Categorical Type Theory

PhD. thesis of

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Preface

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Prerequisites

Familiarity with basic category theory is asssumed. The reader is supposed to have a working knowledge of functors, adjunctions, (locally) cartesian closed categories, Yoneda, etc. Let's say that the first five chapters in Mac Lane [1971] form the starting point. A good introduction would be Barr & Wells [1990]; the parts about sketches are not relevant though. Two points may go beyond this basic category theory.

In the first chapter 2-categories are mentioned occasionally. Briefly, a 2-category is a category where the morphisms between any two objects are objects for a category again; this yields two sorts of composition — "vertical" and "horizontal" — which should satisfy certain interchange laws, see e.g. Mac Lane [1971]. The basic thing used is that adjointness and equivalence are 2-categorical notions. More information may be found in Kelly and Street [1974].

In some examples toposes occur. The expositions there are not self-contained and the reader is referred to Johnstone [1977], Barr & Wells [1985] or Bell [1988] for more information.

Information for reading

One of the main concerns in this work is the connection between two relations: type theoretical "dependence on" and categorical "being fibred over". Before plunging into technical expositions, the reader may want to see this main line and take a look at sections 2.1 and 5.1 first.

The category theory needed to describe calculi with type dependency is definitely more advanced (and interesting) than the one for calculi without such dependency. The latter "propositional" systems are described categorically in chapter 3 and the prerequisites may be found in chapter 1, esp. sections 1,2 and 5. This organization has been chosen to enable reading only these propositional parts. The subsequent chapter 4 contains the technical work on type dependency.

Contents

Introduction and summary

Categorical type theory is understood here as the field concerned both with category theory and type theory and especially with their interplay. As such it grew out of categorical logic. Roughly, we view a logic as a type theory in which propositions can have at most one proof-object. Indeed, one finds that the propositional part of the structures used in categorical logic are preordered categories (where one has at most one arrow between two objects). Thus type theory exhibits more categorical structure than logic. A logician might want to point out that there are no small complete categories other than preorders. Quite reassuringly, one does have small complete *fibred* categories which are not preordered, see 4.2.4 and further. These give interesting examples in categorical type theory.

Having mentioned these differences between categorical logic and type theory, we stress the historic continuity: the basic notions used in categorical type theory have been developed before in categorical logic. In this thesis one finds forms of indexing, quantification by adjoints, comprehension and algebraic theories, which are all based on previous work in logic (especially by F. Lawvere, see e.g. Lawvere [1963], [1969], [1970] or Kock & Reyes [1977]). We want to emphasize that these notions require some refinements and adjustments to make them suitable for type theoretical expositions. For example, we describe quantification by adjoints to weakening functors and not to substitution functors; therefore, a general form of weakening functor will be introduced, see 4.1.1 and 4.1.2.

Typed lambda calculus started with Curry & Feys [1958] and Howard [1970], who considered propositional aspects. Type dependency was brought in by de Bruijn (with the AUTOMATH project, see e.g. de Bruijn [1970]) followed by Martin-Löf (with his intuitionistic type theory, see e.g. Martin-Löf [1984]). In the 1980's the field grew rapidly, mainly by the interest shown from the computer science community.

Categorically, propositional calculi are straightforward; except maybe, for higher order quantification, but that is not what we want to focus on now. Contexts are simply cartesian products of the constituent types, since there is no type dependency involved. In case such dependencies may occur, things become categorically more interesting: contexts are no longer cartesian products, but a form of disjoint sum is needed to model such depending chains of types. The first studies are Cartmell [1978] and Seely [1984].

It thus turned out that the main operation which had to be explained categorically was "context extension" (or "context comprehension" as we sometimes like to call it): given a context Γ and a type $\Gamma \vdash \sigma : Type$, what is the meaning of the context $\Gamma, x{:}\sigma$ (i.e. Γ extended with an extra variable declaration). For this purpose, various notions have been introduced: contextual categories (Cartmell [1978], Streicher [1989]), categories with attributes (Cartmell [1978], Moggi [1991]), display-map categories (Taylor [1987], Hyland & Pitts [1989], Lamarche [1988]), D-categories (Ehrhard [1988a], [1988b]), IC of IC's (Obtułowicz [1989]), categories with fibrations (Pitts [1989]), comprehensive fibrations (Pavlović [1990]) and comprehension categories (Jacobs [1990]). In fact, there are so many notions around that almost everyone working in the field can cherish a private one.

In this thesis we work exclusively with comprehension categories to describe type dependency. Among the above alternatives, comprehension categories are in our opinion at the right level of generality and abstraction: once the notion is fully understood, closure properties (like under change-of-base) or generalizations (like over a fibration) suggest themselves in an obvious way. Much of this work can be read as a systematic exposition of categorical type theory in terms of comprehension categories.

We briefly outline the contents of the five chapters. The first one is about indexing of categories; it contains the basic definitions and results, mainly about fibrations, but also about indexed and internal categories. These are well-established, either in the literature or in the "folklore".

Type theory is the subject of the next chapter. The main innovation here is the description of type systems in terms of "settings plus features". A setting describes the dependencies which may occur, like whether or not a proposition may depend on a type (i.e. contain a variable of a certain type). Features — like products, sums, exponents, axioms or constants — are added on top of a specific setting. In such a way, one obtains individual systems.

The subsequent three chapters show how type theoretical settings can be translated into categorical settings and how type theoretical features can be tranlated into categorical features on top of the translated settings. A categorical setting can be understood as a generalization of Lawvere's notion of algebraic theory. For the settings without type dependency, the translation can be done in a relatively easy way; it may be found directly in chapter 3. There, one finds the standard descriptions for the "left plane" of the cube of typed lambda calculi from Barendregt [1991]. Translations in general are postponed until section 5.1.

Inbetween, the categorical description of type dependency is the subject of chapter 4. It consists of a thorough investigation of comprehension categories and quantification. It is the basis for the translation of settings and features in the beginning of chapter 5 and for the categorical description of some individual systems later in that chapter. Finally, we close with a revision of the semantics of the untyped lambda calculus. Appropriate comprehension categories yield a new notion of "categorical λ-algebra". These are related to set theoretical λ-algebras via an adjunction — which forms an improvement with respect to the categorical structures used by Scott and Koymans.

As already mentioned, this work can be seen as a survey of categorical type theory. It seems therefore appropriate to point out what we consider to be our own contributions.

- The notion of a comprehension category and the related results, see sections 4.1 – 4.4. More specifically the double role these categories play: one time as a model and one time as a domain of quantification. Also the notion of a *closed* comprehension category; it can be seen as a syntax-free description of a structure with dependent products and sums, which has good closure properties.

- The notion of a setting (see 2.1.1), which formalizes the type theoretical relation of dependency. The exposition that "being fibred over" is the categorical counterpart of this relation.

- The translation from type theoretical settings and features to categorical settings and features, using (constant) fibrations and (constant) comprehension categories. Constant fibrations or comprehension categories are used if the relevant dependency does not occur, see section 5.1.

- A number of free constructions linking the most important notions, see 3.3.5, 4.3.10, 4.4.13 and 4.4.16.

- A categorical description of type theoretical exponents without assuming cartesian product types, see 4.2.6.

- The description of a topos as a "split" model of the calculus of constructions, i.e. as a model in which all the relevant structure exists up-to-equality, see 4.3.5 and 5.2.6 (i).

- The revision of the semantics of the untyped lambda calculus.

- A systematic exposition of categorical type theory in terms of fibrations and comprehension categories.

We state that there is no claim to completeness in our survey. Here are two topics which are not covered. First there is nothing about coherence of the various mediating isomorphisms which occur when dealing with "non-split" structures. Although coherence problems have an established categorical interest, we don't think they are really important from a type theoretical point of view (at least not with respect to the type theories considered here): every concrete example of a model we know of can be presented in a "split" way. Indeed, we are particularly keen on presenting them in such a way. In order to obtain this we use "family"-models in which one has "substitution by composition" instead of "substitution by pullbacks".

Secondly, there is nothing about the interpretation of the various typed λ-calculi in their corresponding categories. A bit more categorical, we don't describe the various term models as free constructions. This omission seems more serious; it is motivated by the following two reasons. (1) Writing out interpretations is very laborious; it certainly requires technical skills but it does not seem to bring much conceptually. (2) With the growth of experience in this field, the necessity of having interpretations diminishes: from a certain point on, one doesn't really see much difference anymore between the type theoretical or categorical description of a specific system.

This brings us to the relation between type theoretical and categorical descriptions. We like to see the latter as description at the "assembly" level: categorical formulations require far more attention for details, like substitution or coherence. Programming in type theory is much smoother and proceeds at a level where many of these aspects are trivialized. Thus one can view typed lambda calculi as higher level languages for certain categorical structures.

Chapter 1

Basic Fibred Category Theory

In typed and untyped lambda calculus, contexts play an important structural role. They can be seen as *indices* for the terms and types derivable in that context. It is for this reason that the categorical study of λ-calculi which we are about to undertake starts with the investigation of "indexing". Fibrations form the appropriate categorical concept; they provide a framework for describing categories parametrized by some base category.

In order to understand how the indexing of categories takes place, it is instructive to take a look at indexing of sets first. Indexed sets are described basically in two ways. (1) As a family $\{X_i\}_{i\in I}$, which roughly means, as a map $X : I \to \mathbf{Sets}$, the universe of sets. (2) As a map $f : Y \to I$, where I is still the index-set; the indexed sets are then given by the *fibres* $f^{-1}(\{i\})$. There are obvious translations between these two approaches and the indexing works well in either case, see 1.1.6 for a more mathematical formulation of this statement. For technical reasons however, indexing of categories can best be done in the second way, i.e. with a functor $p : \mathbf{E} \to \mathbf{B}$ satisfying certain properties, which make it a fibration. Every object $A \in \mathbf{B}$ determines a *fibre* category $p^{-1}(A)$ — written usually as $\mathbf{E}_A$ — consisting of objects $E \in \mathbf{E}$ with $pE = A$ and morphism f in $\mathbf{E}$ with $pf = id_A$. In more type-theoretical formulation, one can think of objects $A \in \mathbf{B}$ as contexts and of objects and arrows in $\mathbf{E}_A$ as types and terms in context A. Arrows between contexts in the base category $\mathbf{B}$ can then be seen as substitutions, like in the abstract syntax used by Curien [1989], [1990]. The categorical counterpart of (1) is given by so-called indexed categories, which will be investigated in section 3 below.

This introductory chapter contains only "folklore" material, developed mostly by A. Grothendieck and J. Bénabou. Hence there is no claim to originality.

Although the definition of a fibration is not so difficult, it appears that one does not obtain a practical "working knowledge" of fibrations so easily. Readers unfamiliar with this field are urged to take ample time for this first chapter.

1.1 Fibrations

1.1.1. Basics. Suppose we have a functor $p : \mathbf{E} \to \mathbf{B}$. An object $E \in \mathbf{E}$ (resp. a morphism f in $\mathbf{E}$) is said to be *above* $A \in \mathbf{B}$ (resp. u in $\mathbf{B}$) if $pE = A$ (resp. $pf = u$). A morphism above an identity is called *vertical*. Every object $A \in \mathbf{B}$ thus determines a so-called "fibre" category $\mathbf{E}_A$ consisting of objects above A and vertical morphisms. It is useful to write $\mathbf{E}_u(E, D) = \{f : E \to D \mid pf = u\}$, where it is assumed that $u : pE \to pD$ in $\mathbf{B}$. One often calls $\mathbf{B}$ the *base* category and $\mathbf{E}$ the *total* category.

A morphism $f : D \to E$ in $\mathbf{E}$ is called *cartesian* over a morphism u in $\mathbf{B}$ if f is above u and every $f' : D' \to E$ with $pf' = u \circ v$ in $\mathbf{B}$, uniquely determines a $\phi : D' \to D$ above v with $f \circ \phi = f'$. The functor $p : \mathbf{E} \to \mathbf{B}$ is called a *fibration* if for every $E \in \mathbf{E}$ and $u : A \to pE$ in $\mathbf{B}$, there is a cartesian morphism with codomain E above u. Alternative names are *fibred category* or *category over* $\mathbf{B}$. Dually, $f : D \to E$ is *cocartesian* over u if every $f' : D \to E'$ with $pf' = v \circ u$, uniquely determines a $\phi : E \to E'$ above v with $\phi \circ f = f'$. And: p is a *cofibration* if every morphism $pE \to A$ in $\mathbf{B}$ has a "cocartesian lifting" with domain E; it is called a *bifibration* if it is at the same time a fibration and a cofibration.

These notions are due to A. Grothendieck.

1.1.2. Examples. Let $\mathbf{B}$ be an arbitrary category and let $\mathbf{B}^{\to}$ be the functor category from the partial order $\cdot \to \cdot$ to $\mathbf{B}$. Alternatively, one can think of $\mathbf{B}^{\to}$ as the comma category $(\mathbf{B} \downarrow \mathbf{B})$. This "arrow category" $\mathbf{B}^{\to}$ has morphism of $\mathbf{B}$ as objects and commuting squares as morphisms. Similarly, there is a category $\mathbf{B}^{\to\to}$.

The functor $dom : \mathbf{B}^{\to} \to \mathbf{B}$ forms an example of a fibration. Also, for every $A \in \mathbf{B}$ one has a fibration $dom_A : \mathbf{B}/A \to \mathbf{B}$, where $\mathbf{B}/A$ is the *slice* category having arrows with codomain A as objects and commuting triangles as morphisms.

In case the category $\mathbf{B}$ has pullbacks, the functor $cod : \mathbf{B}^{\to} \to \mathbf{B}$ forms an example of a fibration; cartesian morphisms in $\mathbf{B}^{\to}$ are given by pullback squares. The fibres are (isomorphic to) the slice categories $\mathbf{B}/A$. This functor cod is in fact a bifibration. The (obvious) functor $cod^{\to} : \mathbf{B}^{\to\to} \to \mathbf{B}^{\to}$ is a fibration as well. Even more, the composition $\mathbf{B}^{\to\to} \longrightarrow \mathbf{B}^{\to} \longrightarrow \mathbf{B}$ yields an example of a fibration. The latter fact can be checked by hand, but it actually follows from lemma 1.1.5 below, which says that fibrations are closed under composition.

Every category $\mathbf{C}$ gives rise to a "family fibration" $Fam(\mathbf{C}) \to \mathbf{Sets}$. The total category $Fam(\mathbf{C})$ has families $\{X_i\}_{i \in I}$ of $\mathbf{C}$-objects as objects; these may be described by a pair (I, X) with $X : I \to \mathbf{C}$. Morphisms $(u, \{f_i\}_{i \in I}) : (I, X) \to (J, Y)$ in $Fam(\mathbf{C})$ are given by a function $u : I \to J$ such that for every $i \in I$ one has $f_i : X_i \to Y_{u(i)}$ in $\mathbf{C}$. The first projection $Fam(\mathbf{C}) \to \mathbf{Sets}$ then forms a fibration; one has that $(u, \{f_i\}_{i \in I})$ is cartesian iff every f_i is an isomorphism.

Let **Top** be the category of topological spaces with continuous maps. The forgetful functor $U : \mathbf{Top} \to \mathbf{Sets}$ is a fibration since a function $f : I \to U(X)$ can be lifted to a continuous map $f : f^*(X) \to X$, where $f^*(X)$ is the set I provided

with the topology induced by f, i.e. with opens $\{f^{-1}(U) \mid U \subseteq X \text{ open}\}$. It is the weakest topology on I which makes f continuous.

Some trivial examples of fibrations are given by the identity functor $\mathbf{C} \to \mathbf{C}$ and the unique functor $\mathbf{C} \to \mathbf{1}$ to the terminal category. These are both instances of the "constant" fibration $Fst : \mathbf{B} \times \mathbf{C} \to \mathbf{B}$.

Finally, here are two constructions to form a new fibration from a given one. Let $p : \mathbf{E} \to \mathbf{B}$ be a fibration. The category $Cart(\mathbf{E})$ is described by objects $E \in \mathbf{E}$ and *cartesian* morphisms between them — using that cartesian morphisms are closed under composition. We write $|p| : Cart(\mathbf{E}) \to \mathbf{B}$ for the obvious functor obtained by restriction. All fibre categories of $|p|$ are groupoids, since a morphism which is at the same time vertical and cartesian is an isomorphism.

For the second construction, we write $V(\mathbf{E})$ to denote the full subcategory of $\mathbf{E}^{\to}$ with *vertical* arrows as objects. More explicitly, objects of $V(\mathbf{E})$ are vertical arrows $\alpha : E' \to E$ and morphisms $(f, g) : (\alpha : E' \to E) \to (\beta : D' \to D)$ are $f : E \to D$ and $g : E' \to D'$ in $\mathbf{E}$ satisfying $f \circ \alpha = \beta \circ g$. One obtains an "arrow fibration" $p^{\to} : V(\mathbf{E}) \longrightarrow \mathbf{E} \longrightarrow \mathbf{B}$ by first applying the "codomain" functor and then p. One has that (f, g) is $p^{\to}$-cartesian iff both f and g are p-cartesian. Notice that the fibre $V(\mathbf{E})_A$ is $(\mathbf{E}_A)^{\to}$.

1.1.3. Further investigation. If $p : \mathbf{E} \to \mathbf{B}$ is a fibration and $f : D \to E$ and $f' : D' \to E$ are both cartesian morphisms over u, then $f \cong f'$ in $\mathbf{E}/E$ by a vertical isomorphism. Hence given $u : A \to B$ in $\mathbf{B}$ and E above B, it makes sense to *choose* a cartesian lifting of u with codomain E; we often write $\overline{u}(E) : u^*(E) \to E$ for such a choice. A collection of choices — for every appropriate u and E — is called a *cleavage*. It induces for every $u : A \to B$ a functor $u^* : \mathbf{E}_B \to \mathbf{E}_A$, called *inverse image, reindexing, relabelling* or *substitution* functor. Different cleavages give rise to different, but naturally isomorphic, reindexing functors. In general, one obtains vertical natural isomorphisms $(u \circ v)^* \cong v^* \circ u^*$ and $id^* \cong Id$, as for pullbacks in case of $cod : \mathbf{B}^{\to} \to \mathbf{B}$. If one happens to have identities here (for a certain cleavage), one says that the fibration can be *split*. Notice that one can always choose $id^* = Id$. A *split* fibration is understood here as a fibration which is given together with such a "splitting". The fibration $Fam(\mathbf{C}) \to \mathbf{Sets}$ mentioned in the examples above, has a splitting: for $u : I \to J$ and $\{X_j\}_{j \in J}$ one can take as cartesian lifting $(u, \{id_{X_{u(i)}}\}_{i \in I})$. Similarly, one says that a fibration is *cloven* if it is given together with a cleavage. For every fibration, one can use a suitable form of the axiom of choice to obtain a cleavage.

It is important to notice that such reindexing functors u^* are *implicitly* determined in the definition of a fibration. As is often stressed by J. Bénabou, only *intrinsic* properties of fibrations are of interest, i.e. properties which do not depend in any way on choices of inverse images. A subtle example is the following. Let's say that a fibration $p : \mathbf{E} \to \mathbf{B}$ satisfies property $(*)$ if every reindexing functor u^* has a left adjoint Σ_u. Then $(*)$ is an intrinsic property: it does not depend on the choice of the functors u^* for a given u in $\mathbf{B}$, since these are determined up-to-isomorphism and so are adjoints. Side-remark: it is a standard result that p satisfies $(*)$ iff p is a bifibration, see e.g. Jacobs [1990].

A morphism between fibrations p and q is given by a commuting square as below, in which the functor H preserves cartesian morphisms, i.e. f is p-cartesian implies that Hf is q-cartesian.

$$\begin{tikzcd} \mathbf{E} \arrow[r, "H"'] \arrow[d, "p"'] & \mathbf{D} \arrow[d, "q"] \\ \mathbf{B} \arrow[r, "K"] & \mathbf{A} \end{tikzcd}$$

Given a fibration $q: \mathbf{D} \to \mathbf{A}$ and an arbitrary functor $K: \mathbf{B} \to \mathbf{A}$ one can form the pullback

$$\begin{tikzcd} \mathbf{B} \underset{K,q}{\times} \mathbf{D} \arrow[r, "K'"'] \arrow[d, "K^*(q)"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathbf{D} \arrow[d, "q"] \\ \mathbf{B} \arrow[r, "K"] & \mathbf{A} \end{tikzcd}$$

and verify that $K^*(q)$ is a fibration again. Notice that

$$(\mathbf{B} \underset{K,q}{\times} \mathbf{D})\big((B, D),\ (B', D')\big) \;=\; \dot{\bigcup}_{u \in \mathbf{B}(B,B')}.\ \mathbf{D}_{Ku}\big(D,\ D'\big),$$

where $\dot{\bigcup}$ denotes disjoint union. One easily verifies that (u, f) is $K^*(q)$-cartesian iff f is q-cartesian. As a result, a splitting or cleavage of q can be transferred to $K^*(q)$. Moreover, the above pullback diagram forms a morphism of fibrations. This construction is called *change-of-base* (for fibrations). As a result, the "functor" sending a fibration to its base, can be understood as a fibration itself. Usually, one writes $\mathit{Fib}(\mathbf{B})$ for the "fibre" category of fibrations with base $\mathbf{B}$; morphisms in $\mathit{Fib}(\mathbf{B})$ are called *cartesian functors* or *functors over* $\mathbf{B}$. We use $\mathit{Fib}(\mathbf{B})$ as a "category" only in a suggestive way, since we don't consider aspects of size. The "category" $\mathit{Fib}_{split}(\mathbf{B})$ contains split fibrations and morphisms which preserve the splitting on-the-nose (i.e. up-to-equality and not up-to-isomorphism).

The proofs of the next two elementary results are left to the reader.

1.1.4. Lemma. *Let $p: \mathbf{E} \to \mathbf{B}$ be a functor. One can form the pullback*

$$\begin{tikzcd} \mathbf{E} \underset{p,cod}{\times} \mathbf{B}^{\to} \arrow[r] \arrow[d, "p^*(cod)"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathbf{B}^{\to} \arrow[d, "cod"] \\ \mathbf{E} \arrow[r, "p"'] & \mathbf{B} \end{tikzcd}$$

and define a functor $\mathcal{I}: \mathbf{E}^{\to} \longrightarrow \mathbf{E} \underset{p,cod}{\times} \mathbf{B}^{\to}$ by $[f: E' \to E] \mapsto (E, pf)$. Then

$$p \textit{ is a cloven fibration} \quad \Leftrightarrow \quad \mathcal{I} \textit{ has a full and faithful right adjoint.} \ \square$$

1.1.5. Lemma. *Let $p: \mathbf{E} \to \mathbf{B}$ and $r: \mathbf{B} \to \mathbf{A}$ be fibrations.*

(i) *The functor $rp: \mathbf{E} \to \mathbf{A}$ is a fibration, with*

$$f \textit{ is } rp\textit{-cartesian} \quad \Leftrightarrow \quad f \textit{ is } p\textit{-cartesian and } pf \textit{ is } r\textit{-cartesian.}$$

(ii) *The functor p is cartesian from rp to r.*

(iii) *If $q: \mathbf{D} \to \mathbf{B}$ is another fibration, then*

$$F: p \to q \textit{ in } \mathit{Fib}(\mathbf{B}) \quad \Rightarrow \quad F: rp \to rq \textit{ in } \mathit{Fib}(\mathbf{A}). \ \square$$

1.1.6. Fibred 2-cells. Assume (K, H) and (L, G) are morphisms of fibrations (1-cells) as below.

$$\begin{tikzcd} \mathbf{E} \arrow[r, shift left=1.5ex, "H"] \arrow[r, shift right=1.5ex, "G"'] \arrow[r, phantom, "\Downarrow\tau"] \arrow[d, "p"'] & \mathbf{D} \arrow[d, "q"] \\ \mathbf{B} \arrow[r, shift left=1.5ex, "K"] \arrow[r, shift right=1.5ex, "L"'] \arrow[r, phantom, "\Downarrow\sigma"] & \mathbf{A} \end{tikzcd}$$

A 2-cell from (K, H) to (L, G) is a pair of natural transformations $(\sigma: K \overset{\cdot}{\longrightarrow} L, \tau: H \overset{\cdot}{\longrightarrow} G)$ such that τ is above σ. More precisely, every component τ_E is above σ_{pE}. In the same way, one obtains 2-structure for split fibrations.

As an application of these notions, one may verify that there is an equivalence — which is a 2-categorical notion — over **Sets**,

$$\begin{tikzcd} \mathit{Fam}(\mathbf{Sets}) \arrow[rr, shift left] \arrow[dr] & & \mathbf{Sets}^{\to} \arrow[ll, shift left] \arrow[dl, "cod"] \\ & \mathbf{Sets}, & \end{tikzcd}$$

see 1.1.2 for definitions of the fibrations involved. This equivalence forms the proper mathematical expression of the statement that the two ways of indexing sets, as mentioned in the introduction of this chapter, are essentially the same. Remember that the fibration $\mathit{Fam}(\mathbf{Sets}) \to \mathbf{Sets}$ is split, whereas $\mathbf{Sets}^{\to} \to \mathbf{Sets}$ is not. In general, split fibrations are more pleasant to work with.

Change-of-base as described above also has 2-categorical aspects, as will be shown in the next two lemmas. The first lemma deals with the 2-structure in the fibres and the second one with 2-structure on the base level. The latter one is essentially proposition 3 in Ehrhard [1988a].

1.1.7. Lemma. *Every functor $K:\mathbf{B}\to\mathbf{A}$ induces a "change-of-base" 2-functor $K^*: Fib(\mathbf{A})\to Fib(\mathbf{B})$. This 2-functor restricts to $Fib_{split}(\mathbf{A})\to Fib_{split}(\mathbf{B})$.*

Proof. Straightforward. □

1.1.8. Lemma. *Let $q:\mathbf{D}\to\mathbf{A}$ be a fibration and $K, L:\mathbf{B}\to\mathbf{A}$ (arbitrary) functors with a natural transformation $\sigma: K \stackrel{\cdot}{\longrightarrow} L$ between them. Then there is an (up-to-isomorphism) unique cartesian functor $\langle\sigma\rangle: L^*(q)\to K^*(q)$ provided with a natural transformation $\sigma': K'\circ\langle\sigma\rangle \stackrel{\cdot}{\longrightarrow} L'$,*

$$\begin{tikzcd}
\mathbf{B}\underset{L,q}{\times}\mathbf{D} \arrow[rr, "L'"] \arrow[dr, "\langle\sigma\rangle"] \arrow[dd, "L^*(q)"'] & \Uparrow \sigma' & \mathbf{D} \arrow[dd, "q"] \\
& \mathbf{B}\underset{K,q}{\times}\mathbf{D} \arrow[ur, "K'"'] \arrow[dl, "K^*(q)"] & \\
\mathbf{B} \arrow[rr, bend left=15, "L"] \arrow[rr, bend right=15, "K"'] \arrow[rr, phantom, "\Uparrow\sigma"] & & \mathbf{A}
\end{tikzcd}$$

such that the pair (σ,σ') is a 2-cell $(K, K'\circ\langle\sigma\rangle)\Longrightarrow(L, L')$ from $L^*(q)$ to q and σ' has cartesian components.

Proof. Because $\langle\sigma\rangle$ goes from $L^*(q)$ to $K^*(q)$ one must have that $\langle\sigma\rangle(B,D)$ is of the form $(B,\underline{D})$. Since $\sigma'_{(B,D)}:\underline{D}\to D$ is cartesian over σ_B, one has that $\underline{D}\cong\sigma_B^*(D)$. This determines the object-part of $\langle\sigma\rangle$ up-to-isomorphism. Similarly, the arrow-part is determined: for $(u,f):(B,D)\to(B',D')$ in $\mathbf{B}\underset{L,q}{\times}\mathbf{D}$ one has $\langle\sigma\rangle(u,f) = (u,\underline{f})$, where $\underline{f}:\underline{D}\to\underline{D'}$ is above Ku and makes by naturality of σ' the following diagram commute

$$\begin{tikzcd}
\underline{D} \arrow[r, "\sigma'_{(B,D)}"] \arrow[d, "\underline{f} = K'\circ\langle\sigma\rangle(u,f)"'] & D \arrow[d, "f = L'(u,f)"] \\
\underline{D'} \arrow[r, "\sigma'_{(B',D')}"] & D'.
\end{tikzcd}$$

Since $\sigma'_{(B',D')}$ is cartesian, there can be only one such arrow. This description gives at the same time a recipe for the construction of $\langle\sigma\rangle$ and σ'. □

1.1.9. Lemma (Fibred Yoneda). *Let $p:\mathbf{E}\to\mathbf{B}$ be a fibration and $A\in\mathbf{B}$.*

(i) *There is an equivalence of categories*

$$\mathbf{E}_A \;\simeq\; Fib(\mathbf{B})\big(dom_A,\ p\big).$$

A suitable formulation of the naturality involved may be found in the proof of proposition 1.3.6 below.

(ii) *In case p is a* split *fibration, one obtains an isomorphism*

$$\mathbf{E}_A \;\cong\; Fib_{split}(\mathbf{B})\big(dom_A,\ p\big).$$

The fibration $dom_A:\mathbf{B}/A\to\mathbf{B}$ is mentioned in 1.1.2 and $Fib(\mathbf{B})\big(-,\ -\big)$ denotes the "Hom"-category described in 1.1.6.

Proof. (i) One first uses a suitable version of the axiom of choice to obtain a cleavage for p. An object $E\in\mathbf{E}_A$ then determines a cartesian functor $Yon(E):\mathbf{B}/A\to\mathbf{E}$ by $u\mapsto u^*(E)$ and $[\phi:u\to v]\mapsto$ [the unique $\alpha:u^*(E)\to v^*(E)$ above ϕ satisfying $\overline{v}(E)\circ\alpha = \overline{u}(E)$]. A morphism $f:E\to E'$ in $\mathbf{E}_A$ determines a vertical natural transformation $Yon(f):Yon(E)\stackrel{\cdot}{\longrightarrow}Yon(E')$ with components $Yon(f)_u = u^*(f)$.

One obtains a functor $\Psi: Fib(\mathbf{B})\big(dom_A,\ p\big)\to\mathbf{E}_A$ by $F\mapsto F(id_A)$ and $\sigma\mapsto\sigma_{id_A}$. This yields the required equivalence.

(ii) The construction from (i) now yields an isomorphism, since

$$\begin{array}{rcll}
(\Psi\circ Yon)(E) & = & id_A^*(E) = E & \\
(Yon(E)\circ\Psi)(F)(u) & = & u^*(F(id_A)) & \\
& = & F(u^*(id_A)) & \text{since } F \text{ preserves the splitting} \\
& = & F(id_A\circ u) & \\
& = & F(u). & \square
\end{array}$$

1.1.10. Definition. A fibration $p:\mathbf{E}\to\mathbf{B}$ is called *representable* if it is equivalent to a fibration of the form $dom_A:\mathbf{B}/A\to\mathbf{B}$ for some $A\in\mathbf{B}$.

