Matrix Calculations: Linear Equations

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Outline

First, some admin...

Lectures

- Weekly: Tuesdays 10:45-12:30
- Presence not compulsory...
  - But if you are going to come, actually be here! (This means laptops shut, phones away.)
- The course material consists of:
  - these slides, available via the web
  - *Linear Algebra* lecture notes by Bernd Souvignier ('LNBS')
- Course URL:

  www.cs.ru.nl/A.Kissinger/teaching/matrixrekenen2017b/

  (Link exists in blackboard, under 'course information').
- Generally, things appear on course website (and not on blackboard!). Check there before you ask a question.

First, some admin...

Assignments

- You can work together, but exercises must be handed in individually
- Handing in is not compulsory (except for 3rd-chancers), **but**:
  - It's a tough exam. If you don't do the excercises, you are unlikely to pass.
  - Exercises give up to 1 point (out of 10) bonus on exam.
  - This could be the difference between a 5 and a 6 (...or a 9 and a 10 ☺)

First, some admin...

Werkcollege's

- Werkcollege on Friday, 8:45.
  - Presence not compulsory (except for 3rd-chancers)
  - Answers (for old assignments) & Questions (for new ones)
- Schedule:
  - New assignments on the web by Tuesday evening
  - Next exercise meeting (Friday) you can ask questions
  - Hand-in: **Monday before 4pm**, handwritten or typed, on paper in the delivery boxes, ground floor Mercator 1.
  - You should **NOT** hand in via Blackboard, but it's a good idea to make photos of your work before handing in paper copies.
- There is a separate Exercises web-page (see URL on course webpage).

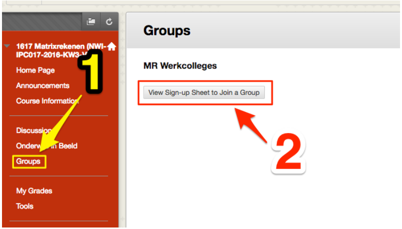
First, some admin...

Werkcollege's

- There will be a werkcollege every Friday (including this one!)
- 4 Groups:
  - **Group A:** John van de Wetering. HG00.114
  - **Group B:** Justin Reniers. HG01.058
  - **Group C:** Justin Hende. HG02.028
  - **Group D:** Bart Gruppen. HG03.632
- Each assistant has a delivery box on the ground floor of the **Mercator 1 building**

First, some admin...

- Register for a class on Blackboard. Click 'Groups' in the sidebar, then the 'View Sign-up Sheet' button:



- Registration **must** be done by tomorrow (Wednesday) at 12:00. (Do it today, if possible.)
- I may shift some people to other groups. This will be finalised by **Thursday**, so check your group assignment then.

First, some admin...

Examination

- Final mark is computed from:
  - Average of markings of assignments: A
  - Written exam (November 6): E
  - Final mark: $F = E + \frac{A}{10}$.
- Second chance for written exam on January 3.
  - you keep the outcome (average) of the assignments.
- If you fail again, you must start all over next year (including re-doing new exercises, and additional requirements)

First, some admin...

If you fail more than twice ...

- Additional requirements will be imposed
- You will have to talk to the study advisor
  - if you have not done so yet, make an appointment
  - **compulsory:** presence at all lectures, werkcollege's, handing in of all exercises
  - **sign in today** during the break (and in future lectures)
  - you exercise mark must be ≥ 5 to take the exam.

Next, some advice...

How to pass this course

- Learn by doing, not just staring at the slides (or video, or lecturer)
- Pro tip: exam questions will look a lot like the exercises
- Give this course the time it needs!
- 3ec means $3 \times 28 = 84$ hours in total
  - Let's say 20 hours for exam
  - 64 hours for 8 weeks means: 8 hours per week!
  - 4 hours in lecture and werkcollege leaves...
  - ...another 4 hours for studying & doing exercises
- Coming up-to-speed is your own responsibility
  - if you feel like you are missing some background knowledge: use Wim Gielen's notes...or wikipedia

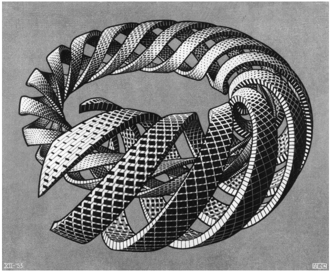
Finally, on to the good stuff...

