Matrix Calculations: Eigenvalues and Eigenvectors

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Version: spring 2016

Outline

Political swingers re-revisited, part I

- Recall the political transisition matrix

$$\mathbf{P} = \begin{pmatrix} 0.8 & 0.1 \\ 0.2 & 0.9 \end{pmatrix} = \frac{1}{10}\begin{pmatrix} 8 & 1 \\ 2 & 9 \end{pmatrix}$$

- with some iterations:

$$\mathbf{P} \cdot \begin{pmatrix} 100 \\ 150 \end{pmatrix} = \begin{pmatrix} 95 \\ 155 \end{pmatrix} \quad \mathbf{P}^2 \cdot \begin{pmatrix} 100 \\ 150 \end{pmatrix} = \begin{pmatrix} 91.5 \\ 158.5 \end{pmatrix} \quad \mathbf{P}^3 \cdot \begin{pmatrix} 100 \\ 150 \end{pmatrix} = \begin{pmatrix} 89.05 \\ 160.95 \end{pmatrix} \quad \cdots$$

- Does this converge to a stable lefty-righty division? If so, what is a stable division?
- Check for yourself: $\mathbf{P} \cdot \begin{pmatrix} 83\frac{1}{3} \\ 166\frac{2}{3} \end{pmatrix} = \begin{pmatrix} 83\frac{1}{3} \\ 166\frac{2}{3} \end{pmatrix}$

Political swingers re-revisited, part II

- When do we have $\mathbf{P} \cdot \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$?
- This involves:

$$\begin{cases} 0.8x + 0.1y &= x \\ 0.2x + 0.9y &= y \end{cases} \quad \text{so} \quad \begin{cases} (0.8-1)x + 0.1y &= 0 \\ 0.2x - (0.9-1)y &= 0 \end{cases}$$

$$\text{ie.} \begin{cases} -0.2x + 0.1y &= 0 \\ 0.2x - 0.1y &= 0 \end{cases} \quad \text{thus} \quad \begin{cases} -2x + y &= 0 \\ 2x - y &= 0 \end{cases} \quad \text{so} \quad y = 2x$$

- Indeed, $\mathbf{P} \cdot \begin{pmatrix} x \\ 2x \end{pmatrix} = \begin{pmatrix} x \\ 2x \end{pmatrix}$ Twice as many righties is stable!
- We found it by solving (homogeneous) equations given by the matrix:

$$\mathbf{P} - \mathbf{I}_2 = \begin{pmatrix} -0.2 & 0.1 \\ 0.2 & -0.1 \end{pmatrix} = \frac{1}{10} \begin{pmatrix} -2 & 1 \\ 2 & -1 \end{pmatrix}$$

Eigenvector and eigenvalues

Definition

Assume an $n \times n$ matrix $\mathbf{A}$.

An eigenvector for $\mathbf{A}$ is a non-null vector $\mathbf{v} \neq 0$ for which there is an eigenvalue $\lambda \in \mathbb{R}$ with:

$$\mathbf{A} \cdot \mathbf{v} = \lambda \cdot \mathbf{v}$$

Example

$\begin{pmatrix} 100 \\ 200 \end{pmatrix}$ is an eigenvector for $\mathbf{P} = \frac{1}{10}\begin{pmatrix} 8 & 1 \\ 2 & 9 \end{pmatrix}$ with eigenvalue $\lambda = 1$.

Two basic results

Lemma

An eigenvector has at most one eigenvalue

Proof: Assume $\mathbf{A} \cdot \mathbf{v} = \lambda_1 \mathbf{v}$ and $\mathbf{A} \cdot \mathbf{v} = \lambda_2 \mathbf{v}$. Then:

$$0 = \mathbf{A} \cdot \mathbf{v} - \mathbf{A} \cdot \mathbf{v} = \lambda_1 \mathbf{v} - \lambda_2 \mathbf{v} = (\lambda_1 - \lambda_2)\mathbf{v}$$

Since $\mathbf{v} \neq 0$ we must have $\lambda_1 - \lambda_2 = 0$, and thus $\lambda_1 = \lambda_2$.

Lemma

If $\mathbf{v}$ is an eigenvector, then so is $a\mathbf{v}$, for each $a \neq 0$.

