



Matrix Calculations: Basis Transformation, Determinants, and Eigenvalues

A. Kissinger

Institute for Computing and Information Sciences
Radboud University Nijmegen

Version: autumn 2018





Outline

Change of basis

Determinants

Eigenvectors and Eigenvalues





Last time

- Any linear map can be **represented** as a matrix:

$$f(\mathbf{v}) = \mathbf{A} \cdot \mathbf{v} \qquad g(\mathbf{v}) = \mathbf{B} \cdot \mathbf{v}$$

- Last time, we saw that **composing** linear maps could be done by **multiplying** their matrices:

$$f(g(\mathbf{v})) = \mathbf{A} \cdot \mathbf{B} \cdot \mathbf{v}$$

- Matrix multiplication is **pretty easy**:

$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \cdot \begin{pmatrix} 1 & -1 \\ 0 & 4 \end{pmatrix} = \begin{pmatrix} 1 \cdot 1 + 2 \cdot 0 & 1 \cdot (-1) + 2 \cdot 4 \\ 3 \cdot 1 + 4 \cdot 0 & 3 \cdot (-1) + 4 \cdot 4 \end{pmatrix} = \begin{pmatrix} 1 & 7 \\ 3 & 13 \end{pmatrix}$$

...so if we can solve other stuff by matrix multiplication, we are **pretty happy**.



Last time

- For example, we can solve systems of linear equations:

$$\mathbf{A} \cdot \mathbf{x} = \mathbf{b}$$

...by finding the **inverse** of a matrix:

$$\mathbf{x} = \mathbf{A}^{-1} \cdot \mathbf{b}$$

- There is an easy shortcut formula for 2×2 matrices:

$$\mathbf{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \implies \mathbf{A}^{-1} = \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

...as long as $ad - bc \neq 0$.

- We'll see today that " $ad - bc$ " is an example of a special number we can compute for any square matrix (not just 2×2) called the **determinant**.



Vectors in a basis

Recall: a basis for a vector space V is a set of vectors

$\mathcal{B} = \{\mathbf{v}_1, \dots, \mathbf{v}_n\}$ in V such that:

- 1 They **uniquely span** V , i.e. for all $\mathbf{v} \in V$, there exist **unique** a_i such that:

$$\mathbf{v} = a_1 \mathbf{v}_1 + \dots + a_n \mathbf{v}_n$$

Because of this, we use a special **notation** for this linear combination:

$$\begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix}_{\mathcal{B}} := a_1 \mathbf{v}_1 + \dots + a_n \mathbf{v}_n$$



Same vector, different outfits

The *same vector* can look different, depending on the choice of basis. Consider the standard basis: $\mathcal{S} = \{(1, 0), (0, 1)\}$ vs. another basis:

$$\mathcal{B} = \left\{ \begin{pmatrix} 100 \\ 0 \end{pmatrix}, \begin{pmatrix} 100 \\ 1 \end{pmatrix} \right\}$$

Is this a basis? Yes...

- It's **independent** because: $\begin{pmatrix} 100 & 100 \\ 0 & 1 \end{pmatrix}$ has 2 pivots.
- It's **spanning** because... we can make every vector in \mathcal{S} using linear combinations of vectors in \mathcal{B} :

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{1}{100} \begin{pmatrix} 100 \\ 0 \end{pmatrix} \qquad \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 100 \\ 1 \end{pmatrix} - \begin{pmatrix} 100 \\ 0 \end{pmatrix}$$

...so we can also make any vector in \mathbb{R}^2 .



Same vector, different outfits

$$\mathcal{S} = \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}$$

$$\mathcal{B} = \left\{ \begin{pmatrix} 100 \\ 0 \end{pmatrix}, \begin{pmatrix} 100 \\ 1 \end{pmatrix} \right\}$$

Examples:

$$\begin{pmatrix} 100 \\ 0 \end{pmatrix}_{\mathcal{S}} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}_{\mathcal{B}}$$

$$\begin{pmatrix} 300 \\ 1 \end{pmatrix}_{\mathcal{S}} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}_{\mathcal{B}}$$

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix}_{\mathcal{S}} = \begin{pmatrix} \frac{1}{100} \\ 0 \end{pmatrix}_{\mathcal{B}}$$

$$\begin{pmatrix} 0 \\ 1 \end{pmatrix}_{\mathcal{S}} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}_{\mathcal{B}}$$



Why???

