - b_i = 0$, by independence, and thus $a_i = b_i$. ∎

Maps

- A map (or 'function') f is an operation that sends elements of one set X to another set Y.
  - in that case we write $f\colon X \to Y$ or sometimes $X \xrightarrow{f} Y$
  - this f sends $x \in X$ to $f(x) \in Y$
  - X is called the domain and Y the codomain of the map f
- Example. $f(n) = \frac{1}{n+1}$ can be seen as map $\mathbb{N} \to \mathbb{Q}$, that is from the *natural* numbers $\mathbb{N}$ to the *rational* numbers $\mathbb{Q}$
- A map is sometimes also called a mapping or a function
- On each set X there is the identity map $\mathrm{id}\colon X \to X$ that does nothing: $\mathrm{id}(x) = x$.
- Also one can compose 2 maps $X \xrightarrow{f} Y \xrightarrow{g} Z$ to a map:

$$g \circ f\colon X \longrightarrow Z \quad \text{given by} \quad (g \circ f)(x) = g(f(x))$$

Linear maps

We have seen that the two relevant operations of a vector space are addition and scalar multiplication. A linear map is required to preserve these two.

Definition

Let V, W be two vector spaces, and $f : V \to W$ a map between them; f is called linear if it preserves both:

- addition: for all $v, v' \in V$,
$$f(\underbrace{v + v'}_{\text{in } V}) = \underbrace{f(v) + f(v')}_{\text{in } W}$$
- scalar multiplication: for each $v \in V$ and $a \in \mathbb{R}$,
$$f(\underbrace{a \cdot v}_{\text{in } V}) = \underbrace{a \cdot f(v)}_{\text{in } W}$$

Linear maps preserve zero and minus

Lemma

Each linear map $f : V \to W$ preserves:

- *zero:* $f(0) = 0$.
- *minus:* $f(-v) = -f(v)$

Proof: Nice illustration of axiomatic reasoning:

$$\begin{aligned}
f(0) &= f(0) + 0 \\
&= f(0) + \big(f(0) - f(0)\big) \\
&= \big(f(0) + f(0)\big) - f(0) \\
&= f(0+0) - f(0) \\
&= f(0) - f(0) \\
&= 0
\end{aligned}$$

$$\begin{aligned}
f(-v) &= f(-v) + 0 \\
&= f(-v) + \big(f(v) - f(v)\big) \\
&= \big(f(-v) + f(v)\big) - f(v) \\
&= f(-v + v) - f(v) \\
&= f(0) - f(v) \\
&= 0 - f(v) \\
&= -f(v)
\end{aligned}$$

$\blacksquare$

Linear map examples I

First we consider maps $f : \mathbb{R} \to \mathbb{R}$. Most of them are *not linear*, like, for instance:

- $f(x) = 1 + x$,since $f(0) = 1 \neq 0$
- $f(x) = x^2$, since $f(-1) = 1 = f(1) \neq -f(1)$.

So: linear maps $\mathbb{R} \to \mathbb{R}$ can only be very simple.

Lemma

Each linear map $f : \mathbb{R} \to \mathbb{R}$ is of the form $f(x) = c \cdot x$, for some $c \in \mathbb{R}$ (this constant c depends on f)

Proof: Scalar multiplication on $\mathbb{R}$ is ordinary multiplication. Hence:

$$f(x) = f(x \cdot 1) = x \cdot f(1) = f(1) \cdot x = c \cdot x, \quad \text{for } c = f(1). \quad \blacksquare$$

Linear map examples II

Consider the map $f\colon \mathbb{R}^3 \to \mathbb{R}^2$ given by

$$f(x_1, x_2, x_3) = (x_1 - x_2,\ x_2 + x_3)$$

We show in detail that this f is linear, following the definition.