Proof: If $\mathbf{A} \cdot \mathbf{v} = \lambda \mathbf{v}$, then:

$$\begin{aligned} \mathbf{A} \cdot (a\mathbf{v}) &= a(\mathbf{A} \cdot \mathbf{v}) && \text{since matrix application is linear} \\ &= a(\lambda \mathbf{v}) = (a\lambda)\mathbf{v} = (\lambda a)\mathbf{v} = \lambda(a\mathbf{v}). \end{aligned}$$

Finding eigenvectors and eigenvalues

- We seek a eigenvector $\mathbf{v}$ and eigenvalue $\lambda \in \mathbb{R}$ with $\mathbf{A} \cdot \mathbf{v} = \lambda \mathbf{v}$
- That is: λ and $\mathbf{v}$ ($\mathbf{v} \neq 0$) such that $(\mathbf{A} - \lambda \cdot \mathbf{I}) \cdot \mathbf{v} = 0$
- Thus, we seek λ for which the system of equations corresponding to the matrix $\mathbf{A} - \lambda \cdot \mathbf{I}$ has a non-zero solution
- Hence we seek $\lambda \in \mathbb{R}$ for which the matrix $\mathbf{A} - \lambda \cdot \mathbf{I}$ does not have n pivots in its echelon form
- This means: we seek $\lambda \in \mathbb{R}$ such that $\mathbf{A} - \lambda \cdot \mathbf{I}$ is not-invertible
- So we need: $\det(\mathbf{A} - \lambda \cdot \mathbf{I}) = 0$
- This can be seen as an equation, with λ as variable
- This $\det(\mathbf{A} - \lambda \cdot \mathbf{I})$ is called the characteristic polynomial of the matrix $\mathbf{A}$

Eigenvalue example I

- **Task**: find eigenvalues of matrix $\mathbf{A} = \begin{pmatrix} 1 & 5 \\ 3 & 3 \end{pmatrix}$
- Note: $\mathbf{A} - \lambda \cdot \mathbf{I} = \begin{pmatrix} 1 & 5 \\ 3 & 3 \end{pmatrix} - \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} = \begin{pmatrix} 1-\lambda & 5 \\ 3 & 3-\lambda \end{pmatrix}$
- Thus:

$$
\begin{aligned}
\det(\mathbf{A} - \lambda \cdot \mathbf{I}) = 0 &\iff \begin{vmatrix} 1-\lambda & 5 \\ 3 & 3-\lambda \end{vmatrix} = 0 \\
&\iff (1-\lambda)(3-\lambda) - 5 \cdot 3 = 0 \\
&\iff \lambda^2 - 4\lambda - 12 = 0 \\
&\iff (\lambda - 6)(\lambda + 2) = 0 \\
&\iff \lambda = 6 \ \text{ or } \ \lambda = -2.
\end{aligned}
$$

Recall: abc-formula

- Consider a second-degree (quadratic) equation

$$ax^2 + bx + c = 0 \qquad \text{(for } a \neq 0\text{)}$$

- Its solutions are:

$$s_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

- These solutions coincide (ie. $s_1 = s_2$) if $b^2 - 4ac = 0$
- Real solutions do not exist if $b^2 - 4ac < 0$

  (But "complex number" solutions do exist in this case.)
- [Recall, if s_1 and s_2 are solutions of $ax^2 + bx + c = 0$, then we can write $ax^2 + bx + c = a(x - s_1)(x - s_2)$]

Higher degree polynomial equations

- For third and fourth degree polynomial equations there are (complicated) formulas for the solutions.
- For degree ≥ 5 no such formulas exist (proved by Abel)
- In those cases one can at most use approximations.
- In the examples in this course the solutions will typically be "obvious".

Eigenvalue example II

- **Task**: find eigenvalues of $\mathbf{A} = \begin{pmatrix} 3 & -1 & -1 \\ -12 & 0 & 5 \\ 4 & -2 & -1 \end{pmatrix}$
- Characteristic polynomial is $\begin{vmatrix} 3-\lambda & -1 & -1 \\ -12 & -\lambda & 5 \\ 4 & -2 & -1-\lambda \end{vmatrix}$

$$\begin{aligned}
&= (3-\lambda)\begin{vmatrix} -\lambda & 5 \\ -2 & -1-\lambda \end{vmatrix} + 12\begin{vmatrix} -1 & -1 \\ -2 & -1-\lambda \end{vmatrix} + 4\begin{vmatrix} -1 & -1 \\ -\lambda & 5 \end{vmatrix} \\
&= (3-\lambda)\Big(\lambda(1+\lambda)+10\Big) + 12\Big(1+\lambda-2\Big) + 4\Big(-5-\lambda\Big) \\
&= (3-\lambda)(\lambda^2+\lambda+10) + 12(\lambda-1) - 20 - 4\lambda \\
&= 3\lambda^2 + 3\lambda + 30 - \lambda^3 - \lambda^2 - 10\lambda + 12\lambda - 12 - 20 - 4\lambda \\
&= -\lambda^3 + 2\lambda^2 + \lambda - 2.
\end{aligned}$$