1.1.11. Opposite fibration (Bénabou [1975]). Let $p:\mathbf{E}\to\mathbf{B}$ be a fibration. A fibration $p^{op}:\mathbf{E}^{(op)}\to\mathbf{B}$ will be described which is "fibrewise" the opposite of p. A little care is needed to do this intrinsically. Let $CV = \{(f_1,f_2)\mid f_1$ is cartesian, f_2 is vertical and $dom(f_1) = dom(f_2)\}$. An equivalence relation is defined on the collection CV by $(f_1,f_2)\sim(g_1,g_2)\ \Leftrightarrow$ there is a vertical map h with $g_1\circ h = f_1$ and $g_2\circ h = f_2$. The equivalence class of (f_1,f_2) will be written as $[f_1,f_2]$.

The total category $\mathbf{E}^{(op)}$ of p^{op} has $E\in\mathbf{E}$ as objects. Morphisms $[f_1,f_2]:E\to D$ are given by

$$\begin{tikzcd}
E & \\
\bullet \arrow[u, "f_2"] \arrow[r, "f_1"] & D
\end{tikzcd}$$

Composition is described by

$$
\begin{tikzcd}
\bullet & & \\
\bullet \arrow[u] \arrow[r] & \bullet & \\
\bullet \arrow[u, dashed] \arrow[r, dashed] & \bullet \arrow[u] \arrow[r] & \bullet
\end{tikzcd}
$$

The functor $p^{op}: \mathbf{E}^{(op)} \to \mathbf{B}$ is then defined by $E \mapsto pE$ and $[f_1, f_2] \mapsto pf_1$. It is left to the reader to verify that

(i) p^{op} is a fibration, with $[f_1, f_2]$ cartesian iff f_2 is an isomorphism;

(ii) p^{op} is the fibrewise opposite, i.e. $\left(\mathbf{E}^{(op)}\right)_A \cong \left(\mathbf{E}_A\right)^{op}$;

(iii) $(p^{op})^{op} \cong p$.

Let $\mathbf{B}$ be a category with pullbacks. The total category $(\mathbf{B}^{\to})^{(op)}$ of the opposite of the fibration $cod: \mathbf{B}^{\to} \to \mathbf{B}$ is sometimes called the "inverse arrow category" and denoted by $Inv(\mathbf{B})$.

Taking the opposite of a *split* fibration can be done without taking equivalence classes as above.

1.2 Category theory over a base category

In the introduction of this chapter we stated that categories varying over a base category form the subject of study in fibred category theory. In the present section we describe how such variable categories can be provided with certain structure, like terminals or cartesian products.

The concept one needs to obtain such structure in fibre categories is that of a *fibred* adjunction; it is an adjunction in the 2-category of fibrations (with the same base category). Let's describe adjunctions explicitly; equivalences are then also well-understood.

1.2.1. Definition. Let $p: \mathbf{E} \to \mathbf{B}$ and $q: \mathbf{D} \to \mathbf{B}$ be fibrations. A *fibred adjunction* from p to q consists of a pair of cartesian functors $F: \mathbf{E} \to \mathbf{D}$ and $G: \mathbf{D} \to \mathbf{E}$ forming an adjunction $F \dashv G$ with vertical unit and counit.

$$
\begin{tikzcd}
\mathbf{E} \arrow[rr, shift left, "F"] \arrow[rr, phantom, "\perp"] \arrow[dr, "p"'] & & \mathbf{D} \arrow[ll, shift left, "G"] \arrow[dl, "q"] \\
& \mathbf{B} &
\end{tikzcd}
$$

Using the triangular identities of an adjunction, one easily verifies that the unit is vertical iff the counit is vertical. It is also worth noticing that change-of-base preserves fibred adjunctions, see 1.1.7.

Cartesian functors F and G as above determine for every object $A \in \mathbf{B}$ "fibrewise" functors $F \restriction A: \mathbf{E}_A \to \mathbf{D}_A$ and $G \restriction A: \mathbf{D}_A \to \mathbf{E}_A$ by restriction. Since unit and counit are vertical, one obtains an adjunction $F \restriction A \dashv G \restriction A$. These "fibrewise" adjunctions are preserved under reindexing. The precise meaning of the latter statement can be found in Jacobs [1990]. There, one also finds some more information about the following quite useful result.

1.2.2. Lemma. *Let $p: \mathbf{E} \to \mathbf{B}$ and $q: \mathbf{D} \to \mathbf{B}$ be fibrations and $H: \mathbf{E} \to \mathbf{D}$ a cartesian functor. The functor H has a fibred left (resp. right) adjoint if and only if both*

- *For every $A \in \mathbf{B}$, the functor $H \restriction A$ has a left (resp. right) adjoint $K(A)$.*
- *For every $u: A \to B$ in $\mathbf{B}$ and for every pair of reindexing functor $u^*: \mathbf{E}_B \to \mathbf{E}_A$ and $u^{\#}: \mathbf{D}_B \to \mathbf{D}_A$, the canonical natural transformation*

  $$K(A)\, u^{\#} \overset{\cdot}{\longrightarrow} u^*\, K(B) \qquad \text{(resp. } u^*\, K(B) \overset{\cdot}{\longrightarrow} K(A)\, u^{\#}\text{)}$$

  is an isomorphism. $\square$

The canonical map $K(A)\, u^{\#} \overset{\cdot}{\longrightarrow} u^*\, K(B)$ is the transpose of $u^{\#} \xrightarrow{u^{\#}(\eta)} u^{\#} H \restriction B K(B) \cong H \restriction A\, u^* K(B)$. Similarly, one obtains the other one.

Of the two equivalent formulations in the above lemma, the second "fibrewise" one is often closer to one's intuition, because it describes the structure induced by a fibred adjunction as structure in the fibres which is preserved under reindexing. Moreover it has practical advantages and therefore it will be used most of the time. The first formulation however, is more important from a theoretical point of view.

1.2.3. Definition. Let $\diamond \in$ {terminal (initial) object, binary (co-) product, (co-) equalizer, exponent}. We say that a fibration p has *fibred* $\diamond$'s if every fibre category has $\diamond$'s and all reindexing functors preserve the $\diamond$'s.

It is then clear what a "fibred CCC" or a "fibred LEX category" is. Sometimes this predicate "fibred" will be omitted. In Jacobs [1990] one may find definitions of these notions in terms of fibred adjunctions.

1.2.4. Examples. (i) Let $\diamond$ be as in the above definition. One has

$$\mathbf{C} \text{ has } \diamond\text{'s} \quad \Leftrightarrow \quad \mathit{Fam}(\mathbf{C}) \to \mathbf{Sets} \text{ has fibred } \diamond\text{'s}.$$

Bi-implications like these will occur also for other notions $\diamond$ which are transferred to the fibred context, see 4.2.5 (i), 4.4.8 (iii) and 4.5.3 (i).

(ii) The "fibration" sending a fibration to its basis (mentioned at the end of 1.1.3) has fibred finite products: the fibration $Id : \mathbf{B} \to \mathbf{B}$ is terminal in $Fib(\mathbf{B})$ and as product of $p : \mathbf{E} \to \mathbf{B}$ and $q : \mathbf{D} \to \mathbf{B}$ one can take $p \circ p^*(q) : \mathbf{E} \underset{p,q}{\times} \mathbf{D} \to \mathbf{B}$ (using 1.1.3 and 1.1.5).

(iii) Let $\mathbf{B}$ be a category with finite limits; it is easy to see that the fibration $cod : \mathbf{B}^{\to} \to \mathbf{B}$ has fibred finite limits. There is something more, every pullback functor u^* has a left adjoint Σ_u given by composition. By a standard result (see e.g. Jacobs [1990]) one obtains that cod is a bifibration.

This $\mathbf{B}$ is called a *locally cartesian closed category* (LCCC) if every fibre (or slice) category $\mathbf{B}/A$ is a CCC. Since the category $\mathbf{B}$ is isomorphic to the fibre above the terminal object, it is then cartesian closed itself. In case $\mathbf{B}$ is an LCCC one has that $cod : \mathbf{B}^{\to} \to \mathbf{B}$ is a fibred CCC, since exponents are automatically preserved: for $u : A \to B$ in $\mathbf{B}$, one has

$$\begin{array}{rcll}
\mathbf{B}/A\big(\,h,\ u^*(f \Rightarrow g)\big) & \cong & \mathbf{B}/B\big(\,\Sigma_u(h),\ f \Rightarrow g\,\big) & \\
& \cong & \mathbf{B}/B\big(\,f \times \Sigma_u(h),\ g\,\big) & \\
& \cong & \mathbf{B}/B\big(\,\Sigma_u(u^*(f) \times h),\ g\,\big) & \text{by composition of pullbacks} \\
& \cong & \mathbf{B}/A\big(\,u^*(f) \times h,\ u^*(g)\,\big) & \\
& \cong & \mathbf{B}/A\big(\,h,\ u^*(f) \Rightarrow u^*(g)\,\big). &
\end{array}$$

Hence an LCCC can also be defined as a category $\mathbf{B}$ having a terminal object and satisfying the property that the functor $cod : \mathbf{B}^{\to} \to \mathbf{B}$ is a fibred CCC. Later we shall come across other characterizations, see 4.2.5 (iii) and 4.5.3 (ii). The category $\mathbf{Sets}$ is an example of an LCCC; in fact, every topos is an LCCC.

1.2.5. Remarks. When working with fibred finite products, it is often quite convenient to have also a global description at hand. Let $p : \mathbf{E} \to \mathbf{B}$ therefore be a fibration with fibred finite products. The two constructions below make use of an arbitrary cleavage, but they don't depend on it.

(i) Having a fibred terminal object, implies that for every object $A \in \mathbf{B}$, there is a terminal object, say $1A$, in the fibre category $\mathbf{E}_A$. Suppose $E \in \mathbf{E}$ above A and $u : A \to B$ in $\mathbf{B}$ are given. Since $u^*(1B) \cong 1A$ one has that $\mathbf{E}_u(E, 1B)$ contains exactly one arrow. Hence we obtain a functor $1 : \mathbf{B} \to \mathbf{E}$ such that $p \circ 1 = Id$. Moreover, one can show that $1 : Id_{\mathbf{B}} \to p$ is a fibred right adjoint to p in $Fib(\mathbf{B})$. We often assume that fibred terminal objects are described by such a functor 1 from the base to the total category.

(ii) Preservation of fibred cartesian products by reindexing functors means that for every $u : A \to B$ in $\mathbf{B}$ and $E, E' \in \mathbf{E}_B$ one has that the canonical map

$$\langle u^*(\pi), u^*(\pi') \rangle \;:\; u^*(E \times E') \;\longrightarrow\; u^*(E) \times u^*(E')$$

is an isomorphism. Hence for any pair of maps $f : D \to E$ and $g : D' \to E'$ in $\mathbf{E}$ with $pf = pg = u$, say, there is a unique $h : D \times D' \to E \times E'$ above u with $\pi \circ h = f \circ \pi$ and $\pi' \circ h = g \circ \pi'$. This property leads us to denote h by $prod(f, g)$. We obtain a cartesian functor $prod : p \times p \to p$ which is a fibred right adjoint to an obvious diagonal functor.

1.2.6. Definition. Let $\diamond$ be as in definition 1.2.3. Suppose that $(K : \mathbf{B} \to \mathbf{B}',\ L : \mathbf{E} \to \mathbf{E}')$ is a morphism between fibrations $p : \mathbf{E} \to \mathbf{B}$ and $p' : \mathbf{E}' \to \mathbf{B}'$ (cf. 1.1.3). We say that (K, L) preserves fibred $\diamond$'s if L is fibrewise a $\diamond$-preserving functor.

1.2.7. A fundamental construction. Suppose a fibration $p : \mathbf{E} \to \mathbf{B}$ is given which has fibred finite products. A new fibration $\overline{p} : \overline{\mathbf{E}} \to \mathbf{E}$ is constructed in the following way. The category $\overline{\mathbf{E}}$ has pairs $E, E' \in \mathbf{E}$ with $pE = pE'$ as objects; morphisms $(f, g) : (E, E') \to (D, D')$ in $\overline{\mathbf{E}}$ are given by arrows $f : E \to D$ and $g : E \times E' \to D'$ in $\mathbf{E}$ with $pf = pg$. Composition in $\overline{\mathbf{E}}$ is given by $(f, g) \circ (h, k) = (f \circ h,\ g \circ \langle h \circ \pi, k \rangle)$ — using the global product from remark 1.2.5 (ii) — and identity by (id, π'). The first projection $\overline{p} : \overline{\mathbf{E}} \to \mathbf{E}$ is then a fibration with $(f, g) : (E, E') \to (D, D')$ is $\overline{p}$-cartesian iff there is a vertical isomorphism,

$$\begin{tikzcd}
E & & E \times u^*(D') \arrow[ll, "\pi"'] \arrow[r, "\pi'"] & u^*(D') \arrow[r] & D' \\
& & \wr\| & & \\
& & E \times E' \arrow[uull, "\pi"] \arrow[uurr, "g"'] & &
\end{tikzcd}$$

where $u = pf = pg$. One easily verifies that $\overline{p}$ has fibred finite products again. Moreover that there is a change-of-base situation,

$$\begin{tikzcd}
\mathbf{E} \arrow[r, "H"] \arrow[d, "p"'] \arrow[dr, phantom, "\lrcorner", very near start] & \overline{\mathbf{E}} \arrow[d, "\overline{p}"] \\
\mathbf{B} \arrow[r, "1"] & \mathbf{E}
\end{tikzcd}$$

in which both 1 (for terminals) and H are full and faithful functors. Further, $(1, H)$ preserves the fibred finite products. In case p is a fibred CCC, also $\overline{p}$ is fibred CCC and the above map preserves the CCC-structure.

The fibration $\overline{p}:\overline{\mathbf{E}} \to \mathbf{E}$ has a clear logical significance: for $E \in \mathbf{E}$ above A, one has that the fibre category $\overline{\mathbf{E}}_E$ is the polynomial category $\mathbf{E}_A[x:1A \to E]$ obtained from the fibre category $\mathbf{E}_A$ by adjoining a variable x of type E, see Lambek and Scott [1986], part I, 5 and 7. It is readily established that $\overline{\mathbf{E}}_E$ is the Kleisli category of the comonad $E \times -$ mentioned there.

In case we additionally assume that p has fibred equalizers (i.e. that it is a fibred LEX category), then the codomain functor $V(\mathbf{E}) \to \mathbf{E}$ mentioned at the end of 1.1.2 yields a similar situation. First of all, we notice that $cod: V(\mathbf{E}) \to \mathbf{E}$ is now a fibration with $(f, g): \alpha \to \beta$ in $V(\mathbf{E})$ cartesian iff it is a pullback square in $\mathbf{E}$. This new fibration has fibred finite limits again; further, there is a change-of-base situation,

$$\begin{tikzcd} \mathbf{E} \arrow[r, "L"] \arrow[d, "p"'] \arrow[dr, phantom, "\lrcorner", very near start] & V(\mathbf{E}) \arrow[d, "cod"] \\ \mathbf{B} \arrow[r, "1"] & \mathbf{E}, \end{tikzcd}$$

in which 1 and L are full and faithful functors; this map $(1, L): p \to cod$ preserves fibred finite limits. Notice that for $E \in \mathbf{E}$ above A, the fibre category $V(\mathbf{E})_E$ is the slice category $\mathbf{E}_A/E$, which is — in the presence of equalizers — the polynomial category $\mathbf{E}_A[x:1A \to E]$. The latter insight is attributed to A. Joyal in Lambek [1989], see also Lambek and Scott [1986], part II, 16 exercise 2.

1.2.8. Lemma. *Let $p: \mathbf{E} \to \mathbf{B}$ be a fibration and $\diamond \in$ {terminal object, cartesian product, equalizer}. Suppose the category $\mathbf{B}$ has $\diamond$'s; then*

$$p \text{ has fibred } \diamond\text{'s} \quad \Leftrightarrow \quad \mathbf{E} \text{ has } \diamond\text{'s and } p \text{ preserves them.}$$

Proof. We shall do the case of cartesian products.
($\Rightarrow$) Suppose $E \in \mathbf{E}$ above A and $D \in \mathbf{E}$ above B are given. Then $E \,\&\, D = \pi_{A,B}^*(E) \times \pi'^*_{A,B}(D)$ — where $\times$ denotes the product in the fibre $\mathbf{E}_{A\times B}$ — forms a product in the category $\mathbf{E}$.
($\Leftarrow$) For $E, E' \in \mathbf{E}$ above A, take $E \times E' = \delta^*(E \,\&\, E')$, where $\delta: A \to A \times A$ is the diagonal. □

1.2.9. Definition. (i) A fibration $p: \mathbf{E} \to \mathbf{B}$ is said to have a *generic object* if there is an object $T \in \mathbf{E}$ such that for every $E \in \mathbf{E}$ there is a cartesian arrow $E \to T$.

In view of the fibred Yoneda lemma 1.1.9, this means that the induced functor $\mathbf{B}/pT \to \mathbf{E}$ is essentially surjective on objects.

(ii) A morphism $(K: \mathbf{B} \to \mathbf{B}', L: \mathbf{E} \to \mathbf{E}')$ between fibrations $p: \mathbf{E} \to \mathbf{B}$ and $p': \mathbf{E}' \to \mathbf{B}'$ with generic objects $T \in \mathbf{E}$ and $T' \in \mathbf{E}'$ preserves these generic objects if there is an isomorphism $LT \cong T'$.

1.2.10. Examples. (i) Let $\mathbf{C}$ be a category with a *small* collection of objects, denoted by $\Omega = Obj(\mathbf{C})$. The fibration $Fam(\mathbf{C}) \to \mathbf{Sets}$ then has generic object $T = (\Omega, id_\Omega) = \{c\}_{c\in\Omega} \in Fam(\mathbf{C})$ above Ω. For every object $\{X_i\}_{i\in I} \in Fam(\mathbf{C})$, one has $X: I \to \Omega$ in $\mathbf{Sets}$ satisfying $X^*(T) = X^*(\Omega, id_\Omega) = (I, id_\Omega \circ X) = (I, X)$.

(ii) Let $\mathbf{B}$ be a category with pullbacks. We write $Sub(\mathbf{B})$ for the full subcategory of $\mathbf{B}^\to$ with monic arrows as objects. Since monics are preserved by pullback functors, the functor $cod: Sub(\mathbf{B}) \to \mathbf{B}$ is a fibration. In case $\mathbf{B}$ is a topos, this fibration has a generic object, viz. the subobject classifier.

(iii) Suppose $p: \mathbf{E} \to \mathbf{B}$ is a fibration with finite products and a generic object. We claim that the fibration $\overline{p}: \overline{\mathbf{E}} \to \mathbf{E}$ from 1.2.7 then also has a generic object and that the above map $p \to \overline{p}$ is a morphism of generic objects. To prove this, we assume that $T \in \mathbf{E}$ above Ω forms a generic object for p. Then $(1\Omega, T) \in \overline{\mathbf{E}}$ above 1Ω is generic for $\overline{p}$, since for an object $(E, E') \in \overline{\mathbf{E}}$, we can find an arrow $u: pE \to \Omega$ in $\mathbf{B}$ satisfying $u^*(T) \cong E'$. By remark 1.2.5 (i), one obtains a (unique) arrow $f: E \to 1\Omega$ above u in $\mathbf{E}$. Then $f^*(1\Omega, T) = (E, u^*(T)) \cong (E, E')$.

The above notion of generic object is clearly intrinsic (i.e. it does not depend on a choice of inverse images). Since we want this property, we are forced to use such a weak notion. For split fibrations one can do better. First we mention that a split fibration $p: \mathbf{E} \to \mathbf{B}$ determines an obvious fibration $|p|: Split(\mathbf{E}) \to \mathbf{B}$, where $Split(\mathbf{E})$ has all objects from $\mathbf{E}$, but only the cartesian morphisms given by the splitting between them. The fibres of $|p|$ are then discrete categories. For non-split fibrations, a similar construction yields the groupoid fibration $|p|: Cart(\mathbf{E}) \to \mathbf{B}$ as described in 1.1.2.

1.2.11. Definition. (i) We say that a split fibration $p: \mathbf{E} \to \mathbf{B}$ has a *split generic object* if the (discrete) fibration $|p|: Split(\mathbf{E}) \to \mathbf{B}$ is representable. More explicitly, if there is an object $\Omega \in \mathbf{B}$ and a collection of isomorphisms $\phi_B: \mathbf{B}(B, \Omega) \to Obj(\mathbf{E}_B)$ natural in B: for $u: B \to B'$ one has $\phi_B(v \circ u) = u^*(\phi_{B'}(v))$. In that case, $T = \phi_\Omega(id_\Omega)$ yields a generic object as in the previous definition.

(ii) A pair $(K: \mathbf{B} \to \mathbf{B}', L: \mathbf{E} \to \mathbf{E}')$ of functors forming a morphism of split fibrations from $p: \mathbf{E} \to \mathbf{B}$ to $p': \mathbf{E}' \to \mathbf{B}'$ is a map of split generic objects $\phi_B: \mathbf{B}(B, \Omega) \to Obj(\mathbf{E}_B)$ and $\phi'_A: \mathbf{B}'(A, \Omega') \to Obj(\mathbf{E}'_A)$, if there is an isomorphism $\alpha: K\Omega \xrightarrow{\sim} \Omega'$ such that $\phi'_{KB}(\alpha \circ u) = L\phi_B(u)$.

In the first example above, one has a split generic object.

1.2.12. Extended example (Realizability Models).
The category ω-$\mathbf{Set}$ has objects $A = (|A|, \vdash_A)$, where $|A|$ is a set and $\vdash_A \subseteq \mathbb{N} \times |A|$ is a relation satisfying $\forall a \in |A|.\ \exists n \in \mathbb{N}.\ n \vdash_A a$. Morphisms $f: A \to B$ in ω-$\mathbf{Set}$ are given by functions $f: |A| \to |B|$ for which there is a *realizer* $n \in \mathbb{N}$ such that $\forall a \in |A|.\ \forall m \in \mathbb{N}.\ m \vdash_A a \Rightarrow n \cdot m \vdash_B f(a)$, where $n \cdot m$ denotes the result of n-th partial recursive function applied to m. It is left to the reader to verify that ω-$\mathbf{Set}$ is an LCCC. There is a full and faithful functor $\Delta : \mathbf{Sets} \hookrightarrow \omega$-$\mathbf{Set}$

given by $X \mapsto (X, \mathbb{N} \times X)$. It induces a morphism of fibred CCC's between the relevant codomain fibrations. This functor Δ is right adjoint to the global sections (or forgetful) functor $\Gamma : \omega\text{-}\mathbf{Set} \to \mathbf{Sets}$.

The full subcategory $\mathbf{M}$ of so-called "modest ω-sets" has objects $A = (|A|, \vdash_A)$ satisfying $\forall a, a' \in |A|.\ \forall n \in \mathbb{N}.\ n \vdash_A a \ \&\ n \vdash_A a' \ \Rightarrow\ a = a'$. As shown in Ehrhard [1989], the inclusion functor $\mathbf{M} \hookrightarrow \omega\text{-}\mathbf{Set}$ has a left adjoint Θ — which constitutes a reflection. For $A = (|A|, \vdash_A) \in \omega\text{-}\mathbf{Set}$, one first defines a relation $\smile$ on $|A|$ by $a \smile a' \ \Leftrightarrow\ \exists n \in \mathbb{N}.\ n \vdash_A a \ \&\ n \vdash_A a'$. Then one takes $\sim$ to be the transitive closure of $\smile$. Finally, one can put $\Theta A = (|A|/\sim, \vdash_{\Theta A})$, with $n \vdash_{\Theta A} [a] \Leftrightarrow \exists a' \in [a].\ n \vdash_A a'$. As a consequence of this reflection, the category $\mathbf{M}$ has finite limits, which are preserved by the inclusion. It is easy to verify that $\mathbf{M}$ is also an LCCC and that the inclusion $\mathbf{M} \hookrightarrow \omega\text{-}\mathbf{Set}$ induces a morphism of fibred CCC's (between the codomain fibrations).

Let $\mathit{PER} = \{R \subseteq \mathbb{N} \times \mathbb{N} \mid R \text{ is a symmetric and transitive relation}\}$ be the set of "partial equivalence relations". For $R \in \mathit{PER}$, one writes $Q(R) = \{[n]_R \mid n \in \mathit{dom}(R)\}$, where $[n]_R = \{m \in \mathbb{N} \mid mRn\}$ and $\mathit{dom}(R) = \{n \in \mathbb{N} \mid nRn\}$. Notice that $\bigcup Q(R) \subseteq \mathit{dom}(R)$. One obtains a category $\mathbf{PER}$ with objects $R \in \mathit{PER}$ and morphism $f : R \to S$ given by functions $f : Q(R) \to Q(S)$ which have a *realizer* $n \in \mathbb{N}$ such that for every $m \in \mathit{dom}(R)$, one has $f([m]_R) = [n \cdot m]_S$. Interestingly, there is an equivalence of categories,

$$\begin{tikzcd} \mathbf{M} \arrow[r, shift left, "\Phi"] & \mathbf{PER}, \arrow[l, shift left, "\Psi"] \end{tikzcd}$$

given as follows. For $A = (|A|, \vdash_A) \in \mathbf{M}$, take $\Phi(A) = \{(n, m) \mid \exists a \in |A|.\ n \vdash_A a \ \&\ m \vdash_A a\}$. For $R \in \mathbf{PER}$, put $\Psi(R) = (Q(R), \in)$.

Let $\mathbf{C}$ be $\omega\text{-}\mathbf{Set}$ or $\mathbf{M}$. The category $\mathit{Fam}_{\mathrm{eff}}(\mathbf{C})$ has pairs (A, X) with $A \in \omega\text{-}\mathbf{Set}$ and $X : |A| \to \mathbf{C}$ as objects. A morphism $(f, \alpha) : (A, X) \to (B, Y)$ consists of a map $f : A \to B$ in $\omega\text{-}\mathbf{Set}$ and an *effective* family $\alpha = \{\alpha_a\}_{a \in |A|}$ of functions $\alpha_a : |X_a| \to |Y_{f(a)}|$; effectivity here means that the family itself has a realizer, i.e. $\exists n \in \mathbb{N}.\ \forall a \in |A|.\ \forall m \in \mathbb{N}.\ m \vdash_A a \ \Rightarrow\ n \cdot m$ realizes α_a. The first projection $\mathit{Fam}_{\mathrm{eff}}(\mathbf{C}) \to \omega\text{-}\mathbf{Set}$ is then a split fibration. There are three things worth noticing.

(i) The object $T = \{\Psi(R)\}_{R \in \mathit{PER}}$ above $\Omega = \Delta(\mathit{PER}) \in \omega\text{-}\mathbf{Set}$ provides the fibration $\mathit{Fam}_{\mathrm{eff}}(\mathbf{M}) \to \omega\text{-}\mathbf{Set}$ with a generic object: for $(A, X) \in \mathit{Fam}_{\mathrm{eff}}(\mathbf{M})$, one has a map $|A| \xrightarrow{X} \mathit{Obj}(\mathbf{M}) \xrightarrow{\Phi} \mathit{PER}$, which yields a morphism $\Phi \circ X : A \to \Omega$ in $\omega\text{-}\mathbf{Set}$ satisfying $(\Phi \circ X)^*(T) = (\Phi \circ X)^*(\Omega, \Psi) = (A, \Phi \circ \Psi \circ X) \cong (A, X)$.

(ii) Similarly to the example in 1.1.6, there is a fibred equivalence,

$$\begin{tikzcd} \mathit{Fam}_{\mathrm{eff}}(\omega\text{-}\mathbf{Set}) \arrow[rr, "\mathcal{Q}", "\sim"'] \arrow[dr] & & \omega\text{-}\mathbf{Set}^{\to} \arrow[dl, "\mathit{cod}"] \\ & \omega\text{-}\mathbf{Set}. & \end{tikzcd}$$

We first define a functor $\mathcal{Q}_0 : \mathit{Fam}_{\mathrm{eff}}(\omega\text{-}\mathbf{Set}) \to \omega\text{-}\mathbf{Set}$ by $(A, X) \mapsto (\dot{\bigcup}_{a \in |A|} . |X_a|, \vdash)$, with $n \vdash (a, x) \ \Leftrightarrow\ \mathit{fst}(n) \vdash_A a \ \&\ \mathit{snd}(n) \vdash_{X_a} x$. On morphisms $\mathcal{Q}_0$ is described by $(f, \alpha) \mapsto \boldsymbol{\lambda}(a, x).(f(a), \alpha_a(x))$; the latter has a realizer because α is an effective family. Finally, $\mathcal{Q}(A, X)$ becomes the projection $\mathcal{Q}_0(A, X) \to A$ in $\omega\text{-}\mathbf{Set}^{\to}$ and $\mathcal{Q}(f, \alpha)$ becomes $(f, \mathcal{Q}_0(f, \alpha))$. Notice that $\mathcal{Q}_0 = \mathit{dom} \circ \mathcal{Q}$.

(iii) The reflection $\mathbf{M} \overset{\leftarrow}{\hookrightarrow} \omega\text{-}\mathbf{Set}$ lifts to a fibred reflection

$$\begin{tikzcd} \mathit{Fam}_{\mathrm{eff}}(\mathbf{M}) \arrow[rr, hook, shift right, "\mathcal{I}"'] \arrow[dr] & & \mathit{Fam}_{\mathrm{eff}}(\omega\text{-}\mathbf{Set}) \arrow[ll, shift right] \arrow[dl] \\ & \omega\text{-}\mathbf{Set}, & \end{tikzcd}$$

by a pointwise construction. Later, in 5.2.7 (i) we shall see that these data imply that $\mathit{Fam}_{\mathrm{eff}}(\mathbf{M}) \to \omega\text{-}\mathbf{Set}$ is a fibred CCC. Of course, this can also be verified directly.

1.3 Indexed categories and split fibrations

As we have seen so far, fibrations describe variable categories. We shall consider two other descriptions of categories varying over a base category: *indexed* categories in this section and *internal* categories in the next one. Below, we understand an indexed category as a *functor* $\Psi : \mathbf{B}^{op} \to \mathbf{Cat}$ and not as a *pseudo-functor*. The latter would mean that one allows isomorphisms $\Psi(id) \cong id$ and $\Psi(u \circ v) \cong \Psi(v) \circ \Psi(u)$, in a coherent way, see Paré and Schumacher [1978]. Such pseudo-functoriality is better captured in fibred category theory, where it is left implicit. This saves a lot of trouble.

Here again, we loosely speak about very large "categories" like $\mathbf{Cat}$, ICat or Fib_{split}. In this way we avoid rather cumbersome formulations.