- Many find the idea of *multiple bases* confusing at first...
- $\mathcal{S} = \{(1, 0), (0, 1)\}$ is a perfectly good basis for \mathbb{R}^2 . Why bother with others?
 - 1 Some vector spaces don't have one "obvious" choice of basis. Example: subspaces $S \subseteq \mathbb{R}^n$.
 - 2 Sometimes it is way more efficient to write a vector with respect to a different basis, e.g.:

$$\begin{pmatrix} 93718234 \\ -438203 \\ 110224 \\ -5423204980 \\ \vdots \end{pmatrix}_S = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \\ \vdots \end{pmatrix}_B$$

- 3 The choice of basis for *vectors* affects how we write *matrices* as well. Often this can be done cleverly. Example: JPEGs, MP3s, search engine rankings, ...



Transforming bases, part I

- **Problem:** given a vector written in $\mathcal{B} = \{(100, 0), (100, 1)\}$, how can we write it in the standard basis? Just use the definition:

$$\begin{pmatrix} x \\ y \end{pmatrix}_{\mathcal{B}} = x \cdot \begin{pmatrix} 100 \\ 0 \end{pmatrix} + y \cdot \begin{pmatrix} 100 \\ 1 \end{pmatrix} = \begin{pmatrix} 100x + 100y \\ y \end{pmatrix}_{\mathcal{S}}$$

- Or, as matrix multiplication:

$$\underbrace{\begin{pmatrix} 100 & 100 \\ 0 & 1 \end{pmatrix}}_{\mathbf{T}_{\mathcal{B} \Rightarrow \mathcal{S}}} \cdot \underbrace{\begin{pmatrix} x \\ y \end{pmatrix}}_{\text{in basis } \mathcal{B}} = \underbrace{\begin{pmatrix} 100x + 100y \\ y \end{pmatrix}}_{\text{in basis } \mathcal{S}}$$

- Let $\mathbf{T}_{\mathcal{B} \Rightarrow \mathcal{S}}$ be the matrix whose *columns* are the basis vectors \mathcal{B} . Then $\mathbf{T}_{\mathcal{B} \Rightarrow \mathcal{S}}$ *transforms* a vector written in \mathcal{B} into a vector written in \mathcal{S} .



Transforming bases, part II

- How do we transform back? Need $\mathbf{T}_{\mathcal{S} \Rightarrow \mathcal{B}}$ which **undoes** the matrix $\mathbf{T}_{\mathcal{B} \Rightarrow \mathcal{S}}$.
- Solution: use the inverse! $\mathbf{T}_{\mathcal{S} \Rightarrow \mathcal{B}} := (\mathbf{T}_{\mathcal{B} \Rightarrow \mathcal{S}})^{-1}$
- Example:

$$(\mathbf{T}_{\mathcal{B} \Rightarrow \mathcal{S}})^{-1} = \begin{pmatrix} 100 & 100 \\ 0 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} \frac{1}{100} & -1 \\ 0 & 1 \end{pmatrix}$$

- ...which indeed gives:

$$\begin{pmatrix} \frac{1}{100} & -1 \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} \frac{a-100b}{100} \\ b \end{pmatrix}$$



Matrices in other bases

- Since *vectors* can be written with respect to different bases, so too can *matrices*.
- For example, let g be the linear map defined by:

$$g\left(\begin{pmatrix} 1 \\ 0 \end{pmatrix}_S\right) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}_S \quad g\left(\begin{pmatrix} 0 \\ 1 \end{pmatrix}_S\right) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}_S$$

- Then, naturally, we would represent g using the matrix:

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}_S$$

- Because indeed:

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

(the columns say where each of the vectors in S go, **written in the basis S**)



On the other hand...