Eigenvalue example II (cntd)

- We need to solve $-\lambda^3 + 2\lambda^2 + \lambda - 2 = 0$
- We try a few "obvious" values: $\lambda = 1$ YES!
- Reduce from degree 3 to 2, by separating $(\lambda - 1)$ in:

$$\begin{aligned} -\lambda^3 + 2\lambda^2 + \lambda - 2 &= (\lambda - 1)(a\lambda^2 + b\lambda + c) \\ &= a\lambda^3 + (b-a)\lambda^2 + (c-b)\lambda - c \end{aligned}$$

- This works for $a = -1$, $b = 1$, $c = 2$
- Now we use "abc" for the equation $-\lambda^2 + \lambda + 2 = 0$
- Solutions: $\lambda = \dfrac{-1 \pm \sqrt{1 + 4 \cdot 2}}{-2} = \dfrac{-1 \pm 3}{-2}$ giving $\lambda = 2, -1$
- All three eigenvalues: $\lambda = 1, \lambda = -1, \lambda = 2$

Getting eigenvectors

- Once we have eigenvalues λ_i for a matrix $\mathbf{A}$ we can find corresponding eigenvectors $\mathbf{v}_i$, with $\mathbf{A} \cdot \mathbf{v}_i = \lambda_i \mathbf{v}_i$
- These $\mathbf{v}_i$ appear as the solutions of $(\mathbf{A} - \lambda_i \cdot \mathbf{I}) \cdot \mathbf{v} = 0$
  - We can make a convenient choice, using that scalar multiplications $a \cdot \mathbf{v}_i$ are also a solution
- We use standard techniqes for solving such equations.

Eigenvector example I

Recall the eigenvalues $\lambda = -2, \lambda = 6$ for $\mathbf{A} = \begin{pmatrix} 1 & 5 \\ 3 & 3 \end{pmatrix}$

$\boxed{\lambda = -2}$ gives matrix $\mathbf{A} - \lambda\mathbf{I} = \begin{pmatrix} 1+2 & 5 \\ 3 & 3+2 \end{pmatrix} = \begin{pmatrix} 3 & 5 \\ 3 & 5 \end{pmatrix}$

- Corresponding system of equations $\begin{cases} 3x + 5y & = & 0 \\ 3x + 5y & = & 0 \end{cases}$
- Solution choice $x = -5, y = 3$, so $(-5, 3)$ is eigenvector (of matrix $\mathbf{A}$ with eigenvalue $\lambda = -2$)
- Check:

$$\begin{pmatrix} 1 & 5 \\ 3 & 3 \end{pmatrix} \cdot \begin{pmatrix} -5 \\ 3 \end{pmatrix} = \begin{pmatrix} -5+15 \\ -15+9 \end{pmatrix} = \begin{pmatrix} 10 \\ -6 \end{pmatrix} = -2 \begin{pmatrix} -5 \\ 3 \end{pmatrix} \checkmark$$

Eigenvector example I (cntd)

$\boxed{\lambda = 6}$ gives matrix $\mathbf{A} - \lambda \mathbf{I} = \begin{pmatrix} 1-6 & 5 \\ 3 & 3-6 \end{pmatrix} = \begin{pmatrix} -5 & 5 \\ 3 & -3 \end{pmatrix}$

- Corresponding system of equations $\begin{cases} -5x + 5y = 0 \\ 3x - 3y = 0 \end{cases}$
- Solution choice $x = 1, y = 1$, so $(1,1)$ is eigenvector (of matrix $\mathbf{A}$ with eigenvalue $\lambda = 6$)
- Check:

$$\begin{pmatrix} 1 & 5 \\ 3 & 3 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1+5 \\ 3+3 \end{pmatrix} = \begin{pmatrix} 6 \\ 6 \end{pmatrix} = 6 \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad \checkmark$$

