Matrix Calculations: Basis Transformation, Determinants, and Eigenvalues

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Version: autumn 2018

Outline

Last time

- Any linear map can be represented as a matrix:

$$f(\boldsymbol{v}) = \boldsymbol{A} \cdot \boldsymbol{v} \qquad\qquad g(\boldsymbol{v}) = \boldsymbol{B} \cdot \boldsymbol{v}$$

- Last time, we saw that composing linear maps could be done by multiplying their matrices:

$$f(g(\boldsymbol{v})) = \boldsymbol{A} \cdot \boldsymbol{B} \cdot \boldsymbol{v}$$

- Matrix multiplication is pretty easy:

$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \cdot \begin{pmatrix} 1 & -1 \\ 0 & 4 \end{pmatrix} = \begin{pmatrix} 1 \cdot 1 + 2 \cdot 0 & 1 \cdot (-1) + 2 \cdot 4 \\ 3 \cdot 1 + 4 \cdot 0 & 3 \cdot (-1) + 4 \cdot 4 \end{pmatrix} = \begin{pmatrix} 1 & 7 \\ 3 & 13 \end{pmatrix}$$

...so if we can solve other stuff by matrix multiplication, we are pretty happy.

Last time

- For example, we can solve systems of linear equations:

$$\boldsymbol{A} \cdot \boldsymbol{x} = \boldsymbol{b}$$

...by finding the inverse of a matrix:

$$\boldsymbol{x} = \boldsymbol{A}^{-1} \cdot \boldsymbol{b}$$

- There is an easy shortcut formula for 2×2 matrices:

$$\boldsymbol{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \qquad \Longrightarrow \qquad \boldsymbol{A}^{-1} = \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

...as long as $ad - bc \neq 0$.

- We'll see today that "$ad - bc$" is an example of a special number we can compute for any square matrix (not just 2×2) called the determinant.

Vectors in a basis

Recall: a basis for a vector space V is a set of vectors $\mathcal{B} = \{\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n\}$ in V such that:

1. They **uniquely** *span* V, i.e. for all $\boldsymbol{v} \in V$, there exist **unique** a_i such that:

$$\boldsymbol{v} = a_1\boldsymbol{v}_1 + \ldots + a_n\boldsymbol{v}_n$$

Because of this, we use a special notation for this linear combination:

$$\begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix}_{\mathcal{B}} := a_1\boldsymbol{v}_1 + \ldots + a_n\boldsymbol{v}_n$$

Same vector, different outfits

The *same vector* can look different, depending on the choice of basis. Consider the standard basis: $\mathcal{S} = \{(1,0),(0,1)\}$ vs. another basis:

$$\mathcal{B} = \left\{ \begin{pmatrix} 100 \\ 0 \end{pmatrix}, \begin{pmatrix} 100 \\ 1 \end{pmatrix} \right\}$$

Is this a basis? Yes...

- It's independent because: $\begin{pmatrix} 100 & 100 \\ 0 & 1 \end{pmatrix}$ has 2 pivots.
- It's spanning because... we can make every vector in $\mathcal{S}$ using linear combinations of vectors in $\mathcal{B}$:

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{1}{100} \begin{pmatrix} 100 \\ 0 \end{pmatrix} \qquad \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 100 \\ 1 \end{pmatrix} - \begin{pmatrix} 100 \\ 0 \end{pmatrix}$$

...so we can also make any vector in $\mathbb{R}^2$.

Same vector, different outfits

$$\mathcal{S} = \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\} \qquad \mathcal{B} = \left\{ \begin{pmatrix} 100 \\ 0 \end{pmatrix}, \begin{pmatrix} 100 \\ 1 \end{pmatrix} \right\}$$

Examples:

$$\begin{pmatrix} 100 \\ 0 \end{pmatrix}_{\mathcal{S}} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}_{\mathcal{B}} \qquad \begin{pmatrix} 300 \\ 1 \end{pmatrix}_{\mathcal{S}} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}_{\mathcal{B}}$$

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix}_{\mathcal{S}} = \begin{pmatrix} \frac{1}{100} \\ 0 \end{pmatrix}_{\mathcal{B}} \qquad \begin{pmatrix} 0 \\ 1 \end{pmatrix}_{\mathcal{S}} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}_{\mathcal{B}}$$

Why???