Q: What is ~~matrix calculation~~ all about?
linear algebra

A: It depends on who you ask...

What is linear algebra all about?

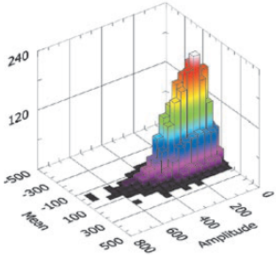
To a mathematician: linear algebra is the mathematics of **geometry** and **transformation**...



It asks: How can we represent a problem in 2D, 3D, 4D (or infinite-dimensional!) space, and **transform** it into a solution?

What is linear algebra all about?

To an engineer: linear algebra is about **numerics**...

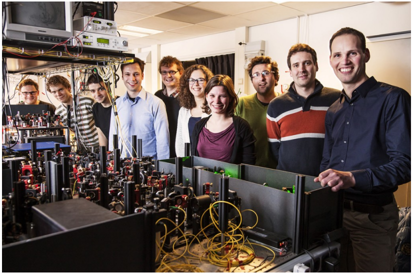


⇒

It asks: Can we encode a complicated question (e.g. 'Will my bridge fall down?') as a big matrix and compute the answer?

What is linear algebra all about?

To an quantum physicist (or quantum computer scientist!): linear algebra is just the way **nature behaves**...



It asks: How can we explain things that can be in **many states** at the same time, or **entangled** to distant things?

A simple example...

Let's start with something everybody knows how to do:

- Suppose I went to the pub last night, but I can't remember how many, umm...'sodas' I had.
- I remember taking out 20 EUR from the cash machine.
- Sodas cost 3 EUR.
- I discover a half-eaten kapsalon in my kitchen. That's 5 EUR.
- I have no money left. (Typical...)

By now, most people have (hopefully) figured out I had...5 sodas. That's because you can solve simple **linear equations**:

$$3x + 5 = 20 \qquad \Longrightarrow \qquad x = 5$$

An (only slightly less) simple example

I have two numbers in mind, but I don't tell you which ones

- if I add them up, the result is 12
- if I subtract, the result is 4

Which two numbers do I have in mind?

Now we have a **system** of linear equations, in two variables:

$$\begin{cases} x + y = 12 \\ x - y = 4 \end{cases} \qquad \text{with solution} \qquad x = 8, \; y = 4.$$

An (only slightly less) simple example

Let's try to find a solution, in general, for:

$$\begin{aligned} x + y &= a \\ x - y &= b \end{aligned}$$

i.e. find the values of x and y in terms of a and b.

- adding the two equations yields:

$$a + b = (x + y) + (x - y) = 2x, \qquad \text{so} \qquad \boxed{x = \frac{a+b}{2}}$$

- subtracting the two equations yields:

$$a - b = (x + y) - (x - y) = 2y, \qquad \text{so} \qquad \boxed{y = \frac{a-b}{2}}$$

Example (from the previous slide)

$a = 12, b = 4$, so $x = \frac{12+4}{2} = \frac{16}{2} = 8$ and $y = \frac{12-4}{2} = \frac{8}{2} = 4$. Yes!

A more difficult example

I have two numbers in mind, but I don't tell you which ones!

- if I add them up, the result is 12
- if I *multiply*, the result is 35

Which two number do I have in mind?

It is easy to check that $x = 5, y = 7$ is a solution.

The system of equations however, is non-linear:

$$\begin{aligned} x + y &= 12 \\ x \cdot y &= 35 \end{aligned}$$

This is already too difficult for this course. (If you don't believe me, try $x^5 + x = -1$ ...on second thought, maybe wait till later.)

We only do linear equations.

Basic definitions

Definition (linear equation and solution)

A linear equation in n variables $x_1, \cdots, x_n$ is an expression of the form:

$$a_1x_1 + \cdots + a_nx_n = b,$$

where $a_1, \ldots, a_n, b$ are given numbers (possibly zero).