Eigenvector independence theorem I

Theorem

Let $\mathbf{A}$ *be an* $n \times n$ *matrix, represented wrt. a basis* $\mathcal{B}$*. Assume* $\mathbf{A}$ *has* n *(pairwise) different eigenvalues* $\lambda_1, \ldots, \lambda_n$*, with corresponding eigenvectors* $\mathcal{C} = \{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$*.* **Then***:*

1. *These* $\mathbf{v}_1, \ldots, \mathbf{v}_n$ *are linearly independent (and thus a basis)*
2. *There is an invertible "basis transformation" matrix* $\mathbf{T}_{\mathcal{C}\Rightarrow\mathcal{B}}$ *giving a diagonalisation:*

$$\mathbf{A} = \mathbf{T}_{\mathcal{C}\Rightarrow\mathcal{B}} \cdot \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & & 0 \\ 0 & & \ddots & 0 \\ 0 & \cdots & 0 & \lambda_n \end{pmatrix} \cdot \left(\mathbf{T}_{\mathcal{C}\Rightarrow\mathcal{B}}\right)^{-1}$$

Thus, this diagonal matrix is the representation of $\mathbf{A}$ *wrt. the eigenvector basis* $\mathcal{C}$*.*

Eigenvector independence theorem II

We specialize the theorem by taking $\mathcal{S}$ to be the standard basis.

Theorem

Let $\mathbf{A}$ *be* $n \times n$ *matrix, represented wrt. the standard basis of* $\mathbb{R}^n$*. Assume* $\mathbf{A}$ *has* n *(pairwise) different eigenvalues* $\lambda_1, \ldots, \lambda_n$*, with corresponding eigenvectors* $\mathcal{C} = \{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$*.* **Then***:*

1. *These* $\mathbf{v}_1, \ldots, \mathbf{v}_n$ *are linearly independent (and thus a basis)*
2. *The vectors* $\mathbf{v}_1, \ldots, \mathbf{v}_n$ *are the columns of the invertible "basis transformation" matrix* $\mathbf{T}_{\mathcal{C}\Rightarrow\mathcal{S}}$
3. *This gives a diagonalisation of* $\mathbf{A}$*:*

$$\mathbf{A} = \mathbf{T}_{\mathcal{C}\Rightarrow\mathcal{S}} \cdot \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & & 0 \\ 0 & & \ddots & 0 \\ 0 & \cdots & 0 & \lambda_n \end{pmatrix} \cdot \left(\mathbf{T}_{\mathcal{C}\Rightarrow\mathcal{S}}\right)^{-1}$$

Multiple eigenvalues

- It may happen that a particular eigenvalue occurs multiple times for a matrix
  - eg. the charachterstic polynomial of $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ has $\lambda = 1$ twice as a zero.
  - for this $\lambda = 1$ there are two independent eigenvectors, namely $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$
- In general, if an eigenvalue λ occurs n times, then there are at most n independent eigenvectors for this λ
  - this number of independent eigenvectors for λ is called the the geometric multiplicity for λ
  - it is the dimension of the eigenspace associated with λ.
  - this is not a further topic of this course; in our examples, eigenvalues have unique eigenvectors

Where are eigenvalues/vectors used?

- In principal component analysis in statistics (implemented in SPSS)
  - generalisation of mean value and covariance to multi-dimensional data analysis
  - eigenvalues of covariance matrix reveal key characteristics
  - sketched in LNBS, but skipped here
    (a brief explanation without statistical setting is useless)
  - applied in speech recognition, data compression, data mining
- In quantum mechanics/computation
  - eigenvalues/vectors represent the special states that appear in measurements
  - cool topic, but also skipped here
- In (probabilistic) transition systems (Markov chains)
  - illustrated already in political swingers example
  - another illustration (car rental) will be elaborated
    (copied from: Johnson, Dean Riess, Arnold: Linear Algebra)

Political swingers re-re-revisited, part I

- Recall the political transisition matrix

$$\mathbf{P} = \begin{pmatrix} 0.8 & 0.1 \\ 0.2 & 0.9 \end{pmatrix} = \frac{1}{10}\begin{pmatrix} 8 & 1 \\ 2 & 9 \end{pmatrix}$$