- Many find the idea of *multiple bases* confusing at first...
- $\mathcal{S} = \{(1,0),(0,1)\}$ is a perfectly good basis for $\mathbb{R}^2$. Why bother with others?
  1. Some vector spaces don't have one "obvious" choice of basis. Example: subspaces $S \subseteq \mathbb{R}^n$.
  2. Sometimes it is way more efficient to write a vector with respect to a different basis, e.g.:

$$\begin{pmatrix} 93718234 \\ -438203 \\ 110224 \\ -5423204980 \\ \vdots \end{pmatrix}_{\mathcal{S}} = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \\ \vdots \end{pmatrix}_{\mathcal{B}}$$

  3. The choice of basis for *vectors* affects how we write *matrices* as well. Often this can be done cleverly. Example: JPEGs, MP3s, search engine rankings, ...

Transforming bases, part I

- **Problem:** given a vector written in $\mathcal{B} = \{(100, 0), (100, 1)\}$, how can we write it in the standard basis? Just use the definition:

$$\begin{pmatrix} x \\ y \end{pmatrix}_{\mathcal{B}} = x \cdot \begin{pmatrix} 100 \\ 0 \end{pmatrix} + y \cdot \begin{pmatrix} 100 \\ 1 \end{pmatrix} = \begin{pmatrix} 100x + 100y \\ y \end{pmatrix}_{\mathcal{S}}$$

- Or, as matrix multiplication:

$$\underbrace{\begin{pmatrix} 100 & 100 \\ 0 & 1 \end{pmatrix}}_{\boldsymbol{T}_{\mathcal{B}\Rightarrow\mathcal{S}}} \cdot \underbrace{\begin{pmatrix} x \\ y \end{pmatrix}}_{\text{in basis } \mathcal{B}} = \underbrace{\begin{pmatrix} 100x + 100y \\ y \end{pmatrix}}_{\text{in basis } \mathcal{S}}$$

- Let $\boldsymbol{T}_{\mathcal{B}\Rightarrow\mathcal{S}}$ be the matrix whose *columns* are the basis vectors $\mathcal{B}$. Then $\boldsymbol{T}_{\mathcal{B}\Rightarrow\mathcal{S}}$ *transforms* a vector written in $\mathcal{B}$ into a vector written in $\mathcal{S}$.

Transforming bases, part II

- How do we transform back? Need $\boldsymbol{T}_{\mathcal{S}\Rightarrow\mathcal{B}}$ which undoes the matrix $\boldsymbol{T}_{\mathcal{B}\Rightarrow\mathcal{S}}$.
- Solution: use the inverse! $\boldsymbol{T}_{\mathcal{S}\Rightarrow\mathcal{B}} := (\boldsymbol{T}_{\mathcal{B}\Rightarrow\mathcal{S}})^{-1}$
- Example:

$$(\boldsymbol{T}_{\mathcal{B}\Rightarrow\mathcal{S}})^{-1} = \begin{pmatrix} 100 & 100 \\ 0 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} \frac{1}{100} & -1 \\ 0 & 1 \end{pmatrix}$$

- ...which indeed gives:

$$\begin{pmatrix} \frac{1}{100} & -1 \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} \frac{a-100b}{100} \\ b \end{pmatrix}$$

Matrices in other bases

- Since *vectors* can be written with respect to different bases, so too can *matrices*.
- For example, let g be the linear map defined by:

$$g\left(\begin{pmatrix} 1 \\ 0 \end{pmatrix}_{\mathcal{S}} \right) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}_{\mathcal{S}} \qquad\qquad g\left(\begin{pmatrix} 0 \\ 1 \end{pmatrix}_{\mathcal{S}} \right) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}_{\mathcal{S}}$$

- Then, naturally, we would represent g using the matrix:

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}_{\mathcal{S}}$$

- Because indeed:

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

(the columns say where each of the vectors in $\mathcal{S}$ go, written in the basis $\mathcal{S}$)

On the other hand...

