

Quantum Processes and Computation

Assignment 4, Monday, March 5, 2018

Exercise teachers:

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Handing in your answers: There are two options:

1. Deliver a hard copy to the mailbox of John van de Wetering. Mercator 1, 3rd floor.
2. E-mail a PDF to wetering@cs.ru.nl. Please include your name and the exercise number in the filename, e.g. ACHTERNAAM-qpc-exercise1.pdf.

Deadline: Friday, March 9, 17:00

Goals: After completing these exercises you know about orthonormal bases, composition of linear maps and logic gates as linear maps. The total number of points is 100, distributed over 5 exercises. Material covered in book: Sections 5.1, 5.2, 5.3.4 and a bit of 5.3.5

Exercise 1 (5.4) (20 points): We saw in the lecture that for a set A with n elements in relations the singletons:

$$\mathcal{B}_A := \left\{ \begin{array}{c} \downarrow \\ \nabla_a \end{array} \mid a \in A \right\}$$

form a basis, that is, that no element can be removed from \mathcal{B}_A without losing the property of being a basis. This basis is also orthonormal. Show that this is the *only* orthonormal basis of A .

Bonus exercise: The orthonormality condition is actually not necessary for proving the uniqueness of the basis. Show that any basis (not necessarily orthonormal) of A must be the singleton basis.

Exercise 2 (5.54) (20 points): Let

$$\begin{array}{c} \downarrow \\ \nabla_\psi \end{array} \leftrightarrow \begin{pmatrix} \psi^0 \\ \psi^1 \end{pmatrix} \quad \text{and} \quad \begin{array}{c} \downarrow \\ \nabla_\phi \end{array} \leftrightarrow (\phi_0 \quad \phi_1)$$

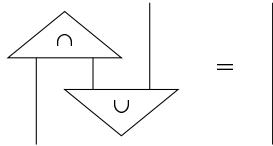
be respectively a 2-dimensional state, and 2-dimensional effect. Let λ be a number. Write the matrices for the processes

$$(i) \quad \begin{array}{c} \lambda \\ \nabla \end{array} \quad (ii) \quad \begin{array}{c} \downarrow \\ \nabla_\psi \end{array} \quad (iii) \quad \begin{array}{c} \downarrow \\ \nabla_\phi \end{array} \quad (iv) \quad \begin{array}{c} \phi \\ \psi \end{array}$$

Exercise 3 (5.58) (20 points): The matrices for cups and caps in 2 dimensions are:

$$\begin{array}{c} \cup \\ \cap \end{array} \leftrightarrow \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} \quad \begin{array}{c} \cap \\ \cup \end{array} \leftrightarrow (1 \quad 0 \quad 0 \quad 1)$$

- (i) First, verify the yanking equation



directly using the matrices of the 2-dimensional cup and cap by using the rules for sequential and parallel composition of matrices, i.e. show that $(\cap \otimes \text{id}) \circ (\text{id} \otimes \cup) = \text{id}$ (where id is the 2×2 identity matrix).

(ii) Second, give the matrices for the cup and cap in 3 dimensions.

The next two exercises are about encoding classical logic gates in the theory of **linear maps**, as explained in Section 5.3.4. Recall that a classical logic gate F can be encoded as a linear map via:

$$\begin{array}{c} \text{Diagram 1} \\ \text{Diagram 2} \end{array} = \sum_{(a_1 \dots a_m \mapsto b_1 \dots b_n) \in F} \dots$$

Using this encoding, we defined:

$$\begin{aligned}
 \text{XOR} &= \begin{array}{c} \text{XOR} \\ \text{gate} \end{array} = \begin{array}{c} \text{triangle} \\ 0 \end{array} \oplus \begin{array}{c} \text{triangle} \\ 1 \end{array} \oplus \begin{array}{c} \text{triangle} \\ 1 \end{array} \oplus \begin{array}{c} \text{triangle} \\ 0 \end{array} \oplus \begin{array}{c} \text{triangle} \\ 1 \end{array} \oplus \begin{array}{c} \text{triangle} \\ 1 \end{array} \\
 \text{CNOT} &:= \begin{array}{c} \text{CNOT} \\ \text{gate} \end{array} := \begin{array}{c} \text{triangle} \\ 0 \end{array} \oplus \begin{array}{c} \text{triangle} \\ 0 \end{array} \oplus \begin{array}{c} \text{triangle} \\ 0 \end{array} \oplus \begin{array}{c} \text{triangle} \\ 1 \end{array} \oplus \begin{array}{c} \text{triangle} \\ 1 \end{array} \oplus \begin{array}{c} \text{triangle} \\ 1 \end{array} \oplus \begin{array}{c} \text{triangle} \\ 0 \end{array} \oplus \begin{array}{c} \text{triangle} \\ 1 \end{array} \oplus \begin{array}{c} \text{triangle} \\ 1 \end{array} \\
 \text{COPY} &:= \begin{array}{c} \text{COPY} \\ \text{gate} \end{array} := \begin{array}{c} \text{triangle} \\ 0 \end{array} \oplus \begin{array}{c} \text{triangle} \\ 0 \end{array} \oplus \begin{array}{c} \text{triangle} \\ 1 \end{array} \oplus \begin{array}{c} \text{triangle} \\ 1 \end{array}
 \end{aligned}$$

Exercise 4 (5.86) (20 points): Show that

The diagram illustrates the decomposition of a CNOT gate into a COPY and an XOR gate. On the left, a standard CNOT gate symbol is shown. An equals sign follows it, and on the right, a circuit diagram is shown. This diagram consists of two rectangular boxes. The top box is labeled 'XOR' and the bottom box is labeled 'COPY'. The COPY box has a single input line on its left and a single output line on its right. The XOR box has a single input line on its left and a single output line on its right. The output line of the COPY box is connected to the input line of the XOR box. Both the COPY and XOR boxes have a small triangle symbol on their right side.

(Hint: try comparing the LHS to the RHS on all basis states, rather than writing out a big sum.)

Next, find ψ and ϕ such that the following equation holds:

The diagram illustrates the decomposition of a CNOT gate into a sequence of operations. On the left, a CNOT gate is shown as a box with 'XOR' above it and 'COPY' below it. A vertical line connects the top of the XOR box to the top of the COPY box. To the right of the boxes is an equals sign (=). To the right of the equals sign is a sequence of three boxes: a triangle with a vertical line pointing down labeled ϕ , a triangle with a vertical line pointing up labeled ψ , and a triangle with a diagonal line labeled ψ .

Although it might not look like much now, this equation will turn out to lie at the heart of the notion of *complementarity* which we will cover in great depth in the coming lectures.

Exercise 5 (based on 5.83) (20 points): Show that if a logic gate has an inverse, its associated linear map is unitary.