- Eigenvalues λ are obtained via $\det(\mathbf{P} - \lambda\,\mathbf{I}_2) = 0$:

$$(\tfrac{8}{10} - \lambda)(\tfrac{9}{10} - \lambda) - \tfrac{1}{10} \cdot \tfrac{2}{10} = \lambda^2 - \tfrac{17}{10}\lambda + \tfrac{7}{10} = 0$$

- Solutions via "abc"

$$\begin{aligned} \tfrac{1}{2}\left(\tfrac{17}{10} \pm \sqrt{\left(\tfrac{17}{10}\right)^2 - \tfrac{28}{10}}\right) &= \tfrac{1}{2}\left(\tfrac{17}{10} \pm \sqrt{\tfrac{289}{100} - \tfrac{280}{100}}\right) \\ &= \tfrac{1}{2}\left(\tfrac{17}{10} \pm \sqrt{\tfrac{9}{100}}\right) \\ &= \tfrac{1}{2}\left(\tfrac{17}{10} \pm \tfrac{3}{10}\right) \end{aligned}$$

- Hence $\lambda = \frac{1}{2} \cdot \frac{20}{10} = 1$ or $\lambda = \frac{1}{2} \cdot \frac{14}{10} = \frac{7}{10}$.

Political swingers re-re-revisited, part II

$\boxed{\lambda = 1}$ solve: $\begin{cases} -0.2x + 0.1y = 0 \\ 0.2x + -0.1y = 0 \end{cases}$ giving $(1, 2)$ as eigenvector

- Indeed $\begin{pmatrix} 0.8 & 0.1 \\ 0.2 & 0.9 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 0.8 + 0.2 \\ 0.2 + 1.8 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} = 1 \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ ✓

$\boxed{\lambda = 0.7}$ solve: $\begin{cases} 0.1x + 0.1y = 0 \\ 0.2x + 0.2y = 0 \end{cases}$ giving $(1, -1)$ as eigenvector

- Check:

$$\begin{pmatrix} 0.8 & 0.1 \\ 0.2 & 0.9 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 0.8 - 0.1 \\ 0.2 - 0.9 \end{pmatrix} = \begin{pmatrix} 0.7 \\ -0.7 \end{pmatrix} = 0.7 \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$ ✓

Political swingers re-re-revisited, part III

- The eigenvalues 1 and 0.7 are different, and indeed the eigenvectors $(1, 2)$ and $(1, -1)$ are independent
- The coordinate-translation $\mathbf{T}_{\mathcal{C}\Rightarrow\mathcal{S}}$ from the eigenvector basis $\mathcal{C} = \{(1, 2), (1, -1)\}$ to the standard basis $\mathcal{S} = \{(1, 0), (0, 1)\}$ consists of the eigenvectors:

$$\mathbf{T}_{\mathcal{C}\Rightarrow\mathcal{S}} = \begin{pmatrix} 1 & 1 \\ 2 & -1 \end{pmatrix}$$

- In the reverse direction:

$$\mathbf{T}_{\mathcal{S}\Rightarrow\mathcal{C}} = \left(\mathbf{T}_{\mathcal{C}\Rightarrow\mathcal{S}}\right)^{-1} = \frac{1}{-1-2}\begin{pmatrix} -1 & -1 \\ -2 & 1 \end{pmatrix} = \frac{1}{3}\begin{pmatrix} 1 & 1 \\ 2 & -1 \end{pmatrix}$$

Political swingers re-re-revisited, part IV

We explicitly check the diagonalisation equation:

$$\begin{aligned}
\mathbf{T}_{\mathcal{C}\Rightarrow\mathcal{S}} \cdot \begin{pmatrix} 1 & 0 \\ 0 & 0.7 \end{pmatrix} \cdot \mathbf{T}_{\mathcal{S}\Rightarrow\mathcal{C}} &= \begin{pmatrix} 1 & 1 \\ 2 & -1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ 0 & 0.7 \end{pmatrix} \cdot \tfrac{1}{3} \begin{pmatrix} 1 & 1 \\ 2 & -1 \end{pmatrix} \\
&= \tfrac{1}{3} \begin{pmatrix} 1 & 0.7 \\ 2 & -0.7 \end{pmatrix} \cdot \begin{pmatrix} 1 & 1 \\ 2 & -1 \end{pmatrix} \\
&= \tfrac{1}{3} \begin{pmatrix} 2.4 & 0.3 \\ 0.6 & 2.7 \end{pmatrix} \\
&= \begin{pmatrix} 0.8 & 0.1 \\ 0.2 & 0.9 \end{pmatrix} \\
&= \mathbf{P}, \quad \text{the original political transition matrix!}
\end{aligned}$$