1.3.1. Definition. (i) An *indexed category* is a functor of the form $\Psi : \mathbf{B}^{op} \to \mathbf{Cat}$.

(ii) A *morphism of indexed categories* from $\Psi : \mathbf{B}^{op} \to \mathbf{Cat}$ to $\Phi : \mathbf{A}^{op} \to \mathbf{Cat}$ is a pair (K, α) where $K : \mathbf{B} \to \mathbf{A}$ is a functor and $\alpha : \Psi \stackrel{\cdot}{\longrightarrow} \Phi K^{op}$ is a natural transformation. Notice that the components of α are functors $\Psi B \to \Phi(KB)$. This determines a "category" ICat.

(iii) A 2-cell $(K, \alpha) \Rightarrow (L, \beta)$ between morphisms (K, α) and (L, β) from $\Psi : \mathbf{B}^{op} \to \mathbf{Cat}$ to $\Phi : \mathbf{A}^{op} \to \mathbf{Cat}$ is a pair (σ, τ) where $\sigma : K \stackrel{\cdot}{\longrightarrow} L$ is a natural transformation and $\tau : \alpha \Rightarrow (\Phi\sigma \circ \beta)$ is a *modification*. The latter means that τ is a family $\{\tau(B)\}_{B \in \mathbf{B}}$ of natural transformations $\tau(B) : \alpha_B \stackrel{\cdot}{\longrightarrow} (\Phi(\sigma_B) \circ \beta_B) : \Psi B \longrightarrow \Phi(KB)$ subject to the condition that for $u : B \to B'$ in $\mathbf{B}$ one has that

$\tau(B)\Psi(u) = \Phi(Ku)\tau(B')$ as in the diagram below.

$$\begin{tikzcd}
\Psi(B') \arrow[rr, "\Psi(u)"] \arrow[d, shift right=4, "\alpha'_B"'] \arrow[d, shift left=4, "\Phi(\sigma_{B'}) \circ \beta_{B'}"] \arrow[d, phantom, "\overset{\tau(B')}{\Longrightarrow}"] & & \Psi(B) \arrow[d, shift right=4, "\alpha_B"'] \arrow[d, shift left=4, "\Phi(\sigma_B) \circ \beta_B"] \arrow[d, phantom, "\overset{\tau(B)}{\Longrightarrow}"] \\
\Phi(KB') \arrow[rr, "\Phi(Ku)"'] & & \Phi(KB)
\end{tikzcd}$$

1.3.2. Proposition. *The functor $ICat \to \mathbf{Cat}$, sending an indexed category to its base, is a split fibration. The fibre above a category $\mathbf{B}$ is denoted by $ICat(\mathbf{B})$.*

Proof. For an indexed category $\Psi : \mathbf{B}^{op} \to \mathbf{Cat}$ and an arbitrary functor $K : \mathbf{A} \to \mathbf{B}$, put $K^*(\Psi) = \Psi \circ K^{op} : \mathbf{A}^{op} \to \mathbf{Cat}$ and $\overline{K}(\Psi) = (K, \{id_{\Psi(KA)}\}_{A \in \mathbf{A}})$ in $ICat$. □

An indexed category $\Psi : \mathbf{B}^{op} \to \mathbf{Cat}$ can be turned into a split fibration with basis $\mathbf{B}$ in a standard way, called the "Grothendieck construction". To obtain the total category $\int_{\mathbf{B}} \Psi$, one takes pairs (A, X) with $X \in \Psi A$ as objects. Morphisms $(A, X) \to (B, Y)$ in $\int_{\mathbf{B}} \Psi$ are pairs (u, f) with $u : A \to B$ in $\mathbf{B}$ and $f : X \to \Psi(u)(Y)$ in ΨA. The first projection $\mathcal{G}(\Psi) : \int_{\mathbf{B}} \Psi \to \mathbf{B}$ is then a fibration which admits an obvious splitting.

This construction forms the basis for the following result.

1.3.3. Theorem (Grothendieck). *Indexed categories are essentially the same as split fibrations, in the sense that there is a fibred equivalence*

$$\begin{tikzcd}
ICat \arrow[rr, shift left, "\mathcal{G}"] \arrow[dr] & & Fib_{split} \arrow[ll, shift left, "\mathcal{I}"] \arrow[dl] \\
& \mathbf{Cat}. &
\end{tikzcd}$$

This gives a categorical version of the equivalence mentioned in 1.1.6.

Proof. The functor $\mathcal{G}$ on objects is described above. For a morphism $(K, \alpha) : (\Psi : \mathbf{B}^{op} \to \mathbf{Cat}) \longrightarrow (\Phi : \mathbf{A}^{op} \to \mathbf{Cat})$ in $ICat$, one defines $\mathcal{G}(K, \alpha) = (K, \int \alpha)$, where $\int \alpha : \int_{\mathbf{B}} \Psi \to \int_{\mathbf{A}} \Phi$ is layed down by $(A, X) \mapsto (KA, \alpha_A(X))$ and $(u, f) \mapsto (Ku, \alpha_A(f))$.

The functor $\mathcal{I} : Fib_{split} \to ICat$ maps a split fibration $p : \mathbf{E} \to \mathbf{B}$ to the functor $\mathcal{I}(p) : \mathbf{B}^{op} \to \mathbf{Cat}$ described by $A \mapsto \mathbf{E}_A$ and $u \mapsto u^*$. Clearly, for a morphism $(K : \mathbf{B} \to \mathbf{A}, H : \mathbf{E} \to \mathbf{D})$ from $p : \mathbf{E} \to \mathbf{B}$ to $q : \mathbf{D} \to \mathbf{A}$ in Fib_{split}, one takes $\mathcal{I}(K, H) = (K, \{H \restriction A\}_{A \in \mathbf{A}})$, where $H \restriction A : \mathbf{E}_A \to \mathbf{D}_{KA}$ is the obvious restriction to the fibres. Naturality in A is obtained because H preserves the splitting on the nose. The required fibred equivalence follows readily. □

The above passages between $ICat$ and Fib_{split} form in fact 2-categorical functors; we take a look at the fibres only.

1.3.4. Proposition. *The Grothendieck construction yields for every category $\mathbf{B}$ a 2-functor*

$$ICat(\mathbf{B}) \quad \longrightarrow \quad Fib_{split}(\mathbf{B})$$

which is full and faithful, both on 1-cells and on 2-cells.

Proof. This functor is full and faithful on 1-cells due to the previous result. The 2-categorical matters are left to the interested reader. □

In view of the previous theorem, indexed categories are not really needed, because one can work with split fibrations instead. An advantage of indexed categories however, is that they are often easier to describe. For example, the (split) fibration $Fam(\mathbf{C}) \to \mathbf{Sets}$ from 1.1.2 is obtained by applying the Grothendieck construction to the functor $\mathbf{Sets}^{op} \to \mathbf{Cat}$ given by $I \mapsto \mathbf{C}^I$. Similarly, one obtains $Fam(\mathbf{C}) \to \mathbf{Cat}$ (cf. Jacobs [1990]) from $\mathbf{A} \mapsto \mathbf{C}^{\mathbf{A}}$. But also $Fam_{\text{eff}}(\mathbf{C}) \to \omega\text{-}\mathbf{Set}$ in 1.2.12 is constructed in such a way. In the sequel, we often describe split fibrations by simply exhibiting the corresponding indexed category.

At this point one can also see that the fibred Yoneda lemma 1.1.9 is a generalization of the ordinary one. For a locally small category $\mathbf{B}$ and a functor $H : \mathbf{B}^{op} \to \mathbf{Sets}$, the Grothendieck construction yields a discrete fibration $\mathcal{G}(H)$ with basis $\mathbf{B}$. Notice that $\mathcal{G}(\mathbf{B}(-, A)) = dom_A : \mathbf{B}/A \to \mathbf{B}$. Using 1.1.9 (ii), one obtains,

$$\begin{array}{rcll}
HA \;=\; \mathcal{G}(H)_A & \cong & Fib_{split}(\mathbf{B})\big(\, dom_A,\ \mathcal{G}(H)\,\big) & \\
& = & Fib_{split}(\mathbf{B})\big(\, \mathcal{G}(\mathbf{B}(-, A)),\ \mathcal{G}(H)\,\big) & \\
& \cong & ICat(\mathbf{B})\big(\, \mathbf{B}(-, A),\ H\,\big) & \text{by the previous proposition} \\
& = & \mathbf{Sets}^{\mathbf{B}^{op}}\big(\, \mathbf{B}(-, A),\ H\,\big). &
\end{array}$$

Notice also that $\mathcal{G}(H)$ is representable in the fibred sense iff H is representable in the ordinary sense.

The next lemma states that the fibred structure appropriate for split fibrations can be described as structure in the fibres which is preserved on-the-nose by reindexing functors.

1.3.5. Lemma. *Let $p : \mathbf{E} \to \mathbf{B}$ and $q : \mathbf{D} \to \mathbf{B}$ be split fibrations and $H : \mathbf{E} \to \mathbf{D}$ a splitting-preserving functor. One has a split fibred adjunction $F \dashv H$ (resp. $H \dashv G$), i.e. an adjunction in the 2-category $Fib_{split}(\mathbf{B})$, if and only if both*

- *For every $A \in \mathbf{B}$, the functor $H \restriction A$ has a left (resp. right) adjoint $K(A)$.*

- *For every $u: A \to B$, the canonical natural transformation*

$$K(A)\, u^{\#} \stackrel{\cdot}{\longrightarrow} u^*\, K(B) \qquad\qquad (\textit{resp. } u^*\, K(B) \stackrel{\cdot}{\longrightarrow} K(A)\, u^{\#}\ )$$

is an identity. *Here $u^*: \mathbf{E}_B \to \mathbf{E}_A$ and $u^{\#}: \mathbf{D}_B \to \mathbf{D}_A$ are the reindexing functors induced by the splittings of p and q.* □

One should be aware of the fact that in the above formulation the *canonical* transformation $K(A)\, u^{\#} \stackrel{\cdot}{\longrightarrow} u^*\, K(B)$ should be the identity and not just $K(A)\, u^{\#} = u^*\, K(B)$. The formulation we use expresses that the pair $(u^{\#}, u^*)$ is a map of adjunctions from $K(B) \dashv H \restriction B$ to $K(A) \dashv H \restriction A$ — see Mac Lane [1971], IV 7 — resp. $(u^*, u^{\#})$ from $H \restriction B \dashv K(B)$ to $H \restriction A \dashv K(A)$.

Thus it is clear what a *split* fibred CCC is. For example, if $\mathbf{C}$ is a CCC, then $\mathit{Fam}(\mathbf{C}) \to \mathbf{Sets}$ is such a split fibred CCC.

1.3.6. Proposition (Bénabou). *Every fibration is equivalent to a split one.*

Proof. Let $p: \mathbf{E} \to \mathbf{B}$ be an arbitrary fibration. Applying the Grothendieck construction to the functor $\mathbf{B}^{op} \to \mathbf{Cat}$ given by $A \mapsto \mathit{Fib}_{split}(\mathbf{B})\big(\mathit{dom}_A,\ p\big)$ yields a split fibration equivalent to p; this gives the naturality we spoke about in the Yoneda lemma 1.1.9 (ii). □

1.4 Internal categories

As a second alternative way of describing variable categories, we now consider *internal* categories. Such categories are described by a number of commuting diagrams in a base category, which correspond to the defining equations of a category. The base category provides the universe in which one is working; it is often called the *ambient* category.

1.4.1. Basics. Let $\mathbf{B}$ be a category; for the time being, we assume that $\mathbf{B}$ has finite limits, see remark 1.4.3 below. An internal category $\mathbf{C}$ in $\mathbf{B}$ is given by the following data. First, there are objects C_0 and C_1, which should be understood as the object of objects and the object of morphisms of $\mathbf{C}$. Secondly, there is a commuting diagram,

$$\begin{tikzcd} & C_0 \arrow[dl, "id"'] \arrow[d, "i"] \arrow[dr, "id"] & \\ C_0 & C_1 \arrow[l, "\partial_0"] \arrow[r, "\partial_1"'] & C_0, \end{tikzcd}$$

where ∂_0 and ∂_1 are domain and codomain maps and i provides the internal category $\mathbf{C}$ with identity maps. From this one constructs pullback diagrams,

$$\begin{tikzcd} C_2 \arrow[r, "\pi_1"] \arrow[d, "\pi_0"'] \arrow[dr, phantom, "\lrcorner", very near start] & C_1 \arrow[d, "\partial_0"] \\ C_1 \arrow[r, "\partial_1"'] & C_0 \end{tikzcd} \qquad\qquad \begin{tikzcd} C_3 \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & C_1 \arrow[d, "\partial_0"] \\ C_2 \arrow[r, "\partial_1 \circ \pi_1"'] & C_0, \end{tikzcd}$$

where C_2 and C_3 are the objects of composable pairs and triples of morphisms in $\mathbf{C}$. Thirdly, there is a "composition" morphism $m: C_2 \to C_1$ satisfying

$$\begin{array}{rcllcl} \partial_0 \circ m & = & \partial_0 \circ \pi_0 & : & C_2 & \to\ C_0 \\ \partial_1 \circ m & = & \partial_1 \circ \pi_1 & : & C_2 & \to\ C_0 \\ m \circ i \times id & = & \pi_1 & : & C_0 \underset{id,\partial_0}{\times} C_1 & \to\ C_1 \\ m \circ id \times i & = & \pi_0 & : & C_1 \underset{\partial_1,id}{\times} C_0 & \to\ C_1 \\ m \circ m \times id & = & m \circ id \times m & : & C_3 & \to\ C_1. \end{array}$$

Summing up, an internal category $\mathbf{C}$ in $\mathbf{B}$ is given by a 6-tuple $\langle C_0, C_1, \partial_0, \partial_1, i, m\rangle$ satisfying the above requirements.

An *internal functor* F between two internal categories $\mathbf{C} = \langle C_0, C_1, \partial_0, \partial_1, i, m\rangle$ and $\mathbf{C}' = \langle C_0', C_1', \partial_0', \partial_1', i', m'\rangle$ consists of a pair of maps $F_0: C_0 \to C_0'$ and $F_1: C_1 \to C_1'$ satisfying

$$\begin{array}{rcl} F_0 \circ \partial_0 & = & \partial_0' \circ F_1 \\ F_0 \circ \partial_1 & = & \partial_1' \circ F_1 \\ F_1 \circ i & = & i' \circ F_0 \\ F_1 \circ m & = & m' \circ F_1 \times F_1. \end{array}$$

In this way, a category $\mathit{Cat}(\mathbf{B})$ is obtained. One easily verifies that $\mathit{Cat}(\mathbf{B})$ has finite products. The category $\mathit{Cat}(\mathbf{B})$ is in fact a 2-category: a 2-cell in $\mathit{Cat}(\mathbf{B})$ is given as follows. One has $\sigma: F \stackrel{\cdot}{\longrightarrow} G: \mathbf{C} \to \mathbf{C}'$ iff σ is a morphism $C_0 \to C_1'$ making the following two diagrams commute.

$$\begin{tikzcd} & C_0 \arrow[dl, "F_0"'] \arrow[d, "\sigma"] \arrow[dr, "G_0"] & \\ C_0' & C_1' \arrow[l, "\partial_0'"] \arrow[r, "\partial_1'"'] & C_0' \end{tikzcd} \qquad \begin{tikzcd} C_1 \arrow[r, "\langle \sigma \circ \partial_0, G_1\rangle"] \arrow[d, "\langle F_1, \sigma \circ \partial_1\rangle"'] & C_2' \arrow[d, "m'"] \\ C_2' \arrow[r, "m'"] & C_1' \end{tikzcd}$$

The 2-categorical structure determines what *internal* adjunctions are. Thus we can define internal structure in the usual way. But first we need an auxiliary notion: an internal category is called *discrete* if its identity map is an isomorphism. Every object $A \in \mathbf{B}$ yields a discrete $|A| \in Cat(\mathbf{B})$ with A both as object of objects and as object of morphisms. Next one can say that $\mathbf{C} \in Cat(\mathbf{B})$ has an *internal terminal object* if the unique internal functor $\mathbf{C} \to |t|$ has an internal right adjoint. Here $t \in \mathbf{B}$ is terminal and hence $|t| \in Cat(\mathbf{B})$ as well. Similarly, $\mathbf{C}$ has *internal cartesian products* if the obvious diagonal $\mathbf{C} \to \mathbf{C} \times \mathbf{C}$ has an internal right adjoint, say $prod : \mathbf{C} \times \mathbf{C} \to \mathbf{C}$. A bit less trivially, $\mathbf{C}$ has *internal exponents* if the functor $\widetilde{prod} : |C_0| \times \mathbf{C} \to |C_0| \times \mathbf{C}$ has an internal right adjoint. This functor $\widetilde{prod}$ is constructed from $prod$ and an obvious inclusion $|C_0| \to \mathbf{C}$. Thus one obtains the notion of an internal CCC.

1.4.2. Examples. (i) Let $\mathbf{C}$ be a small category, i.e. a category with small collections both of objects and of morphisms. Then $\mathbf{C}$ is internal in **Sets** and it forms an internal CCC iff it is an ordinary CCC.

(ii) The category **PER** from the realizability example 1.2.12 is internal in ω-**Set**. One takes $\mathbf{PER}_0 = \Delta PER$ and $\mathbf{PER}_1 = (\dot{\bigcup}_{R,S \in PER}. Q(R \to S), \vdash)$, where $R \to S$ is the exponent object in the category **PER** described by $n(R \to S)m \Leftrightarrow \forall k, l \in \mathbb{N}.\ kRl \Rightarrow m \cdot kSn \cdot l$. The realizability relation $\vdash$ of $\mathbf{PER}_1$ is described by $m \vdash (R, S, [n]_{R \to S}) \Leftrightarrow m(R \to S)n$. This category **PER** forms an internal CCC in ω-**Set**.

1.4.3. Remark. The above description of categories internal in an ambient (or base) category $\mathbf{B}$ started from the assumption that $\mathbf{B}$ has finite limits. Careful inspection shows that one actually needs only two pullbacks, viz. C_2 and C_3. From now on, we allow ourselves the liberty to say the $\mathbf{C}$ is internal in an arbitrary category $\mathbf{B}$ if there is just enough structure around to formulate the above requirements. This matter will be of relevance for example in theorem 3.3.3.

1.4.4. Definition (Externalization). There is a 2-functor

$$[-] \;:\; Cat(\mathbf{B}) \longrightarrow Fib_{split}(\mathbf{B}).$$

(i) For $\mathbf{C} \in Cat(\mathbf{B})$, let $\sum(\mathbf{C})$ be the total category with objects (A, X) such that $X : A \to C_0$ in $\mathbf{B}$. Morphisms $(A, X) \to (B, Y)$ in $\sum(\mathbf{C})$ are pairs (u, f) with $u : A \to B$ in $\mathbf{B}$ and $f : A \to C_1$ satisfying $\partial_0 \circ f = X$ and $\partial_1 \circ f = Y \circ u$. Composition in $\sum(\mathbf{C})$ is defined using composition both in $\mathbf{B}$ and in $\mathbf{C}$. The first projection $[\mathbf{C}] : \sum(\mathbf{C}) \to \mathbf{B}$ is then a split fibration.

(ii) For $F : \mathbf{C} \to \mathbf{D}$ in $Cat(\mathbf{B})$ one defines $[F] : \sum(\mathbf{C}) \to \sum(\mathbf{D})$ by $(A, X) \mapsto (A, F_0 \circ X)$ and $(u, f) \mapsto (u, F_1 \circ f)$.

(iii) For $\sigma : F \stackrel{\cdot}{\longrightarrow} G : \mathbf{C} \to \mathbf{D}$ in $Cat(\mathbf{B})$, one obtains $[\sigma] : [F] \stackrel{\cdot}{\longrightarrow} [G]$ with components $[\sigma]_{(A,X)} = (id_A,\ \sigma \circ X)$.

Notice that for $\mathbf{C} \in Cat(\mathbf{B})$, $C_0 \in \mathbf{B}$ yields a split generic object for the fibration $[\mathbf{C}] : \sum(\mathbf{C}) \to \mathbf{B}$, see definition 1.2.11.

1.4.5. Proposition. *The externalization functor* $[-] : Cat(\mathbf{B}) \to Fib_{split}(\mathbf{B})$ *is*

(i) *finite product preserving;*

(ii) *full and faithful, both on 1-cells and on 2-cells.*

Proof. (i) Straightforward.

(ii) We shall do fulness on 2-cells, which is the most complicated case. Assume therefore that $\tau : [F] \stackrel{\cdot}{\longrightarrow} [G] : \mathbf{C} \to \mathbf{D}$ in $Cat(\mathbf{B})$ is given. We take $\sigma = snd(\tau_{(C_0, id_{C_0})})$ and must show that $[\sigma]_{(A,X)} = \tau_{(A,X)}$. Notice that $(X, i \circ X) : (A, X) \to (C_0, id_{C_0})$ is cartesian in $\sum(\mathbf{C})$. It is not hard to prove that $[G](X, i \circ X) \circ \tau_{(A,X)} = [G](X, i \circ X) \circ [\sigma]_{(A,X)}$. But then the result follows from the fact that $[G]$ is a cartesian functor. $\square$

1.4.6. Corollary.

$$\mathbf{C} \textit{ is an internal CCC} \quad \Leftrightarrow \quad [\mathbf{C}] \textit{ is a split fibred CCC.}$$

Proof. By the previous proposition, since the CCC-structure is defined 2-categorically using finite products. $\square$

1.4.7. Definition (Bénabou [1975]). A fibration is called *small* if it is equivalent to a fibration of the form $[\mathbf{C}]$ for some $\mathbf{C}$ internal in the base category.

The fibration $Fam_{\text{eff}}(\mathbf{M}) \to \omega$-**Set** from 1.2.12 forms an example of a small fibration: as one might have expected, there is an equivalence of categories $\sum(\mathbf{PER}) \simeq Fam_{\text{eff}}(\mathbf{M})$ over ω-**Set**, see 1.4.2 (ii). Further on in 4.5.8, one can see that small fibrations can also be described without reference to internal categories, viz. in terms of "locally small" fibrations and generic objects.

1.4.8. Proposition (Internalization). *Let* $p : \mathbf{E} \to \mathbf{B}$ *be a split fibration, where* $\mathbf{B}$ *is locally small and all fibres are small. Then there is an internal category* $\hat{p}$ *in* $\widehat{\mathbf{B}} = \mathbf{Sets}^{\mathbf{B}^{op}}$ *and a change-of-base situation,*

$$\begin{tikzcd}
\mathbf{E} \arrow[r, "H"] \arrow[d, "p"'] \arrow[dr, phantom, "\lrcorner", very near start] & \sum(\hat{p}) \arrow[d, "{[\hat{p}]}"] \\
\mathbf{B} \arrow[r, "Y"] & \widehat{\mathbf{B}},
\end{tikzcd}$$

where $Y : \mathbf{B} \to \widehat{\mathbf{B}}$ *is the Yoneda embedding. The functors* Y *and* H *in this diagram are both full and faithful. Moreover, one has*

$$p \textit{ is a split fibred CCC} \quad \Leftrightarrow \quad \hat{p} \textit{ is an internal CCC}$$

and this structure is preserved by the map (Y, H).
(The above size restrictions could be avoided by working in a suitably larger universe than **Sets**.)

Proof. Define $\hat{p}_0 : \mathbf{B}^{op} \to \mathbf{Sets}$ by $A \mapsto Obj(\mathbf{E}_A)$ and $\hat{p}_1 : \mathbf{B}^{op} \to \mathbf{Sets}$ by $A \mapsto Mor(\mathbf{E}_A)$. It is then obvious that one obtains an internal category. The functor $H : \mathbf{E} \to \sum(\hat{p})$ is described by $E \mapsto (Y_{pE}, \tilde{E})$, where $\tilde{E} : Y_{pE} \stackrel{\cdot}{\longrightarrow} \hat{p}_0$ is defined by $\tilde{E}_A(u) = u^*(E)$. Similarly, for $f : E \to D$ one defines $Hf = (Y_{pf}, \tilde{f})$, where $\tilde{f} : Y_{pE} \stackrel{\cdot}{\longrightarrow} \hat{p}_1$ is described by $\tilde{f}_A(u) = u^*(f')$ in which the vertical map $f' : E \to (pf)^*(D)$ is such that $\overline{pf}(D) \circ f' = f$. The rest is straightforward. □

1.5 Quantification along cartesian projections

This last section contains basically only two definitions. Examples will be given in the third chapter. Throughout, base categories are supposed to have finite products.

1.5.1. Definition. Let $p : \mathbf{E} \to \mathbf{B}$ be a fibration.

(i) Let A be an object of $\mathbf{B}$. We say that p admits $Cons_A$-products (resp. sums) if both

- for every $B \in \mathbf{B}$, every reindexing functor $\pi^*_{B,A} : \mathbf{E}_B \to \mathbf{E}_{B\times A}$ has a right adjoint Π_B (resp. a left adjoint Σ_B).
- for every morphism $u : B \to B'$ in $\mathbf{B}$, the canonical natural transformation

$$u^* \, \Pi_{B'} \stackrel{\cdot}{\longrightarrow} \Pi_B \, (u \times id)^* \qquad\qquad (\text{resp. } \Sigma_B \, (u \times id)^* \stackrel{\cdot}{\longrightarrow} u^* \, \Sigma_{B'} \;)$$

  is an isomorphism.

(ii) We say that p admits $Cons_{\mathbf{B}}$-products/sums if it admits $Cons_A$-products/ sums for every $A \in \mathbf{B}$.

In the fourth chapter we shall see that $Cons_{\mathbf{B}}$ and $Cons_{\Omega}$ form so-called "comprehension categories". Using these, a general notion of quantification for fibrations will be given. At this point however, the above elementary description is more suitable.

Next we introduce morphisms of fibrations with the above forms of quantification.

1.5.2. Definition. Assume $(K : \mathbf{B} \to \mathbf{B}',\ L : \mathbf{E} \to \mathbf{E}')$ is a morphism of fibrations from $p : \mathbf{E} \to \mathbf{B}$ and $p' : \mathbf{E}' \to \mathbf{B}'$ such that K preserves finite products. We write $\gamma_{B,B'}$ for the inverse of the canonical map $K(B \times B') \to KB \times KB'$.

(i) Suppose p has $Cons_{\mathbf{B}}$-products via $\pi^*_{B,B'} \dashv \Pi_{(B,B')}$ and p' has $Cons_{\mathbf{B}'}$-products via $\pi^*_{A,A'} \dashv \Pi'_{(A,A')}$. Then (K, L) preserves $Cons_{\mathbf{B}}$-products if the canonical natural transformation

$$L \circ \Pi_{(B,B')} \quad \stackrel{\cdot}{\longrightarrow} \quad \Pi'_{(KB,KB')} \circ \gamma^*_{B,B'} \circ L$$

is an isomorphism. Similarly, preservation of $Cons_{\mathbf{B}}$-sums means that

$$\Sigma'_{(KB,KB')} \circ \gamma^*_{B,B'} \circ L \quad \stackrel{\cdot}{\longrightarrow} \quad L \circ \Sigma_{(B,B')}$$

is an isomorphism, where this time $\Sigma_{(B,B')} \dashv \pi^*_{B,B'}$ in $\mathbf{E}$ and $\Sigma'_{(A,A')} \dashv \pi^*_{A,A'}$ in $\mathbf{E}'$.

(ii) Assume p has $Cons_{\Omega}$-products via $\pi^*_{B,\Omega} \dashv \Pi_B$ and p' has $Cons_{\Omega'}$-products via $\pi^*_{A,\Omega'} \dashv \Pi'_A$. Additionally, we assume that there is an isomorphism $\beta : \Omega' \stackrel{\sim}{\longrightarrow} K\Omega$ and use it to form $\gamma'_B = \gamma_{B,\Omega} \circ id \times \beta : KB \times \Omega' \to K(B \times \Omega)$. Then (K, L) preserves $Cons_{\Omega}$-products if the canonical

$$L \circ \Pi_B \quad \stackrel{\cdot}{\longrightarrow} \quad \Pi'_{KB} \circ \gamma'^*_B \circ L$$

is an isomorphism. Similarly, (K, L) preserves $Cons_{\Omega}$-sums if

$$\Sigma'_{KB} \circ \gamma'^*_B \circ L \quad \stackrel{\cdot}{\longrightarrow} \quad L \circ \Sigma_B$$

is an isomorphism.

1.5.3. Definition. A *split* fibration admits $Cons_{...}$-products/sums if it admits this structure in such a way that the above intermediary natural transformations are identities. Analogously for corresponding morphisms.

1.5.4. Quantification for internal categories. Let $\mathbf{B}$ be a cartesian closed category. For every $\mathbf{C} \in Cat(\mathbf{B})$ and $A \in \mathbf{B}$, one can form an internal category $\mathbf{C}^A = \langle C_0^A, C_1^A, \ldots \rangle$ and an obvious internal diagonal functor $\Delta_A : \mathbf{C} \to \mathbf{C}^A$. We say that $\mathbf{C}$ admits *internal* $Cons_A$-products (resp. sums) if this functor Δ_A has an internal right (resp. left) adjoint. Internal $Cons_{\mathbf{B}}$-products/sums are of course given by internal $Cons_A$-products/sums for every $A \in \mathbf{B}$. It is left to the reader to verify that

$$\begin{array}{ll} & \mathbf{C} \text{ admits internal } Cons_A\text{-products/sums} \\ \Leftrightarrow & [\mathbf{C}] \text{ admits split } Cons_A\text{-products/sums;} \end{array}$$

analogously to 1.4.6.

1.5.5. Lemma. *Suppose* $p : \mathbf{E} \to \mathbf{B}$ *is a split fibration as in 1.4.8. If* p *admits a* split *generic object — i.e.* $\hat{p}_0 \cong Y_{\Omega}$ *see definition 1.2.11 — then one has for every* $A \in \mathbf{B}$,

$$\begin{array}{ll} & p : \mathbf{E} \to \mathbf{B} \textit{ admits split } Cons_A\textit{-products/sums} \\ \Leftrightarrow & \hat{p} \textit{ admits internal } Cons_{Y_A}\textit{-products/sums.} \end{array}$$

Moreover, externalization yields a morphism $p \to [\tilde{p}]$ *which preserves this structure.*

Proof. By the fact that

$$\begin{array}{rcl} \hat{p}_0(B) & = & Obj(\mathbf{E}_B) \\ (Y_A \Rightarrow \hat{p}_0)(B) & = & \widehat{\mathbf{B}}\,(Y_B \times Y_A,\ \hat{p}_0) \\ & \cong & \widehat{\mathbf{B}}\,(Y_{B\times A},\ Y_{\Omega}) \\ & \cong & \mathbf{B}\,(B \times A,\ \Omega) \\ & \cong & Obj(\mathbf{E}_{B\times A}). \end{array}$$ □

Chapter 2

Type Systems

Generalized Type Systems (abbr. GTS's) have been introduced in Barendregt [1991] and [199?]. They provide an abstract way of describing typed λ-calculi by specifying collections of *sorts, axioms* and *rules*. Although this description is a major step forward in the classification of various systems, there are certain drawbacks.