Preservation of scalar multiplication (from $\mathbb{R}^3$ to $\mathbb{R}^2$):

$$\begin{aligned}
f\Big(a \cdot (x_1, x_2, x_3)\Big) &= f\Big(a \cdot x_1, a \cdot x_2, a \cdot x_3\Big) \\
&= \Big(a \cdot x_1 - a \cdot x_2,\ a \cdot x_2 + a \cdot x_3\Big) \\
&= \Big(a \cdot (x_1 - x_2),\ a \cdot (x_2 + x_3)\Big) \\
&= a \cdot (x_1 - x_2,\ x_2 + x_3) \\
&= a \cdot f(x_1, x_2, x_3).
\end{aligned}$$

Linear map examples II (cntd)

Preservation of addition of f from $\mathbb{R}^3$ to $\mathbb{R}^2$ given by:

$$f(x_1, x_2, x_3) = (x_1 - x_2,\ x_2 + x_3)$$

$$\begin{aligned}
& f\Big((x_1, x_2, x_3) + (y_1, y_2, y_3)\Big) \\
& = f\Big(x_1 + y_1, x_2 + y_2, x_3 + y_3\Big) \\
& = \Big((x_1 + y_1) - (x_2 + y_2),\ (x_2 + y_2) + (x_3 + y_3)\Big) \\
& = \Big((x_1 - x_2) + (y_1 - y_2),\ (x_2 + x_3) + (y_2 + y_3)\Big) \\
& = \Big(x_1 - x_2, x_2 + x_3\Big) + \Big(y_1 - y_2, y_2 + y_3\Big) \\
& = f(x_1, x_2, x_3) + f(y_1, y_2, y_3).
\end{aligned}$$

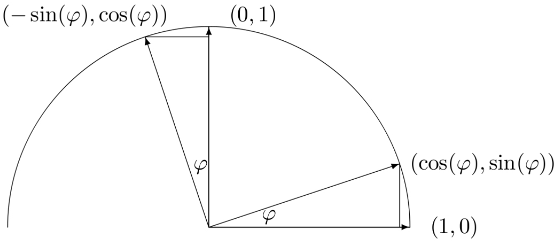
■

Linear map examples III

Consider the map $f\colon \mathbb{R}^2 \to \mathbb{R}^2$ given by

$$f(x,y) = \big(\, x\cos(\varphi) - y\sin(\varphi),\ x\sin(\varphi) + y\cos(\varphi)\,\big)$$

This map describes rotation in the plane, with angle φ:



In the same way one can show that f is linear [Do it yourself!]

Linear maps and bases, example I

- Recall the linear map $f(x_1, x_2, x_3) = (x_1 - x_2,\ x_2 + x_3)$
- **Claim**: this map is entirely determined by what it does on the base vectors $(1,0,0), (0,1,0), (0,0,1) \in \mathbb{R}^3$, namely:

$$f(1,0,0) = (1,0) \quad f(0,1,0) = (-1,1) \quad f(0,0,1) = (0,1).$$

- Indeed, using linearity:

$$\begin{aligned}
& f(x_1, x_2, x_3) \\
&= f\Big((x_1,0,0) + (0,x_2,0) + (0,0,x_3)\Big) \\
&= f\Big(x_1 \cdot (1,0,0) + x_2 \cdot (0,1,0) + x_3 \cdot (0,0,1)\Big) \\
&= f\Big(x_1 \cdot (1,0,0)\Big) + f\Big(x_2 \cdot (0,1,0)\Big) + f\Big(x_3 \cdot (0,0,1)\Big) \\
&= x_1 \cdot f(1,0,0) + x_2 \cdot f(0,1,0) + x_3 \cdot f(0,0,1) \\
&= x_1 \cdot (1,0) + x_2 \cdot (-1,1) + x_3 \cdot (0,1) \\
&= (x_1 - x_2,\ x_2 + x_3)
\end{aligned}$$

Linear maps and bases, example I (cntd)

- Our $f(x_1, x_2, x_3) = (x_1 - x_2,\ x_2 + x_3)$ is thus determined by:

  $$f(1,0,0) = (1,0) \qquad f(0,1,0) = (-1,1) \qquad f(0,0,1) = (0,1)$$

- We can organise these data in a 2×3 matrix:

  $$\begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & 1 \end{pmatrix}$$

  The $f(v_i)$, for base vector v_i, appears as the i-the column.