Political swingers re-re-revisited, part V

This diagonalisation $\mathbf{P} = \mathbf{T} \cdot \begin{pmatrix} 1 & 0 \\ 0 & 0.7 \end{pmatrix} \cdot \mathbf{T}^{-1}$ is useful for iteration

- $$\begin{aligned} \mathbf{P}^2 &= \mathbf{T} \cdot \begin{pmatrix} 1 & 0 \\ 0 & 0.7 \end{pmatrix} \cdot \mathbf{T}^{-1} \cdot \mathbf{T} \cdot \begin{pmatrix} 1 & 0 \\ 0 & 0.7 \end{pmatrix} \cdot \mathbf{T}^{-1} \\ &= \mathbf{T} \cdot \begin{pmatrix} 1 & 0 \\ 0 & 0.7 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ 0 & 0.7 \end{pmatrix} \cdot \mathbf{T}^{-1} \\ &= \mathbf{T} \cdot \begin{pmatrix} 1^2 & 0 \\ 0 & (0.7)^2 \end{pmatrix} \cdot \mathbf{T}^{-1} \end{aligned}$$
- $$\mathbf{P}^n = \mathbf{T} \cdot \begin{pmatrix} (1)^n & 0 \\ 0 & (0.7)^n \end{pmatrix} \cdot \mathbf{T}^{-1}$$
- $$\begin{aligned} \lim_{n\to\infty} \mathbf{P}^n &= \mathbf{T} \cdot \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \cdot \mathbf{T}^{-1} \qquad \text{since } \lim_{n\to\infty} (0.7)^n = 0 \\ &= \begin{pmatrix} 1 & 1 \\ 2 & -1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \cdot \tfrac{1}{3} \begin{pmatrix} 1 & 1 \\ 2 & -1 \end{pmatrix} = \tfrac{1}{3} \begin{pmatrix} 1 & 1 \\ 2 & 2 \end{pmatrix} \end{aligned}$$

Political swingers re-re-revisited, part VI

- In an earlier lecture we wondered how to compute $\mathbf{P}^n \cdot \begin{pmatrix} 100 \\ 150 \end{pmatrix}$
- We can now see that in the limit it goes to:

$$\begin{aligned} \lim_{n\to\infty} \mathbf{P}^n \cdot \begin{pmatrix} 100 \\ 150 \end{pmatrix} &= \tfrac{1}{3}\begin{pmatrix} 1 & 1 \\ 2 & 2 \end{pmatrix} \cdot \begin{pmatrix} 100 \\ 150 \end{pmatrix} \\ &= \tfrac{1}{3}\begin{pmatrix} 250 \\ 500 \end{pmatrix} = \begin{pmatrix} 83\frac{1}{3} \\ 166\frac{2}{3} \end{pmatrix} \end{aligned}$$

(This was already suggested earlier, but now we can calculate it!)

Recall the useful limit result

$$\lim_{n\to\infty} a^n = 0, \qquad \text{for } |a| < 1.$$

Rental car returns, part I

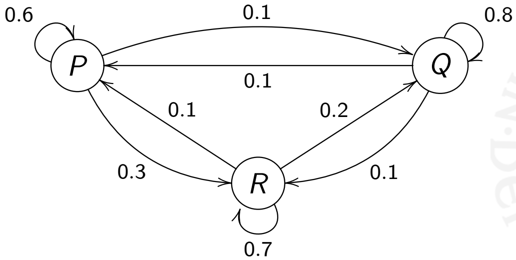
- Assume a car rental company with three locations, for picking up and returning cars, written as P, Q, R
- The weekly distribution history shows:

Location P	60% stay at P	10% go to Q	30% go to R
Location Q	10% go to P	80% stay at Q	10% go to R
Location R	10% go to P	20% go to Q	70% stay at R

Rental car returns, part II

Two possible representations of these return distributions

1. As probabilistic transition system



Rental car returns, part III

2. As a transition matrix

$$\mathbf{A} = \begin{pmatrix} 0.6 & 0.1 & 0.1 \\ 0.1 & 0.8 & 0.2 \\ 0.3 & 0.1 & 0.7 \end{pmatrix} = \frac{1}{10}\begin{pmatrix} 6 & 1 & 1 \\ 1 & 8 & 2 \\ 3 & 1 & 7 \end{pmatrix}$$

