Quantum Processes and Computation

Assignment 12, Wednesday, May 15, 2019

Exercise teachers:
Aleks Kissinger (`aleks@cs.ru.nl`)
John van de Wetering (`wetering@cs.ru.nl`)

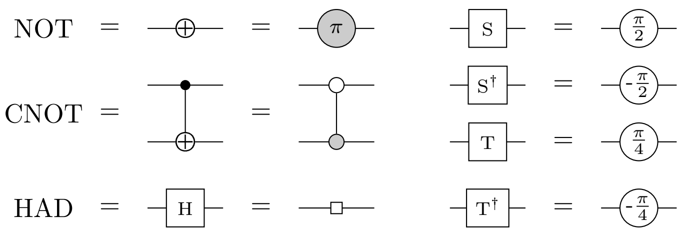
Handing in your answers: There are two options:

1. Deliver a hard copy to the mailbox of John van de Wetering. Mercator 1, 3rd floor.
2. E-mail a PDF to `wetering@cs.ru.nl`. Please include your name and the exercise number in the filename, e.g. `ACHTERNAAM-qpc-exercise1.pdf`.

Deadline: Tuesday, May 21, 12:00

Goals: After doing these exercises you can reason about quantum circuits using the ZX-calculus. The total number of points is 100, distributed over 3 exercises.
Material covered in book: sections 9.4, 12.1.

The usage of ZX-diagrams to represent quantum circuits is not (yet) standard. A more commonly used notation is known just as *quantum circuit notation*. In this notation, the atomic building blocks are gates instead of spiders. A set of commonly used gates and their notation is:

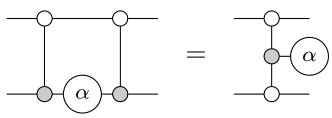


(1)

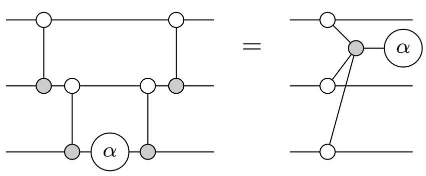
All of these gates are Clifford except for the T gate and its adjoint $\mathrm{T}^\dagger$. To make sure you get familiar with this notation, we will use it a few times throughout this exercise sheet.

Exercise 1 (30 points):

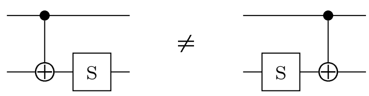
(i) In the last exercise sheet you had to find a circuit representation of the controlled-phase gate. The important part of this gate is what we call a *phase-gadget*:



Show that this equation generalises by proving the following equation in the ZX-calculus:

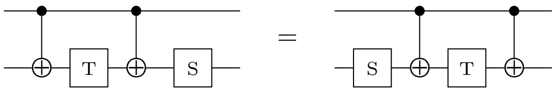


(ii) Show that a phase gate does not commute past the target of a CNOT:

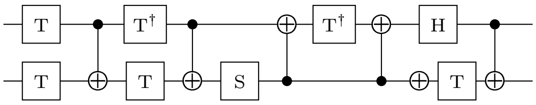


Hint: Find some input state under which the two sides aren't equal.

(iii) Show that a phase gate *does* commute past a phase-gadget:



Exercise 2 (40 points): Consider the following quantum circuit:



The goal is to find a more optimal implementation of this circuit. Write this circuit as a ZX-diagram, simplify it using diagrammatic reasoning, and then write it again as a quantum circuit in quantum circuit notation using the gate-set of (1). Use whatever rewrite rules you know to minimize the *T-count* of the circuit. This is the amount of T and T† gates in the circuit (or in terms of a ZX-diagram, the amount of spiders that have odd multiples of $\frac{\pi}{4}$ on them). For full marks, find an equivalent circuit that has a T-count of zero.

T-count optimisation is an important problem in large-scale quantum computation as in most fault-tolerant architectures, Clifford gates are cheap to implement, while T gates can require a lot of resources and time to implement (sometimes up to 100 times more than a CNOT).

Exercise 3 (30 points): This exercise asks you to construct a classical oracle for a given classical boolean function.

(i) Find a 3-input, 2-output ZX-diagram that acts on the computational basis states as the boolean function $f : \{0,1\}^3 \to \{0,1\}^2$ given by $f(x_1, x_2, x_3) = (x_1 \oplus x_2, x_2 \oplus x_3)$.

(ii) This diagram is of course not unitary (it doesn't even have the same amount of inputs as outputs). Find an extension of the diagram such that $x_1 \oplus x_2$ and $x_2 \oplus x_3$ are still available as outputs, but such that the diagram is unitary.

Hint: If you follow Proposition 12.15 you will end up with a 5 qubit unitary, but it is possible to do this with 3 qubits. Can you find it?

(iii) **BONUS:** Write the diagram you got in the previous point as a circuit using quantum circuit notation.