- Lets look at what g does to another basis:

$$\mathcal{B} = \left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \end{pmatrix} \right\}$$

- First $(1, 1) \in \mathcal{B}$:

$$g\left(\begin{pmatrix} 1 \\ 0 \end{pmatrix}_{\mathcal{B}}\right) = g\left(\begin{pmatrix} 1 \\ 1 \end{pmatrix}\right) = g\left(\begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix}\right) \equiv \dots$$

- Then, by linearity:

$$\dots = g\left(\begin{pmatrix} 1 \\ 0 \end{pmatrix}\right) + g\left(\begin{pmatrix} 0 \\ 1 \end{pmatrix}\right) = \begin{pmatrix} 0 \\ 1 \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}_{\mathcal{B}}$$



On the other hand...

$$\mathcal{B} = \left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \end{pmatrix} \right\}$$

- Similarly $(1, -1) \in \mathcal{B}$:

$$g\left(\begin{pmatrix} 0 \\ 1 \end{pmatrix}_{\mathcal{B}}\right) = g\left(\begin{pmatrix} 1 \\ -1 \end{pmatrix}\right) = g\left(\begin{pmatrix} 1 \\ 0 \end{pmatrix} - \begin{pmatrix} 0 \\ 1 \end{pmatrix}\right) = \dots$$

- Then, by linearity:

$$\dots = g\left(\begin{pmatrix} 1 \\ 0 \end{pmatrix}\right) - g\left(\begin{pmatrix} 0 \\ 1 \end{pmatrix}\right) = \begin{pmatrix} 0 \\ 1 \end{pmatrix} - \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \end{pmatrix}_{\mathcal{B}}$$



A new matrix

- From this:

$$g\left(\begin{pmatrix} 1 \\ 0 \end{pmatrix}_{\mathcal{B}}\right) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}_{\mathcal{B}} \qquad g\left(\begin{pmatrix} 0 \\ 1 \end{pmatrix}_{\mathcal{B}}\right) = \begin{pmatrix} 0 \\ -1 \end{pmatrix}_{\mathcal{B}}$$

- It follows that we should instead use *this* matrix to represent g :

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}_{\mathcal{B}}$$

- Because indeed:

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \end{pmatrix}$$

(the columns say where each of the vectors in \mathcal{B} go, **written in the basis \mathcal{B}**)



A new matrix

- So on different bases, g acts in a totally different way!

$$g\left(\begin{pmatrix} 1 \\ 0 \end{pmatrix}_S\right) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}_S \qquad g\left(\begin{pmatrix} 0 \\ 1 \end{pmatrix}_S\right) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}_S$$

$$g\left(\begin{pmatrix} 1 \\ 0 \end{pmatrix}_B\right) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}_B \qquad g\left(\begin{pmatrix} 0 \\ 1 \end{pmatrix}_B\right) = \begin{pmatrix} 0 \\ -1 \end{pmatrix}_B$$

- ...and hence gets a totally different matrix:

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}_S \qquad \text{vs.} \qquad \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}_B$$



Transforming bases, part II

Theorem

Let \mathcal{B} be a basis. If a linear map $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$ has a matrix \mathbf{A} computed using the standard basis S , and a matrix \mathbf{A}' computed using \mathcal{B} , then:

$$\mathbf{A} = \mathbf{T}_{\mathcal{B} \Rightarrow S} \cdot \mathbf{A}' \cdot \mathbf{T}_{S \Rightarrow \mathcal{B}}$$

n.b. the matrices \mathbf{A} and \mathbf{A}' are called *similar*, because they represent the *same* linear map in *different* ways.



For example

$$\mathcal{S} = \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\} \quad \mathcal{B} = \left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \end{pmatrix} \right\}$$

- Two bases give two different matrices for g :

$$\mathbf{A} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}_{\mathcal{S}} \quad \mathbf{A}' = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}_{\mathcal{B}}$$

- Then, the theorem says we have: $\mathbf{A} = \mathbf{T}_{\mathcal{B} \Rightarrow \mathcal{S}} \cdot \mathbf{A}' \cdot \mathbf{T}_{\mathcal{S} \Rightarrow \mathcal{B}}$
- Indeed, we can check:

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \cdot \frac{1}{-2} \begin{pmatrix} -1 & -1 \\ -1 & 1 \end{pmatrix}$$



There and back again

- To *translate* from a matrix written in \mathcal{B} to a matrix written in \mathcal{S} , we do this:

$$\underbrace{\mathbf{A}}_{\text{in } \mathcal{S}} = \mathbf{T}_{\mathcal{B} \Rightarrow \mathcal{S}} \cdot \underbrace{\mathbf{A}'}_{\text{in } \mathcal{B}} \cdot \mathbf{T}_{\mathcal{S} \Rightarrow \mathcal{B}}$$

- Since $\mathbf{T}_{\mathcal{S} \Rightarrow \mathcal{B}} = (\mathbf{T}_{\mathcal{B} \Rightarrow \mathcal{S}})^{-1}$, we can go the other way like this:

$$\underbrace{\mathbf{A}'}_{\text{in } \mathcal{B}} = \mathbf{T}_{\mathcal{S} \Rightarrow \mathcal{B}} \cdot \underbrace{\mathbf{A}}_{\text{in } \mathcal{S}} \cdot \mathbf{T}_{\mathcal{B} \Rightarrow \mathcal{S}}$$

- How to remember? Look at the **blue** matrix.