A solution for such an equation is given by n numbers $s_1, \ldots, s_n$ such that $a_1s_1 + \cdots + a_ns_n = b$.

Example

The linear equation $3x_1 + 4x_2 = 11$ has many solutions, eg. $x_1 = 1, x_2 = 2$, or $x_1 = -3, x_2 = 5$.

More basic definitions

Definition

A $(m \times n)$ system of linear equations consists of m equations with n variables, written as:

$$\begin{aligned} a_{11}x_1 + \cdots + a_{1n}x_n &= b_1 \\ &\vdots \\ a_{m1}x_1 + \cdots + a_{mn}x_n &= b_m \end{aligned}$$

A solution for such a system consists of n numbers $s_1, \ldots, s_n$ forming a solution for **each** of the equations.

Example solution

Example

Consider the system of equations

$$\begin{aligned} x_1 + x_2 + 2x_3 &= 9 \\ 2x_1 + 4x_2 - 3x_3 &= 1 \\ 3x_1 + x_2 + x_3 &= 8. \end{aligned}$$

- How to find solutions, if any?
- Finding solutions requires some work.
- But checking solutions is easy, and you should always do so, just to be sure.
- Solution: $x_1 = 1, x_2 = 2, x_3 = 3$. ✓

Easy and hard

- General systems of equations are hard to solve. But what kinds of systems are easy?
- How about this one?

$$\begin{aligned} x_1 &= 7 \\ x_2 &= -2 \\ x_3 &= 2 \end{aligned}$$

- ...this one's not too shabby either:

$$\begin{aligned} x_1 + 2x_2 - x_3 &= 1 \\ x_2 + 2x_3 &= 2 \\ x_3 &= 2 \end{aligned}$$

Transformation

So, why don't we take something hard, and **transform** it into something easy?

$$\begin{cases} 2x_2 + x_3 = -2 \\ 3x_1 + 5x_2 - 5x_3 = 1 \\ 2x_1 + 4x_2 - 2x_3 = 2 \end{cases} \Rightarrow \begin{cases} x_1 + 2x_2 - x_3 = 1 \\ x_2 + 2x_3 = 2 \\ x_3 = 2 \end{cases} \Rightarrow \begin{cases} x_1 = 7 \\ x_2 = -2 \\ x_3 = 2 \end{cases}$$

Sound like something **linear algebra** might be good for?

Gaussian elimination

Gaussian elimination is the 'engine room' of all computer algebra.

It was named after this guy:

Carl Friedrich Gauss (1777-1855)

(famous for inventing: like half of mathematics)

Gaussian elimination

Gaussian elimination is the 'engine room' of all computer algebra.
...but it was probably actually invented by this guy:

Liu Hui (ca. 3rd century AD)

Variable names are inessential

The following programs are equivalent:

```
for(int i=0; i<10; i++){
    P(i);
}
```

```
for(int j=0; j<10; j++){
    P(j);
}
```

Similarly, the following systems of equations are equivalent:

$$\begin{aligned} 2x + 3y + z &= 4 \\ x + 2y + 2z &= 5 \\ 3x + y + 5z &= -1 \end{aligned} \qquad \begin{aligned} 2u + 3v + w &= 4 \\ u + 2v + 2w &= 5 \\ 3u + v + 5w &= -1 \end{aligned}$$

Matrices

The essence of the system

$$\begin{aligned} 2x + 3y + z &= 4 \\ x + 2y + 2z &= 5 \\ 3x + y + 5z &= -1 \end{aligned}$$

is not given by the variables, but by the numbers, written as:

coefficient matrix

$$\begin{pmatrix} 2 & 3 & 1 \\ 1 & 2 & 2 \\ 3 & 1 & 5 \end{pmatrix}$$

augmented matrix

$$\left(\begin{array}{ccc|c} 2 & 3 & 1 & 4 \\ 1 & 2 & 2 & 5 \\ 3 & 1 & 5 & -1 \end{array}\right)$$

