Quantum Processes and Computation

Assignment 5, Monday, March 12, 2018

Exercise teachers:
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Handing in your answers: There are two options:

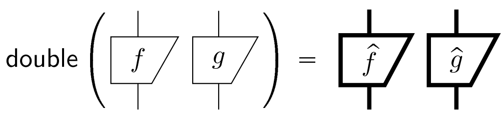
1. Deliver a hard copy to the mailbox of John van de Wetering. Mercator 1, 3rd floor.
2. E-mail a PDF to `wetering@cs.ru.nl`. Please include your name and the exercise number in the filename, e.g. `ACHTERNAAM-qpc-exercise1.pdf`.

Deadline: Friday, March 16, 17:00

Goals: After completing these exercises you will know how to work with diagonalisations of processes, and gain an understanding of the relationship between the process theories of **linear maps** and **quantum maps**. The total number of points is 100, distributed over 3 exercises. Material covered in book: sections 5.3.3 and 6.1.

Exercise 1 (6.10 & 6.22) (20 points):

(i) Show that doubling preserves parallel composition:

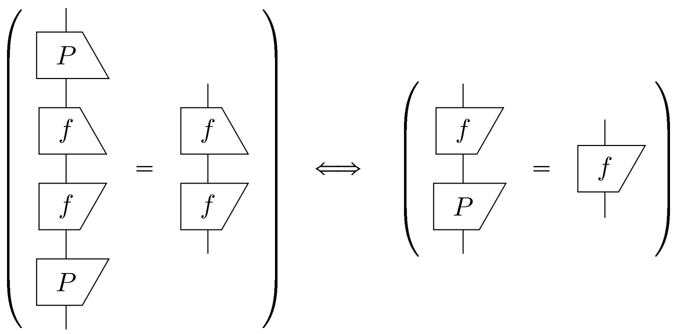


(ii) Show that doubling preserves normalisation: that a state ψ is normalised if and only if its doubled state $\widehat{\psi}$ is normalised.

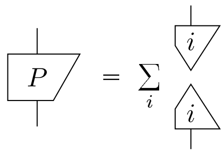
(iii) Show that doubling preserves orthogonality: that states ψ and ϕ are orthogonal if and only if $\widehat{\psi}$ and $\widehat{\phi}$ are orthogonal.

Hint: Use theorem 6.17 for the latter two points.

Exercise 2 (5.77) (40 points): We call a process P a *projector* if it is self-adjoint and *idempotent*: $P \circ P = P$. Show that $f^\dagger \circ f$ detects whether f is invariant under a projector, that is, show that for any projector P:

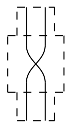


Hint: Look at the proof of proposition 5.74 for inspiration. More concretely: write P using the spectral theorem as

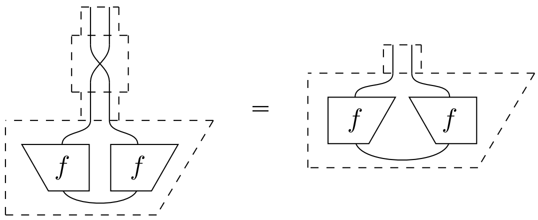


for some orthonormal set of states $\{|i\rangle\}$ (i.e. not necessarily a full ONB). This orthonormal set can be completed to a full ONB (see proposition 5.79). How does P act on these additional orthogonal states? As a result how will $f^\dagger \circ f$ act on these orthogonal states? Use positive definiteness of the inner product to conclude the same for f, and then you're almost there.

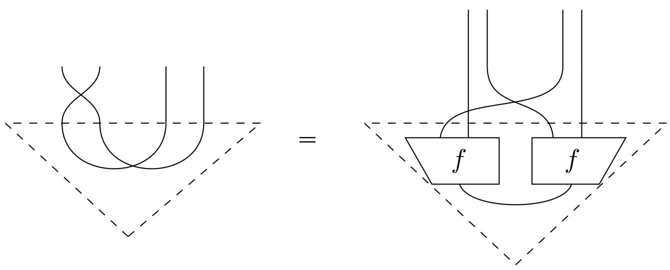
The transpose of a positive process is again a positive process and by bending some wires we can also take the 'transpose' of a ⊗-positive state, i.e. of a quantum state (see **Corollary 6.36**). This transpose acts as a swap of wires on the doubled system:



and it indeed sends quantum states to quantum states:



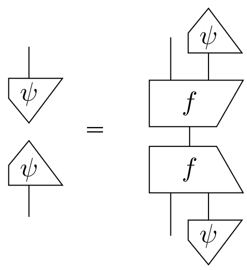
In the next exercise we will show that nevertheless, this swap of wires is *not* a quantum operation.
Exercise 3 (40 points): In this exercise we will show that a swap applied to one pair of the wires of the doubled cup state will result in a state that is no longer ⊗-positive, and therefore not a quantum state. We will do this by contradiction. So suppose:



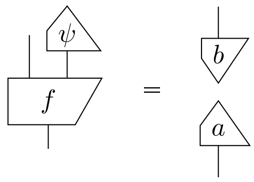
(1)

for some process f.

(i) Let ψ be a normalised state. Show that the equation above implies that



and hence, by Proposition 5.74, that there exist states a and b such that:



(2)

(ii) Plug ψ into equation 1 and use equation 2 to show that the identity wire disconnects. Conclude that therefore the swap can't be a quantum map.

Note: In proposition 6.48 it is also shown that the swap is not a quantum operation, but it uses a specific counter-example found in **linear maps**. The proof above only uses string diagrams and the property implied by proposition 5.74.

Subnote: We came up with this proof just last week, and as far as we are aware it is the first time someone has shown that the transpose is not a quantum operation using diagrammatic operations, so by completing this proof you really are on the bleeding edge of quantum research.