- Lets look at what g does to another basis:

$$\mathcal{B} = \{ \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \end{pmatrix} \}$$

- First $(1, 1) \in \mathcal{B}$:

$$g(\begin{pmatrix} 1 \\ 0 \end{pmatrix}_{\mathcal{B}}) = g(\begin{pmatrix} 1 \\ 1 \end{pmatrix}) = g(\begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix}) = \dots$$

- Then, by linearity:

$$\dots = g(\begin{pmatrix} 1 \\ 0 \end{pmatrix}) + g(\begin{pmatrix} 0 \\ 1 \end{pmatrix}) = \begin{pmatrix} 0 \\ 1 \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}_{\mathcal{B}}$$

On the other hand...

$$\mathcal{B} = \left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \end{pmatrix} \right\}$$

- Similarly $(1, -1) \in \mathcal{B}$:

$$g\left(\begin{pmatrix} 0 \\ 1 \end{pmatrix}_{\mathcal{B}} \right) = g\left(\begin{pmatrix} 1 \\ -1 \end{pmatrix} \right) = g\left(\begin{pmatrix} 1 \\ 0 \end{pmatrix} - \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right) = \dots$$

- Then, by linearity:

$$\dots = g\left(\begin{pmatrix} 1 \\ 0 \end{pmatrix} \right) - g\left(\begin{pmatrix} 0 \\ 1 \end{pmatrix} \right) = \begin{pmatrix} 0 \\ 1 \end{pmatrix} - \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \end{pmatrix}_{\mathcal{B}}$$

A new matrix

- From this:

$$g\left(\begin{pmatrix} 1 \\ 0 \end{pmatrix}_{\mathcal{B}} \right) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}_{\mathcal{B}} \qquad g\left(\begin{pmatrix} 0 \\ 1 \end{pmatrix}_{\mathcal{B}} \right) = \begin{pmatrix} 0 \\ -1 \end{pmatrix}_{\mathcal{B}}$$

- It follows that we should instead use *this* matrix to represent g:

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}_{\mathcal{B}}$$

- Because indeed:

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \end{pmatrix}$$

(the columns say where each of the vectors in $\mathcal{B}$ go, written in the basis $\mathcal{B}$)

A new matrix

- So on different bases, g acts in a totally different way!

$$g\left(\begin{pmatrix} 1 \\ 0 \end{pmatrix}_{\mathcal{S}} \right) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}_{\mathcal{S}} \qquad g\left(\begin{pmatrix} 0 \\ 1 \end{pmatrix}_{\mathcal{S}} \right) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}_{\mathcal{S}}$$

$$g\left(\begin{pmatrix} 1 \\ 0 \end{pmatrix}_{\mathcal{B}} \right) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}_{\mathcal{B}} \qquad g\left(\begin{pmatrix} 0 \\ 1 \end{pmatrix}_{\mathcal{B}} \right) = \begin{pmatrix} 0 \\ -1 \end{pmatrix}_{\mathcal{B}}$$

- ...and hence gets a totally different matrix:

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}_{\mathcal{S}} \quad \text{vs.} \quad \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}_{\mathcal{B}}$$

Transforming bases, part II

Theorem

Let $\mathcal{B}$ be a basis. If a linear map $f : \mathbb{R}^n \to \mathbb{R}^n$ has a matrix $\boldsymbol{A}$ computed using the standard basis $\mathcal{S}$, and a matrix $\mathbf{A}'$ computed using $\mathcal{B}$, then:

$$\boldsymbol{A} = \boldsymbol{T}_{\mathcal{B}\Rightarrow\mathcal{S}} \cdot \boldsymbol{A}' \cdot \boldsymbol{T}_{\mathcal{S}\Rightarrow\mathcal{B}}$$

n.b. the matrices $\boldsymbol{A}$ and $\boldsymbol{A}'$ are called *similar*, because they represent the *same* linear map in *different* ways.