Easy and hard matrices

So, the question becomes, how to we turn a *hard* matrix:

$$\left(\begin{array}{ccc|c} 0 & 2 & 1 & -2 \\ 3 & 5 & -5 & 1 \\ 2 & 4 & -2 & 2 \end{array}\right) \leftrightarrow \begin{cases} 2x_2 + x_3 = -2 \\ 3x_1 + 5x_2 - 5x_3 = 1 \\ 2x_1 + 4x_2 - 2x_3 = 2 \end{cases}$$

...into an easy one:

$$\left(\begin{array}{ccc|c} 1 & 2 & -1 & 1 \\ 0 & 1 & 2 & 2 \\ 0 & 0 & 1 & 2 \end{array}\right) \leftrightarrow \begin{cases} x_1 + 2x_2 - x_3 = 1 \\ x_2 + 2x_3 = 2 \\ x_3 = 2 \end{cases}$$

...or an *even easier* one:

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & 7 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 2 \end{array}\right) \leftrightarrow \begin{cases} x_1 = 7 \\ x_2 = -2 \\ x_3 = 2 \end{cases}$$

Solving equations by row operations

- Operations on **equations** become operations on **rows**, e.g.

$$\left(\begin{array}{cc|c} 1 & 1 & -2 \\ 3 & -1 & 2 \end{array}\right) \leftrightarrow \begin{cases} x_1 + x_2 = -2 \\ 3x_1 - x_2 = 2 \end{cases}$$

- Multiply row 1 by 3, giving:

$$\left(\begin{array}{cc|c} 3 & 3 & -6 \\ 3 & -1 & 2 \end{array}\right) \leftrightarrow \begin{cases} 3x_1 + 3x_2 = -6 \\ 3x_1 - x_2 = 2 \end{cases}$$

- Subtract the first row from the second, giving:

$$\left(\begin{array}{cc|c} 3 & 3 & -6 \\ 0 & -4 & 8 \end{array}\right) \leftrightarrow \begin{cases} 3x_1 + 3x_2 = -6 \\ -4x_2 = 8 \end{cases}$$

- So $x_2 = \frac{8}{-4} = -2$. The first equation becomes: $3x_1 - 6 = -6$, so $x_1 = 0$. Always check your answer. ✓

Relevant operations & notation

	on equations	**on matrices**	**LNBS**
exchange of rows	$E_i \leftrightarrow E_j$	$R_i \leftrightarrow R_j$	$W_{i,j}$
multiplication with $c \neq 0$	$E_i := cE_i$	$R_i := cR_i$	$V_i(c)$
addition with $c \neq 0$	$E_i := E_i + cE_j$	$R_i := R_i + cR_j$	$O_{i,j}(c)$

These operations on equations/matrices:

- help to find solutions
- but do not change solutions (introduce/delete them)
- **The goal:** put matrices in **Echelon form**

Pivots

- Echelon form = all the **pivots** are in a convenient place
- A **pivot** is the first non-zero entry of a row:

$$\left(\begin{array}{ccc|c} 0 & \boxed{2} & 1 & -2 \\ \boxed{3} & 5 & -5 & 1 \\ 0 & 0 & \boxed{\text{-}2} & 2 \end{array}\right)$$

- If a row is all zeros, it **has no pivot**:

$$\left(\begin{array}{ccc|c} 0 & \boxed{2} & 1 & -2 \\ \boxed{3} & 5 & -5 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

We call this a *zero row*.

Echelon form

A matrix is in **Echelon form** (a.k.a. *rijtrapvorm*) if:

1. All of the rows with pivots occur before zero rows, and
2. Pivots always occur to the right of previous pivots

$$\left(\begin{array}{cccc|c} \boxed{3} & 2 & 5 & -5 & 1 \\ 0 & 0 & \boxed{2} & 1 & -2 \\ 0 & 0 & 0 & \boxed{\text{-}2} & 2 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right) \checkmark$$

$$\left(\begin{array}{cccc|c} \boxed{3} & 2 & 5 & -5 & 1 \\ 0 & 0 & \boxed{2} & 1 & -2 \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ 0 & 0 & 0 & \boxed{\text{-}2} & 2 \end{array}\right) ☠ \qquad \left(\begin{array}{cccc|c} \boxed{3} & 2 & 5 & -5 & 1 \\ 0 & 0 & \boxed{\mathbf{4}} & -2 & 2 \\ 0 & \boxed{\mathbf{2}} & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right) ☠ \qquad \left(\begin{array}{cccc|c} \boxed{3} & 2 & 5 & -5 & 1 \\ 0 & 0 & \boxed{\mathbf{4}} & -2 & 2 \\ 0 & 0 & \boxed{\mathbf{2}} & 1 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right) ☠$$