Example (from \mathcal{S} to \mathcal{B})

- Consider the standard basis $\mathcal{S} = \{(1, 0), (0, 1)\}$ for \mathbb{R}^2 , and as alternative basis $\mathcal{B} = \{(-1, 1), (0, 2)\}$
- Let the linear map $f : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ be represented in the standard basis \mathcal{S} by the following matrix:

$$\mathbf{A} = \begin{pmatrix} 1 & -1 \\ 2 & 3 \end{pmatrix}$$

- What is the representation \mathbf{A}' of f in the basis \mathcal{B} ?
- Since \mathcal{S} is the standard basis, $\mathbf{T}_{\mathcal{B} \Rightarrow \mathcal{S}} = \begin{pmatrix} -1 & 0 \\ 1 & 2 \end{pmatrix}$ contains the \mathcal{B} -vectors as its columns



Example (from \mathcal{S} to \mathcal{B} , cont'd)

- The basis transformation matrix $\mathbf{T}_{\mathcal{S} \Rightarrow \mathcal{B}}$ in the other direction is obtained as **matrix inverse**:

$$\mathbf{T}_{\mathcal{S} \Rightarrow \mathcal{B}} = (\mathbf{T}_{\mathcal{B} \Rightarrow \mathcal{S}})^{-1} = \begin{pmatrix} -1 & 0 \\ 1 & 2 \end{pmatrix}^{-1} = \frac{1}{-2-0} \begin{pmatrix} 2 & 0 \\ -1 & -1 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} -2 & 0 \\ 1 & 1 \end{pmatrix}$$

- Hence:

$$\begin{aligned} \mathbf{A}' &= \mathbf{T}_{\mathcal{S} \Rightarrow \mathcal{B}} \cdot \mathbf{A} \cdot \mathbf{T}_{\mathcal{B} \Rightarrow \mathcal{S}} \\ &= \frac{1}{2} \begin{pmatrix} -2 & 0 \\ 1 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & -1 \\ 2 & 3 \end{pmatrix} \cdot \begin{pmatrix} -1 & 0 \\ 1 & 2 \end{pmatrix} \\ &= \frac{1}{2} \begin{pmatrix} -2 & 2 \\ 3 & 2 \end{pmatrix} \cdot \begin{pmatrix} -1 & 0 \\ 1 & 2 \end{pmatrix} \\ &= \frac{1}{2} \begin{pmatrix} 4 & 4 \\ -1 & 4 \end{pmatrix} \\ &= \begin{pmatrix} 2 & 2 \\ -\frac{1}{2} & 2 \end{pmatrix} \end{aligned}$$

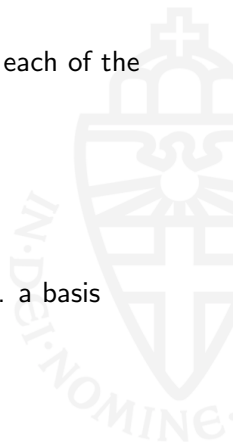


The magic basis

- Recall: diagonal matrices correspond to rescaling each of the axes:

$$D = \begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{pmatrix}$$

- These matrices are really easy to work with
- Most matrices (secretly) have their *own* basis, i.e. a basis where they are diagonal
- This is called the *eigenbasis* of the matrix





The magic basis

- Scaling each of the basis vectors means:

$$\mathbf{A} \cdot \mathbf{v} = \lambda \mathbf{v}$$

for each $\mathbf{v} \in \mathcal{B}$.