For example

$$\mathcal{S} = \{\begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix}\} \qquad \mathcal{B} = \{\begin{pmatrix}1\\1\end{pmatrix}, \begin{pmatrix}1\\-1\end{pmatrix}\}$$

- Two bases give two different matrices for g:

$$\boldsymbol{A} = \begin{pmatrix}0 & 1\\1 & 0\end{pmatrix}_{\mathcal{S}} \qquad \boldsymbol{A}' = \begin{pmatrix}1 & 0\\0 & -1\end{pmatrix}_{\mathcal{B}}$$

- Then, the theorem says we have: $\boldsymbol{A} = \boldsymbol{T}_{\mathcal{B}\Rightarrow\mathcal{S}} \cdot \boldsymbol{A}' \cdot \boldsymbol{T}_{\mathcal{S}\Rightarrow\mathcal{B}}$
- Indeed, we can check:

$$\begin{pmatrix}0 & 1\\1 & 0\end{pmatrix} = \begin{pmatrix}1 & 1\\1 & -1\end{pmatrix} \cdot \begin{pmatrix}1 & 0\\0 & -1\end{pmatrix} \cdot \frac{1}{-2}\begin{pmatrix}-1 & -1\\-1 & 1\end{pmatrix}$$

There and back again

- To *translate* from a matrix written in $\mathcal{B}$ to a matrix written in $\mathcal{S}$, we do this:

$$\underbrace{\boldsymbol{A}}_{\text{in } \mathcal{S}} = \boldsymbol{T}_{\mathcal{B} \Rightarrow \mathcal{S}} \cdot \underbrace{\boldsymbol{A}'}_{\text{in } \mathcal{B}} \cdot \boldsymbol{T}_{\mathcal{S} \Rightarrow \mathcal{B}}$$

- Since $\boldsymbol{T}_{\mathcal{S} \Rightarrow \mathcal{B}} = (\boldsymbol{T}_{\mathcal{B} \Rightarrow \mathcal{S}})^{-1}$, we can go the other way like this:

$$\underbrace{\boldsymbol{A}'}_{\text{in } \mathcal{B}} = \boldsymbol{T}_{\mathcal{S} \Rightarrow \mathcal{B}} \cdot \underbrace{\boldsymbol{A}}_{\text{in } \mathcal{S}} \cdot \boldsymbol{T}_{\mathcal{B} \Rightarrow \mathcal{S}}$$

- How to remember? Look at the blue matrix.

Example (from $\mathcal{S}$ to $\mathcal{B}$)

- Consider the standard basis $\mathcal{S} = \{(1,0),(0,1)\}$ for $\mathbb{R}^2$, and as alternative basis $\mathcal{B} = \{(-1,1),(0,2)\}$
- Let the linear map $f : \mathbb{R}^2 \to \mathbb{R}^2$ be represented in the standard basis $\mathcal{S}$ by the following matrix:

$$\boldsymbol{A} = \begin{pmatrix} 1 & -1 \\ 2 & 3 \end{pmatrix}$$

- What is the representation $\boldsymbol{A}'$ of f in the basis $\mathcal{B}$?
- Since $\mathcal{S}$ is the standard basis, $\boldsymbol{T}_{\mathcal{B}\Rightarrow\mathcal{S}} = \begin{pmatrix} -1 & 0 \\ 1 & 2 \end{pmatrix}$ contains the $\mathcal{B}$-vectors as its columns

Example (from $\mathcal{S}$ to $\mathcal{B}$, cont'd)

- The basis transformation matrix $\boldsymbol{T}_{\mathcal{S}\Rightarrow\mathcal{B}}$ in the other direction is obtained as matrix inverse:

$$\boldsymbol{T}_{\mathcal{S}\Rightarrow\mathcal{B}} = \left(\boldsymbol{T}_{\mathcal{B}\Rightarrow\mathcal{S}}\right)^{-1} = \begin{pmatrix} -1 & 0 \\ 1 & 2 \end{pmatrix}^{-1} = \frac{1}{-2-0}\begin{pmatrix} 2 & 0 \\ -1 & -1 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} -2 & 0 \\ 1 & 1 \end{pmatrix}$$