This matrix **A** describes what is called a Markov chain:

- all entries are in the unit interval $[0, 1]$ of probabilities
- in each column, the entries add up to 1

Rental car returns, part IV

Task:

- Start from the following division of cars:

$$P = Q = R = 200 \qquad \text{ie.} \qquad \begin{pmatrix} P \\ Q \\ R \end{pmatrix} = \begin{pmatrix} 200 \\ 200 \\ 200 \end{pmatrix}$$

- Determine the division of cars after two weeks
- Determine the equilibrium division, reached as the number of weeks goes to infinity

Rental car returns, part V

- After one week we have:

$$\mathbf{A} \cdot \begin{pmatrix} 200 \\ 200 \\ 200 \end{pmatrix} = \frac{1}{10} \begin{pmatrix} 6 & 1 & 1 \\ 1 & 8 & 2 \\ 3 & 1 & 7 \end{pmatrix} \cdot \begin{pmatrix} 200 \\ 200 \\ 200 \end{pmatrix}$$

$$= \frac{1}{10} \begin{pmatrix} 1200 + 200 + 200 \\ 200 + 1600 + 400 \\ 600 + 200 + 1400 \end{pmatrix} = \begin{pmatrix} 160 \\ 220 \\ 220 \end{pmatrix}$$

- After two weeks we have:

$$\mathbf{A} \cdot \begin{pmatrix} 160 \\ 220 \\ 220 \end{pmatrix} = \frac{1}{10} \begin{pmatrix} 6 & 1 & 1 \\ 1 & 8 & 2 \\ 3 & 1 & 7 \end{pmatrix} \cdot \begin{pmatrix} 160 \\ 220 \\ 220 \end{pmatrix}$$

$$= \frac{1}{10} \begin{pmatrix} 960 + 220 + 220 \\ 160 + 1760 + 440 \\ 480 + 220 + 1540 \end{pmatrix} = \begin{pmatrix} 140 \\ 236 \\ 224 \end{pmatrix}$$

Rental car returns, part VI

- For the equilibrium we first compute eigenvalues and eigenvectors of the transition matrix **A**
- The characteristic polynomial is:

$$\begin{aligned}
&\begin{vmatrix} 0.6-\lambda & 0.1 & 0.1 \\ 0.1 & 0.8-\lambda & 0.2 \\ 0.3 & 0.1 & 0.7-\lambda \end{vmatrix} = \frac{1}{1000}\begin{vmatrix} 6-10\lambda & 1 & 1 \\ 1 & 8-10\lambda & 2 \\ 3 & 1 & 7-10\lambda \end{vmatrix} \\
&= \tfrac{1}{1000}\Big[(6-10\lambda)\Big((8-10\lambda)(7-10\lambda)-2\Big) \\
&\qquad -1\Big((7-10\lambda)-1\Big)+3\Big(2-1(8-10\lambda)\Big)\Big] \\
&= \cdots \\
&= \tfrac{1}{1000}\Big[-1000\lambda^3+2100\lambda^2-1400\lambda+300\Big] \\
&= -\lambda^3+2.1\lambda^2-1.4\lambda+0.3.
\end{aligned}$$

Rental car returns, part VII

- Next we solve $-\lambda^3 + 2.1\lambda^2 - 1.4\lambda + 0.3 = 0$.
- We seek a trivial solution; again $\lambda = 1$ works!
- Now we can write

$$-\lambda^3 + 2.1\lambda^2 - 1.4\lambda + 0.3 \;=\; (\lambda - 1)(-\lambda^2 + 1.1\lambda - 0.3)$$

- We can apply the "abc" formula to the second part:

$$\begin{aligned} \frac{-1.1 \pm \sqrt{(1.1)^2 - 4 \cdot 0.3}}{-2} &= \frac{-1.1 \pm \sqrt{1.21 - 1.2}}{-2} \\ &= \frac{-1.1 \pm \sqrt{0.01}}{-2} \\ &= \frac{-1.1 \pm 0.1}{-2} \end{aligned}$$

- This yields additional eigenvalues: $\lambda = 0.5$ and $\lambda = 0.6$.