- Rewritten, that's:

$$\mathbf{A} \cdot \mathbf{v} - \lambda \mathbf{v} = \mathbf{0} \quad \Rightarrow \quad \mathbf{A} \cdot \mathbf{v} - \lambda \mathbf{I} \cdot \mathbf{v} = \mathbf{0} \quad \Rightarrow \quad (\mathbf{A} - \lambda \mathbf{I}) \cdot \mathbf{v} = \mathbf{0}$$

- Need to find *non-zero* solutions to $(\mathbf{A} - \lambda \mathbf{I}) \cdot \mathbf{v} = \mathbf{0}$.
- That happens whenever $(\mathbf{A} - \lambda \mathbf{I})$ has $< n$ pivots.
- If only there was a way to find a λ where that happens....



Determinants

What a determinant does

For an $n \times n$ matrix \mathbf{A} , the determinant $\det(\mathbf{A})$ is a number (in \mathbb{R})

It satisfies:

$$\begin{aligned}\det(\mathbf{A}) = 0 &\iff \mathbf{A} \text{ is not invertible} \\ &\iff \mathbf{A}^{-1} \text{ does not exist} \\ &\iff \mathbf{A} \text{ has } < n \text{ pivots in its echolon form}\end{aligned}$$

Determinants have useful properties, but calculating determinants involves some work.



Determinant of a 2×2 matrix

- Assume $\mathbf{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$
- Recall that the inverse \mathbf{A}^{-1} exists if and only if $ad - bc \neq 0$, and in that case is:

$$\mathbf{A}^{-1} = \frac{1}{ad-bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

- In this 2×2 -case we **define**:

$$\det \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

- Thus, indeed: $\det(\mathbf{A}) = 0 \iff \mathbf{A}^{-1}$ does not exist.



Determinant of a 2×2 matrix: example

- Example:

$$\mathbf{A} = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$$

- Then:

$$\det(\mathbf{A}) = 1 \cdot 4 - 2 \cdot 3 = 4 - 6 = -2$$

- $\det(\mathbf{A}) = -2 \neq 0 \implies \mathbf{A}$ is invertible!
- Indeed, we can compute:

$$\mathbf{A}^{-1} = \frac{1}{\det(\mathbf{A})} \begin{pmatrix} 4 & -2 \\ -3 & 1 \end{pmatrix} = \frac{1}{-2} \begin{pmatrix} 4 & -2 \\ -3 & 1 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} -4 & 2 \\ 3 & -1 \end{pmatrix}$$





Determinant of a 3×3 matrix

- Assume $\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$
- Then one defines:

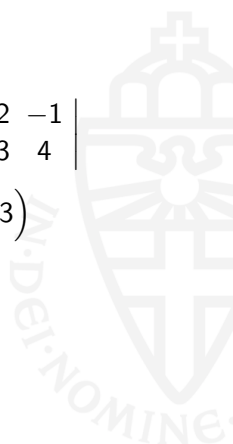
$$\begin{aligned} \det \mathbf{A} &= \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} \\ &= +a_{11} \cdot \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{21} \cdot \begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix} + a_{31} \cdot \begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix} \end{aligned}$$

- Methodology:
 - take entries a_{i1} from first column, with alternating signs (+, -)
 - take determinant from square submatrix obtained by deleting the first column and the i -th row



Determinant of a 3×3 matrix, example

$$\begin{aligned} \begin{vmatrix} 1 & 2 & -1 \\ 5 & 3 & 4 \\ -2 & 0 & 1 \end{vmatrix} &= 1 \begin{vmatrix} 3 & 4 \\ 0 & 1 \end{vmatrix} - 5 \begin{vmatrix} 2 & -1 \\ 0 & 1 \end{vmatrix} + -2 \begin{vmatrix} 2 & -1 \\ 3 & 4 \end{vmatrix} \\ &= (3 - 0) - 5(2 - 0) - 2(8 + 3) \\ &= 3 - 10 - 22 \\ &= -29 \end{aligned}$$





The general, $n \times n$ case

$$\begin{vmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{n1} & \cdots & a_{nn} \end{vmatrix} = +a_{11} \cdot \begin{vmatrix} a_{22} & \cdots & a_{2n} \\ \vdots & & \vdots \\ a_{n2} & \cdots & a_{nn} \end{vmatrix} - a_{21} \cdot \begin{vmatrix} a_{12} & \cdots & a_{1n} \\ a_{32} & \cdots & a_{3n} \\ \vdots & & \vdots \\ a_{n2} & \cdots & a_{nn} \end{vmatrix} \\
 + a_{31} \begin{vmatrix} \cdots \\ \cdots \\ \cdots \end{vmatrix} \cdots \pm a_{n1} \begin{vmatrix} a_{12} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{(n-1)2} & \cdots & a_{(n-1)n} \end{vmatrix}$$