- Not all systems can be described; Martin-Löf's type theory, for example, is not covered by the GTS-formalism.
- Handling of constants is quite problematic, certainly if they may contain variables as parameters.
- Occurrence of certain dependencies is an outcome of the axioms and rules. This is both conceptually and technically problematic.

Below we shall define *Type Systems* (abbr. TS's) in such a way that the above drawbacks disappear. Our approach is more structural and closer to a categorical way of thinking. We first introduce a so-called "TS-setting" which determines the dependencies that may arise in a system based on that setting. On top of such a setting one can put "features", like axioms, constants or products. Hence the new picture gives us

$$\text{TS's} \quad = \quad \text{settings} \quad + \quad \text{features},$$

where the features depend on the setting. Later, we shall show that TS-settings correspond to certain "categorical settings". The features can be described categorically as certain extras which can be added on top of such structures (often by adding certain adjunctions). The main ideas underlying Type Systems will be described in the first section below. There, we mention the features only to give the intuition of what is going on. A more detailed treatment of (some of) these may be found in the second section. Finally, in the third section some known systems are redescribed in the new TS-framework.

2.1 Informal description

Like in the GTS-description, we start with a set of sorts having as typical elements $*$, $\Box$, $\triangle$ etc. Some authors write *prop, type, kind, set* etc. for sorts, but in the GTS-tradition there is no intended meaning. Meta-variables for sorts are denoted by $s, s', s_1, s_2, \ldots$ The basic aspect of sorts is described by the rule

$$\frac{\Gamma \vdash A : s}{\Gamma, \alpha : A \vdash \alpha : A}$$

Hence if something is in a sort, it may be put in the context and serve as a range for a variable.

2.1.1. Definition. A *TS-setting*, or simply a *setting* is a pair $(\mathit{Sort}, \prec)$, where *Sort* is a non-empty set and $\prec \subseteq \mathit{Sort} \times \mathit{Sort}$ is a transitive relation. It is called the *relation of dependency*: in case $s_1 \prec s_2$ (or equivalently $s_2 \succ s_1$), we say that s_2 depends on s_1; the intuition is that if a derivation has produced statements

$$\frac{}{\Gamma \vdash A : s_1 \qquad \Gamma, \alpha : A \vdash B : s_2,}$$

then α may occur as a free variable in B. More informally, $s_1 \prec s_2$ means that "grandchildren" of s_1 may occur in "children" of s_2 — where P is called a child of Q if $P : Q$. This explains the transitivity requirement.

2.1.2. Examples. In the setting with one sort $*$ and no dependencies (i.e. $\prec = \emptyset$) one can only have constant types $\emptyset \vdash A : *$. This setting underlies "simply typed λ-calculus", or $\lambda 1$ as it is called in section 2.3. In case one has $* \succ *$, then one can have statements of the form $x : A \vdash B(x) : *$ for $A : *$. Such dependency underlies Martin-Löf's type theory.

In the system $\lambda\!\!\to$ one has two sorts $*$ and $\Box$ and an axiom $* : \Box$, see section 2.3 or Barendregt [1991]. In $\lambda\!\!\to$ one has statements like

$$\alpha : * \vdash \alpha \to \alpha : *.$$

These require a dependency $\Box \prec *$.

2.1.3. Remarks. (i) If s_2 depends on s_1, the notation $s_2 \succ s_1$ is preferred to $s_1 \prec s_2$. Roughly, the categorical intuition is that s_2 is fibred over s_1. The transitivity requirement corresponds to the fact that fibrations are closed under composition, see lemma 1.1.5. A more detailed exposition of the categorical understanding of these dependencies may be found in section 5.1.

(ii) An expression A in $\Gamma \vdash A : s$ will be called an s-*type*, or simply a *type*, when the sort s is not of much relevance. Similarly, an expression M in $\Gamma \vdash M : A$, where A is an s-type, will be called an (s-)*term of type A*. Notice that types and terms are "relative" notions: in presence of an axiom $s_1 : s_2$, one has that s_1-types are s_2-terms.

(iii) In the literature one can find "type dependency" to name the possibility of s-term variables occurring in s-types, like in Martin-Löf's Type Theory. In our TS-framework, this is possible in case there is a dependency of the form $s \succ s$. Henceforth, this will be called *s-type dependency*.

(iv) Specifying a setting means specifying one's type theoretical "universe of discourse".

2.1.4. TS-features. We now proceed to describe informally what kind of features can be added to a given setting $(\mathit{Sort}, \prec)$. In this thesis we consider

(i) axioms
(ii) constants
(iii) s-closure
(iv) (s_1, s_2)-quantification
(v) (s_1, s_2)-identity
(vi) (s_1, s_2)-inclusion

but one could consider additional features.

Ad (i) An *axiom* is an ordered pair of sorts, usually written as $s_1 : s_2$. Such an axiom may be added to the given setting only if s_1 depends on s_2 — i.e. $s_1 \succ s_2$ — since it enables statements like

$$\frac{\vdash s_1 : s_2}{\alpha : s_1 \vdash \alpha : s_1}$$

in which a grandchild of s_2 (viz. α on the LHS) occurs in a child of s_1 (viz. α on the RHS).

Ad (ii) A setting determines which kind of dependencies may occur in a type system. Hence it also determines what kind of parameters a constant may have. In general, one would like to be able to use both constant types and terms of a given sort, possibly provided with conversions. Let's first look at some examples.

$$\begin{array}{rcl}
 & \vdash & \mathbb{N} : s \\
 & \vdash & \mathit{Zero} : \mathbb{N} \\
n{:}\,\mathbb{N} & \vdash & \mathit{Succ}(n) : \mathbb{N} \\
n{:}\,\mathbb{N} & \vdash & \mathit{List}(n) : s \\
n{:}\,\mathbb{N}, m{:}\,\mathbb{N} & \vdash & \mathit{Matrix}(n,m) : s \\
n{:}\,\mathbb{N}, m{:}\,\mathbb{N}, A{:}\,\mathit{Matrix}(n,m) & \vdash & \mathit{row}(n,m,A) : \mathit{List}(n)
\end{array}$$

In these examples, $\mathit{List}(n)$ is the type of lists of length n and $\mathit{Matrix}(n,m)$ is the type of $n \times m$ matrices. The intended meaning of the term $\mathit{row}(n,m,A)$ is then clear. Notice that the constant type $\mathit{List}(n)$ forms a child of s in which a grandchild of s occurs. Such things may be used only if we have $s \succ s$. Hence we come to the following stipulations. A constant type C may be introduced by

$$\alpha_1 : A_1, \ldots, \alpha_n : A_n \;\vdash\; C(\vec{\alpha}) : s,$$

where $A_i : s_i$, only if $s \succ s_i$. In that case one can introduce constant terms of type C by

$$\alpha_1 : A_1, \ldots, \alpha_n : A_n, \beta_1 : B_1, \ldots, \beta_m : B_m \;\vdash\; M(\vec{\alpha}, \vec{\beta}) : C(\vec{\alpha}),$$

possibly with conversions

$$\alpha_1 : A_1, \ldots, \alpha_n : A_n, \beta_1 : B_1, \ldots, \beta_m : B_m \;\vdash\; M(\vec{\alpha}, \vec{\beta}) = N(\vec{\alpha}, \vec{\beta}) : C(\vec{\alpha}).$$

Implicitly, we understand what substitution (just filling-up an open space) and weakening are for such constants. The idea is to have *generalized algebraic theories* (in the sense of Cartmell [1986]) on an arbitrary setting.

Ad (iii) With the feature "s-closure" we express that s-types are closed under cartesian products, exponents, units etc. These may always be added to a setting, because a rule of the form

$$\frac{\Gamma \vdash A : s \qquad \Gamma \vdash B : s}{\Gamma \vdash A \to B : s}$$

does not create new situations with respect to occurrences of variables.

Ad (iv) The feature "(s_1, s_2)-quantification" is used to describe dependent products and sums of the following form.

$$\frac{\Gamma \vdash A : s_1 \qquad \Gamma, x{:}A \vdash B : s_2}{\Gamma \vdash \Pi x{:}A.B : s_2}$$

This rule may be used if the relevant dependency really occurs, i.e. if $s_2 \succ s_1$.

Ad (v) The "(s_1, s_2)-identity" feature describes the rule

$$\frac{\Gamma \vdash A : s_1 \qquad \Gamma \vdash M : A \qquad \Gamma \vdash N : A}{\Gamma \vdash I_A(M, N) : s_2}$$

which may be used if $s_2 \succ s_1$.

Ad (vi) The feature "(s_1, s_2)-inclusion" gives the possibility to embed s_1-types and terms in s_2-types and terms, in the following way.

$$\frac{\Gamma \vdash A : s_1}{\Gamma \vdash \mathit{In}(A) : s_2} \qquad \frac{\Gamma \vdash M : A}{\Gamma \vdash \mathit{In}(M) : \mathit{In}(A)} \qquad \frac{\Gamma \vdash N : \mathit{In}(A)}{\Gamma \vdash \mathit{Out}(N) : A}$$

Tacitly, we assume that $s_1 \neq s_2$. In some versions of the Calculus of Constructions one can find $(\mathit{prop}, \mathit{type})$-inclusion. Also in Pavlović [1990], a similar operation occurs, under the name "extent". Let us consider the implications of these rules for the dependencies. Suppose one has a sort s with $s_1 \succ s$; then

$$\frac{\Gamma \vdash B : s \qquad \Gamma, y : B \vdash A(y) : s_1}{\Gamma, y : B \vdash \mathit{In}(A(y)) : s_2}$$

which creates a $s_2 \succ s$ dependency. Similarly, if one has $s \succ s_1$, say occurring in

$$\Gamma \vdash A : s_1 \qquad \Gamma, x{:}A \vdash B(x) : s.$$

Then one can derive $\Gamma, x'{:}\mathit{In}(A) \vdash B[x := \mathit{Out}(x')] : s$, which creates a $s \succ s_2$ dependency. Hence, use of (s_1, s_2)-inclusion requires that $\forall s \in \mathit{Sort}.\ (s_1 \succ s \Rightarrow s_2 \succ s)\ \&\ (s \succ s_1 \Rightarrow s \succ s_2)$.

2.2 Rules

In this section we describe rules for the TS-features s-closure, (s_1, s_2)-quantification, (s_1, s_2)-identity and (s_1, s_2)-inclusion. We proceed mostly by first giving the formation, introduction and elimination rules for a certain type operation. Then the conversion rules and the behaviour under substitution will be described. The relevant setting will be left implicit, but is supposed to be such that the feature under consideration may be used. In the substitution rules, the variable involved may be a grandchild of any available sort.

2.2.1. Rules for contexts. Contexts are ordered lists of variable declarations. First of all, the empty list is a context; next, there is what we like to call the *context comprehension* rule: if Γ is a context and A is a type in context Γ — i.e. $\Gamma \vdash A : s$ for some $s \in \mathit{Sort}$ — then one can add a declaration of a (fresh) variable of type A to context Γ. The result is denoted by $\Gamma, x : A$. It comes with the following rules.

$$\frac{\Gamma \vdash A : s}{\Gamma, x : A \vdash x : A} \ (\mathit{projection})$$

$$\frac{\Gamma, x : A, y : B, \Delta \vdash \ldots}{\Gamma, y : B, x : A, \Delta \vdash \ldots} \ \text{if } x \notin \mathit{FV}(B) \quad (\mathit{exchange})$$

$$\frac{\Gamma \vdash A : s \qquad \Gamma \vdash \ldots}{\Gamma, x : A \vdash \ldots} \ (\mathit{weakening})$$

$$\frac{\Gamma \vdash M : A \qquad \Gamma, x : A, \Delta \vdash \ldots}{\Gamma, \Delta[x := M] \vdash (\ldots)[x := M]} \ (\mathit{substitution})$$

In the end, this last substitution rule may turn out to be derivable. We like to mention it explicitly, since substitution will play an important categorical role.

2.2.2. Start rules. In order to get off the ground, certain basic types have to be available; the above context projection rule then gives the possibility to form terms. To obtain such types, one can use either axioms or constant types (if described previously). These set the whole machinery in motion.

2.2.3. Rules for s-closure. For a given sort s, we consider consecutively *units*, *cartesian products* and *exponents*. A unit-type can be understood as a singleton.

Unit.

$$\frac{}{\vdash 1_s : s} \qquad \frac{}{\vdash \langle\rangle : 1_s}$$

with conversion

$$\frac{\Gamma \vdash M : 1_s}{\Gamma \vdash M = \langle\rangle : 1_s}$$

and substitutions

$$\begin{aligned} (1_s)[z := R] &\equiv 1_s \\ \langle\rangle[z := R] &\equiv \langle\rangle. \end{aligned}$$

Cartesian product.

$$\frac{\Gamma \vdash A : s \qquad \Gamma \vdash B : s}{\Gamma \vdash A \times B : s}$$

$$\frac{\Gamma \vdash M : A \qquad \Gamma \vdash N : B}{\Gamma \vdash \langle M, N\rangle : A \times B} \qquad \frac{\Gamma \vdash L : A \times B}{\Gamma \vdash \pi L : A \qquad \Gamma \vdash \pi' L : B}$$

with conversions

$$\frac{\Gamma \vdash M : A \qquad \Gamma \vdash N : B}{\Gamma \vdash \pi\langle M, N\rangle = M : A \qquad \Gamma \vdash \pi'\langle M, N\rangle = N : B} \qquad \frac{\Gamma \vdash L : A \times B}{\Gamma \vdash \langle \pi L, \pi' L\rangle = L : A \times B}$$

and substitutions

$$\begin{aligned} (A \times B)[z := R] &\equiv (A[z := R]) \times (B[z := R]) \\ \langle M,\ N\rangle[z := R] &\equiv \langle M[z := R],\ N[z := R]\rangle \\ (\pi L)[z := R] &\equiv \pi(L[z := R]) \\ (\pi' L)[z := R] &\equiv \pi'(L[z := R]). \end{aligned}$$

Exponent.

$$\frac{\Gamma \vdash A : s \qquad \Gamma \vdash B : s}{\Gamma \vdash A \rightarrow B : s}$$

$$\frac{\Gamma, x : A \vdash L : B}{\Gamma \vdash \lambda x{:}\, A.L : A \rightarrow B} \qquad \frac{\Gamma \vdash M : A \rightarrow B \qquad \Gamma \vdash N : A}{\Gamma \vdash MN : B}$$

with conversions

$$\frac{\Gamma, x : A \vdash L : B \qquad \Gamma \vdash N : A}{\Gamma \vdash (\lambda x{:}\, A.L)N = L[x := N] : B} \qquad \frac{\Gamma \vdash M : A \rightarrow B}{\Gamma \vdash \lambda x{:}\, A.Mx = M : A \rightarrow B}$$

and

$$\begin{aligned} (A \rightarrow B)[z := R] &\equiv (A[z := R]) \rightarrow (B[z := R]) \\ (\lambda x{:}\, A.L)[z := R] &\equiv \lambda x{:}\, (A[z := R]).(L[z := R]) \\ (MN)[z := R] &\equiv (M[z := R])(N[z := R]) \end{aligned}$$

where substitution under the variable-binding λ is done with the usual care.

2.2.4. Rules for (s_1, s_2)-quantification.

Dependent product.

$$\frac{\Gamma \vdash A : s_1 \qquad \Gamma, x : A \vdash B : s_2}{\Gamma \vdash \Pi x{:}\, A.B : s_2}$$

$$\frac{\Gamma, x : A \vdash L : B}{\Gamma \vdash \lambda x{:}\, A.L : \Pi x{:}\, A.B} \qquad \frac{\Gamma \vdash M : \Pi x{:}\, A.B \qquad \Gamma \vdash N : A}{\Gamma \vdash MN : B[x := N]}$$

with conversions

$$\frac{\Gamma, x : A \vdash L : B \qquad \Gamma \vdash N : A}{\Gamma \vdash (\lambda x{:}\, A.L)N = L[x := N] : B[x := N]} \qquad \frac{\Gamma \vdash M : \Pi x{:}\, A.B}{\Gamma \vdash \lambda x{:}\, A.Mx = M : \Pi x{:}\, A.B}$$

and substitutions

$$\begin{aligned} (\Pi x{:}\, A.B)[z := R] &\equiv \Pi x{:}\, (A[z := R]).(B[z := R]) \\ (\lambda x{:}\, A.L)[z := R] &\equiv \lambda x{:}\, (A[z := R]).(L[z := R]) \\ (MN)[z := R] &\equiv (M[z := R])(N[z := R]). \end{aligned}$$

Dependent sums.

$$\frac{\Gamma \vdash A : s_1 \qquad \Gamma, x : A \vdash B : s_2}{\Gamma \vdash \Sigma x{:}\, A.B : s_2} \qquad \frac{\Gamma \vdash M : A \qquad \Gamma \vdash N : B[x := M]}{\Gamma \vdash \langle M, N\rangle : \Sigma x{:}\, A.B}$$

There are two sum elimination rules; the first one is usually called “weak”, to distinguish it from a “strong” version to be mentioned afterwards.

$$\frac{\Gamma \vdash P : \Sigma x{:}\, A.B \qquad \Gamma \vdash C : s_2 \qquad \Gamma, x : A, y : B \vdash Q : C}{\Gamma \vdash Q \ \textbf{\textit{where}}\ \langle x, y\rangle := P : C}\ (\textit{weak}\ \Sigma)$$

with conversions

$$\frac{\Gamma \vdash M : A \qquad \Gamma \vdash N : B[x := M] \qquad \Gamma, x : A, y : B \vdash Q : C}{\Gamma \vdash Q \ \textbf{\textit{where}}\ \langle x, y\rangle := \langle M, N\rangle \ = \ Q[x := M][y := N] : C}$$

$$\frac{\Gamma \vdash P : \Sigma x{:}\, A.B \qquad \Gamma \vdash C : s_2 \qquad \Gamma, w : \Sigma x{:}\, A.B \vdash Q : C}{\Gamma \vdash Q[w := \langle x, y\rangle] \ \textbf{\textit{where}}\ \langle x, y\rangle := P \ = \ Q[w := P] : C}$$

and substitutions

$$\begin{aligned}
(\Sigma x{:}\, A.B)[z := R] &\ \equiv\ \Sigma x{:}\,(A[z := R]).(B[z := R]) \\
\langle M,\ N\rangle[z := R] &\ \equiv\ \langle M[z := R],\ N[z := R]\rangle \\
(Q\ \textbf{\textit{where}}\ \langle x, y\rangle := P)[z := R] &\ \equiv\ Q[z := R]\ \textbf{\textit{where}}\ \langle x, y\rangle := P[z := R].
\end{aligned}$$

Notice that the variables x and y become bound in $Q\ \textbf{\textit{where}}\ \langle x, y\rangle := P$. There seems to be no standard notation for the term obtained in the sum elimination rules. We adopt the Miranda-like block expression $Q\ \textbf{\textit{where}}\ \langle x, y\rangle := P$, because it is quite intuitive and puts Q, as the most important part, in front position. Alternative notation is $\textbf{\textit{let}}\ \langle x, y\rangle := P\ \textbf{\textit{in}}\ Q$ or $\boldsymbol{\mathcal{E}}_{x,y}(P, Q)$.

For these (s_1, s_2)-sums, one requires the dependency $s_2 \succ s_1$. In case one also has $s_2 \succ s_2$, then one can formulate a *strong* sum elimination rule. The difference concerns the fact that the s_2-type C as used before may now contain a variable of the s_2-type $\Sigma x{:}\, A.B$.

$$\frac{\Gamma \vdash P : \Sigma x{:}\, A.B \qquad \Gamma, w : \Sigma x{:}\, A.B \vdash C : s_2 \qquad \Gamma, x : A, y : B \vdash Q : C[w := \langle x, y\rangle]}{\Gamma \vdash Q\ \textbf{\textit{where}}\ \langle x, y\rangle := P : C[w := P]}\ (\textit{strong}\ \Sigma)$$

The strong conversion rules are slightly different form the weak ones.

$$\frac{\Gamma \vdash M : A \qquad \Gamma \vdash N : B[x := M] \qquad \Gamma, x : A, y : B \vdash Q : C[w := \langle x, y\rangle]}{\Gamma \vdash Q\ \textbf{\textit{where}}\ \langle x, y\rangle := \langle M, N\rangle \ = \ Q[x := M][y := N] : C[w := \langle M, N\rangle]}$$

$$\frac{\Gamma \vdash P : \Sigma x{:}\, A.B \qquad \Gamma, w : \Sigma x{:}\, A.B \vdash Q : C}{\Gamma \vdash Q[w := \langle x, y\rangle]\ \textbf{\textit{where}}\ \langle x, y\rangle := P \ = \ Q[w := P] : C[w := P]}$$

The substitutions are the same.

2.2.5. Rules for (s_1, s_2)-identity.

$$\frac{\Gamma \vdash A : s_1 \qquad \Gamma \vdash M : A \qquad \Gamma \vdash N : A}{\Gamma \vdash I_A(M, N) : s_2}$$

$$\frac{\Gamma \vdash M = N : A}{\Gamma \vdash r_{M,N} : I_A(M, N)} \qquad\qquad \frac{\Gamma \vdash L : I_A(M, N)}{\Gamma \vdash M = N : A}$$

with conversion

$$\frac{\Gamma \vdash L : I_A(M, N)}{\Gamma \vdash L = r_{M,N} : I_A(M, N)}$$

and substitutions

$$\begin{aligned}
I_A(M, N)[z := R] &\ \equiv\ I_{A[z:=R]}(M[z := R],\ N[z := R]) \\
(r_{M,N})[z := R] &\ \equiv\ r_{M[z:=R],N[z:=R]}
\end{aligned}$$

where the latter can in fact be deduced from the former.

The above formulation of identity rules follows Martin-Löf [1984] (where one has $s_1 = s_2$). Identity types will only play a marginal role in this thesis.

2.2.6. Rules for (s_1, s_2)-inclusion.

$$\frac{\Gamma \vdash A : s_1}{\Gamma \vdash \mathit{In}(A) : s_2}$$

$$\frac{\Gamma \vdash M : A}{\Gamma \vdash \mathit{In}(M) : \mathit{In}(A)} \qquad\qquad \frac{\Gamma \vdash N : \mathit{In}(A)}{\Gamma \vdash \mathit{Out}(N) : A}$$

with conversions

$$\frac{\Gamma \vdash M : A}{\Gamma \vdash \mathit{Out}(\mathit{In}(M)) = M : A} \qquad\qquad \frac{\Gamma \vdash N : \mathit{In}(A)}{\Gamma \vdash \mathit{In}(\mathit{Out}(N)) = N : \mathit{In}(A)}$$

and substitutions

$$\begin{aligned}
\mathit{In}(A)[z := R] &\ \equiv\ \mathit{In}(A[z := R]) \\
\mathit{In}(M)[z := R] &\ \equiv\ \mathit{In}(M[z := R]) \\
\mathit{Out}(N)[z := R] &\ \equiv\ \mathit{Out}(N[z := R])
\end{aligned}$$

2.2.7. Rules for conversion. Above, we only mentioned the main points of the intended conversion relation and omitted the rather obvious rules to produce a so-called “compatible equivalence relation”. Notice that the conversion relation is initially only defined on terms, but since terms may occur in types and hence in contexts, one may also have conversion on types and contexts. The conversion relation on contexts is given by componentwise conversion. The following rule is of relevance.

$$\frac{\Gamma \vdash M : A \qquad \Gamma \vdash A = A' : s \qquad \Gamma = \Gamma'}{\Gamma' \vdash M : A'} \; (Eq)$$

Later, in constructing term models of type systems, we have to consider terms, types and contexts modulo conversion. Equivalence classes of these will be denoted by $[\Gamma]$, $[A]$, $[M]$ etc. In doing so, variables present some difficulties; these can be handled by either being very precise — and using de Bruijn's nameless notation — or by being very sloppy. We choose the latter approach.

In the rest of this section some relations between the above features are established. The first two results are standard.

2.2.8. Lemma. (i) *Weak (s, s)-sums gives s-cartesian products.*
(ii) *(s, s)-products gives s-exponents.*

Proof. (i) For $\Gamma \vdash A, B : s$, put $A \times B \equiv \Sigma x{:}\, A.B$ with x fresh. For $\Gamma \vdash L : A \times B$, take $\pi L \equiv x$ ***where*** $\langle x, y\rangle := L$ and $\pi' L \equiv y$ ***where*** $\langle x, y\rangle := L$; the latter may be defined because $x \notin FV(B)$. Then obviously $\pi\langle M, N\rangle = M$ and $\pi'\langle M, N\rangle = N$, but also

$$\begin{aligned}\langle \pi L,\ \pi' L\rangle &= \langle \pi\langle x, y\rangle,\ \pi'\langle x, y\rangle\rangle \ \textbf{\textit{where}}\ \langle x, y\rangle := L\\ &= \langle x, y\rangle \ \textbf{\textit{where}}\ \langle x, y\rangle := L\\ &= L.\end{aligned}$$

(ii) Obvious, using weakening as in (i). □

2.2.9. Lemma. *For strong (s, s)-sums, the elimination and conversion rules mentioned above are equivalent to the following rules with explicit projections.*

$$\frac{\Gamma \vdash P : \Sigma x{:}\, A.B}{\Gamma \vdash \pi P : A \qquad \Gamma \vdash \pi' P : B[x := \pi P]}$$

$$\frac{\Gamma \vdash M : A \qquad \Gamma \vdash N : B[x := N]}{\Gamma \vdash \pi\langle M, N\rangle = M : A \qquad \Gamma \vdash \pi'\langle M, N\rangle = N : B[x := M]}$$

$$\frac{\Gamma \vdash P : \Sigma x{:}\, A.B}{\Gamma \vdash \langle \pi P, \pi' P\rangle = P : \Sigma x{:}\, A.B}$$

Proof. In one direction, one takes for a term $\Gamma \vdash P : \Sigma x{:}\, A.B$ as projections $\pi P \equiv x$ ***where*** $\langle x, y\rangle := P$ and $\pi' P \equiv y$ ***where*** $\langle x, y\rangle := P$; The latter is obtained by using $C(w) \equiv B[x := \pi w]$ in the above strong Σ-rule. Surjectivity of pairing is obtained as in the previous proof. The other way, one defines Q ***where*** $\langle x, y\rangle := P$ as $Q[x := \pi P][y := \pi' P]$. □

In the rest of this work, strong (s, s)-sums will be used in this form with explicit projections.

2.2.10. Lemma (Jacobs, Moggi & Streicher [1991]).

weak (s_1, s_2)-sums $+$ strong (s_2, s_2)-sums $\Rightarrow$ strong (s_1, s_2)-sums.

Proof. Let's use "$\exists$" for the (s_1, s_2)-sums and "Σ" for the (strong) (s_2, s_2)-sums. Assume that types $\Gamma \vdash A : s_1$ and $\Gamma, x : A \vdash B : s_2$ are given together with terms $\Gamma \vdash P : \exists x{:}\, A.B$ and $\Gamma, x : A, y : B \vdash Q : C[w := \langle x, y\rangle]$, where $\Gamma, w : \exists x{:}\, A.B \vdash C : s_2$. Write $C' \equiv \Sigma w{:}\,(\exists x{:}\, A.B).\, C$ and $Q' \equiv \langle\langle x, y\rangle, Q\rangle$. Then $\Gamma \vdash C' : s_2$ and $\Gamma, x : A, y : B \vdash Q' : C'$. Using the weak (s_1, s_2)-elimination rule, one obtains $\Gamma \vdash Q'$ ***where*** $\langle x, y\rangle := P : C' \equiv \Sigma w{:}\,(\exists x{:}\, A.B).\, C$. Hence one can take as new term Q ***with*** $\langle x, y\rangle := P \equiv \pi'\{Q'$ ***where*** $\langle x, y\rangle := P\}$, which is of type $C[w := P]$, since

$$\begin{aligned}\pi\{Q' \ \textbf{\textit{where}}\ \langle x, y\rangle := P\} &= \pi\{Q' \ \textbf{\textit{where}}\ \langle x, y\rangle := \langle x', y'\rangle\} \ \textbf{\textit{where}}\ \langle x', y'\rangle := P\\ &= \pi\{\langle\langle x', y'\rangle,\ Q[x := x'][y := y']\rangle\} \ \textbf{\textit{where}}\ \langle x', y'\rangle := P\\ &= \langle x', y'\rangle \ \textbf{\textit{where}}\ \langle x', y'\rangle := P\\ &= P.\ \square\end{aligned}$$

2.2.11. Lemma. *The following holds in a type system with (s_1, s_2)-inclusion.*
(i) *s_1-unit $\Rightarrow$ s_2-unit.*
(ii) *(s_2, s)-products $\Rightarrow$ (s_1, s)-products;*
(strong) (s_2, s)-sums $\Rightarrow$ (strong) (s_1, s)-sums.
(iii) *Suppose one additionally has strong (s_2, s_2)-sums and weak (s_2, s_1)-sums. The following statements are then equivalent.*

(1) The (s_2, s_1)-sums are strong.
(2) The induced (s_1, s_1)-sums $\widetilde{\Sigma}$ are strong.
(3) The (s_1, s_2)-inclusion In preserves strong sums, i.e. for $\Gamma \vdash A : s_1$ and $\Gamma, x : A \vdash B : s_1$, the operation

$$\frac{\Gamma \vdash M : \Sigma x'{:}\, In(A).\, In(B[x := Out(x')])}{\Gamma \vdash In(\langle Out(\pi M),\ Out(\pi' M)\rangle) : In(\widetilde{\Sigma} x{:}\, A.B)}$$

is invertible.

Proof. (i) The type $1_{s_2} \equiv In(1_{s_1})$ with term $In(\langle\rangle)$ works as s_2-unit: if $\Gamma \vdash M : 1_{s_2}$, then $\Gamma \vdash Out(M) : 1_{s_1}$ which gives $\Gamma \vdash Out(M) = \langle\rangle$. Hence $\Gamma \vdash M = In(Out(M)) = In(\langle\rangle) : 1_{s_2}$.
(ii) We do the sum-case. For types $\Gamma \vdash A : s_1$ and $\Gamma, x : A \vdash B : s$, take $\exists x{:}\, A.B \equiv \Sigma x'{:}\, In(A).\, B[x := Out(x')]$. For terms $\Gamma \vdash M : A$ and $\Gamma \vdash N : B[x := M] \equiv B[x := Out(x')][x' := In(M)]$, one has a $\exists$-pairing $\langle\langle M, N\rangle\rangle \equiv \langle In(M), N\rangle$. The corresponding elimination is given by Q ***with*** $\langle\langle x, y\rangle\rangle := P \equiv Q[x := Out(x')]$ ***where*** $\langle x', y\rangle := P$.

(iii) The implication (a) $\Rightarrow$ (b) results from (ii); the reverse follows from the previous lemma. The equivalence (b) $\Leftrightarrow$ (c) is easy. $\square$

The last three results of this section are based on category theoretical ideas.