Rental car returns, part VIII

$\lambda = 1$ has eigenvector $(4, 9, 7)$; indeed:

$$\mathbf{A} \cdot \begin{pmatrix} 4 \\ 9 \\ 7 \end{pmatrix} = \frac{1}{10} \begin{pmatrix} 6 & 1 & 1 \\ 1 & 8 & 2 \\ 3 & 1 & 7 \end{pmatrix} \cdot \begin{pmatrix} 4 \\ 9 \\ 7 \end{pmatrix} = \frac{1}{10} \begin{pmatrix} 24+9+7 \\ 4+72+14 \\ 12+9+49 \end{pmatrix} = 1 \begin{pmatrix} 4 \\ 9 \\ 7 \end{pmatrix}$$

$\lambda = 0.6$ has eigenvector $(0, -1, 1)$:

$$\mathbf{A} \cdot \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix} = \frac{1}{10} \begin{pmatrix} 6 & 1 & 1 \\ 1 & 8 & 2 \\ 3 & 1 & 7 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix} = \frac{1}{10} \begin{pmatrix} -1+1 \\ -8+2 \\ -1+7 \end{pmatrix} = 0.6 \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}$$

$\lambda = 0.5$ has eigenvector $(-1, -1, 2)$:

$$\mathbf{A} \cdot \begin{pmatrix} -1 \\ -1 \\ 2 \end{pmatrix} = \frac{1}{10} \begin{pmatrix} 6 & 1 & 1 \\ 1 & 8 & 2 \\ 3 & 1 & 7 \end{pmatrix} \cdot \begin{pmatrix} -1 \\ -1 \\ 2 \end{pmatrix} = \frac{1}{10} \begin{pmatrix} -6-1+2 \\ -1-8+4 \\ -3-1+14 \end{pmatrix} = 0.5 \begin{pmatrix} -1 \\ -1 \\ 2 \end{pmatrix}$$

Rental car returns, part IX

- Now: eigenvector base $\mathcal{C} = \{(4,9,7),(0,-1,1),(-1,-1,2)\}$ and standard base as $\mathcal{S} = \{(1,0,0),(0,1,0),(0,0,1)\}$.
- Then we can do change-of-coordinates back-and-forth:

$$\mathbf{T}_{\mathcal{C}\Rightarrow\mathcal{S}} = \begin{pmatrix} 4 & 0 & -1 \\ 9 & -1 & -1 \\ 7 & 1 & 2 \end{pmatrix} \qquad \mathbf{T}_{\mathcal{S}\Rightarrow\mathcal{C}} = \frac{1}{20}\begin{pmatrix} 1 & 1 & 1 \\ 25 & -15 & 5 \\ -16 & 4 & 4 \end{pmatrix}$$

- These translation matrices yield a diagonalisation:

$$\mathbf{A} = \mathbf{T}_{\mathcal{C}\Rightarrow\mathcal{S}} \cdot \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0.6 & 0 \\ 0 & 0 & 0.5 \end{pmatrix} \cdot \mathbf{T}_{\mathcal{S}\Rightarrow\mathcal{C}}$$

Rental car returns, part X

- Thus:

$$\lim_{n\to\infty} \mathbf{A}^n = \lim_{n\to\infty} \mathbf{T}_{\mathcal{C}\Rightarrow\mathcal{S}} \cdot \begin{pmatrix} 1^n & 0 & 0 \\ 0 & (0.6)^n & 0 \\ 0 & 0 & (0.5)^n \end{pmatrix} \cdot \mathbf{T}_{\mathcal{S}\Rightarrow\mathcal{C}}$$

$$= \mathbf{T}_{\mathcal{C}\Rightarrow\mathcal{S}} \cdot \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \cdot \mathbf{T}_{\mathcal{S}\Rightarrow\mathcal{C}} = \frac{1}{20} \begin{pmatrix} 4 & 4 & 4 \\ 9 & 9 & 9 \\ 7 & 7 & 7 \end{pmatrix}$$

- Finally, the equilibrium starting from $P = Q = R = 200$ is:

$$\frac{1}{20} \begin{pmatrix} 4 & 4 & 4 \\ 9 & 9 & 9 \\ 7 & 7 & 7 \end{pmatrix} \cdot \begin{pmatrix} 200 \\ 200 \\ 200 \end{pmatrix} = \begin{pmatrix} 120 \\ 270 \\ 210 \end{pmatrix}.$$