Even better: reduced Echelon form

A matrix in **reduced Echelon form** if it is in Echelon form, and each row contains *at most* one '1' to the left of the line.

$$\left(\begin{array}{ccc|c} \boxed{1} & 0 & 0 & 7 \\ 0 & \boxed{1} & 0 & -2 \\ 0 & 0 & \boxed{1} & 2 \end{array}\right) \checkmark$$

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & 7 \\ 0 & 2 & 0 & -2 \\ 0 & 0 & 1 & 2 \end{array}\right) ☠ \qquad \left(\begin{array}{ccc|c} 1 & 0 & 0 & 7 \\ 0 & 1 & 1 & -2 \\ 0 & 0 & 1 & 2 \end{array}\right) ☠ \qquad \left(\begin{array}{ccc|c} 0 & 1 & 0 & -2 \\ 1 & 0 & 0 & 7 \\ 0 & 0 & 1 & 2 \end{array}\right) ☠$$

Reduced Echelon form lets us read off the solutions directly from the matrix. The big matrix above gives:

$$x_1 = 7 \qquad x_2 = -2 \qquad x_3 = 2$$

Transformations example, part I

equations	**matrix**
$\begin{aligned} 2x_2 + x_3 &= -2 \\ 3x_1 + 5x_2 - 5x_3 &= 1 \\ 2x_1 + 4x_2 - 2x_3 &= 2 \end{aligned}$	$\left(\begin{array}{ccc\|c} 0 & 2 & 1 & -2 \\ 3 & 5 & -5 & 1 \\ 2 & 4 & -2 & 2 \end{array}\right)$
$E_1 \leftrightarrow E_3$	$R_1 \leftrightarrow R_3$
$\begin{aligned} 2x_1 + 4x_2 - 2x_3 &= 2 \\ 3x_1 + 5x_2 - 5x_3 &= 1 \\ 2x_2 + x_3 &= -2 \end{aligned}$	$\left(\begin{array}{ccc\|c} 2 & 4 & -2 & 2 \\ 3 & 5 & -5 & 1 \\ 0 & 2 & 1 & -2 \end{array}\right)$
$E_1 := \frac{1}{2} E_1$	$R_1 := \frac{1}{2} R_1$
$\begin{aligned} x_1 + 2x_2 - 1x_3 &= 1 \\ 3x_1 + 5x_2 - 5x_3 &= 1 \\ 2x_2 + x_3 &= -2 \end{aligned}$	$\left(\begin{array}{ccc\|c} 1 & 2 & -1 & 1 \\ 3 & 5 & -5 & 1 \\ 0 & 2 & 1 & -2 \end{array}\right)$

Transformations example, part II

equations	**matrix**
$\begin{aligned} x_1 + 2x_2 - 1x_3 &= 1 \\ 3x_1 + 5x_2 - 5x_3 &= 1 \\ 2x_2 + x_3 &= -2 \end{aligned}$	$\left(\begin{array}{ccc\|c} 1 & 2 & -1 & 1 \\ 3 & 5 & -5 & 1 \\ 0 & 2 & 1 & -2 \end{array}\right)$
$E_2 := E_2 - 3E_1$	$R_2 := R_2 - 3R_1$
$\begin{aligned} x_1 + 2x_2 - 1x_3 &= 1 \\ -x_2 - 2x_3 &= -2 \\ 2x_2 + x_3 &= -2 \end{aligned}$	$\left(\begin{array}{ccc\|c} 1 & 2 & -1 & 1 \\ 0 & -1 & -2 & -2 \\ 0 & 2 & 1 & -2 \end{array}\right)$
$E_2 := -E_2$	$R_2 := -R_2$
$\begin{aligned} x_1 + 2x_2 - 1x_3 &= 1 \\ x_2 + 2x_3 &= 2 \\ 2x_2 + x_3 &= -2 \end{aligned}$	$\left(\begin{array}{ccc\|c} 1 & 2 & -1 & 1 \\ 0 & 1 & 2 & 2 \\ 0 & 2 & 1 & -2 \end{array}\right)$