2.2.12. Reflection Lemma. *In a type system with (s_1, s_2)-inclusion, weak (s_2, s_1)-sums and an s_1-unit one has that s_2-types and terms can be "reflected" back into s_1 in the following way.*

$$\frac{\Gamma \vdash B : s_2}{\Gamma \vdash \underline{In}(B) \equiv \exists y\colon B.1_{s_1} : s_1} \qquad \frac{\Gamma \vdash M : B}{\Gamma \vdash \underline{In}(M) \equiv \langle M, \langle\rangle\rangle : \underline{In}(B)}$$

However, one cannot define something like $\underline{Out}$ on s_1-terms $N : \underline{In}(B)$. A bit weaker, one has an $\underline{Out}$-operation in the following way.

$$\frac{\Gamma \vdash A : s_1 \qquad \Gamma \vdash N : \underline{In}(In(A))}{\Gamma \vdash \underline{Out}(N) \equiv Out(y) \;\boldsymbol{where}\; \langle y, z\rangle := N : A}$$

Then $\underline{Out}$ is inverse of $\underline{In} \circ In$ in the sense that for $\Gamma \vdash A : s_1$ one has

$$\frac{\Gamma \vdash M : A}{\Gamma \vdash \underline{Out}(\underline{In}(In(M))) = M : A} \qquad \frac{\Gamma \vdash N : \underline{In}(In(A))}{\Gamma \vdash \underline{In}(In(\underline{Out}(N))) = N : \underline{In}(In(A)).}$$

In this way one obtains that

(i) *s_2-cartesian products/exponents $\Rightarrow$ s_1-cartesian products/exponents;*

(ii) *(s_1, s_2)-products $\Rightarrow$ (s_1, s_1)-products.*

Proof. We first establish that $\underline{Out}$ is inverse of $\underline{In} \circ In$.

$$\begin{array}{rcl}
\underline{Out}(\underline{In}(In(M))) & \equiv & Out(y) \;\boldsymbol{where}\; \langle y, z\rangle := \langle In(M), \langle\rangle\rangle \\
& = & Out(In(M)) \quad = \quad M. \\
\underline{In}(In(\underline{Out}(N))) & \equiv & \underline{In}\, In\{Out(y) \;\boldsymbol{where}\; \langle y, z\rangle := N\} \\
& = & \underline{In}\, In\{Out(y) \;\boldsymbol{where}\; \langle y, z\rangle := \langle y', z'\rangle\} \;\boldsymbol{where}\; \langle y', z'\rangle := N \\
& = & \underline{In}(In(Out(y'))) \;\boldsymbol{where}\; \langle y', z'\rangle := N \\
& = & \langle y', \langle\rangle\rangle \;\boldsymbol{where}\; \langle y', z'\rangle := N \\
& = & \langle y', z'\rangle \;\boldsymbol{where}\; \langle y', z'\rangle := N, \quad \text{since } z' \colon 1_{s_1} \\
& = & N.
\end{array}$$

(i) For types $\Gamma \vdash A_1, A_2 : s_1$, one takes $\Gamma \vdash A_1 \,\&\, A_2 \equiv \underline{In}(In(A_1 \times In(A_2)) : s_1$ with pairing given by $\langle\langle M, N\rangle\rangle \equiv \underline{In}(\langle In(M), In(N)\rangle)$ and first projection by $\mathit{fst}\, L \equiv Out(\pi y) \;\boldsymbol{where}\; \langle y, z\rangle := L$. The rest is left to the reader.

(ii) For types $\Gamma \vdash A : s_1$ and $\Gamma, x : A \vdash B : s_1$, put $\forall x{:}A.B \equiv \underline{In}(\Pi x\colon A.In(B))$. For $\Gamma, x : A \vdash L : B$, an abstraction term $\Lambda x\colon A.L \equiv \underline{In}(\lambda x\colon A.In(L))$ is obtained; application is given by $App(M, N) \equiv Out(yN) \;\boldsymbol{where}\; \langle y, z\rangle := M$. $\square$

The next lemma gives a type theoretical version of a result about LCCC's from Freyd [1972].

2.2.13. Proposition. *A type system with strong (s, s)-sums and (s, s)-identities has s-exponents if and only if it has (s, s)-products.*

Proof. (if) Obvious from 2.2.8 (ii).
(only if) Assume that types $\Gamma \vdash A : s$ and $\Gamma, x : A \vdash B : s$ are given; one has to construct $\Gamma \vdash \Pi x\colon A.B : s$. The argument follows the dependent product construction in set-theory. One takes $A' \equiv A \to (\Sigma x\colon A.B)$ and $\Pi x\colon A.B \equiv \Sigma f\colon A'.\ I_{A \to A}(\lambda x\colon A.x,\ \ \lambda x\colon A.\,\pi(fx))$. Then for a term $\Gamma, x : A \vdash L : B$ one has $L' \equiv \lambda x\colon A.\langle x, L\rangle$ of type A' and so one obtains an abstraction term $\Lambda x\colon A.L \equiv \langle L', r\rangle$ of type $\Pi x\colon A.B$, where r has an obvious identity type. For terms $\Gamma \vdash M : \Pi x\colon A.B$ and $\Gamma \vdash N : A$ one has $\Gamma \vdash \pi M : A \to (\Sigma x\colon A.B)$ and thus one can take $App(M, N) \equiv \pi'((\pi M)N)$. $\square$

In Freyd's categorical proof of this result, equalizers play an important role. To extract these from the above type theoretical proof, we must anticipate the categorical description of Type Systems. Remember from the introduction to the first chapter that contexts can be seen as indices for the fibre categories of types and terms (of a fixed sort) derivable in that context. More explicitly, for every sort s and every context Γ, one obtains a category with types $\Gamma \vdash A : s$ as objects. A morphism $\Gamma \vdash A : s \longrightarrow \Gamma \vdash B : s$ is a term $\Gamma, x : A \vdash M : B$. Composition is done by substitution and context projection yields identity morphisms; we don't write this here, but everything should be considered up-to-conversion.

2.2.14. Lemma. *In a type system with strong (s, s)-sums and (s, s)-identities, one has "fibred equalizers".*

Proof. For types $\Gamma \vdash A, B : s$ and morphisms $\Gamma, x : A \vdash M, N : B$ one takes $Eq(M, N) \equiv \Sigma x\colon A.\ I_B(M, N)$. Then we have a map $\Gamma \vdash Eq(M, N) : s \longrightarrow \Gamma \vdash A : s$, given by the first projection, such that composition with M and N yields convertible terms: $\Gamma, z : Eq(M, N) \vdash M[x := \pi z] = N[x := \pi z] : B$. Moreover, for an object $\Gamma \vdash C : s$ and a morphism $\Gamma, y : C \vdash L : A$ which also equalizes M and N, i.e. $\Gamma, y : C \vdash M[x := L] = N[x := L] : B$, there is a unique term $\Gamma, y : C \vdash L' : Eq(M, N)$ such that $\Gamma, y : C \vdash \pi L' = L : A$. $\square$

2.3 Examples of type systems

Before we come to the actual description of various systems, some conventions about our use of the TS-features have to be mentioned.

First, constants are considered as rather ad hoc and are omitted from the system-descriptions below.

Secondly, by requiring the presence of the feature s-closure, we require an s-unit, s-cartesian products and s-exponents. Of course, there is no necessity to do so — one can equally well require only s-exponents, even with β-conversion only — but this choice we make provides us with a syntax which contains usual categorical constructions.

Thirdly, the feature (s_1, s_2)-quantification requires some stipulations.

- In case the setting allows us to use strong sums, we want to do so, unless explicitly stated otherwise. Hence the feature (s_1, s_2)-quantification in a setting with $s_2 \succ s_2$ besides $s_2 \succ s_1$ includes products and *strong* sums. In case we want products and weak sums, we require *weak* (s_1, s_2)-quantification. Hence, requiring (s_1, s_2)-quantification in s TS-setting with $s_2 \not\succ s_2$ amounts to the same as requiring weak (s_1, s_2)-quantification.

- Besides products and (strong) sums, the requirement of (weak) (s_1, s_2)-quantification also includes an s_2-unit. This stipulation has practical advantages, since it gives that (weak) (s, s)-quantification implies s-closure (see lemma 2.2.8).

Summarizing the dependencies necessary for the features, we obtain

$$\begin{array}{rll}
\text{axiom } s_1 : s_2 & \text{only if} & s_1 \succ s_2 \\
s\text{-closure} & \text{only if} & \qquad\qquad\text{(no restriction)} \\
(s_1, s_2)\text{-quantification} & \text{only if} & s_2 \succ s_1 \\
(s_1, s_2)\text{-identity} & \text{only if} & s_2 \succ s_1 \\
(s_1, s_2)\text{-inclusion} & \text{only if} & \forall s \in \mathit{Sort}.\ s \succ s_2 \Rightarrow s \succ s_1 \ \&\ s_1 \succ s \Rightarrow s_2 \succ s
\end{array}$$

with the remark that (s_1, s_2)-quantification in a TS-setting with $s_2 \succ s_1$ and additionally $s_2 \succ s_2$ includes *strong* sums.

In the tables below we put features which come for free as a consequence of others between square brackets. The first three settings receive explicit names. It may help understanding these systems to read $*$ as propositions and $\Box$ as types.

"Minimal" setting:		$\mathit{Sort} = \{*\}$ $\quad \prec = \emptyset$			
System	axiom	closure	quantification	identity	inclusion
$\lambda 1$		$*$			

"Propositions as Types" setting:		$\mathit{Sort} = \{*\}$ $\quad * \succ *$			
System	axiom	closure	quantification	identity	inclusion
λP1		$[\,*\,]$	$(*, *)$		
λPi		$[\,*\,]$	$(*, *)$	$(*, *)$	
$\lambda *$	$* : *$	$[\,*\,]$	$(*, *)$		

λPi denotes Martin-Löf's Type Theory.

"Propositional" setting:			$\mathit{Sort} = \{*, \Box\}$ $\quad * \succ \Box$		
System	axiom	closure	quantification	identity	inclusion
$\lambda\!\rightarrow$	$* : \Box$	$*$			
$\lambda 2$	$* : \Box$	$*$	$(\Box, *)$		
$\lambda\underline{\omega}$	$* : \Box$	$*$ $\Box$			
$\lambda\omega$	$* : \Box$	$*$ $\Box$	$(\Box, *)$		

The systems $\lambda 2$ and $\lambda\omega$ are Girard's second and higher order λ-calculus F and $F\omega$. These four systems consitute the "left plane" of Barendregt's cube.

The next setting combines the previous two.

Setting:		$\mathit{Sort} = \{*, \Box\}$ $\quad * \succ \Box,\ * \succ *,\ \Box \succ \Box$			
System	axiom	closure	quantification	identity	inclusion
HML	$* : \Box$	$[\,*\ \ \Box\,]$	$(\Box, *)$ $(*, *)$ $(\Box, \Box)$		
weak HML	$* : \Box$	$[\,\Box\,]$ $[\,*\,]$	$(\Box, \Box)$ weak $(*, *)$ weak $(\Box, *)$		

In Moggi [1991] a slightly different system called Higher Order ML (HML) is defined; it is set up as a system with great expressive power in which one has a "compile-time" $\Box$-part which does not depend on a "run-time" $*$-part. Here we add an axiom $* : \Box$. A comparable system called "Theory of Predicates" is studied in Pavlović [1990].

In the next setting the previously missing dependency $\Box \succ *$ is added.

Setting:		$\mathit{Sort} = \{*, \Box\}$ $\quad * \succ \Box,\ * \succ *,\ \Box \succ \Box,\ \Box \succ *$			
System	axiom	closure	quantification	identity	inclusion
λP	$* : \Box$	$[\,*\,]$	$(*, *)$ $(*, \Box)$		
λP2	$* : \Box$	$[\,*\,]$	$(*, *)$ $(*, \Box)$ $(\Box, *)$		
λP$\overline{\omega}$	$* : \Box$	$[\,*\,]$ $[\,\Box\,]$	$(*, *)$ $(*, \Box)$ $(\Box, \Box)$		
λC	$* : \Box$	$[\,*\,]$ $[\,\Box\,]$	$(*, *)$ $(\Box, *)$ $(\Box, \Box)$ $(\Box, *)$		
CC	$* : \Box$	$[\,*\,]$ $[\,\Box\,]$	$[\,(*, *)$ $(*, \Box)\,]$ $(\Box, \Box)$ $(\Box, *)$		$(*, \Box)$
weak CC	$* : \Box$	$[\,\Box\,]$ $[\,*\,]$	$(\Box, \Box)$ $[\,(*, \Box)\,]$ $[$ weak $(*, *)\,]$ weak $(\Box, *)$		$(*, \Box)$

The first four of these systems consitute the "right plane" of Barendregt's cube. The last three systems are different versions of the Calculus of Constructions, due to Th. Coquand and G. Huet. The next two settings have three sorts; the first one is due to H. Geuvers.

Setting:	$Sort = \{*, \Box, \triangle\}$ $\quad * \succ \Box,\ \Box \succ \triangle,\ * \succ \triangle$				
System	axiom	closure	quantification	identity	inclusion
λHOL	$* : \Box \quad \Box : \triangle$	$* \quad \Box$	$(\Box, *)$		

In Barendregt [1991], various rather complicated systems for predicate logic are considered (based on work of S. Berardi). Use of (parametrized) constants makes a simplification possible. The basic form which we present below gives rise to many ramifications. They provide various ways to do predicate logic with deductions as proof-objects. Below, the sort $\triangle$ should be understood as sets.

Setting:	$Sort = \{*, \triangle, \Box\}$ $\quad * \succ \triangle,\ * \succ \Box$				
System	axiom	closure	quantification	identity	inclusion
λPRED	$* : \Box$	$* \quad \triangle$	$(\triangle, *)$		

In this chapter one may have noticed that in classifying type systems, the emphasis concerns not so much the individual systems but their underlying settings. This remains important in later chapters.

Chapter 3

The Propositional Setting

In the previous chapter, we briefly mentioned that our categorical description of type systems follows the pattern of "setting + features". Settings involving type dependency are the most difficult ones and the description of these will find its place in the last two chapters 4 and 5. Here we focus our attention on the systems $\lambda\!\rightarrow$, $\lambda 2$, $\lambda\underline{\omega}$ and $\lambda\omega$ which form the "left plane" of Barendregt's cube. The exposition below serves at the same time as an introduction to our approach and as an overview of examples and results, most of which are known (except the last two results of section 3.3).

3.1 Type theoretical and category theoretical settings

On close inspection one may find that a TS-setting concerns the organization of contexts. The role of "categorical settings" will be the same. Basically, we follow Lawvere's [1963] use of algebraic theories (see also Kock and Reyes [1977]). For example, the minimal TS-setting $Sort = \{*\}$ with $\prec\; = \emptyset$ gives rise to a *cartesian* category of contexts (i.e. a category with finite products): take contexts $\Gamma \equiv x_1{:}\,\sigma_1, \ldots, x_n{:}\,\sigma_n$ as objects. Morphisms $\Gamma \to \Delta$, where $\Delta \equiv y_1{:}\,\tau_1, \ldots, y_m{:}\,\tau_m$, are m-tuples of equivalence classes of terms $([M_1], \ldots, [M_m])$ such that $\Gamma \vdash M_i : \tau_i$. Composition is done by substitution and identities are given by (equivalence classes of) variables. The empty context is then terminal and concatenation of contexts yields cartesian products. This structure is independent from (admissible) TS-features. In fact, it is presupposed by such features.

The other way round, one may consider a cartesian category as providing constants for the abovementioned setting. Objects form types (without free variables), which combine to contexts using products. Morphisms form terms, which may contain free variables. Notice that the structural rules concerning contexts (including substitution and weakening) can be performed in this categorical setting. One of the aims in this thesis is to describe for a TS-setting a corresponding categorical setting in which one has the same "expressive power". Because we don't work out interpretations, this statement remains a bit intuitive — but it should become quite clear in the course of this work — especially in the first section of chapter 5. Two differences between the type theoretical and the categorical approach are important.

- In type theory substitution is an (inductively) *defined* operation. In fact also weakening is such an operation, but one needs an explicit syntax using a shift ($\uparrow$), like in Curien [1990] to express this fact. In categorical settings however, substitution and weakening are *primitive* operations, handled both by reindexing (with composition as a special case).

- Substitution (and weakening) in type theory preserves all available operations; this is required by definition. In categorical settings, reindexing generally preserves operations only up-to-isomorphism, unless explicit "split"-conditions are satisfied. Coherence of all these isomorphisms is a topic outside the scope of this work. The interested reader may consult Curien [1990].

Now we turn our attention to one specific setting.

In the propositional setting one has $Sort = \{*, \Box\}$ with $* \succ \Box$. Hence $*$-types may contain $\Box$-terms, but not the other way round. The exchange rule (see 2.2.1) enables us to separate contexts into a sequence of $\Box$-termvariable declarations, followed by a sequence of $*$-termvariable declarations. Notationally, we exploit this fact in the use of the following statements.

$$\Gamma \vdash B : \Box \qquad \text{and} \qquad \Gamma + \Theta \vdash \tau : *$$
$$\Gamma \vdash \sigma : B \qquad \text{and} \qquad \Gamma + \Theta \vdash M : \tau$$

where $\Gamma \equiv \langle \alpha_1 : A_1, \ldots, \alpha_n : A_n \rangle$ is a "$\Box$-context" and $\Theta \equiv \langle x_1 : \sigma_1, \ldots, x_m : \sigma_m \rangle$ with $\Gamma \vdash \sigma_i : *$ is a "$*$-context"; $+$ denotes concatenation of sequences.

A categorical setting corresponding to this propositional setting consists of a "CC fibred over a CC", i.e. of a fibration with fibred finite products over a base category which also has finite products. As an illustration, we describe again how the contexts of this propositional setting form such a fibred CC over a CC, denoted by $p : \mathbf{E} \to \mathbf{B}$.

B obj. $\Box$-contexts Γ.

mor. $\Gamma \to \Gamma' \equiv \langle \beta_1 : B_1, \ldots, \beta_n : B_n \rangle$ are n-tuples $(\sigma_1, \ldots, \sigma_n)$ of $\Box$-types with $\Gamma \vdash \sigma_i : B_i$.

E obj. $\Gamma + \Theta$, where $\Theta \equiv \langle x_1 : \sigma_1, \ldots, x_m : \sigma_m \rangle$, is a $*$-context with $\Gamma \vdash \sigma_i : *$.

mor. $\Gamma + \Theta \to \Gamma' + \Theta'$, with $\Gamma' \equiv \langle \beta_1 : B_1, \ldots, \beta_n : B_n \rangle$ and $\Theta' \equiv \langle y_1 : \tau_1, \ldots, y_m : \tau_m \rangle$ are pairs consisting of an n-tuple of $\Box$-types $\vec{\sigma} = (\sigma_1, \ldots, \sigma_n) : \Gamma \to \Gamma'$ in **B** and an m-tuple of equivalence classes of $*$-terms $([M_1], \ldots, [M_m])$ such that $\Gamma + \Theta \vdash M_i : \tau_i[\vec{\beta} := \vec{\sigma}]$.

Notice that both substitution and weakening (concerning $\Box$) are handled by reindexing.

The other way round, the reader may want to convince him/herself that a (split) fibred CC over a CC can be seen as the propositional setting, dealing with all the context structure.

3.2 Definitions and examples

In this section, categorical versions of the type systems $\lambda{\to}, \lambda 2, \lambda\underline{\omega}$ and $\lambda\omega$ will be described. To construct such categories for these four systems, we simply add approriate categorical features to a propositional setting consisting of a fibred CC over a base CC.

3.2.1. Warning. The description below is based on a rather harmless simplification, which is used throughout the literature. A more subtle account requires techniques which will be developed in the next chapter. The simplification is based one the presence of cartesian product types in our stipulation about closure in the beginning of section 2.3. This makes it possible to deal with $\Box$- and $*$-contexts at the "type" level and so we can dispense categorically with two extra levels, see 5.1.1 and 5.3.4 for a full account.

Let $p : \mathbf{E} \to \mathbf{B}$ be a fibration with fibred finite products where $\mathbf{B}$ is a category with finite products. We think of the objects of $\mathbf{B}$ and $\mathbf{E}$ as $\Box$-types and $*$-types respectively. The feature $\Box$-closure corresponds to $\mathbf{B}$ being a CCC. The feature $*$-closure corresponds to p being a fibred CCC. The axiom $* : \Box$ corresponds to p having a generic object. Finally, the $(\Box, *)$-quantification corresponds either to $Cons_\Omega$-products and sums or to $Cons_\mathbf{B}$-products and (plus a terminal object for p), depending on how many $\Box$-types one has. Hence we come to the notions described below. Essentially, they are all contained in Seely [1987]; see also Pitts [1987] and Coquand & Ehrhard [1987].

3.2.2. Definition. (i) A $\lambda{\to}$-category is a fibred CCC with a generic object over a base CC.

(ii) A$\lambda\underline{\omega}$-category is a fibred CCC with a generic object over a base CCC.

(iii) A $\lambda 2$-category is a fibred CCC with a generic object T over a base CC; additionally, the fibration admits $Cons_\Omega$-products and sums, where $\Omega = pT$.

(iv) A $\lambda\omega$-category is a fibred CCC with a generic object over a base CCC $\mathbf{B}$; additionally, the fibration admits $Cons_\mathbf{B}$-products and sums.

In the literature, a $\lambda\omega$-category is mostly called a PL-category, after <u>P</u>olymorphic <u>L</u>ambda calculus, see Seely [1987].

3.2.3. Definition. Let $\Diamond$ be $\to, 2, \underline{\omega}$ or ω.

(i) A morphism of $\lambda\diamond$-categories is a morphism of fibrations which preserves the relevant structure (see chapter 1).

(ii) A *split* $\lambda\diamond$-category is a $\lambda\diamond$-category in which the fibration and all the relevant structure is split. A morphism of split $\lambda\diamond$-categories preserves the structure on-the-nose.

(iii) A $\lambda\diamond$-category will be called *small* if the fibration involved is small.

Finally, internal versions of the above notions will be mentioned. They have a slightly more simple definition, but the actual description of internal examples is more involved, see e.g. Asperti and Martini [199?].

3.2.4. Definition. Let $\mathbf{B}$ be a category with finite products and $\mathbf{C}$ an internal category in $\mathbf{B}$.

(i) $\mathbf{C}$ is an internal $\lambda\!\rightarrow$-category if it is an internal CCC.

(ii) $\mathbf{C}$ is an internal $\lambda\underline{\omega}$-category if $\mathbf{B}$ is a CCC and $\mathbf{C}$ is an internal CCC.

(iii) $\mathbf{C}$ is an internal $\lambda 2$-category if $\mathbf{C}$ is an internal CCC with internal Cons_{C_0}-products and sums.

(iv) $\mathbf{C}$ is an internal $\lambda\omega$-category if $\mathbf{B}$ is a CCC and $\mathbf{C}$ is an internal CCC which admits internal $\mathit{Cons}_{\mathbf{B}}$-products and sums.

Implicitly in (iii), we assume that $\mathbf{B}$ is a CCC, or at least that the exponent object $C_0^{C_0}$ exists, see the definition of internal Cons_{C_0}-quantification in 1.5.4.

Before describing examples of the above notions, a useful technical result will be mentioned.

3.2.5. Lemma (Frobenius). *Let $p\colon \mathbf{E} \to \mathbf{B}$ be a fibred CCC over a CC admitting Cons_A-sums. The transpose of $id \times \eta : \pi^*_{B,A}(E) \times E' \to \pi^*_{B,A}(E) \times \pi^*_{B,A}(\Sigma_B.E') \cong \pi^*_{B,A}(E \times \Sigma_B.E')$ yields a vertical isomorphism $\Sigma_B.(\pi^*_{B,A}(E) \times E') \cong E \times \Sigma_B.E'$.*

Proof. By Yoneda:

$$\begin{aligned}
\mathbf{E}_B\big(\,\Sigma_B.(\pi^*_{B,A}(E) \times E'),\ E''\,\big) &\cong \mathbf{E}_{B\times A}\big(\,\pi^*_{B,A}(E) \times E',\ \pi^*_{B,A}(E'')\,\big)\\
&\cong \mathbf{E}_{B\times A}\big(\,E',\ \pi^*_{B,A}(E) \Rightarrow \pi^*_{B,A}(E'')\,\big)\\
&\cong \mathbf{E}_{B\times A}\big(\,E',\ \pi^*_{B,A}(E \Rightarrow E'')\,\big)\\
&\cong \mathbf{E}_B\big(\,\Sigma_B.E',\ E \Rightarrow E''\,\big)\\
&\cong \mathbf{E}_B\big(\,E \times \Sigma_B.E',\ E''\,\big). \qquad \square
\end{aligned}$$

3.2.6. Examples. (i) If $\mathbf{C}$ is a small CCC, then it is an internal $\lambda\underline{\omega}$-category in $\mathbf{Sets}$. If $\mathbf{C}$ only has a small collection of objects, then the family fibration $\mathit{Fam}(\mathbf{C}) \to \mathbf{Sets}$ is a split $\lambda\underline{\omega}$-category.

(ii) The fibration $\mathit{Fam}_{\mathrm{eff}}(\mathbf{M}) \to \omega\text{-}\mathbf{Set}$ from 1.2.12 is a split $\lambda\omega$-category. In fact it is a small one. For $A, B \in \omega\text{-}\mathbf{Set}$, one has a right adjoint $\Pi_{(B,A)} : \mathit{Fam}_{\mathrm{eff}}(\mathbf{M})_{B\times A} \to \mathit{Fam}_{\mathrm{eff}}(\mathbf{M})_B$ to $\pi^*_{B,A}$ by $[X : B \times A \to \mathbf{M}] \mapsto \boldsymbol{\lambda} b \in |B|.\,(\Pi_{a\in|A|}.\,|X_{(b,a)}|, \vdash)$, where $\vdash$ is described by $m \vdash f \;\Leftrightarrow\; \forall a \in |A|.\ \forall k \in \mathbb{N}.\ k \vdash_A a \;\Rightarrow\; m \cdot k \vdash_{X_{(b,a)}} f(k)$. Indeed, this yields a collection of modest sets again. Sums are described by $\Sigma_{(B,A)}(X) = \boldsymbol{\lambda} b \in |B|.\,\Theta(\dot{\bigcup}_{a\in|A|}.\,|X_{(b,a)}|, \vdash)$, where Θ is left adjoint to the inclusion $\mathbf{M} \hookrightarrow \omega\text{-}\mathbf{Set}$ and $\vdash$ is given by $m \vdash (a, x) \;\Leftrightarrow\; \mathit{fst}(m) \vdash_A a \;\&\; \mathit{snd}(m) \vdash_{X_{(b,a)}} x$. Actually, all reindexing functors of $\mathit{Fam}_{\mathrm{eff}}(\mathbf{M}) \to \omega\text{-}\mathbf{Set}$ — and not just the cartesian projections — have both a left and a right adjoint.

After definition 1.4.7 the equivalence $\mathit{Fam}_{\mathrm{eff}}(\mathbf{M}) \simeq \sum(\mathbf{PER})$ over $\omega\text{-}\mathbf{Set}$ was mentioned. It yields that the externalization $\sum(\mathbf{PER}) \to \omega\text{-}\mathbf{Set}$ is also a $\lambda\omega$-category. As mentioned in Hyland [1989], change-of-base along the functor $\Delta\colon \mathbf{Sets} \to \omega\text{-}\mathbf{Set}$ (see 1.2.12) yields another $\lambda\omega$-category, which will be denoted by $\mathit{Fam}_{\mathrm{com}}(\mathbf{PER}) \to \mathbf{Sets}$. Objects of $\mathit{Fam}_{\mathrm{com}}(\mathbf{PER})$ are functions X from I to the *PER*. Vertical morphisms $\alpha\colon X \to Y$ over I are collections $\alpha = \{\alpha_i\}_{i\in I}$ of maps $\alpha_i\colon X_i \to Y_i$ in $\mathbf{PER}$ which have a *common* realizer, i.e. $\exists n \in \mathbb{N}.\ \forall i \in I.\ n$ realizes α_i. This $\lambda\omega$-category was first described in Girard [1972].

(iii) The two basic examples from tripos theory (see Hyland, Johnstone & Pitts [1980] and Pitts [1981]) are as follows. It is easily verified that a category $\mathbf{C}$ has infinite products (resp. coproducts) iff every reindexing functor of $\mathit{Fam}(\mathbf{C}) \to \mathbf{Sets}$ has a right (resp. left) adjoint and the Beck-Chevalley condition holds (see also 4.2.5 (i)). Hence if $\mathbf{C}$ is a complete Heyting algebra (considered as a preorder category which is complete and cocomplete and cartesian closed), then the family fibration $\mathit{Fam}(\mathbf{C}) \to \mathbf{Sets}$ is a split $\lambda\omega$-category.

Let $\mathbf{B}$ be a topos. In 1.2.10 (ii) it was already mentioned that the fibration $\mathit{cod}\colon \mathit{Sub}(\mathbf{B}) \to \mathbf{B}$ has a generic object. It forms in fact a $\lambda\omega$-category. (This result follows from applying theorem 5.2.8 to example 5.2.6 (i).)

In both these examples one has a *preorder* fibration, i.e. a fibration with preorder categories as fibres. These provide so-called *proof-irrelevance* or *truth-value* semantics of type theories. As explained in "Introduction and summary", we see them as "logical" models.

In Jacobs [1991] one can find ramifications of the notion of a split $\lambda 2$-category dealing with non-extensionality. Further examples can be found there, including a simple PER model which forms a split $\lambda 2$-category as defined above.

3.2.7. Extended example (Domain models).
The following exposition is based mainly on Coquand, Gunter & Winskel [1989]. A partial order $\langle I, \leq\ \rangle$ is called *directed* if the set I is non-empty and satisfies $\forall i, j \in I.\ \exists k \in I.\ i \leq k \;\&\; j \leq k$. A *directed system* over a category $\mathbf{B}$ is a functor from a directed set to $\mathbf{B}$. In detail it is given by a family $\{B_i\}_{i\in I}$ of objects of

$\mathbf{B}$, indexed by a directed set, together with a collection of maps $\{u_{ij}: B_i \to B_j\}_{i \le j}$ satisfying $u_{ii} = id$ and $i \le j \le k \;\Rightarrow\; u_{jk} \circ u_{ij} = u_{ik}$. The category $\mathbf{B}$ is called *directed complete* if every directed system has a colimit. In the above case this means that there is a collection $\{v_i : B_i \to B\}_{i \in I}$ satisfying $v_j \circ u_{ij} = v_i$; moreover, for every other collection $\{w_i : B_i \to C\}_{i \in I}$ with $w_j \circ u_{ij} = w_i$, there is a unique $\alpha : B \to C$ in $\mathbf{B}$ with $w_i = \alpha \circ v_i$. We write $\mathbf{DcCat}$ for the "category" of (not necessarily small) directed complete categories and continuous (i.e. directed colimit preserving) functors. It is not hard to verify that $\mathbf{DcCat}$ has finite products.