- Hence:

$$\begin{aligned} \boldsymbol{A}' &= \boldsymbol{T}_{\mathcal{S}\Rightarrow\mathcal{B}} \cdot \boldsymbol{A} \cdot \boldsymbol{T}_{\mathcal{B}\Rightarrow\mathcal{S}} \\ &= \frac{1}{2}\begin{pmatrix} -2 & 0 \\ 1 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & -1 \\ 2 & 3 \end{pmatrix} \cdot \begin{pmatrix} -1 & 0 \\ 1 & 2 \end{pmatrix} \\ &= \frac{1}{2}\begin{pmatrix} -2 & 2 \\ 3 & 2 \end{pmatrix} \cdot \begin{pmatrix} -1 & 0 \\ 1 & 2 \end{pmatrix} \\ &= \frac{1}{2}\begin{pmatrix} 4 & 4 \\ -1 & 4 \end{pmatrix} \\ &= \begin{pmatrix} 2 & 2 \\ -\frac{1}{2} & 2 \end{pmatrix} \end{aligned}$$

The magic basis

- Recall: diagonal matrices correspond to rescaling each of the axes:

$$\boldsymbol{D} = \begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{pmatrix}$$

- These matrices are really easy to work with
- Most matrices (secretly) have their *own* basis, i.e. a basis where they are diagonal
- This is called the *eigenbasis* of the matrix

The magic basis

- Scaling each of the basis vectors means:

$$\boldsymbol{A} \cdot \boldsymbol{v} = \lambda \boldsymbol{v}$$

  for each $\boldsymbol{v} \in \mathcal{B}$.

- Rewritten, thats:

$$\boldsymbol{A} \cdot \boldsymbol{v} - \lambda \boldsymbol{v} = \boldsymbol{0} \quad \Rightarrow \quad \boldsymbol{A} \cdot \boldsymbol{v} - \lambda \boldsymbol{I} \cdot \boldsymbol{v} = \boldsymbol{0} \quad \Rightarrow \quad (\boldsymbol{A} - \lambda \boldsymbol{I}) \cdot \boldsymbol{v} = \boldsymbol{0}$$

- Need to find *non-zero* solutions to $(\boldsymbol{A} - \lambda \boldsymbol{I}) \cdot \boldsymbol{v} = \boldsymbol{0}$.
- That happens whenever $(\boldsymbol{A} - \lambda \boldsymbol{I})$ has $< n$ pivots.
- If only there was a way to find a λ where that happens....

Determinants

What a determinant does

For an $n \times n$ matrix $\boldsymbol{A}$, the determinant $\det(\boldsymbol{A})$ is a number (in $\mathbb{R}$)

It satisfies:

$$\begin{aligned} \det(\boldsymbol{A}) = 0 &\iff \boldsymbol{A} \text{ is not invertible} \\ &\iff \boldsymbol{A}^{-1} \text{ does not exist} \\ &\iff \boldsymbol{A} \text{ has } < n \text{ pivots in its echolon form} \end{aligned}$$

Determinants have useful properties, but calculating determinants involves some work.

Determinant of a 2×2 matrix

- Assume $\boldsymbol{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$
- Recall that the inverse $\boldsymbol{A}^{-1}$ exists if and only if $ad - bc \neq 0$, and in that case is:

$$\boldsymbol{A}^{-1} = \tfrac{1}{ad-bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

- In this 2×2-case we define:

$$\det \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

- Thus, indeed: $\det(\boldsymbol{A}) = 0 \iff \boldsymbol{A}^{-1}$ does not exist.

Determinant of a 2×2 matrix: example

- Example:

$$\boldsymbol{A} = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$$

- Then:

$$\det(\boldsymbol{A}) = 1 \cdot 4 - 2 \cdot 3 = 4 - 6 = -2$$

- $\det(\boldsymbol{A}) = -2 \neq 0 \quad \Longrightarrow \quad \boldsymbol{A}$ is invertible!
- Indeed, we can compute:

$$\boldsymbol{A}^{-1} = \frac{1}{\det(A)} \begin{pmatrix} 4 & -2 \\ -3 & 1 \end{pmatrix} = \frac{1}{-2} \begin{pmatrix} 4 & -2 \\ -3 & 1 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} -4 & 2 \\ 3 & -1 \end{pmatrix}$$

Determinant of a 3×3 matrix

- Assume $\boldsymbol{A} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$
- Then one defines:

$$\det \boldsymbol{A} = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$
$$= +a_{11} \cdot \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{21} \cdot \begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix} + a_{31} \cdot \begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix}$$

