Quantum Processes and Computation

Assignment 6, Monday, March 26, 2018

Exercise teachers:
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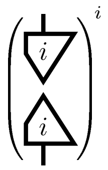
Handing in your answers: There are two options:

1. Deliver a hard copy to the mailbox of John van de Wetering. Mercator 1, 3rd floor.
2. E-mail a PDF to `wetering@cs.ru.nl`. Please include your name and the exercise number in the filename, e.g. `ACHTERNAAM-qpc-exercise1.pdf`.

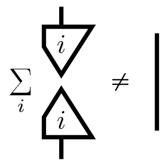
Deadline: Friday, March 30, 17:00

Goals: After completing these exercises you will know how to work with diagonalisations of processes, and gain an understanding of the relationship between the process theories of **linear maps** and **quantum maps**. The total number of points is 100, distributed over 3 exercises. Material covered in book: sections 5.3.3 and 6.1.

Exercise 1 (7.13) (20 points): Let $|i\rangle$ form an ONB, so that it forms a non-demolition ONB measurement in the following way:



If we sum over all the branches we get a process that is called *decoherence*. Show that decoherence is not equal to the identity:

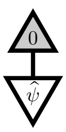
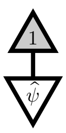


Exercise 2 (50 points): In this exercise our system is a qubit. Let $|0\rangle$ and $|1\rangle$ denote the standard basis vectors of $\mathbb{C}^2$, and let $|+\rangle := \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$ and $|-\rangle := \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$ denote the *X-basis*. We write the associated quantum states as:

 $:= \mathsf{double}(|+\rangle)$,  $:= \mathsf{double}(|-\rangle)$

(i) Show explicitly that $\{|+\rangle, |-\rangle\}$ forms a basis for $\mathbb{C}^2$ and thus that we can construct an ONB measurement out of their associated doubled pure effects.

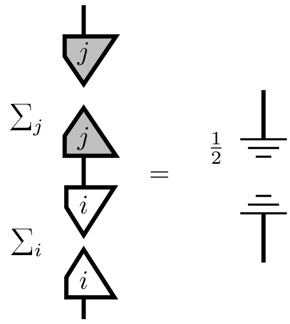
(ii) Let $r, s \in \mathbb{R}$ be such that $r^2 + s^2 = 1$ so that the vector $|\psi\rangle = r\,|0\rangle + s\,|1\rangle$ is normalised. Calculate the probabilities associated to the X-basis measurement of the pure quantum state $\hat{\psi}$ in terms of r and s. I.e., calculate the numbers:

 and 

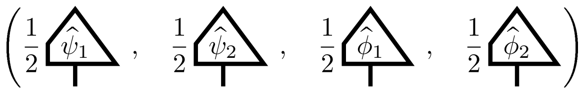
(iii) Show that the two possible outcomes of an ONB measurement when applied to the maximally mixed state $\frac{1}{2}$  have equal probability (regardless of the chosen ONB).

(iv) By the previous point, when we measure the maximally mixed state in the X-basis, both outcomes are equally likely. Show that however, if we do a non-demolition measurement in the X-basis *and then* we do another (demolition) measurement in the X-basis, that we will never get a different outcome than the one we've seen in the first (non-demolition) measurement. So if the outcome was $|+\rangle$ the first time, it will never be $|-\rangle$ the second time around.

(v) Show that if we apply the decoherence channel of the X-basis, and then we apply the decoherence channel of the standard basis, that we get the *completely decohering* channel (also known as the *noise channel*):



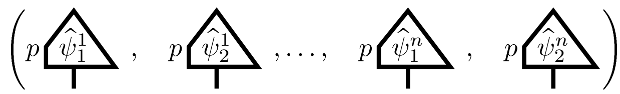
Exercise 3 (30 points): Suppose we have two different ONB measurements on a qubit. Before we do anything with the quantum system we can throw a coin. If it turns up heads we can perform the first ONB measurement, and if it turns up tails we can do the other one. This is modelled by the quantum process



where $(\widehat{\psi}_1, \widehat{\psi}_2)$ and $(\widehat{\phi}_1, \widehat{\phi}_2)$ both form ONB measurements.

(i) Show that the above set of 4 effects indeed forms a quantum measurement (i.e. that it satisfies the causality condition).

(ii) We can generalise the above construction. Suppose we now have n different ONB measurements which we call $(\widehat{\psi}_1^j, \widehat{\psi}_2^j)$ for $j = 1, \ldots, n$. Find the number p such that



is a quantum measurement. Would the number p change if instead of ONB measurements on a qubit, we would consider ONB measurements on a bigger system?