- Applying f can be done by a new kind of multiplication:

$$\begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & 1 \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \stackrel{\text{def}}{=} \begin{pmatrix} 1 \cdot x_1 + -1 \cdot x_2 + 0 \cdot x_3 \\ 0 \cdot x_1 + 1 \cdot x_2 + 1 \cdot x_3 \end{pmatrix} = \begin{pmatrix} x_1 - x_2 \\ x_2 + x_3 \end{pmatrix}$$

The general case

The aim is to obtain a matrix for an arbitrary linear map.

- Assume a linear map $f\colon V \to W$, where:
  - the vector space V has basis $\{v_1, \ldots, v_n\} \subseteq V$;
  - W has basis $\{w_1, \ldots, w_m\}$
- Each $x \in V$ can be written as $x = a_1v_1 + \cdots + a_nv_n$. Hence:

$$\begin{array}{rcll} f(x) & = & f\big(a_1v_1 + \cdots + a_nv_n\big) & \\ & = & a_1f(v_1) + \cdots + a_nf(v_n) & \qquad \text{by linearity of } f \end{array}$$

  Thus, f is determined by its values $f(v_1), \ldots, f(v_n)$ on base vectors $v_j \in V$.
- By writing $f(v_j) = b_{1j}w_1 + \cdots + b_{mj}w_m$ we obtain an $m \times n$ matrix with entries $\big(b_{ij}\big)_{i \leq m, j \leq n}$

Towards matrix-vector multiplication

In this setting, we have:

$$
\begin{aligned}
& f(x) \\
& = f(a_1 v_1 + \cdots + a_n v_n) \\
& = a_1 f(v_1) + \cdots + a_n f(v_n) \\
& = a_1 \big(b_{11} w_1 + \cdots + b_{m1} w_m\big) + \cdots + a_n \big(b_{1n} w_1 + \cdots + b_{mn} w_m\big) \\
& = \big(a_1 b_{11} + \cdots + a_n b_{1n}\big) w_1 + \cdots + \big(a_1 b_{m1} + \cdots + a_n b_{mn}\big) w_m \\
& = \big(b_{11} a_1 + \cdots + b_{1n} a_n\big) w_1 + \cdots + \big(b_{m1} a_1 + \cdots + b_{mn} a_n\big) w_m
\end{aligned}
$$

This motivates the definition of matrix-vector multiplication:

$$
\begin{pmatrix} b_{11} & \cdots & b_{1n} \\ \vdots & & \vdots \\ b_{m1} & \cdots & b_{mn} \end{pmatrix} \cdot \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix} = \begin{pmatrix} b_{11} a_1 + \cdots + b_{1n} a_n \\ \vdots \\ b_{m1} a_1 + \cdots + b_{mn} a_n \end{pmatrix}
$$

Matrix-vector multiplication: Definition

Definition

For vectors $v = (x_1, \ldots, x_n), w = (y_1, \ldots, y_n) \in \mathbb{R}^n$ define their inner product (or dot product) as the real number:

$$\langle v, w \rangle \;=\; x_1 y_1 + \cdots + x_n y_n$$

Definition

If $B = \begin{pmatrix} b_{11} & \cdots & b_{1n} \\ \vdots & & \vdots \\ b_{m1} & \cdots & b_{mn} \end{pmatrix}$ and $w = \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix}$, then $B \cdot w$

is the vector whose i-th element is the dot product of the i-th row of matrix B with the (input) vector w.