A *domain* is a bounded complete algebraic cpo. Together with continuous functions, domains form a category $\mathbf{DOM}$. It is a subcategory of $\mathbf{DcCat}$. We write $\mathbf{DEP}$ for the category of domains with "embedding projections" as morphisms: a map $X \to Y$ in $\mathbf{DEP}$ consists of a pair (f^e, f^p) where $f^e : X \to Y$ and $f^p : Y \to X$ are continuous functions satisfying $f^p \circ f^e = id$ and $f^e \circ f^p \le id$. $\mathbf{DEP}$ is a CCC, via continuous functors $\times, \to : \mathbf{DEP} \times \mathbf{DEP} \to \mathbf{DEP}$, and it is directed complete, see Smith & Plotkin [1982] and Coquand, Gunter & Winskel [1989] for the details.

We form an indexed category $\Psi : \mathbf{DcCat}^{op} \to \mathbf{Cat}$ with (continuous) functors $X : \mathbf{A} \to \mathbf{DEP}$ in $\mathbf{DcCat}$ as objects of $\Psi\mathbf{A}$. Morphisms $X \to Y$ in $\Psi\mathbf{A}$ are *continuous* families $\{\alpha_A \in \mathbf{DOM}(XA, YA)\}_{A \in \mathbf{A}}$ where continuity of the family is expressed by the following two conditions.

- for every $u : A \to B$ in $\mathbf{A}$ one has

$$Y(u)^e \circ \alpha_A \circ X(u)^p \;\;\le\;\; \alpha_B$$

- for every directed colimit $\{v_i : A_i \to A\}_{i \in I}$ one has

$$\alpha_A = \bigsqcup_{i \in I} Y(v_i)^e \circ \alpha_{A_i} \circ X(v_i)^p$$

see also Coquand & Ehrhard [1987]. The Grothendieck construction applied to Ψ yields a split fibration $\mathit{Fam}_{\mathrm{cont}}(\mathbf{DEP}) \longrightarrow \mathbf{DcCat}$. Actually, it is a split $\lambda{\to}$-category.

Interestingly, there is a full and faithful functor

$$\begin{tikzcd} \mathit{Fam}_{\mathrm{cont}}(\mathbf{DEP}) \arrow[rr, "\mathcal{P}"] \arrow[dr] & & \mathbf{DcCat}^{\to} \arrow[dl, "cod"] \\ & \mathbf{DcCat} & \end{tikzcd}$$

which maps cartesian arrows to pullbacks. In the next chapter we introduce the name *full comprehension category* for such a functor $\mathcal{P}$. This structure passed without explicit attention in previous work on these categories (but it was used implicitly). As in example 1.2.12, we first define a functor $\mathcal{P}_0 : \mathit{Fam}_{\mathrm{cont}}(\mathbf{DEP}) \to \mathbf{DcCat}$. For

$X : \mathbf{A} \to \mathbf{DEP}$, one obtains a category $\mathcal{P}_0(\mathbf{A}, X)$ with objects (A, x) where $x \in XA$. A morphism $(A, x) \xrightarrow{u} (B, y)$ in $\mathcal{P}_0(\mathbf{A}, X)$ is a map $u : A \to B$ in $\mathbf{A}$ satisfying $X(u)^e(x) \le y$. It is readily established that $\mathcal{P}_0(\mathbf{A}, X)$ is directed complete: given $\{u_{ij} : (A_i, x_i) \to A_j, x_j)\}$, let $\{v_i : A_i \to A\}$ be the colimit of the u_{ij}'s in $\mathbf{A}$ and $x = \bigsqcup_{i \in I} .\, X(v_i)^e(x_i)$ in XA. Then $\{v_i : (A_i, x_i) \to (A, x)\}$ is a colimit in $\mathcal{P}_0(\mathbf{A}, X)$. For a morphism $(F, \alpha) : (\mathbf{A}, X) \to (\mathbf{B}, Y)$ in $\mathit{Fam}_{\mathrm{cont}}(\mathbf{DEP})$, i.e. for a continuous functor $F : \mathbf{A} \to \mathbf{B}$ and a continuous family $\alpha : X \to YF$, one defines $\mathcal{P}_0(F, \alpha) : \mathcal{P}_0(\mathbf{A}, X) \to \mathcal{P}_0(\mathbf{B}, Y)$ by $(A, x) \mapsto (FA, \alpha_A(x))$ and $u \mapsto Fu$. The fact that the family α is continuous guarantees that $\mathcal{P}_0(F, \alpha)$ is well-defined and continuous again. The abovementioned functor $\mathcal{P} : \mathit{Fam}_{\mathrm{cont}}(\mathbf{DEP}) \to \mathbf{DcCat}^{\to}$ is layed down by $(\mathbf{A}, X) \mapsto$ [the (continuous) projection $\mathcal{P}_0(\mathbf{A}, X) \to \mathbf{A}$] and $(F, \alpha) \mapsto (F, \mathcal{P}_0(F, \alpha))$. Actually, this projection is a cofibration.

Finally, we establish that $\mathcal{P}$ is "fibrewise" full and faithful (which is enough). That it is faithful, is easy; hence we only show that it is full. Suppose therefore that for $X, Y : \mathbf{A} \to \mathbf{DEP}$ in $\mathit{Fam}_{\mathrm{cont}}(\mathbf{DEP})$, a continuous functor $H : \mathcal{P}_0(\mathbf{A}, X) \to \mathcal{P}_0(\mathbf{A}, Y)$ is given with $\mathcal{P}(\mathbf{A}, Y) \circ H = \mathcal{P}(\mathbf{A}, X)$; then one can write $H(A, x) = (A, \alpha_A(x))$. For every $A \in \mathbf{A}$, one obtains a continuous function $\alpha_A : XA \to YA$; these functions yield a continuous family $\{\alpha_A\}$: for $u : A \to B$ and $x \in XB$, put $z = X(u)^p(x)$. Then $X(u)^e(z) \le x$, so $u : (A, z) \to (B, x)$ in $\mathcal{P}_0(\mathbf{A}, X)$. Hence $Hu = u : (A, \alpha_A(z)) \to (B, \alpha_B(x))$ in $\mathcal{P}_0(\mathbf{A}, Y)$. But then $\alpha_B(x) \ge Y(u)^e(\alpha_A(z)) = \{Y(u)^e \circ \alpha_A \circ X(u)^p\}(x)$. This settles the first requirement about continuity; the second one is left to the reader.

Next we mention some results about this functor $\mathcal{P}$ (see Coquand, Gunter & Winskel [1989] propositions 7 and 8), which will be useful later in 4.1.6 (vi) and 4.3.2 (iv). Let $X : \mathbf{A} \to \mathbf{DEP}$ be an object of $\mathit{Fam}_{\mathrm{cont}}(\mathbf{DEP})$. Then in case $\mathbf{A}$ is a domain (as a preorder category) one has

(i) $\mathcal{P}_0(\mathbf{A}, X)$ is a domain;

(ii) the collection $|(\mathbf{A}, X)|$ of continuous "sections" $H : \mathbf{A} \to \mathcal{P}_0(\mathbf{A}, X)$ in $\mathbf{DcCat}$ with $\mathcal{P}(\mathbf{A}, X) \circ H = id$ is a domain; the ordering is pointwise (in the second component).

As a consequence of (i) above one obtains by restriction another "comprehension category" $\mathit{Fam}_{\mathrm{cont}}(\mathbf{DEP}) \to \mathbf{DOM}^{\to}$ which will be used in 4.3.2 (v). Here the objects of $\mathit{Fam}_{\mathrm{cont}}(\mathbf{DEP})$ are arrows $\mathbf{A} \to \mathbf{DEP}$ where $\mathbf{A}$ is a domain. We don't bother to give different names to the total categories in $\mathit{Fam}_{\mathrm{cont}}(\mathbf{DEP}) \to \mathbf{DcCat}$ and $\mathit{Fam}_{\mathrm{cont}}(\mathbf{DEP}) \to \mathbf{DOM}$. As long as we consider them together with the base category, there is no confusion.

The main result of Coquand, Gunter & Winskel [1989] is that $\mathit{Fam}_{\mathrm{cont}}(\mathbf{DEP})$ over the full subcategory of $\mathbf{DcCat}$ generated by the objects $\mathbf{DEP}^n, n \in \mathbb{N}$ forms a split $\lambda{\to}$-category with $(\Box, *)$-products.

3.3 Some constructions

Observations from chapter 1 (especially 1.4.6 and 1.5.4) easily bring us to the following result about externalization.

3.3.1. Proposition. *Let $\diamond$ be $\rightarrow, 2, \underline{\omega}$ or ω. Suppose $\mathbf{C} \in Cat(\mathbf{B})$; then*

$$\mathbf{C} \text{ is an internal } \lambda\diamond\text{-category} \quad \Leftrightarrow \quad [\mathbf{C}] \text{ is a split } \lambda\diamond\text{-category.} \qquad \square$$

Under certain size-conditions, internalization is also possible, see 1.4.8 and 1.5.5. The next proposition is the main result of Asperti and Martini [199?].

3.3.2. Proposition. *Let $p : \mathbf{E} \to \mathbf{B}$ be a split fibration where $\mathbf{B}$ is locally small and all fibres are small. Then*
(i) *p is a split fibred CCC $\Rightarrow$ $\hat{p}$ in $\widehat{\mathbf{B}} = \mathbf{Sets}^{\mathbf{B}^{op}}$ is an internal $\lambda\underline{\omega}$-category.*
(ii) *p is a split $\lambda 2$-category $\Rightarrow$ $\hat{p}$ in $\widehat{\mathbf{B}}$ is an internal $\lambda 2$-category.*
Further, the change-of-base situation $p \to [\hat{p}]$ from 1.4.8 is a morphism of these categories.

Proof. (i) Obvious, see 1.4.8.
(ii) By lemma 1.5.5. $\square$

The next result is essentially due to Pitts [1987], although the formulation used there is different. It also yields a form of internalization. The fibration $\overline{p}$ is introduced in 1.2.7.

3.3.3. Theorem. *Let $\diamond$ be $\rightarrow$ or 2.*

$$p : \mathbf{E} \to \mathbf{B} \text{ is a } \lambda\diamond\text{-category} \quad \Rightarrow \quad \overline{p} : \overline{\mathbf{E}} \to \mathbf{E} \text{ is a small } \lambda\diamond\text{-category.}$$

Further, the change-of-base situation $p \to \overline{p}$ from 1.2.7 is a morphism of these categories.

Proof. Let $p : \mathbf{E} \to \mathbf{B}$ be a fibred CCC with generic object $T \in \mathbf{E}$ above Ω. We already know from 1.2.7 and 1.2.10 (iii) that $\overline{p} : \overline{\mathbf{E}} \to \mathbf{E}$ is again a fibred CCC with generic object $(1\Omega, T) \in \overline{\mathbf{E}}$. The fact that $\overline{p}$ is a small fibration follows from a general theorem to be treated in 4.5.8 (using 4.5.5 and 4.4.4 (i)). The details can also be checked in this special case: one can form an appropriate internal category in $\mathbf{E}$ with $\Omega_0 = 1\Omega$ as object of objects and $\Omega_1 = (\Omega_0 \,\&\, \Omega_0) \times (\pi^*_{\Omega,\Omega}(T) \Rightarrow \pi'^*_{\Omega,\Omega}(T))$ as object of morphisms (where & denotes the "global" product in $\mathbf{E}$, see 1.2.8). The first projection in the fibre then yields a pair $\langle \partial_0, \partial_1 \rangle : \Omega_1 \to \Omega_0 \,\&\, \Omega_0$ in $\mathbf{E}$. It is not hard to verify that pullbacks for Ω_2 and Ω_3 exist, see 1.4.1. Hence $\overline{p} : \overline{\mathbf{E}} \to \mathbf{E}$ is *small* $\lambda{\rightarrow}$-category.

Next, let's assume that p is a $\lambda 2$-category via adjunctions $\Sigma_B \dashv \pi^*_{B,\Omega} \dashv \Pi_B$. One defines product and sum functors $\overline{\mathbf{E}}_{E \,\&\, \Omega_0} \to \overline{\mathbf{E}}_E$ for $\overline{p}$ by $(E \,\&\, \Omega_0, E') \mapsto (E, \Pi_{pE}.E')$ and $(E, \Sigma_{pE}.E')$. The Frobenius isomorphism from lemma 3.2.5 is needed to establish the required adjunctions for these new sums. $\square$

Pitts [1987] goes on to embed $\overline{p} : \overline{\mathbf{E}} \to \mathbf{E}$ in the topos of presheaves $\mathbf{Sets}^{\mathbf{E}^{op}}$ — under certain size conditions — which yields a "topos model" of $\lambda 2$. Further details may be found there.

The next construction requires some preliminary work. We consider categories with an explicitly given cartesian closed structure. Morphisms of these are required to preserve this structure on-the-nose. Every split $\lambda{\rightarrow}$-category yields such a CCC by looking only at the fibre above the terminal object in the basis and forgetting the rest. Obviously, a morphism of $\lambda{\rightarrow}$-categories yields a morphism between the corresponding cartesian closed fibre categories. Our intention in the rest of this section is to show that this forgetful functor has a left adjoint, i.e. that every CCC generates a free $\lambda{\rightarrow}$-category. In order to make the presentation more accessible, we first construct a simple (non-free) $\lambda{\rightarrow}$-category from a given CCC. Later, the free one is derived from it. Our construction is clearly inspired by the work in Bainbridge *et al.* [1990], but dinaturality doesn't play a role here.

Let $\mathbf{C}$ be a CCC. We form the category $NP(\mathbf{C})$ — where 'N' stands for negative and 'P' for positive — as follows. Objects are natural numbers $n \in \mathbb{N}$. Morphisms $(F_1, \ldots, F_m) : n \to m$ are functors $F_i : (\mathbf{C}^{op})^n \times \mathbf{C}^n \to \mathbf{C}$. Especially, we have for every object $X \in \mathbf{C}$ a constant functor $K^n_X : n \to 1$; furthermore, we use projections $proj_i : n \to 1$ described by $(\vec{X}, \vec{Y}) \mapsto Y_i$ and $(\vec{f}, \vec{g}) \mapsto g_i$. Given $F : n \to 1$, i.e. $F : (\mathbf{C}^{op})^n \times \mathbf{C}^n \to \mathbf{C}$, we write $F^{tw} : (\mathbf{C}^{op})^n \times \mathbf{C}^n \to \mathbf{C}^{op}$ for the "twisted" version of F obtained as the composite of

$$(\mathbf{C}^{op})^n \times \mathbf{C}^n \cong \mathbf{C}^n \times (\mathbf{C}^{op})^n \cong (\mathbf{C}^{op\,op})^n \times (\mathbf{C}^{op})^n \cong ((\mathbf{C}^{op})^n \times \mathbf{C}^n)^{op} \xrightarrow{F^{op}} \mathbf{C}^{op},$$

see Bainbridge *et al.* [1990] appendix A6. Notice that $F^{tw}(\vec{X}, \vec{Y}) = F(\vec{Y}, \vec{X})$ and $F^{tw}(\vec{f}, \vec{g}) = F(\vec{g}, \vec{f})$: positive occurrences are changed to negative ones and vice-versa. Now one can define composition in $NP(\mathbf{C})$ by $(G_1, \ldots, G_k) \circ (F_1, \ldots, F_m) = (H_1, \ldots, H_k)$, where $H_i = G_i \circ (F^{tw}_1, \ldots, F^{tw}_m, F_1, \ldots, F_m)$. Notice that $id_n = (proj_1, \ldots, proj_n)$. In this way one obtains a category $NP(\mathbf{C})$. It has finite products: 0 is terminal and $n + m$ is the products of n and m. Hence $NP(\mathbf{C})$ is an algebraic theory in the sense of Lawvere [1963]. For arrows $F, G : n \to 1$, we put

$$\begin{array}{rclclclcl} F \times G & = & prod \circ (F, G) & : & (\mathbf{C}^{op})^n \times \mathbf{C}^n & \longrightarrow & \mathbf{C} \times \mathbf{C} & \longrightarrow & \mathbf{C}; \\ F \Rightarrow G & = & exp \circ (F^{tw}, G) & : & (\mathbf{C}^{op})^n \times \mathbf{C}^n & \longrightarrow & \mathbf{C}^{op} \times \mathbf{C} & \longrightarrow & \mathbf{C}. \end{array}$$

Next we define an indexed category $\Psi : NP(\mathbf{C})^{op} \to \mathbf{Cat}$ by giving the fibre categories $\Psi(n)$ morphisms $F : n \to 1$ in $NP(\mathbf{C})$ as objects. Morphisms $F \to G$ in $\Psi(n)$ are families $\sigma = \{\sigma_{\vec{X}}\}_{\vec{X} \in \mathbf{C}^n}$ of arrows $\sigma_{\vec{X}} : F(\vec{X}, \vec{X}) \to G(\vec{X}, \vec{X})$ in $\mathbf{C}$. There is no dinaturality requirement for such families. Especially, we have constant families $K^n_g = \{g\}_{\vec{X} \in \mathbf{C}^n} : K^n_X \to K^n_Y$ for every $g : X \to Y$ in $\mathbf{C}$. For $(H_1, \ldots, H_n) : m \to n$ in $NP(\mathbf{C})$, we define $\vec{H}^* = \Psi(\vec{H}) : \Psi(n) \to \Psi(m)$ by $F \mapsto F \circ \vec{H}$ and $\sigma \mapsto \{\sigma_{H_1(\vec{Y}, \vec{Y}), \ldots, H_n(\vec{Y}, \vec{Y})}\}_{\vec{Y} \in \mathbf{C}^m}$.

This construction of $\Psi : NP(\mathbf{C})^{op} \to \mathbf{Cat}$ is a categorical version of a construction used a few times in Jacobs [1991] section 6, starting from a *set* (of ideals or per's) — instead of from a category — to obtain similar examples. There, the negative and positive occurrences don't play a role. Comparable structures are defined in examples 5.5.6 (i), (ii).

3.3.4. Proposition. *Applying the Grothendieck construction to* $\Psi : NP(\mathbf{C})^{op} \to \mathbf{Cat}$ *yields a (split)* $\lambda{\to}$*-category.*

Proof. Basically one has to show that the fibre categories $\Psi(n)$ are cartesian closed and that this structure is preserved on-the-nose by the reindexing functors. This all holds by the pointwise character of the construction. □

The above construction does not produce the free $\lambda{\to}$-category generated by $\mathbf{C}$ because the categories $NP(\mathbf{C})$ and $\Psi(n)$ are too big. With a term model construction in mind, we now define appropriate subcategories $NP_f(\mathbf{C})$ and $\Psi_f(n)$, where 'f' stands for free.

$NP_f(\mathbf{C})$ still has objects $n \in \mathbb{N}$ and morphisms $(F_1, \ldots, F_m) : n \to m$ are still built from F_i's from n to 1, but these arrows $n \to 1$ are in $NP_f(\mathbf{C})$ given as the smallest collection of functors $(\mathbf{C}^{op})^n \times \mathbf{C}^n \to \mathbf{C}$ satisfying

(i) $K^n_X : n \to 1$;
(ii) $proj_i : n \to 1$;
(iii) $F, G : n \to 1 \;\Rightarrow\; F \times G,\; F \Rightarrow G : n \to 1$;
(iv) $F : n \to 1,\; H_1, \ldots, H_n : m \to 1 \;\Rightarrow\; F \circ (H_1, \ldots, H_n) : m \to 1$.

In the latter case we use composition as defined above. One easily verifies that $NP_f(\mathbf{C}) \hookrightarrow NP(\mathbf{C})$ is a category with finite products.

The fibre categories $\Psi_f(n)$ have arrows $F : n \to 1$ in $NP_f(\mathbf{C})$ as objects. The morphisms in these categories are in the smallest collection satisfying

(i) $K^n_g : K^n_X \to K^n_Y$;
(ii) $id_F : F \to F$;
(iii) $\sigma : F \to G,\; \tau : G \to K \;\Rightarrow\; \tau \circ \sigma : F \to K$;
(iv) $\sigma : F \to G$ in $\Psi_f(n)$, $\vec{H} : m \to n$ in $NP_f(\mathbf{C}) \;\Rightarrow\; \vec{H}^*(\sigma) : \vec{H}^*(F) \to \vec{H}^*(G)$ in $\Psi_f(m)$;
(v) $!_F : F \to K^n_t$ for $F \in \Psi_f(n)$;
(vi) $\pi : F \times G \to F, \quad \pi' : F \times G \to G$;
(vii) $\sigma : K \to F,\; \tau : K \to G \;\Rightarrow\; \langle \sigma, \tau \rangle : K \to F \times G$;
(viii) $ev : (F \Rightarrow G) \times F \to G$;
(ix) $\sigma : K \times F \to G \;\Rightarrow\; \Lambda(\sigma) : K \to F \Rightarrow G$.

One easily verifies again that $\Psi_f(n) \hookrightarrow \Psi(n)$ is a category which is cartesian closed.

3.3.5. Theorem. *The Grothendieck construction applied to* $\Psi_f : NP_f(\mathbf{C})^{op} \to \mathbf{Cat}$ *yields the free* $\lambda{\to}$*-category generated by* $\mathbf{C}$.

Proof. There is a unit functor $\eta_{\mathbf{C}} : \mathbf{C} \to \Psi_f(0)$ given by $X \mapsto K^0_X$ and $g \mapsto K^0_g$. It preserves the CCC structure. Let $p : \mathbf{E} \to \mathbf{B}$ be a split $\lambda{\to}$-category with a split generic object via $\phi_B : \mathbf{B}(B, \Omega) \xrightarrow{\sim} Obj(\mathbf{E}_B)$ and let $L : \mathbf{C} \to \mathbf{E}_t$ preserve the CCC-structure on-the-nose. We have to construct a (unique) morphism $(L_1, L_2) : \mathcal{G}(\Psi_f) \to p$ of split $\lambda{\to}$-categories, such that $\eta_{\mathbf{C}}$ followed by the restriction $\Psi_f(0) \to \mathbf{E}_t$ is L again. There is no choice at all for L_1 and L_2, since their behaviour on the constant families of objects and arrows is described by L and on the rest by the fact that the structure should be preserved. For example $L_1 : NP_f(\mathbf{C}) \to \mathbf{B}$ is given by $n \mapsto \Omega^n$ (since $1 \mapsto \Omega$) and $K^n_X = K^0_X \circ !_n \mapsto \phi_t^{-1}(LX) \circ !_{\Omega^n}$. Similarly one finds L_2. □

As already remarked, the free $\lambda{\to}$-category $\mathcal{G}(\Psi_f)$ is constructed essentially as a term model, starting from objects and arrows of $\mathbf{C}$ as constant types and terms. Describing it as such enables a deeper type theoretical analysis. In this way it is shown in Girard, Scedrov & Scott [1991] that all morphisms in the fibre categories $\Psi_f(n)$ are actually dinatural transformations. The proof makes a detour through Gentzen's sequent calculus.

Using these techniques, one might be able to settle whether the unit functor $\eta_{\mathbf{C}} : \mathbf{C} \to \Psi_f(0)$ in the above proof of 3.3.5 is a full embedding.

Chapter 4

More Fibred Category Theory

The categorical study of type dependency is our next subject. The main notion here is what we call a "comprehension category". Such a category will be used in two different but, closely related ways: first as a categorical setting and secondly as a domain of quantification (for a fibration). These matters can be found in the first and second section. The third one deals with *closed comprehension categories* which can be understood as categories with dependent sums and products. We show that these categories have good closure properties.

The fourth section investigates a technique (due to J. Bénabou) of doing category theory "on top of a given fibration". It gives the possibility to construct more complicated settings having different levels in the next chapter. Finally, we mention some (standard) results about locally small fibrations and (a fibred version of) Freyd's adjoint functor theorem.

4.1 Comprehension categories

4.1.1. Definition (Jacobs [1990]). A *comprehension category* is a functor of the form $\mathcal{P} : \mathbf{E} \to \mathbf{B}^{\to}$ satisfying

(i) $cod \circ \mathcal{P} : \mathbf{E} \to \mathbf{B}$ is a fibration;

(ii) f is cartesian in $\mathbf{E} \Rightarrow \mathcal{P}f$ is a pullback in $\mathbf{B}$.

This $\mathcal{P}$ is called a *full* comprehension category in case $\mathcal{P}$ is a full and faithful functor. It is called *cloven* or *split* whenever the fibration involved is cloven or split.

Notice that we don't require that the base category $\mathbf{B}$ has *all* pullbacks. In case it does, $\mathcal{P}$ is a cartesian functor. It is easy to verify that $\mathcal{P}$ is a full comprehension category if and only if $\mathcal{P}$ is fibrewise a full and faithful functor.

4.1.2. Notation. For a comprehension category $\mathcal{P} : \mathbf{E} \to \mathbf{B}^{\to}$ we standardly write $p = cod \circ \mathcal{P}$ and $\mathcal{P}_0 = dom \circ \mathcal{P}$. The object part of $\mathcal{P}$ then forms a natural transformation $\mathcal{P} : \mathcal{P}_0 \stackrel{\cdot}{\longrightarrow} p$. Similarly, for e.g. $\mathcal{Q} : \mathbf{D} \to \mathbf{A}^{\to}$, we write $q = cod \circ \mathcal{Q}$ and $\mathcal{Q}_0 = dom \circ \mathcal{Q}$. The functors $(-)_0$ do the work of context extension (or comprehension) as can be seen clearly in the term model example below.

The components $\mathcal{P}E$ are often called *projections* (and sometimes *display maps*); reindexing functors of the form $\mathcal{P}E^*$ are called *weakening* functors. For an object $E \in \mathbf{E}$ we write $|E| = \{u : pE \to \mathcal{P}_0E \mid \mathcal{P}E \circ u = id\}$; elements of $|E|$ may be called *terms of type* E. Motivation for this terminology may be found in the term model described next.

It is our claim that a full comprehension category with a terminal object in the basis constitutes a categorical version of the "Propositions as Types"-setting $Sort = \{*\}$ with $* \succ *$. To support this claim, we shall organize the contexts of this setting as such a comprehension category $\mathcal{P} : \mathbf{E} \to \mathbf{B}^{\to}$. The objects of $\mathbf{B}$ are equivalence classes $[\Gamma]$ of contexts. A morphism $[\Gamma] \to [\Delta]$, where $\Delta \equiv y_1 : \tau_1, \ldots, y_n : \tau_n$ consists of an n-tuple of equivalence classes of terms $\langle [M_1], \ldots, [M_n] \rangle$ satisfying $\Gamma \vdash M_i : \tau_i[y_1 := M_1, \ldots, y_{i-1} := M_{i-1}]$. Objects of the category $\mathbf{E}$ are of the form $[\Gamma \vdash \sigma : *]$ and arrows $[\Gamma \vdash \sigma : *] \to [\Delta \vdash \tau : *]$ are pairs $([\vec{M}], [N])$ with $[\vec{M}] : [\Gamma] \to [\Delta]$ in $\mathbf{B}$ and $\Gamma, x : \sigma \vdash N : \tau[\vec{y} := \vec{M}]$. The functor $\mathcal{P}$ is then described by $[\Gamma \vdash \sigma : *] \mapsto$ (the projection $[\Gamma, x : \sigma] \to [\Gamma]$). If Γ is of the form $x_1 : \sigma_1, \ldots, x_m : \sigma_m$, this projection is simply $\langle [x_1], \ldots, [x_m] \rangle$.

Notice that the functor $\mathcal{P}_0$ performs "context comprehension" $[\Gamma \vdash \sigma : *] \mapsto [\Gamma, x : \sigma]$. Similarly, other type theoretical operations can be understood categorically using this specific comprehension category.

Next we introduce a simple, but important construction to obtain so-called "constant" comprehension categories.

4.1.3. Example. Let $\mathbf{B}$ be a category with finite products and T a non-empty collection of objects from $\mathbf{B}$; T is called *non-trivial* if for some $X \in T$, the collection $\mathbf{B}(t, X)$ is non-empty — where $t \in \mathbf{B}$ is terminal. We form a split full comprehension category $Cons_T : \mathbf{B}/\!/T \to \mathbf{B}^{\to}$ as follows. The total category $\mathbf{B}/\!/T$ has pairs (A, X) with $A \in \mathbf{B}$ and $X \in T$ as objects. Morphisms $(u, f) : (A, X) \to (B, Y)$ in $\mathbf{B}/\!/T$ are given by two maps $u : A \to B$ and $f : A \times X \to Y$ in $\mathbf{B}$. The functor $Cons_T$ is then defined by $(A, X) \mapsto \pi : A \times X \to A$ and $(u, f) \mapsto (u, \langle u \circ \pi, f \rangle)$. Notice that the fibre above the terminal object is the full subcategory of $\mathbf{B}$ determined by T. Comprehension categories of this form will be called *constant* because there is no dependency involved. We condider two extremes.

(i) If the collection T consists of a single element, say $T = \{\Omega\}$, then we write $\mathbf{B}/\!/\Omega$ and $Cons_\Omega$ instead of $\mathbf{B}/\!/\{\Omega\}$ and $Cons_{\{\Omega\}}$.

(ii) If T contains all objects from $\mathbf{B}$, we write $\overline{\mathbf{B}}$ for $\mathbf{B}/\!/Obj(\mathbf{B})$ and $Cons_{\mathbf{B}}$ for $Cons_{\mathrm{Obj}(\mathbf{B})}$. This notation coincides with the one introduced in 1.2.7 when the construction described there is applied to the fibration $\mathbf{B} \to \mathbf{1}$ (the terminal category).

The expressions "$Cons_\Omega$"- and "$Cons_{\mathbf{B}}$"-quantification used in section 1.5 are based on these two comprehension categories; this will become clear in the next section when we deal with products and sums.

4.1.4. Definition. A *morphism of comprehension categories* is given by a triple (K, L, γ), where (K, L) is a morphism of fibrations $p \to q$ as in the diagram below

$$\begin{tikzcd}
\mathbf{E} \arrow[rr, "L"] \arrow[d, shift right=3, "\mathcal{P}_0"'] \arrow[d, shift left=3, "p"] \arrow[d, phantom, "\overset{\mathcal{P}}{\Longrightarrow}"] & & \mathbf{D} \arrow[d, shift right=3, "\mathcal{Q}_0"'] \arrow[d, shift left=3, "q"] \arrow[d, phantom, "\overset{\mathcal{Q}}{\Longrightarrow}"] \\
\mathbf{B} \arrow[rr, "K"] & & \mathbf{A}
\end{tikzcd}$$

and $\gamma : \mathcal{Q}_0L \stackrel{\cdot}{\longrightarrow} K\mathcal{P}_0$ is a natural isomorphism satisfying $K\mathcal{P} \circ \gamma = \mathcal{Q}L$.

This notion of morphism is slightly more general than the one used in Jacobs [1990], where one has $\gamma = id$.