Transformations example, part III

equations	**matrix**
$\begin{aligned} x_1 + 2x_2 - 1x_3 &= 1 \\ x_2 + 2x_3 &= 2 \\ 2x_2 + x_3 &= -2 \end{aligned}$	$\left(\begin{array}{ccc\|c} 1 & 2 & -1 & 1 \\ 0 & 1 & 2 & 2 \\ 0 & 2 & 1 & -2 \end{array}\right)$
$E_3 := E_3 - 2E_2$	$R_3 := R_3 - 2R_2$
$\begin{aligned} x_1 + 2x_2 - 1x_3 &= 1 \\ x_2 + 2x_3 &= 2 \\ -3x_3 &= -6 \end{aligned}$	$\left(\begin{array}{ccc\|c} 1 & 2 & -1 & 1 \\ 0 & 1 & 2 & 2 \\ 0 & 0 & -3 & -6 \end{array}\right)$ Echelon (*rijtrap*) form
$E_3 := -\frac{1}{3}E_3$	$R_3 := -\frac{1}{3}R_3$
$\begin{aligned} x_1 + 2x_2 - 1x_3 &= 1 \\ x_2 + 2x_3 &= 2 \\ x_3 &= 2 \end{aligned}$	$\left(\begin{array}{ccc\|c} 1 & 2 & -1 & 1 \\ 0 & 1 & 2 & 2 \\ 0 & 0 & 1 & 2 \end{array}\right)$

Transformations example, part IV

equations	matrix
$\begin{array}{rcl} x_1 + 2x_2 - 1x_3 & = & 1 \\ x_2 + 2x_3 & = & 2 \\ x_3 & = & 2 \end{array}$	$\left(\begin{array}{ccc\|c} 1 & 2 & -1 & 1 \\ 0 & 1 & 2 & 2 \\ 0 & 0 & 1 & 2 \end{array}\right)$ Echelon form
$E_1 := E_1 - 2E_2$	$R_1 := R_1 - 2R_2$
$\begin{array}{rcl} x_1 - 5x_3 & = & -3 \\ x_2 + 2x_3 & = & 2 \\ x_3 & = & 2 \end{array}$	$\left(\begin{array}{ccc\|c} 1 & 0 & -5 & -3 \\ 0 & 1 & 2 & 2 \\ 0 & 0 & 1 & 2 \end{array}\right)$
$E_2 := E_2 - 2E_3$	$R_2 := R_2 - 2R_3$
$\begin{array}{rcl} x_1 - 5x_3 & = & -3 \\ x_2 & = & -2 \\ x_3 & = & 2 \end{array}$	$\left(\begin{array}{ccc\|c} 1 & 0 & -5 & -3 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 2 \end{array}\right)$

Transformations example, part V

equations	**matrix**
$\begin{aligned} x_1 - 5x_3 &= -3 \\ x_2 &= -2 \\ x_3 &= 2 \end{aligned}$	$\left(\begin{array}{ccc\|c} 1 & 0 & -5 & -3 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 2 \end{array}\right)$
$E_1 := E_1 + 5E_3$	$R_1 := R_1 + 5R_3$
$\begin{aligned} x_1 &= 7 \\ x_2 &= -2 \\ x_3 &= 2 \end{aligned}$	$\left(\begin{array}{ccc\|c} 1 & 0 & 0 & 7 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 2 \end{array}\right)$

reduced echelon form

Gauss elimination

- Solutions can be found by mechanically applying simple rules
  - in Dutch this is called *vegen*
  - first produce echelon form (rijtrapvorm), then either (a) finish by substitution, or (b) obtain single-variable equations, reduced echelon form (gereduceerde rijtrapvorm)
  - it is one of the most important algorithms in virtually any computer algebra system
- Applying these operations is actually easier on matrices, than on the equations themselves
- You should be able to do Gauss elimination in your sleep! It is a basic technique used throughout the course.