Matrix-vector multiplication, concretely

- Recall $f(x_1, x_2, x_3) = (x_1 - x_2,\ x_2 + x_3)$ with matrix:
$$\begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & 1 \end{pmatrix}$$
- We can directly calculate
$f(1, 2, -1) = (1 - 2, 2 - 1) = (-1, 1)$
- We can also get the same result by matrix-vector multiplication:
$$\begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} = \begin{pmatrix} 1 \cdot 1 + -1 \cdot 2 + 0 \cdot -1 \\ 0 \cdot 1 + 1 \cdot 2 + 1 \cdot -1 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$
- This multiplication can be understood as: putting the argument values $x_1 = 1, x_2 = 2, x_3 = -1$ in variables of the underlying equations, and computing the outcome.

Another example, to learn the mechanics

$$
\begin{pmatrix} 9 & 3 & 2 & 9 & 7 \\ 8 & 5 & 6 & 6 & 3 \\ 4 & 5 & 8 & 9 & 3 \\ 3 & 4 & 3 & 3 & 4 \end{pmatrix} \cdot \begin{pmatrix} 9 \\ 5 \\ 2 \\ 5 \\ 7 \end{pmatrix}
$$

$$
= \begin{pmatrix} 9\cdot 9 + 3\cdot 5 + 2\cdot 2 + 9\cdot 5 + 7\cdot 7 \\ 8\cdot 9 + 5\cdot 5 + 6\cdot 2 + 6\cdot 5 + 3\cdot 7 \\ 4\cdot 9 + 5\cdot 5 + 8\cdot 2 + 9\cdot 5 + 3\cdot 7 \\ 3\cdot 9 + 4\cdot 5 + 3\cdot 2 + 3\cdot 5 + 4\cdot 7 \end{pmatrix}
$$

$$
= \begin{pmatrix} 81 + 15 + 4 + 45 + 49 \\ 72 + 25 + 12 + 30 + 21 \\ 36 + 25 + 16 + 45 + 21 \\ 27 + 20 + 6 + 15 + 28 \end{pmatrix} = \begin{pmatrix} 194 \\ 160 \\ 143 \\ 96 \end{pmatrix}
$$

Linear map from matrix

- We have seen how a linear map can be described via a matrix
- One can also read each matrix as a linear map

Example

- Consider the matrix $\begin{pmatrix} 2 & 0 & -1 \\ 5 & 1 & -3 \end{pmatrix}$
- It has 3 columns/inputs and two rows/outputs. Hence it describes a map $f\colon \mathbb{R}^3 \to \mathbb{R}^2$
- Namely: $f(x_1, x_2, x_3) = (2x_1 - x_3,\ 5x_1 + x_2 - 3x_3)$.

Examples of linear maps and matrices I

Projections are linear maps. Consider $f : \mathbb{R}^3 \to \mathbb{R}^2$

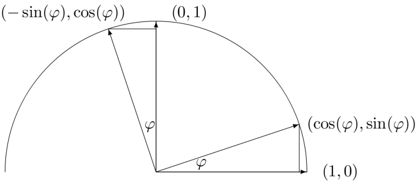
$$f \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}.$$

f maps $3d$ space to the the $2d$ plane.
The matrix of f is the following 2×3 matrix:

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

Examples of linear maps and matrices II

We have already seen: Rotation over an angle φ is a linear map



This rotation is described by $f : \mathbb{R}^2 \to \mathbb{R}^2$ given by

$$f(x, y) = \big(\, x\cos(\varphi) - y\sin(\varphi),\ x\sin(\varphi) + y\cos(\varphi) \,\big)$$

The matrix that describes f is

$$\begin{pmatrix} \cos(\varphi) & -\sin(\varphi) \\ \sin(\varphi) & \cos(\varphi) \end{pmatrix}.$$

Examples of linear maps and matrices III

Reflection through an axis is a linear map

- Reflection through the y-axis: $(x, y) \mapsto (-x, y)$ is given by
$$\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}.$$
- Reflection in a different straight line that goes through $(0, 0)$, for example the line $y = 2x$:
  - We first choose a different basis E for $\mathbb{R}^2$, with one vector orthogonal to the axis and one on the axis.
  - We choose $E = \{(2, -1), (1, 2)\}$.
  - In terms of the basis E, the matrix for f is just
$$\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}.$$
  - We will learn how to transform this back to a matrix for the standard basis!