Another way of understanding a map (K, L, γ) is as a vertical isomorphism in

$$\begin{tikzcd}
\mathbf{E} \arrow[r, "L"] \arrow[d, "\mathcal{P}"'] \arrow[dr, phantom, "\overset{\gamma}{\Longrightarrow}"] & \mathbf{D} \arrow[d, "\mathcal{Q}"] \\
\mathbf{B}^{\to} \arrow[r, "K^{\to}"'] & \mathbf{A}^{\to}
\end{tikzcd}$$

The context comprehension functors $(-)_0$ of a comprehension category "reflect" the total category back into the basis. In case one has a fibration with a terminal object, an obvious way of doing this is by requiring that the fibrewise global-sections functors are representable. A bit more explicitly, let $p : \mathbf{E} \to \mathbf{B}$ be a fibration with terminal via $1 : \mathbf{B} \to \mathbf{E}$; one requires that for $E \in \mathbf{E}$ above $A \in \mathbf{B}$ the map $(\mathbf{B}/A)^{op} \to \mathbf{Ens}$ given by

$$B \xrightarrow{u} A \quad \mapsto \quad \mathbf{E}_B\,(1B,\ u^*(E)\,)$$

is representable (where $\mathbf{Ens}$ is a suitably large universe). Let $\mathcal{P}E$ be representing arrow in $\mathbf{B}$; then

$$\begin{aligned}
\mathbf{E}(1B,\ E) \quad &\cong \quad \dot{\bigcup}_{u : B \to A}.\ \mathbf{E}_B\,(1B,\ u^*(E)) \\
&\cong \quad \dot{\bigcup}_{u : B \to A}.\ \mathbf{B}/A\,(u,\ \mathcal{P}E) \\
&\cong \quad \mathbf{B}(B,\ dom(\mathcal{P}E)).
\end{aligned}$$

Hence one obtains a right adjoint to $1 : \mathbf{B} \to \mathbf{E}$. The following definition captures this situation. This notion is introduced in Ehrhard [1988a] under the name *D-category*.

4.1.5. Definition. (i) A *comprehension category with unit* is given by a fibration $p : \mathbf{E} \to \mathbf{B}$ provided with a terminal object functor $1 : \mathbf{B} \to \mathbf{E}$, which has a right adjoint $\mathcal{P}_0 : \mathbf{E} \to \mathbf{B}$. The ensuing functor $\mathcal{P} : \mathbf{E} \to \mathbf{B}^{\to}$ given by $E \mapsto p(\varepsilon_E)$ — where $\varepsilon : 1\mathcal{P}_0 \stackrel{\cdot}{\longrightarrow} Id$ is counit — then forms a comprehension category (see Jacobs [1990] for the proof).

(ii) A *morphism of comprehension categories with unit* is a morphism (K, L) of fibrations preserving the terminal in

$$\begin{tikzcd}[column sep=huge, row sep=huge]
\mathbf{E} \arrow[rr, "L"] \arrow[d, shift right=6, "p"'] \arrow[d, shift left=6, "\mathcal{P}_0"] & & \mathbf{D} \arrow[d, shift right=6, "q"'] \arrow[d, shift left=6, "\mathcal{Q}_0"] \\
\mathbf{B} \arrow[rr, "K"'] \arrow[u, "1" description] \arrow[u, phantom, shift right=3, "\dashv"] \arrow[u, phantom, shift left=3, "\dashv"] & & \mathbf{A} \arrow[u, "\top" description] \arrow[u, phantom, shift right=3, "\dashv"] \arrow[u, phantom, shift left=3, "\dashv"]
\end{tikzcd}$$

such that the canonical map $K\mathcal{P}_0 \stackrel{.}{\to} \mathcal{Q}_0 L$ is an isomorphism. The latter is obtained by transposing $\top K\mathcal{P}_0 \cong L1\mathcal{P}_0 \xrightarrow{L\varepsilon} L$.

Next, we mention some examples of comprehension categories with units and morphisms of these. Some more examples may be found in Jacobs [1990].

4.1.6. Examples. (i) Let's go back to the constant comprehension categories from 4.1.3. Consider two categories $\mathbf{B}, \mathbf{A}$ with finite products and a functor $K : \mathbf{B} \to \mathbf{A}$ preserving these; we write $\gamma_{B,B'}$ for the inverse of the canonical map $K(B \times B') \to KB \times KB'$. Assume non-empty collections $T \subseteq Obj(\mathbf{B})$ and $S \subseteq Obj(\mathbf{A})$ such that $K[T] \subseteq S$. Then there is a morphism of comprehension categories $(K, K') : Cons_T \to Cons_S$, where $K' : \mathbf{B}/\!/T \to \mathbf{A}/\!/S$ is defined by $(B, X) \mapsto (KB, KX)$ and $[(u, f) : (B, X) \to (B', X')] \mapsto (Ku, Kf \circ \gamma_{B,X})$. The functor K' preserves the splitting.

We also observe that for non-trivial T, the comprehension category $Cons_T$ admits a unit if and only if the collection T contains a terminal object.

(ii) Suppose $\mathbf{C}$ is a category with a terminal object t such that all collections $\mathbf{C}(t, X)$ are small. There is then a comprehension category with unit $Fam(\mathbf{C}) \to \mathbf{Sets}^{\to}$ given by $\{X_i\}_{i \in I} \mapsto$ [the projection $\dot{\bigcup}_{i\in I}.\mathbf{C}(t, X_i) \longrightarrow I$]. The fibration involved is the family fibration from 1.1.2. This comprehension category is full if and only if the functor $\mathbf{C}(t, -) : \mathbf{C} \to \mathbf{Sets}$ is full and faithful, see Jacobs [1990].

A functor $H : \mathbf{C} \to \mathbf{D}$ induces a functor $Fam(H) : Fam(\mathbf{C}) \to Fam(\mathbf{D})$ which preserves the splitting. In case H is full and faithful and preserves the terminal object, it gives rise to a map of comprehension categories with unit.

(iii) Let $\mathbf{B}$ be a category with pullbacks. The identity functor on $\mathbf{B}^{\to}$ is then a full comprehension category with unit. This example involves the adjoint situation

$$\begin{tikzcd}[row sep=huge]
\mathbf{B}^{\to} \arrow[d, shift right=6, "cod"'] \arrow[d, shift left=6, "dom"] \\
\mathbf{B} \arrow[u, "id_{(-)}" description] \arrow[u, phantom, shift right=3, "\dashv"] \arrow[u, phantom, shift left=3, "\dashv"]
\end{tikzcd}$$

(iv) Going back to the term model described before 4.1.3, one finds that if there is a unit (as described in 2.2.3) for $*$, then one can define a functor $1 : \mathbf{B} \to \mathbf{E}$ by $[\Gamma] \mapsto [\Gamma \vdash 1_* : *]$. It is easily established that it is a terminal object functor and a left adjoint to the context comprehension functor $\mathcal{P}_0$.

(v) In 1.2.12 we already described two examples of split full comprehension categories with unit, viz. the equivalence $Fam_{\text{eff}}(\omega\text{-}\mathbf{Set}) \xrightarrow{\mathcal{Q}} \omega\text{-}\mathbf{Set}^{\to}$ and the composition $Fam_{\text{eff}}(\mathbf{M}) \stackrel{\mathcal{I}}{\hookrightarrow} Fam_{\text{eff}}(\omega\text{-}\mathbf{Set}) \xrightarrow{\mathcal{Q}} \omega\text{-}\mathbf{Set}^{\to}$.

(vi) The functors $Fam_{\text{cont}}(\mathbf{DEP}) \to \mathbf{DcCat}^{\to}$ and $Fam_{\text{cont}}(\mathbf{DEP}) \to \mathbf{DOM}^{\to}$ from 3.2.7 are both examples of full split comprehension categories with unit.

Next we consider some technicalities.

4.1.7. Lemma. *Let $\mathcal{P} : \mathbf{E} \to \mathbf{B}^{\to}$ be a comprehension category. For every $E \in \mathbf{E}$ and $u : A \to pE$ in $\mathbf{B}$ one can allways choose a pullback of the following form.*

$$\begin{tikzcd}[row sep=huge, column sep=huge]
\mathcal{P}_0 u^*(E) \arrow[r, "\mathcal{P}_0\overline{u}(E)"] \arrow[d, "\mathcal{P}u^*(E)"'] \arrow[dr, phantom, very near start, "\lrcorner"] & \mathcal{P}_0 E \arrow[d, "\mathcal{P}E"] \\
A \arrow[r, "u"'] & pE
\end{tikzcd}$$

Hence one can choose a pullback functor $\mathcal{P}E^{\#} : \mathbf{B}/pE \to \mathbf{B}/\mathcal{P}_0 E$ by $u \mapsto \mathcal{P}_0\overline{u}(E)$.

Proof. By requirement (ii) in definition 4.1.1. □

4.1.8. Proposition. *Let $p : \mathbf{E} \to \mathbf{B}$ be a (cloven) fibration provided with a functor $\mathcal{P}_0 : \mathbf{E} \to \mathbf{B}$ and a natural transformation $\mathcal{P} : \mathcal{P}_0 \stackrel{.}{\longrightarrow} p$. Then*

$\mathcal{P}$ forms a comprehension category

$\Leftrightarrow$ for every $u : A \to B$ in $\mathbf{B}$ and $E \in \mathbf{E}_B$, the operation

$$\begin{array}{rcll} |u^*(E)| & \longrightarrow & \mathbf{B}/B\,(u, \mathcal{P}E) & \quad \textit{given by} \\ v & \longmapsto & \mathcal{P}_0\overline{u}(E) \circ v & \end{array}$$

is invertible.

Proof. ($\Rightarrow$) By the previous lemma, using that

$$\begin{array}{rcll} |u^*(E)| & = & \{v : A \to \mathcal{P}_0 u^*(E) \mid \mathcal{P}u^*(E) \circ v = id\}, & \text{see 4.1.2} \\ & \cong & \{w : A \to \mathcal{P}_0 E \mid \mathcal{P}E \circ w = u\} & \\ & = & \mathbf{B}/B\,(u, \mathcal{P}E). & \end{array}$$

($\Leftarrow$) Let's write $\Upsilon_{u,E}$ for the inverse of the above operation. We have to show that the diagram $(u, \mathcal{P}_0\overline{u}(E)) : \mathcal{P}u^*(E) \to \mathcal{P}E$ is a pullback in $\mathbf{B}$. Assume therefore that $v_1 : C \to A$ and $v_2 : C \to \mathcal{P}_0 E$ with $\mathcal{P}E \circ v_2 = u \circ v_1$ are given. One has $v_2 \in \mathbf{B}/B\,(u \circ v, \mathcal{P}E)$ and thus $w = \Upsilon_{u \circ v_1, E}(v_2) \in |(u \circ v_1)^*(E)|$. Then $w' = \mathcal{P}_0\overline{v_1}(u^*(E) \circ \varphi \circ w : C \to \mathcal{P}_0 u^*(E)$ is the required mediating arrow — where φ is an obvious iso in $\mathbf{B}$. □

4.1.9. Remarks. (i) The isomorphism $\mathbf{B}/pE\,(u, \mathcal{P}E) \cong |u^*(E)|$ that we just established, can equivalently be expressed by

$$\mathbf{B}(A, \mathcal{P}_0 E) \;\cong\; \dot{\bigcup}_{u : A \to pE} |u^*(E)|$$

using that $\mathbf{B}(A, \mathcal{P}_0 E) \cong \dot{\bigcup}_{u : A \to pE} \mathbf{B}(u, \mathcal{P}E)$. The result above shows that this "disjoint sum" which is encoded in the definition of a comprehension category is the heart of the matter. It is closely related to the context rules in type theory — especially to what we have called "context comprehension" in 2.2.1.

(ii) The previous proposition may serve as a basis for an equational presentation of split comprehension categories.

4.1.10. Lemma. *Let $\mathcal{P} : \mathbf{E} \to \mathbf{B}^{\to}$ be a comprehension category with unit, say via $1 : \mathbf{B} \to \mathbf{E}$. Then*

(i) *for $E \in \mathbf{E}$ above A one has $|E| \cong \mathbf{E}_A(1A, E)$;*
(ii) *for $E \in \mathbf{E}$ and $u : B \to pE$ one has $\mathbf{B}/pE\,(u, \mathcal{P}E) \cong \mathbf{E}_B(1B, u^*(E))$;*
(iii) *$\mathcal{P}1 : \mathcal{P}_0 1 \stackrel{\cdot}{\longrightarrow} Id$ is an isomorphism; hence $\mathcal{P}$ preserves the fibred terminal.*

Proof. (i) By the adjunction $1 \dashv \mathcal{P}_0$.
(ii) By the previous proposition and (i).
(iii) The unit $\eta : Id \stackrel{\cdot}{\longrightarrow} 1\mathcal{P}_0$ is an iso since 1 is full and faithful. But $\mathcal{P}1 \circ \eta = p\varepsilon 1 \circ p1\eta = p(\varepsilon 1 \circ 1\eta) = id$. □

Using (ii) in the previous lemma, one can prove that if $\mathcal{P} : \mathbf{E} \to \mathbf{B}^{\to}$ is comprehension category with unit, then $\mathcal{P}$ preserves (fibred) limits. One can also use this fact to prove (iii).

In the rest of this section we describe a number of ways to obtain new comprehension categories from given ones. The first described below is based on a construction from Ehrhard [1988b]; the third and fifth are based on constructions from Moggi [1991].

4.1.11. Constructions on comprehension categories.

(i) **Full completion.** Given a comprehension category $\mathcal{P} : \mathbf{E} \to \mathbf{B}^{\to}$, one forms a *full* comprehension category $\mathcal{P}^{\heartsuit} : \mathbf{E}^{\heartsuit} \to \mathbf{B}^{\to}$, called by Ehrhard the *heart of* $\mathcal{P}$, as follows. The category $\mathbf{E}^{\heartsuit}$ has objects $E \in \mathbf{E}$ and morphisms $(u, v) : E \to E'$ in $\mathbf{E}^{\heartsuit}$ are given by maps $u : pE \to pE'$ and $v : \mathcal{P}_0 E \to \mathcal{P}_0 E'$ in $\mathbf{B}$ such that $u \circ \mathcal{P}E = v \circ \mathcal{P}E'$. The functor $\mathcal{P}^{\heartsuit} : \mathbf{E}^{\heartsuit} \to \mathbf{B}$ is then given by $E \mapsto \mathcal{P}E$ and $(u, v) \mapsto (u, v)$.

There is a unit morphism $\mathcal{P} \to \mathcal{P}^{\heartsuit}$ by a functor $\eta_{\mathcal{P}} : \mathbf{E} \to \mathbf{E}^{\heartsuit}$ with $E \mapsto E$ and $f \mapsto \langle pf, \mathcal{P}_0 f\rangle$. This arrow is universal: for a map $(K : \mathbf{B} \to \mathbf{A}, L : \mathbf{E} \to \mathbf{D}, \gamma)$ from $\mathcal{P}$ to a full $\mathcal{Q} : \mathbf{D} \to \mathbf{A}^{\to}$ one finds a unique map $(K, L', \gamma) : \mathcal{P}^{\heartsuit} \to \mathcal{Q}$ where $L' : \mathbf{E}^{\heartsuit} \to \mathbf{D}$ is defined by $E \mapsto LE$ and $[E \stackrel{(u,v)}{\to} E'] \mapsto \mathcal{Q}^{-1}(Ku, \gamma_{E'}^{-1} \circ Kv \circ \gamma_E)$.

(ii) **Change-of-base along fibrations.** Starting from a comprehension category $\mathcal{P} : \mathbf{E} \to \mathbf{B}^{\to}$ and a fibration $r : \mathbf{C} \to \mathbf{B}$, a new comprehension category $r^*(\mathcal{P})$ with base category $\mathbf{C}$ can be chosen as follows. First form the fibration $r^*(p)$ by change-of-base

$$\begin{tikzcd}
\mathbf{C} \times_{r,p} \mathbf{E} \arrow[rr] \arrow[d, "r^*(p)"'] \arrow[drr, phantom, "\lrcorner", very near start] & & \mathbf{E} \arrow[d, "p"] \\
\mathbf{C} \arrow[rr, "r"] & & \mathbf{B}
\end{tikzcd}$$

and then choose $r^*(\mathcal{P}) : \mathbf{C} \times_{r,p} \mathbf{E} \longrightarrow \mathbf{C}^{\to}$ by $(C, E) \mapsto \overline{\mathcal{P}E}(C) : \mathcal{P}E^*(C) \to C$. On arrows $(f, g) : (C, E) \to (C', E')$ where $rf = pg$ one defines $r^*(\mathcal{P})(f, g) = (f, h)$, in which $h : \mathcal{P}E^*(C) \to \mathcal{P}E'^*(C')$ is the unique arrow above $\mathcal{P}_0 g$ satisfying $\overline{\mathcal{P}E'}(C') \circ h = f \circ \overline{\mathcal{P}E}(C)$.

An alternative description of this construction involves lemma 1.1.4. Applying the pullback functor r^* to $\mathcal{P} : p \to cod$ yields the comprehension category $r^*(\mathcal{P})$ by composition in:

$$\begin{tikzcd}
\bullet \arrow[r, dashed] \arrow[dr, "r^*(p)"'] & \bullet \arrow[r, dashed, "Cleav."] \arrow[d, "r^*(cod)"] & \mathbf{C}^{\to} \arrow[dl, "cod"] \\
& \mathbf{C} &
\end{tikzcd}$$

The resulting $r^*(\mathcal{P})$ is then determined (by choice) up to an isomorphism of comprehension categories.

The morphism of fibrations $r^*(p) \to p$ in the above diagram is in fact a morphism of comprehension categories $r^*(\mathcal{P}) \to \mathcal{P}$.

It is left to the reader to verify that $r^*(\mathcal{P})$ is full or has a unit in case $\mathcal{P}$ is full or has a unit. Moreover, that the map $r^*(\mathcal{P}) \to \mathcal{P}$ preserves the unit.

This change-of-base is can be extended to maps in the following way: for a morphism of comprehension categories $\mathcal{P} \to \mathcal{P}'$ like in definition 4.1.4 and a morphism of fibrations $r \to r'$, one obtains a morphism $r^*(\mathcal{P}) \to r'^*(\mathcal{P}')$.

(iii) **Juxtaposition.** Given two comprehension categories $\mathbf{E} \stackrel{\mathcal{P}}{\longrightarrow} \mathbf{B}^{\to} \stackrel{\mathcal{Q}}{\longleftarrow} \mathbf{D}$ one constructs another comprehension category $\mathcal{Q} \cdot \mathcal{P}$ with base category $\mathbf{B}$, by first performing change-of-base

$$\begin{tikzcd}
\mathbf{D} \times_{\mathcal{Q}_0, p} \mathbf{E} \arrow[r] \arrow[d, "\mathcal{Q}_0^*(p)"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathbf{E} \arrow[d, "p"] \\
\mathbf{D} \arrow[r, "\mathcal{Q}_0"] & \mathbf{B}
\end{tikzcd}$$

and then defining $\mathcal{Q} \cdot \mathcal{P} : \mathbf{D} \times_{\mathcal{Q}_0, p} \mathbf{E} \longrightarrow \mathbf{B}^{\to}$ by $(D, E) \mapsto \mathcal{Q}D \circ \mathcal{P}E$ and $(f, g) \mapsto (qf, \mathcal{P}_0 g)$. One has $cod \circ \mathcal{Q} \cdot \mathcal{P} = q \circ \mathcal{Q}_0^*(p)$.

(iv) **Localization.** Let $\mathcal{P} : \mathbf{E} \to \mathbf{B}^{\to}$ be a comprehension category. For each object $A \in \mathbf{B}$ one can form a comprehension category $\mathcal{P}[A] : \mathbf{E}[A] \to \mathbf{B}[A]^{\to}$ in which A is used as initial context. The comprehension category $\mathcal{P}[A]$ contains that part of $\mathcal{P}$ that can be seen from A.

A chain of types is a sequence $E_0, \ldots, E_n$ of objects $E_i \in \mathbf{E}$ with $pE_{i+1} = \mathcal{P}_0 E_i$. A chain may be empty. Let $\mathbf{B}[A]$ be the category with chains $E_0, \ldots, E_n$ satisfying $pE_0 = A$, as objects. A morphism u from $E_0, \ldots, E_n$ to $D_0, \ldots, D_m$ in $\mathbf{B}[A]$ is a morphism $u : \mathcal{P}_0 E_n \to \mathcal{P}_0 D_m$ in $\mathbf{B}$ commuting with the chain of projections, i.e. satsifying

$$\mathcal{P}D_0 \circ \ldots \circ \mathcal{P}D_m \circ u \quad = \quad \mathcal{P}E_0 \circ \ldots \circ \mathcal{P}E_n.$$

(A little care is needed here: if one of the chains is empty, one should read A for $\mathcal{P}_0 E_n$ or $\mathcal{P}_0 D_m$.)

The category $\mathbf{E}[A]$ has non-empty chains $E_0, \ldots, E_n$ with $pE_0 = A$ as objects. A morphism f from $E_0, \ldots, E_n$ to $D_0, \ldots, D_n$ in $\mathbf{E}[A]$ is a morphism $f : E_n \to D_m$ in $\mathbf{E}$ such that $pf : E_0, \ldots, E_{n-1} \longrightarrow D_0, \ldots, D_{m-1}$ in $\mathbf{B}[A]$.

The functor $\mathcal{P}[A] : \mathbf{E}[A] \to \mathbf{B}[A]^{\to}$ is defined by

$$\begin{array}{ccl}
E_0, \ldots, E_n & \mapsto & \mathcal{P}E_n \;:\; E_0, \ldots, E_n \;\longrightarrow\; E_0, \ldots, E_{n-1} \\
f & \mapsto & \langle pf, \mathcal{P}_0 f \rangle.
\end{array}$$

Without proof we mention that

(1) $\mathcal{P}[A]$ is a comprehension category;
(2) $\mathcal{P}[A]$ is full (resp. has a unit) in case $\mathcal{P}$ is full (resp. has a unit);
(3) there is a morphism of comprehension categories $\mathcal{P}[A] \to \mathcal{P}$;
(4) Every arrow $B \to A$ in $\mathbf{B}$ gives rise to a morphism of comprehension categories $\mathcal{P}[A] \to \mathcal{P}[B]$. This last point requires a cleavage.

(v) **Multiplication.** Suppose two (cloven) comprehension categories $\mathbf{E} \xrightarrow{\mathcal{P}} \mathbf{B}^{\to} \xleftarrow{\mathcal{Q}} \mathbf{D}$ are given. One forms a new comprehension category $\mathcal{P} \otimes \mathcal{Q}$ with underlying fibration $p \times q = p \circ p^*(q) : \mathbf{E} \times_{p,q} \mathbf{D} \to \mathbf{B}$ as follows. Put $\mathcal{P} \otimes \mathcal{Q}(E, D) = \mathcal{P}E \circ \mathcal{Q}(\mathcal{P}E^*(D))$ and for $(f, g) : (E, D) \to (E', D')$ take $\mathcal{P} \otimes \mathcal{Q}(f, g) = (pf, w)$, where

w is the mediating arrow. This makes $\otimes$ an (up-to-isomorphism) associative and symmetric operation. A unit for $\otimes$ is formed by the identity natural transformation on $Id_{\mathbf{B}}$. Hence cloven comprehension categories on a given base category have the structure of a symmetric monoidal category.

(vi) **Composition.** Given two comprehension categories

$$\begin{tikzcd}
\mathbf{E} \arrow[r, bend left, "\mathcal{P}_0"] \arrow[r, bend right, "p"'] \arrow[r, phantom, "\Downarrow \mathcal{P}"] & \mathbf{B} \arrow[r, bend left, "\mathcal{R}_0"] \arrow[r, bend right, "r"'] \arrow[r, phantom, "\Downarrow \mathcal{R}"] & \mathbf{A}
\end{tikzcd}$$

One obtains a functor $\mathcal{R}\mathcal{P} : \mathbf{E} \to \mathbf{B}^{\to}$ by $E \mapsto \mathcal{R}(pE) \circ \mathcal{R}_0(\mathcal{P}E)$ $(= r(\mathcal{P}E) \circ \mathcal{R}(\mathcal{P}_0 E)$). It forms a comprehension category if $\mathcal{R}$ has a unit: $\mathcal{R}_0$ then preserves pullbacks (see also lemma 1.1.5).

4.2 Quantification along arbitrary projections

A comprehension category $\mathcal{P} : \mathbf{E} \to \mathbf{B}^{\to}$ determines a class of "projection" morphisms $\{\mathcal{P}E \mid E \in \mathbf{E}\}$. Quantification along such projections is described in the next definition by adjoints to the corresponding "weakening" functors — which are the reindexing functors of these projections.

4.2.1. Definition. Let $q : \mathbf{D} \to \mathbf{B}$ be a fibration and $\mathcal{P} : \mathbf{E} \to \mathbf{B}^{\to}$ be a comprehension category; we say that q admits $\mathcal{P}$-products (resp. $\mathcal{P}$-sums) iff both

- for every $E \in \mathbf{E}$ any weakening functor $\mathcal{P}E^* : \mathbf{D}_{pE} \to \mathbf{D}_{\mathcal{P}_0 E}$ has a right adjoint Π_E (resp. a left adjoint Σ_E).
- the "Beck-Chevalley" condition holds, i.e. for every cartesian morphism $f : E \to E'$ in $\mathbf{E}$ one has that the canonical natural transformation

  $$(pf)^* \Pi_{E'} \stackrel{\cdot}{\longrightarrow} \Pi_E (\mathcal{P}_0 f)^* \qquad\qquad (\text{resp. } \Sigma_E (\mathcal{P}_0 f)^* \stackrel{\cdot}{\longrightarrow} (pf)^* \Sigma_{E'})$$

  is an isomorphism.

The first map is the transpose of $\mathcal{P}E^* \, (pf)^* \, \Pi_{E'} \cong (\mathcal{P}_0 f)^* \, \mathcal{P}E'^* \, \Pi_{E'} \xrightarrow{(\mathcal{P}_0 f)^*(\varepsilon)} (\mathcal{P}_0 f)^*$; similarly one obtains the second one.

4.2.2. Remark. Let q and $\mathcal{P}$ be as above. We recall from 1.1.2 that the fibration $p : \mathbf{E} \to \mathbf{B}$ determines a groupoid fibration $|p| : Cart(\mathbf{E}) \to \mathbf{B}$. Similarly the comprehension category $\mathcal{P} : \mathbf{E} \to \mathbf{B}^{\to}$ determines two functors $|p|$, $|\mathcal{P}_0| : Cart(\mathbf{E}) \to \mathbf{B}$ and a natural transformation between them. By change-of-base of q along $\mathcal{P} : |\mathcal{P}_0| \stackrel{\cdot}{\longrightarrow} |p|$ one obtains two fibrations $|p|^*(q)$ and $|\mathcal{P}_0|^*(q)$ and a functor $\langle \mathcal{P} \rangle : |p|^*(q) \to |\mathcal{P}_0|^*(q)$, see lemma 1.1.7. Using lemma 1.2.2, one can prove that *q admits $\mathcal{P}$-products* (resp. *$\mathcal{P}$-sums*) if this functor $\langle \mathcal{P} \rangle$ has a fibred right (resp. left) adjoint. This approach generalizes definition 7 in Ehrhard [1988a]. For practical reasons we chose to work with the fibrewise formulation in used in the definition above.

In some special cases we don't mention the comprehension categories involved.

4.2.3. Definition. (i) Let $p : \mathbf{E} \to \mathbf{B}$ be a fibration on a basis with pullbacks; one says that p has *(fibred) products* (resp. *sums*) iff p has products (resp. sums) with respect to the identity comprehension category on $\mathbf{B}^{\to}$, see 4.1.6 (iii). This is the usual definition in fibred category theory.

(ii) Let $\mathcal{P} : \mathbf{E} \to \mathbf{B}^{\to}$ be a comprehension category; we say that $\mathcal{P}$ has *products* (resp. sums) iff $p = cod \circ \mathcal{P}$ has $\mathcal{P}$-products (resp. $\mathcal{P}$-sums).

4.2.4. Definition. (i) (Bénabou) A fibration is called *complete* if it has fibred products and fibrewise finite limits.

(ii) A fibration will be called *small complete* if it is both small and complete.

In ordinary category theory a category is sometimes called *small-complete* if it is complete, i.e. if every small diagram has a limit. Here a small complete category/fibration is one which is both small and complete.

A result of P. Freyd (see e.g. Mac Lane [1971], V.2, proposition 3) states that there are no small complete categories except preorders. Remarkably, there are small complete *fibred* categories (which are not fibrewise preordered), see Hyland [1989], Hyland, Robinson & Rosolini [1990] or (ii) below.

4.2.5. Examples. (i) It is easily verified that a category $\mathbf{C}$ has infinite products (resp. coproducts) iff the fibration $Fam(\mathbf{C}) \to \mathbf{Sets}$ admits products (resp. sums). This bi-implication extends to completeness.

(ii) After definition 1.4.7 it was already mentioned that the fibration $Fam_{\text{eff}}(\mathbf{M}) \to \omega\text{-}\mathbf{Set}$ is small. It has finite limits because $\mathbf{M}$ has them. Products are obtained in the following way — which generalizes the constructions from 3.2.6 (ii). For $f : A \to B$ in ω-**Set** one has $\Pi_f : Fam_{\text{eff}}(\mathbf{M})_B \to Fam_{\text{eff}}(\mathbf{M})_A$ by $[X : B \to \mathbf{M}] \mapsto [\boldsymbol{\lambda} a \in |A|. (\Pi_{b \in f^{-1}(a)}. |X_b|, \vdash)]$. The realizability relation $\vdash$ is described by $n \vdash \varphi \Leftrightarrow \forall a, m\, \forall b, k.\, m \vdash_A a \;\&\; k \vdash_B b \;\Rightarrow\; n \cdot m \cdot k \vdash_{X_b} \varphi(a)(b)$.

(iii) In 1.2.4 (iii) an LCCC has been defined as a category $\mathbf{B}$ with finite limits such that every slice category $\mathbf{B}/A$ is cartesian closed. Equivalently — as shown in Freyd [1972], see also lemma 2.2.13 — one can require finite limits for $\mathbf{B}$ plus fibred products for the fibration $cod : \mathbf{B}^{\to} \to \mathbf{B}$. This fibration is then complete. Note also that it trivially has sums.

4.2.6. Extended example.
Let $\mathbf{B}$ be a category with finite products and let $T \subseteq Obj(\mathbf{B})$, see 4.1.6. It is not hard to prove that the "constant" comprehension category $Cons_T$ admits products if exponents exist in $\mathbf{B}$ of all objects in T; also that $Cons_T$ admits sums if T is closed under cartesian products. Hence for a constant comprehension category, products are given by exponents and sums by cartesian products. This corresponds in type theory to the fact that Π is $\to$ and Σ is $\times$ in case there is no type dependency.