- Methodology:
  - take entries a_{i1} from first column, with alternating signs (+, -)
  - take determinant from square submatrix obtained by deleting the first column and the i-th row

Determinant of a 3×3 matrix, example

$$
\begin{aligned}
\begin{vmatrix} 1 & 2 & -1 \\ 5 & 3 & 4 \\ -2 & 0 & 1 \end{vmatrix}
&= 1 \begin{vmatrix} 3 & 4 \\ 0 & 1 \end{vmatrix} - 5 \begin{vmatrix} 2 & -1 \\ 0 & 1 \end{vmatrix} + -2 \begin{vmatrix} 2 & -1 \\ 3 & 4 \end{vmatrix} \\
&= \big(3 - 0\big) - 5\big(2 - 0\big) - 2\big(8 + 3\big) \\
&= 3 - 10 - 22 \\
&= -29
\end{aligned}
$$

The general, $n \times n$ case

$$\begin{vmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{n1} & \dots & a_{nn} \end{vmatrix} = +a_{11} \cdot \begin{vmatrix} a_{22} & \cdots & a_{2n} \\ \vdots & & \vdots \\ a_{n2} & \dots & a_{nn} \end{vmatrix} - a_{21} \cdot \begin{vmatrix} a_{12} & \cdots & a_{1n} \\ a_{32} & \cdots & a_{3n} \\ \vdots & & \vdots \\ a_{n2} & \dots & a_{nn} \end{vmatrix}$$

$$+ a_{31} \begin{vmatrix} \dots \\ \dots \\ \dots \end{vmatrix} \quad \cdots \quad \pm a_{n1} \begin{vmatrix} a_{12} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{(n-1)2} & \dots & a_{(n-1)n} \end{vmatrix}$$

(where the last sign $\pm$ is $+$ if n is odd and - if n is even)

Then, each of the smaller determinants is computed recursively.

Summary

- Determinants detect when a matrix is invertible
- Though we showed an inefficient way to compute determinants, there is a more efficient algorithm using, you guessed it…Gaussian elimination!
- Solutions to non-homogeneous systems in inverses of matrices can be expressed directly in terms of determinants using *Cramer's rule* (wiki it!)
- Most importantly: determinants will now be used to calculate *eigenvalues*.

Eigenvectors and eigenvalues

Some matrices have a 'magic basis', which turns them into diagonal matrices. This basis consists of *eigenvectors*:

Definition

Assume an $n \times n$ matrix $\boldsymbol{A}$.

An eigenvector for $\boldsymbol{A}$ is a non-zero vector $\boldsymbol{v} \neq 0$ for which there is an eigenvalue $\lambda \in \mathbb{R}$ with:

$$\boldsymbol{A} \cdot \boldsymbol{v} = \lambda \cdot \boldsymbol{v}$$

Example

$\begin{pmatrix} 1 \\ 2 \end{pmatrix}$ is an eigenvector for $\boldsymbol{P} = \frac{1}{10}\begin{pmatrix} 8 & 1 \\ 2 & 9 \end{pmatrix}$ with eigenvalue $\lambda = 1$.

Finding eigenvalues

- We want an eigenvector $\boldsymbol{v}$ and eigenvalue λ:

$$\boldsymbol{A} \cdot \boldsymbol{v} = \lambda \boldsymbol{v}$$

- So, (from before) we need the matrix $\boldsymbol{A} - \lambda \cdot \boldsymbol{I}$ to have $< n$ pivots in its echelon form
- That means $\boldsymbol{A} - \lambda \cdot \boldsymbol{I}$ is not-invertible, i.e.:

$$\det(\boldsymbol{A} - \lambda \cdot \boldsymbol{I}) = 0$$

Finding eigenvalues

$$\det(\boldsymbol{A} - \lambda \cdot \boldsymbol{I}) = 0$$

- $\det(\boldsymbol{A} - \lambda \cdot \boldsymbol{I})$ is a polynomial, with λ as a variable. It is called the *characteristic polynomial*
- Solving the equation gives us eigenvalues.
- Once we have eigen values, eigenvectors are easy (coming soon...)