Matrix summary

- Assume bases $\{v_1, \ldots, v_n\} \subseteq V$ and $\{w_1, \ldots, w_m\} \subseteq W$
- Each linear map $f: V \to W$ corresponds to an $m \times n$ matrix, and vice-versa.
  We often write the matrix of f as M_f
- The i-th column in this matrix M_f is given by the coefficients of $f(v_i)$, wrt. the basis $w_1, \ldots, w_m$ of W
- Matrix-vector multiplication corresponds to application of a map to an input: $f(v)$ is the same as $M_f \cdot v$.
- This matrix M_f of f depends on the choice of basis: for different bases of V and W a different matrix is obtained
- (Matrix-vector multiplication forms itself a linear map)

The identity matrix

Consider the following $n \times n$ identity matrix with diagonal of 1's:

$$I_n = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & \cdots & 1 \end{pmatrix}$$

- To which map does I_n correspond?

  The identity map $\mathbb{R}^n \to \mathbb{R}^n$.
- To which system of equations does I_n correspond?

$$\begin{cases} x_1 & = & 0 \\ & \vdots & \\ x_n & = & 0 \end{cases}$$

Matrices as vectors I

- Write **Mat**$_{m,n} = \{M \mid M \text{ is an } m \times n \text{ matrix}\}$
- Thus each $M \in \mathbf{Mat}_{m,n}$ can be written as $M = (a_{ij})$, for $1 \leq i \leq m$ and $1 \leq j \leq n$
- We can add two such matrices $M, N \in \mathbf{Mat}_{m,n}$, giving $M + N \in \mathbf{Mat}_{m,n}$.
  - the matrices are added entry-wise, that is:
  - if $M = (a_{ij})$, $N = (b_{ij})$, $M + N = (c_{ij})$, then $c_{ij} = a_{ij} + b_{ij}$
- Similarly, matrices can be multiplied by a scalar $s \in \mathbb{R}$
  - $s \cdot M \in \mathbf{Mat}_{m,n}$ has entries $s \cdot a_{ij}$
- Finally, there is a zero matrix $0_{m,n} \in \mathbf{Mat}_{m,n}$, with only zeros as entries

$\mathbf{Mat}_{m,n}$ is a vector space (of dimension $m \cdot n$).

Matrices as vectors II: example

- Addition:

$$\begin{pmatrix} 2 & 0 & 1 \\ -1 & -3 & 5 \end{pmatrix} + \begin{pmatrix} 1 & 1 & 2 \\ 2 & -2 & 5 \end{pmatrix} = \begin{pmatrix} 3 & 1 & 3 \\ 1 & -5 & 10 \end{pmatrix}$$

- Scalar multiplication:

$$5 \cdot \begin{pmatrix} 2 & 0 & 1 \\ -1 & -3 & 5 \end{pmatrix} = \begin{pmatrix} 10 & 0 & 5 \\ -5 & -15 & 25 \end{pmatrix}$$

Matrices as vectors III: transpose

- For a matrix $M \in \mathbf{Mat}_{m,n}$ write $M^T \in \mathbf{Mat}_{n,m}$ for the transpose of M
- It is obtained by mirroring:
  - if $M = (a_{ij})$ then M^T has entries a_{ji}
  - For example

$$\begin{pmatrix} 2 & 0 & 1 \\ -1 & -3 & 5 \end{pmatrix}^T = \begin{pmatrix} 2 & -1 \\ 0 & -3 \\ 1 & 5 \end{pmatrix}$$

Theorem

Transposition is a linear map $(-)^T : \mathbf{Mat}_{m,n} \to \mathbf{Mat}_{n,m}$. *That is:*

- $(M + N)^T = M^T + N^T$
- $(a \cdot M)^T = a \cdot M^T$