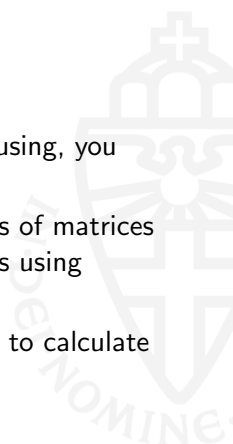
(where the last sign \pm is $+$ if n is odd and $-$ if n is even)

Then, each of the smaller determinants is computed recursively.



Summary

- Determinants detect when a matrix is invertible
- Though we showed an inefficient way to compute determinants, there is a more efficient algorithm using, you guessed it...Gaussian elimination!
- Solutions to non-homogeneous systems in inverses of matrices can be expressed directly in terms of determinants using *Cramer's rule* (wiki it!)
- Most importantly: determinants will now be used to calculate *eigenvalues*.





Eigenvectors and eigenvalues

Some matrices have a 'magic basis', which turns them into diagonal matrices. This basis consists of *eigenvectors*:

Definition

Assume an $n \times n$ matrix \mathbf{A} .

An **eigenvector** for \mathbf{A} is a non-zero vector $\mathbf{v} \neq 0$ for which there is an **eigenvalue** $\lambda \in \mathbb{R}$ with:

$$\mathbf{A} \cdot \mathbf{v} = \lambda \cdot \mathbf{v}$$

Example

$\begin{pmatrix} 1 \\ 2 \end{pmatrix}$ is an eigenvector for $\mathbf{P} = \frac{1}{10} \begin{pmatrix} 8 & 1 \\ 2 & 9 \end{pmatrix}$ with eigenvalue $\lambda = 1$.



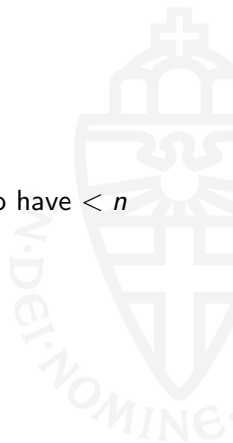
Finding eigenvalues

- We want an **eigenvector** \mathbf{v} and **eigenvalue** λ :

$$\mathbf{A} \cdot \mathbf{v} = \lambda \mathbf{v}$$

- So, (from before) we need the matrix $\mathbf{A} - \lambda \cdot \mathbf{I}$ to have $< n$ pivots in its echelon form
- That means $\mathbf{A} - \lambda \cdot \mathbf{I}$ is **not-invertible**, i.e.:

$$\det(\mathbf{A} - \lambda \cdot \mathbf{I}) = 0$$

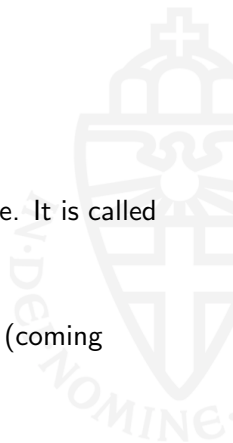




Finding eigenvalues

$$\det(\mathbf{A} - \lambda \cdot \mathbf{I}) = 0$$

- $\det(\mathbf{A} - \lambda \cdot \mathbf{I})$ is a **polynomial**, with λ as a variable. It is called the *characteristic polynomial*
- Solving the equation gives us eigenvalues.
- Once we have eigen values, eigenvectors are easy (coming soon...)

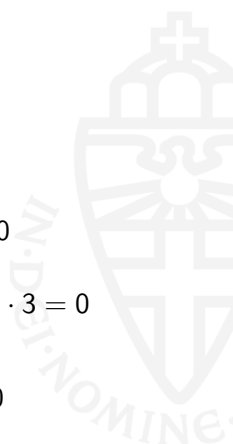




Eigenvalue example I

- **Task:** find eigenvalues of matrix $\mathbf{A} = \begin{pmatrix} 1 & 5 \\ 3 & 3 \end{pmatrix}$
- $\mathbf{A} - \lambda \cdot \mathbf{I} = \begin{pmatrix} 1 & 5 \\ 3 & 3 \end{pmatrix} - \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} = \begin{pmatrix} 1 - \lambda & 5 \\ 3 & 3 - \lambda \end{pmatrix}$
- Thus:

$$\begin{aligned} \det(\mathbf{A} - \lambda \cdot \mathbf{I}) = 0 &\iff \begin{vmatrix} 1 - \lambda & 5 \\ 3 & 3 - \lambda \end{vmatrix} = 0 \\ &\iff (1 - \lambda)(3 - \lambda) - 5 \cdot 3 = 0 \\ &\iff \lambda^2 - 4\lambda - 12 = 0 \\ &\iff (\lambda - 6)(\lambda + 2) = 0 \\ &\iff \lambda = 6 \text{ or } \lambda = -2. \end{aligned}$$





Recall: quadratic formula

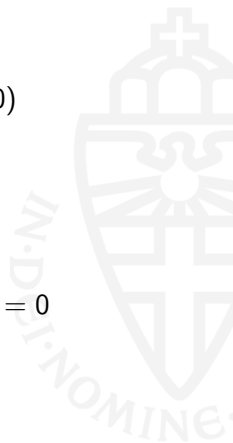
- Consider a **second-degree** (quadratic) equation

$$ax^2 + bx + c = 0 \quad (\text{for } a \neq 0)$$

- Its **solutions** are:

$$s_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

- These solutions **coincide** (ie. $s_1 = s_2$) if $b^2 - 4ac = 0$
- Real solutions **do not exist** if $b^2 - 4ac < 0$
(But complex number solutions do exist in this case.)





Eigenvalue example II

- **Task:** find eigenvalues of matrix $\mathbf{A} = \frac{1}{2} \begin{pmatrix} 5 & -1 \\ -1 & 5 \end{pmatrix}$
- $\mathbf{A} - \lambda \cdot \mathbf{I} = \begin{pmatrix} \frac{5}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{5}{2} \end{pmatrix} - \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} = \begin{pmatrix} \frac{5}{2} - \lambda & -\frac{1}{2} \\ -\frac{1}{2} & \frac{5}{2} - \lambda \end{pmatrix}$
- Thus:

$$\begin{aligned} \det(\mathbf{A} - \lambda \cdot \mathbf{I}) = 0 &\iff \begin{vmatrix} \frac{5}{2} - \lambda & -\frac{1}{2} \\ -\frac{1}{2} & \frac{5}{2} - \lambda \end{vmatrix} = 0 \\ &\iff \left(\frac{5}{2} - \lambda\right)\left(\frac{5}{2} - \lambda\right) - \left(-\frac{1}{2}\right)\left(-\frac{1}{2}\right) = 0 \\ &\iff \lambda^2 - 5\lambda + \frac{25}{4} - \frac{1}{4} = 0 \\ &\iff \lambda^2 - 5\lambda + 6 = 0 \\ &\iff \lambda_{1,2} = \frac{+5 \pm \sqrt{25 - 4 \cdot 1 \cdot 6}}{2} = \frac{5 \pm 1}{2} \\ &\iff \lambda_1 = 2 \quad \text{and} \quad \lambda_2 = 3 \end{aligned}$$



Next time: putting it all together

- Once we know eigenvalues, getting eigenvectors is easy. Find any non-zero solutions for:

$$\mathbf{A} \cdot \mathbf{v} = \lambda \mathbf{v} \quad \Leftrightarrow \quad (\mathbf{A} - \lambda \mathbf{I}) \cdot \mathbf{v} = \mathbf{0}$$

- This is a **homogeneous** system, which we can solve
- Solve using λ_1 to get \mathbf{v}_1 , λ_2 to get \mathbf{v}_2 , and so on. Gives us an *eigenbasis*:

$$\mathcal{B} = \{\mathbf{v}_1, \mathbf{v}_2, \dots\}$$

- Transforming \mathbf{A} to the basis \mathcal{B} gives us a diagonal matrix:

$$\mathbf{T}_{\mathcal{S} \Rightarrow \mathcal{B}} \cdot \mathbf{A} \cdot \mathbf{T}_{\mathcal{B} \Rightarrow \mathcal{S}} = \begin{pmatrix} \lambda_1 & 0 & 0 & \cdots \\ 0 & \lambda_2 & 0 & \cdots \\ \vdots & \vdots & \ddots & \vdots \end{pmatrix}$$

- ...and this is useful!