Eigenvalue example I

- **Task**: find eigenvalues of matrix $\boldsymbol{A} = \begin{pmatrix} 1 & 5 \\ 3 & 3 \end{pmatrix}$
- $\boldsymbol{A} - \lambda \cdot \boldsymbol{I} = \begin{pmatrix} 1 & 5 \\ 3 & 3 \end{pmatrix} - \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} = \begin{pmatrix} 1-\lambda & 5 \\ 3 & 3-\lambda \end{pmatrix}$
- Thus:

$$\begin{aligned} \det(A - \lambda \cdot I) = 0 &\iff \begin{vmatrix} 1-\lambda & 5 \\ 3 & 3-\lambda \end{vmatrix} = 0 \\ &\iff (1-\lambda)(3-\lambda) - 5 \cdot 3 = 0 \\ &\iff \lambda^2 - 4\lambda - 12 = 0 \\ &\iff (\lambda - 6)(\lambda + 2) = 0 \\ &\iff \lambda = 6 \ \text{ or } \ \lambda = -2. \end{aligned}$$

Recall: quadratic formula

- Consider a second-degree (quadratic) equation

$$ax^2 + bx + c \;=\; 0 \qquad\qquad (\text{for } a \neq 0)$$

- Its solutions are:

$$s_{1,2} \;=\; \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

- These solutions coincide (ie. $s_1 = s_2$) if $b^2 - 4ac = 0$
- Real solutions do not exist if $b^2 - 4ac < 0$

  (But complex number solutions do exist in this case.)

Eigenvalue example II

- **Task**: find eigenvalues of matrix $\boldsymbol{A} = \frac{1}{2}\begin{pmatrix} 5 & -1 \\ -1 & 5 \end{pmatrix}$
- $\boldsymbol{A} - \lambda \cdot \boldsymbol{I} = \begin{pmatrix} \frac{5}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{5}{2} \end{pmatrix} - \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} = \begin{pmatrix} \frac{5}{2} - \lambda & -\frac{1}{2} \\ -\frac{1}{2} & \frac{5}{2} - \lambda \end{pmatrix}$
- Thus:

$$
\begin{aligned}
\det(A - \lambda \cdot I) = 0 \iff & \begin{vmatrix} \frac{5}{2} - \lambda & -\frac{1}{2} \\ -\frac{1}{2} & \frac{5}{2} - \lambda \end{vmatrix} = 0 \\
\iff & (\tfrac{5}{2} - \lambda)(\tfrac{5}{2} - \lambda) - (-\tfrac{1}{2})(-\tfrac{1}{2}) = 0 \\
\iff & \lambda^2 - 5\lambda + \tfrac{25}{4} - \tfrac{1}{4} = 0 \\
\iff & \lambda^2 - 5\lambda + 6 = 0 \\
\iff & \lambda_{1,2} = \tfrac{+5 \pm \sqrt{25 - 4 \cdot 1 \cdot 6}}{2} = \tfrac{5 \pm 1}{2} \\
\iff & \lambda_1 = 2 \quad \text{and} \quad \lambda_2 = 3
\end{aligned}
$$

Next time: putting it all together

- Once we know eigen*values*, getting eigenvectors is easy. Find any non-zero solutions for:

$$\boldsymbol{A} \cdot \boldsymbol{v} = \lambda \boldsymbol{v} \qquad \Leftrightarrow \qquad (\boldsymbol{A} - \lambda \boldsymbol{I}) \cdot v = \boldsymbol{0}$$

- This is a homogeneous system, which we can solve
- Solve using λ_1 to get $\boldsymbol{v}_1$, λ_2 to get $\boldsymbol{v}_2$, and so on. Gives us an *eigen*basis:

$$\mathcal{B} = \{\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots\}$$

- Transforming $\boldsymbol{A}$ to the basis $\mathcal{B}$ gives us a diagonal matrix:

$$\boldsymbol{T}_{\mathcal{S}\Rightarrow\mathcal{B}} \cdot \boldsymbol{A} \cdot \boldsymbol{T}_{\mathcal{B}\Rightarrow\mathcal{S}} = \begin{pmatrix} \lambda_1 & 0 & 0 & \cdots \\ 0 & \lambda_2 & 0 & \cdots \\ \vdots & \vdots & \ddots & \vdots \end{pmatrix}$$

- ...and this is useful!