In a sense, this is a remarkable result: it gives the possibility to describe type theoretical exponents without (type theoretical) cartesian products. Let $\mathbf{B}$ be the category of contexts of the minimal setting (see the beginning of 3.1) and let T be the collection of types. One has $T \subseteq Obj(\mathbf{B})$ by identifying a type with a singleton context. This gives a term model in which right adjoints to weakening functors correspond to exponent types and (independently) left adjoints to cartesian product types.

Of course, at a different level (viz. the level of contexts) cartesian products do play a role in the description of these type theoretical exponents. It is a merit of comprehension categories to separate these levels.

It is worth mentioning a mathematical example here. Let D be a cpo. A subset $I \subseteq D$ is called an *ideal* in D iff (i) $\bot \in I$; (ii) $x \leq y \in I \;\Rightarrow\; x \in I$; (iii) directed $X \subseteq I \;\Rightarrow\; \bigsqcup X \in I$. Ideals are the non-empty closed subsets with respect to the Scott topology. With the ordering inherited from D, they form cpos themselves.

One forms a base category $\mathbf{B}$ with ideals $I \subseteq D^n$ (for some $n \in \mathbb{N}$) as objects. A morphism from $I \subseteq D^n$ to $J \subseteq D^m$ is a continuous function $f : D^n \to D^m$ with $f[I] \subseteq J$. The product of ideals $I \subseteq D^n$ and $J \subseteq D^m$ is $I \times J \subseteq D^{n+m}$.

Now let's assume that D is isomorphic to its own space of continuous functions $[D \to D]$, via maps $F : D \to [D \to D]$ and $G : [D \to D] \to D$ satisfying $F \circ G = id$ and $G \circ F = id$. As usal, we write $x \cdot y$ for $F(x)(y)$ and $\lambda x.-$ for $G(\boldsymbol{\lambda} x.-)$. An example of such a cpo is D. Scott's D_∞, see e.g. Barendregt [1984].

In a standard way one forms an exponent of ideals $I, J \subseteq D$ by $I \Rightarrow J = \{x \in D \mid \forall y \in I.\; x \cdot y \in J\}$. In general however, a cartesian product for ideals $I, J \subseteq D$ does not seem to exist in D. Hence we don't have a CCC-structure.

But taking $T \subseteq Obj(\mathbf{B})$ as the collection of ideals in $D^1 = D$, yields a comprehension category $Cons_T$ with products described by exponents. For $(I, J) \in \mathbf{B}//T$ one can define $\Pi_{(I,J)}.(I \times J, K) = (I, J \Rightarrow K)$. In this way we are able to capture these exponent ideals categorically.

4.2.7. Definition. Suppose we have a diagram

$$\begin{tikzcd}
\mathbf{E} \arrow[rr, "L"] \arrow[d, shift right=3] \arrow[d, shift left=3] \arrow[d, phantom, "\overset{\mathcal{P}}{\Longrightarrow}"] & & \mathbf{E}' \arrow[d, shift right=3] \arrow[d, shift left=3] \arrow[d, phantom, "\overset{\mathcal{P}'}{\Longrightarrow}"] \\
\mathbf{B} \arrow[rr, "K"] & & \mathbf{B}' \\
\mathbf{D} \arrow[u, "q"] \arrow[rr, "H"] & & \mathbf{D}' \arrow[u, "q'"']
\end{tikzcd}$$

in which (K, L) together with $\gamma : \mathcal{P}_0{}'L \xrightarrow{\sim} K\mathcal{P}_0$ is a morphism of comprehension categories and (K, H) is a morphism of fibrations.

(i) Suppose that q has $\mathcal{P}$-products via fibrewise adjunctions $\mathcal{P}(-)^* \dashv \Pi_{(-)}$ and that q' has $\mathcal{P}'$-products via $\mathcal{P}'(-)^* \dashv \Pi'_{(-)}$. Then the (K, L, H) diagram forms a *morphism of products* if for each $E \in \mathbf{E}$, the canonical natural transformation

$$H \circ \Pi_E \quad \stackrel{\cdot}{\longrightarrow} \quad \Pi'_{LE} \circ \gamma_E^* \circ H$$

is an isomorphism.

(ii) Similarly the diagram forms a *morphism of sums* — described by $\Sigma_{(-)} \dashv \mathcal{P}(-)^*$ and $\Sigma'_{(-)} \dashv \mathcal{P}'(-)^*$ — if for each $E \in \mathbf{E}$ one has canonically,

$$\Sigma'_{LE} \circ \gamma_E^* \circ H \quad \cong \quad H \circ \Sigma_E.$$

4.2.8. Remarks. (i) An exposition similar to the one above can be given about an appropriate form of quantification for *split* fibrations. Every reindexing functor should then preserve all the structure "on-the-nose".

(ii) The reader may want to verify that the explicit definition of Cons_Ω- and $\mathit{Cons}_\mathbf{B}$-quantification (and corresponding maps) given in section 1.5 coincides with the one presented above, using the comprehension categories Cons_Ω and $\mathit{Cons}_\mathbf{B}$ from 4.1.3.

In chapter 2 we described "weak" and "strong" sums in type theory. The above definition covers the weak case. For the strong one the fibration q must be (part of) a comprehension category. This corresponds to the extra dependency required for strong sums in section 2.2. But first, we need the technical result (i) below. The second point generalizes the Frobenius isomorphism from lemma 3.2.5. Verifications are easy and left to the reader.

4.2.9. Lemma. *Suppose q admits $\mathcal{P}$-sums as described above.*

(i) *For every $E \in \mathbf{E}$ and $D \in \mathbf{D}$ with $qD = \mathcal{P}_0 E$, one has that the morphism $in_{E,D} = \overline{\mathcal{P}E}(\Sigma_E.D) \circ \eta_D : D \to \mathcal{P}E^*(\Sigma_E.D) \to \Sigma_E.D$ is cocartesian.*

(ii) *The transpose of $id \times \eta_{E'} : \mathcal{Q}D^*(E) \times E' \longrightarrow \mathcal{Q}D^*(E) \times \mathcal{Q}D^*(\Sigma_D.E') \cong \mathcal{Q}D^*(E \times \Sigma_D.E')$ yields an isomorphism $\Sigma_D.(\mathcal{Q}D^*(E) \times E') \xrightarrow{\sim} E \times \Sigma_D.E'$.* □

4.2.10. Definition. Given comprehension categories $\mathbf{E} \xrightarrow{\mathcal{P}} \mathbf{B}^{\to} \xleftarrow{\mathcal{Q}} \mathbf{D}$, we say that $\mathcal{Q}$ has *strong* $\mathcal{P}$-sums in case $\mathcal{Q}$ has $\mathcal{P}$-sums in such a way that every morphism $\mathcal{Q}_0(in_{E,D})$ in $\mathbf{B}$ (cf. the previous lemma) is orthogonal to the class $\{\mathcal{Q}D' \mid D' \in \mathbf{D}\}$. The latter means that for every $D' \in \mathbf{D}$ and u, v forming a commuting square,

$$\begin{tikzcd} \bullet \arrow[r, "\mathcal{Q}_0(in_{E,D})"] \arrow[d, "u"'] & \bullet \arrow[d, "v"] \arrow[dl, "w"] \\ \bullet \arrow[r, "\mathcal{Q}D'"'] & \bullet \end{tikzcd}$$

there is a unique w satisfying $\mathcal{Q}D' \circ w = v$ and $w \circ \mathcal{Q}_0(in_{E,D}) = u$.

One easily verifies that a comprehension category $\mathcal{Q}$ has strong sums (i.e. strong $\mathcal{Q}$-sums, see definition 4.2.3) iff the above morphism $\mathcal{Q}_0(in_{E,D})$ is an isomorphism. The latter formulation is used in Jacobs [1990] to define strong sums for comprehension categories. There one also finds the relation between strong sums and indecomposability of terminal objects.

Next we mention some useful results about quantification. More results like these may be found in Jacobs, Moggi & Streicher [1991].

4.2.11. Lemma. *q admits $\mathcal{P}$-products $\Leftrightarrow$ q^{op} admits $\mathcal{P}$-sums.*

Proof. By the fact that the opposite is taken fibrewise, see 1.1.11. □

4.2.12. Lemma. *Let $\mathcal{P} : \mathbf{E} \to \mathbf{B}^{\to}$ be a comprehension category and let $q : \mathbf{D} \to \mathbf{B}, r : \mathbf{C} \to \mathbf{B}$ be fibrations; the fibration $r^*(\mathcal{P})$ obtained by change-of-base is described in 4.1.11 (ii).*

(i) *q admits $\mathcal{P}$-products/sums $\Rightarrow$ $r^*(q)$ admits $r^*(\mathcal{P})$-products/sums; further, the pair of morphisms $r^*(\mathcal{P}) \to \mathcal{P}$ together with $r^*(q) \to q$ forms a morphism of products/sums.*

(ii) *Suppose q has $\mathcal{P}$-products; similarly, q' has $\mathcal{P}'$-products. Let's assume further a product-preserving pair of morphisms $\mathcal{P} \to \mathcal{P}'$ and $q \to q'$ like in definition 4.2.7. A morphism of fibrations $r \to r'$ then induces a product preserving pair of maps $r^*(\mathcal{P}) \to r'^*(\mathcal{P}')$ and $r^*(q) \to r'^*(q')$.*

Similarly for sums.

(iii) *A comprehension category $\mathcal{Q}$ admits strong $\mathcal{P}$-sums $\Rightarrow$ $r^*(\mathcal{Q})$ admits strong $r^*(\mathcal{P})$-sums.*

Proof. (i) Assume $\mathcal{P}E^* \dashv \Pi_E$ in $\mathbf{D}$; we seek $(q^*(\mathcal{P})(C, E))^* \dashv \forall_{(C,E)}$. This is done by defining $\forall_{(C,E)} : (\mathbf{C} \times_{r,q} \mathbf{D})_{\mathcal{P}E^*(C)} \to (\mathbf{C} \times_{r,q} \mathbf{D})_C$ as $(\mathcal{P}E^*(C), D) \mapsto (C, \Pi_E.D)$. Sums are handled similarly.

(ii) Straightforward using the map $r^*(\mathcal{P}) \to r'^*(\mathcal{P}')$ from 4.1.11 (ii).

(iii) Notice that $in = in_{(C,E),(\mathcal{P}E^*(C),D)} = (\overline{\mathcal{P}E}(C), in_{E,D}) : (\mathcal{P}E^*(C), D) \to \exists_{(C,E)}.(\mathcal{P}E^*(C), D)$ and that $q^*(\mathcal{P})_0(in)$ is by definition above $\mathcal{R}_0(in_{E,D})$. Orthogonality can then be lifted. □

The next lemma resembles 2.2.12.

4.2.13. Lemma. *Let $q : \mathbf{D} \to \mathbf{B}$ be a fibration and $\mathcal{P} : \mathbf{E} \to \mathbf{B}^{\to}$ be a comprehension category.*

(i) *If there is a fibred reflection $r \to q$ (i.e. a fibration $r : \mathbf{C} \to \mathbf{B}$ and a full and faithful cartesian functor $\mathbf{C} \to \mathbf{D}$ which has a fibred left adjoint), then*

(1) q has fibred finite limits $\Rightarrow$ r has fibred finite limits.

(2) q has $\mathcal{P}$-products/sums $\Rightarrow$ r has $\mathcal{P}$-products/sums.

Further, the functor $\mathbf{C} \to \mathbf{D}$ is continuous, i.e. preserves the finite limits and products.

(ii) *In case $\mathcal{P}$ is a full comprehension category with unit and sums and q has a fibred terminal object, which is preserved by a full and faithful cartesian functor $G: \mathbf{D} \to \mathbf{E}$, then*

$$G \text{ has a fibred left adjoint} \quad \Leftrightarrow \quad q \text{ has } \mathcal{P}\text{-sums.}$$

Proof. (i) By a standard argument.
(ii) ($\Rightarrow$) By (i).
($\Leftarrow$) Define $F: \mathbf{E} \to \mathbf{D}$ by $E \mapsto \Sigma_E(\top\mathcal{P}_0 E)$, where $\top: \mathbf{B} \to \mathbf{D}$ describes the terminal object for q. By 4.2.9 (i), F extends to a functor, which is cartesian by Beck-Chevalley. Then for $E \in \mathbf{E}$ above A and $D \in \mathbf{D}$ above B, one has

$$\begin{array}{rcll}
\mathbf{D}(FE,\ D) & \cong & \dot{\bigcup}_{u:A\to B}.\ \mathbf{D}_A\big(\Sigma_E.(\top\mathcal{P}_0E),\ u^*(D)\big) & \\
& \cong & \dot{\bigcup}_{u:A\to B}.\ \mathbf{D}_{\mathcal{P}_0E}\big(\top\mathcal{P}_0E,\ \mathcal{P}E^*u^*(D)\big) & \\
& \cong & \dot{\bigcup}_{u:A\to B}.\ \mathbf{E}_{\mathcal{P}_0E}\big(G\top\mathcal{P}_0E,\ G(\mathcal{P}E^*u^*(D))\big) & \\
& \cong & \dot{\bigcup}_{u:A\to B}.\ \mathbf{E}_{\mathcal{P}_0E}\big(1\mathcal{P}_0E,\ \mathcal{P}E^*u^*(GD)\big) & \\
& \cong & \dot{\bigcup}_{u:A\to B}.\ \mathbf{B}/A\big(\mathcal{P}E,\ \mathcal{P}(u^*(GD))\big) & \text{by 4.1.10 (ii)} \\
& \cong & \dot{\bigcup}_{u:A\to B}.\ \mathbf{E}_A\big(E,\ u^*(GD)\big) & \text{because } \mathcal{P} \text{ is full} \\
& \cong & \mathbf{E}\big(E,\ GD\big). & \square
\end{array}$$

4.3 Closed comprehension categories

The notion of a *closed* comprehension category to be introduced next is of great importance: like a CCC, an LCCC or a topos, it forms a module with pleasant properties. It is a category with a unit and dependent products and strong sums. Comparable "closed" versions have been defined for other categorical notions for type dependency as mentioned in "Introduction and summary", see Blanco [1991].

Most of this section will be devoted to examples and properties. At the end we will be able to give categorical versions of the systems λP1, λPi and $\lambda *$ which are based on the "Propositions as Types" setting.

4.3.1. Definition. (i) A *closed* comprehension category (abbr. CCompC) is a full comprehension category with unit, products and strong sums; moreover, the base category is required to have a terminal object. The products and sums are with respect to the comprehension category itself, see 4.2.3 (ii).

A closed comprehension category is *split* if all the structure involved is split.

(ii) A *morphism of CCompC's* is a morphism of comprehension categories with unit, which preserves the products and sums; additionally, preservation of the terminal object in the basis is required.

4.3.2. Examples. (i) Let $\mathbf{B}$ be a category with finite limits. The identity functor on $\mathbf{B}^{\to}$ is then a full comprehension category with unit and strong sums. Moreover,

$$id_{\mathbf{B}^{\to}} \text{ is a CCompC} \quad \Leftrightarrow \quad \mathbf{B} \text{ is an LCCC.}$$

(ii) Let $\mathbf{B}$ be a category with finite products. The full comprehension category $\mathit{Cons}_{\mathbf{B}}$ from 4.1.3 has a unit and strong sums. Moreover,

$$\mathit{Cons}_{\mathbf{B}} \text{ is a (split) CCompC} \quad \Leftrightarrow \quad \mathbf{B} \text{ is a CCC.}$$

These two examples show that finite products and exponents are related like finite limits and local exponentials.

(iii) The comprehension categories $\mathit{Fam}(\mathbf{Sets}) \to \mathbf{Sets}^{\to}$ and $\mathit{Fam}_{\mathrm{eff}}(\omega\text{-}\mathbf{Set}) \to \omega\text{-}\mathbf{Set}^{\to}$ are both closed. This is not surprising because $\mathbf{Sets}$ and ω-$\mathbf{Set}$ are LCCC's. But the interesting point is that all the structure is split.

Similarly, one has by composition a CCompC $\mathit{Fam}_{\mathrm{eff}}(\mathbf{M}) \hookrightarrow \mathit{Fam}_{\mathrm{eff}}(\omega\text{-}\mathbf{Set}) \to \omega\text{-}\mathbf{Set}^{\to}$, see 1.2.12 and 4.1.6 (v).

(iv) The comprehension category $\mathit{Fam}(\mathbf{Sets}) \to \mathbf{Cat}^{\to}$ mentioned in Jacobs [1990] is closed; this example goes back to Lawvere [1970]. The fibration involved is obtained by applying the Grothendieck construction to $\mathbf{C} \mapsto \mathbf{Sets}^{\mathbf{C}}$.

(v) The comprehension category $\mathit{Fam}_{\mathrm{cont}}(\mathbf{DEP}) \to \mathbf{DOM}^{\to}$ introduced at the end of 3.2.7 and 4.1.6 (vi) is also an example. Remember that for a domain $\mathbf{A}$ and a continuous functor $X: \mathbf{A} \to \mathbf{DEP}$ the domain $\mathcal{P}_0X$ has elements (a, x) where $a \in \mathbf{A}$ and $x \in X_a$; the ordering is given by $(a,x) \leq (b,y) \Leftrightarrow a \leq b \;\&\; X^e_{ab}(x) \leq y$. For a continuous functor $Y: \mathcal{P}_0X \to \mathbf{DEP}$ one can define sum and product $\Sigma_X.Y,\ \Pi_X.Y : \mathbf{A} \to \mathbf{DEP}$ by $(\Sigma_X.Y)_a = \mathcal{P}_0(X_a, Y_{(a,-)})$ where $Y_{(a,-)}$ is considered as a functor $X_a \to \mathbf{DEP}$. In a similar way one takes $(\Pi_X.Y)_a = |(X_a, Y_{(a,-)})|$, the domain of sections mentioned at the end of 3.2.7. An extensive treatment of these constructions may be found in Palmgren & Stoltenberg-Hansen [1990].

(vi) The term model of the calculus λP1 (see section 2.3) is an example of a split CCompC: the comprehension category with unit was already described in example 4.1.6 (iv). The type theoretical product and strong sum provide the appropriate categorical structure.

4.3.3. Extended example (Closure model).
The following exposition is based on Scott [1976] and Barendregt & Rezus [1983]; Taylor [1985] is also of relevance. We consider the complete lattice $P\omega$. The set of Scott-continuous functions $[P\omega \to P\omega]$ comes equipped with continuous maps $F : P\omega \to [P\omega \to P\omega]$ and $G : [P\omega \to P\omega] \to P\omega$ satisfying $F \circ G = id$ and $G \circ F \geq id$. As usual we write $x \cdot y$ for $F(x)(y)$ and $\lambda x \ldots$ for $G(\boldsymbol{\lambda} \ldots)$. Further,

we use that there is a continuous surjective pairing $[-,-]\colon P\omega \times P\omega \to P\omega$ with projections π, π'.

A *closure* is an element $a \in P\omega$ satisfying $a \circ a = a \geq \boldsymbol{I}$, where $a \circ a = \lambda x.\, a \cdot (a \cdot x)$ and $\boldsymbol{I} = \lambda x.x$. Closures form a category $\mathbf{CL}$ by the stipulation that a morphism $u\colon a \to b$ between closures is an element $u \in P\omega$ satisfying $b \circ u \circ a = u$ (or equivalently, $b \circ u = b$ and $u \circ a = u$). One easily verifies that $\mathbf{CL}$ is a CCC with $t = \lambda x.\omega$, $a \times b = \lambda x.\,[a \cdot \pi x, b \cdot \pi' x]$ and $b^a = \lambda x.\, b \circ x \circ a$. For $a \in \mathbf{CL}$ we write $im(a) = \{a \cdot x \mid x \in P\omega\}$; then $im(a) = \{x \in P\omega \mid a \cdot x = x\}$ and $im(b^a) = \mathbf{CL}(a,b)$.

A crucial result is the existence of a closure Ω with $im(\Omega) = Obj(\mathbf{CL})$, i.e. $a \in \mathbf{CL} \;\Leftrightarrow\; \Omega \cdot a = a$. It gives us the possibility to define a split fibration $p\colon Fam(\mathbf{CL}) \to \mathbf{CL}$ of "closure-indexed closures". Objects of $Fam(\mathbf{CL})$ are arrows $X\colon a \to \Omega$ in $\mathbf{CL}$. An arrow $(X\colon a \to \Omega) \to (Y\colon b \to \Omega)$ is a pair (u, α) with $u\colon a \to b$ in $\mathbf{CL}$ and $\alpha \in P\omega$ an "a-indexed family of morphisms". The latter means that $\alpha \circ a = \alpha$ and $\alpha \cdot z\colon X \cdot z \to Y \cdot (u \cdot z)$ in $\mathbf{CL}$ (for all $z \in P\omega$). Here we use that $X \cdot z \in im(\Omega) = Obj(\mathbf{CL})$. The first projection $p\colon Fam(\mathbf{CL}) \to \mathbf{CL}$ is then a split fibration; it has a terminal object via $1\colon \mathbf{CL} \to Fam(\mathbf{CL})$ by $a \mapsto (\lambda xy.\omega\colon a \to \Omega)$. A right adjoint $\mathcal{P}_0\colon Fam(\mathbf{CL}) \to \mathbf{CL}$ to 1 is described by $(X\colon a \to \Omega) \mapsto \lambda z.[a \cdot \pi z, X \cdot \pi z \cdot \pi' z]$. In this way one obtains a (full) comprehension category with unit $Fam(\mathbf{CL}) \to \mathbf{CL}^{\to}$.

For $X\colon a \to \Omega$ and $Y\colon \mathcal{P}_0 X \to \Omega$ one defines $\Sigma_X.Y,\ \Pi_X.Y\colon a \to \Omega$ by

$$\begin{array}{rcl} \Sigma_X.Y & = & \lambda zv.\,[X \cdot z \cdot \pi v, Y \cdot [a \cdot z, X \cdot z \cdot \pi v] \cdot \pi' v] \\ \Pi_X.Y & = & \lambda zvw.\, Y \cdot [a \cdot z, X \cdot z \cdot w] \cdot (v \cdot (X \cdot z \cdot w)). \end{array}$$

This yields a (split) CCompC.

Finally it is worth noticing that the fibration $p\colon Fam(\mathbf{CL}) \to \mathbf{CL}$ has a (split) generic object. Hence this example supports a "type of all types".

4.3.4. Extended example (Separated families in a topos).
Let $\mathbf{B}$ be a topos with a topology $j\colon \Omega \to \Omega$. For every object $A \in \mathbf{B}$, the slice category $\mathbf{B}/A$ is a topos again; further there is a functor $A^*\colon \mathbf{B} \to \mathbf{B}/A$ given by $B \mapsto [\pi\colon A \times B \to A]$. In $\mathbf{B}/A$, one has that $A^*(\Omega)$ forms a subobject classifier and that $A^*(j)\colon A^*(\Omega) \to A^*(\Omega)$ is a topology. It is not hard to verify that for a monic m in $\mathbf{B}/A$ one has

$$\begin{pmatrix} X' \\ \downarrow \\ A \end{pmatrix} \overset{m}{\rightarrowtail} \begin{pmatrix} X \\ \downarrow \\ A \end{pmatrix} \text{ is } A^*(j)\text{-closed/dense.} \quad \Leftrightarrow \quad X' \overset{m}{\rightarrowtail} X \text{ is } j\text{-closed/dense.}$$

The full subcategory $\mathcal{SF}_j(\mathbf{B}) \hookrightarrow \mathbf{B}^{\to}$ of "separated families" is defined by

$$\begin{pmatrix} X \\ \downarrow \\ A \end{pmatrix} \in \mathcal{SF}_j(\mathbf{B}) \quad \Leftrightarrow \quad \begin{pmatrix} X \\ \downarrow \\ A \end{pmatrix} \text{ is } A^*(j)\text{-separated in } \mathbf{B}/A$$

We claim that the inclusion $\mathcal{SF}_j(\mathbf{B}) \hookrightarrow \mathbf{B}^{\to}$ is a CCompC. This follows from the following four results.

(i) The composite $\mathcal{SF}_j(\mathbf{B}) \hookrightarrow \mathbf{B}^{\to} \overset{cod}{\to} \mathbf{B}$ is a fibration.
(ii) The inclusion $\mathcal{SF}_j(\mathbf{B}) \hookrightarrow \mathbf{B}^{\to}$ is a full comprehension category with unit.
(iii) The comprehension category $\mathcal{SF}_j(\mathbf{B}) \hookrightarrow \mathbf{B}^{\to}$ has strong sums.
(iv) The fibration $\mathcal{SF}_j(\mathbf{B}) \to \mathbf{B}$ is complete.

Ad (i). For a family $\begin{pmatrix} X \\ \downarrow f \\ A \end{pmatrix}$ and a map $u\colon B \to A$, let's denote the pullback cone by $B \overset{u^*(f)}{\longleftarrow} u^*(X) \overset{u'}{\longrightarrow} X$. We show that if $\begin{pmatrix} X \\ \downarrow f \\ A \end{pmatrix}$ is separated, then also $u^*\begin{pmatrix} X \\ \downarrow f \\ A \end{pmatrix}$.
Therefore, assume one has a dense monic m and a pair φ, ψ with $\varphi \circ m = \psi \circ m$ in

$$\begin{tikzcd} \begin{pmatrix} Y' \\ \downarrow g' \\ B \end{pmatrix} \arrow[r, tail, "m"] & \begin{pmatrix} Y \\ \downarrow g \\ B \end{pmatrix} \arrow[d, shift right, "\varphi"'] \arrow[d, shift left, "\psi"] \\ & \begin{pmatrix} u^*(X) \\ \downarrow \\ B \end{pmatrix} \end{tikzcd}$$

In order to obtain $\varphi = \psi$, it suffices to show that $u^*(f) \circ \varphi = u^*(f) \circ \psi$ and $u' \circ \varphi = u' \circ \psi$ (using the pullback in $\mathbf{B}$). The first equation obviously holds; the second one follows by moving to the fibre $\mathbf{B}/A$. There one has

$$\begin{tikzcd} \begin{pmatrix} Y' \\ \downarrow u \circ g' \\ A \end{pmatrix} \arrow[r, tail, "m"] & \begin{pmatrix} Y \\ \downarrow u \circ g \\ A \end{pmatrix} \arrow[d, shift right, "u' \circ \varphi"'] \arrow[d, shift left, "u' \circ \psi"] \\ & \begin{pmatrix} X \\ \downarrow f \\ A \end{pmatrix} \end{tikzcd}$$

Ad (ii). Obvious, since the identity families are separated.

Ad (iii). For separated families $\begin{pmatrix} X \\ \downarrow f \\ A \end{pmatrix}$ and $\begin{pmatrix} A \\ \downarrow u \\ B \end{pmatrix}$ the composite $\begin{pmatrix} X \\ \downarrow u \circ f \\ B \end{pmatrix}$ is also separated: given a dense monic m into $\begin{pmatrix} Y \\ \downarrow g \\ B \end{pmatrix}$ and two arrows $\varphi, \psi : \begin{pmatrix} Y \\ \downarrow g \\ B \end{pmatrix} \to \begin{pmatrix} X \\ \downarrow u \circ f \\ A \end{pmatrix}$, one obtains $f \circ \varphi = f \circ \psi = h$, say, by using that the family u is separated in $\mathbf{B}/B$. Hence we can consider φ, ψ as maps from the family h to f in $\mathbf{B}/A$. This yields $\varphi = \psi$.

Ad (iv). For a separated family $\begin{pmatrix} X \\ \downarrow f \\ A \end{pmatrix}$ and an arbitrary map $u\colon A \to B$, the family $\begin{pmatrix} \Pi_u(X) \\ \downarrow \Pi_u(f) \\ B \end{pmatrix}$ is separated again. This result follows by an easy argument which makes use of the adjunction $u^* \dashv \Pi_u$. Fibred finite products are obtained in the obvious way.

The full subcategory $Orth(A) \hookrightarrow \mathbf{B}^{\to}$ of families orthogonal to an object A gives rise to a similar situation, see Hyland, Robinson & Rosolini [1991]. One should verify that the comprehension category $Orth(A) \hookrightarrow \mathbf{B}^{\to}$ has strong sums, i.e. that orthogonal families are closed under composition.

4.3.5. Extended example (Split topos models).
Let $\mathbf{B}$ be a topos. In a straightforward way, it gives rise to two CCompC's, namely $Id\colon \mathbf{B}^{\to} \to \mathbf{B}^{\to}$ (see 4.3.2 (i)) and the inclusion of $Sub(\mathbf{B}) \hookrightarrow \mathbf{B}^{\to}$ of monic arrows. The point of this example is to show that there are two *split* CCompC's which are equivalent (over $\mathbf{B}$) to those mentioned above. In split structures there is no need to deal with nasty mediating isomorphisms; this make the effort worthwile. The essential point in the construction below is to replace substitution via pullbacks by substitution via composition.

In the topos $\mathbf{B}$ we begin by choosing for every map φ with codomain Ω a "kernel" $\{\varphi\}$ such that the following diagram is a pullback,

$$\begin{tikzcd}
\bullet \arrow[r, tail, "\{\varphi\}"] \arrow[d, "!"'] \arrow[dr, phantom, "\lrcorner", very near start] & A \arrow[d, "\varphi"] \\
t \arrow[r, "\top"] & \Omega
\end{tikzcd}$$

where $\top\colon t \to \Omega$ is the subobject classifier.

(i) Let $\mathcal{F}(\mathbf{B})$ be the category of "families of $\mathbf{B}$". Objects are maps $X\colon A \times A' \to \Omega$. In informal notation, X can be understood as an A-indexed collection $\{X_a\}_{a\in A}$, where $X_a = \{a' \in A' \mid X(a,a') = \top\}$. A morphism from $X\colon A \times A' \to \Omega$ to

$Y : B \times B' \to \Omega$ in $\mathcal{F}(\mathbf{B})$ is a pair (u, f) forming a commuting square as follows.

$$\begin{tikzcd}
\bullet \arrow[r, "f"] \arrow[d, tail, "\{X\}"'] & \bullet \arrow[d, tail, "\{Y\}"] \\
A \times A' \arrow[d, "\pi"'] & B \times B' \arrow[d, "\pi"] \\
A \arrow[r, "u"] & B
\end{tikzcd}$$

A functor $q\colon \mathcal{F}(\mathbf{B}) \to \mathbf{B}$ is defined by $[A \times A' \xrightarrow{X} \Omega] \mapsto A$ and $(u, f) \mapsto u$. It is a *split* fibration: for $u\colon A \to B$ and $Y : B \times B' \to \Omega$ one can take $u^*(Y) = Y \circ u \times id : A \times B' \to \Omega$. As indicated by the above diagram, one obtains a full comprehension category $\mathcal{Q}\colon \mathcal{F}(\mathbf{B}) \to \mathbf{B}^{\to}$ by $[X : A \times A' \to \Omega] \mapsto [\pi \circ \{X\}]$. The idea of using an inclusion followed by a projection as "display maps" occurs also in Cartmell [1985], but there only for the category of sets. The notational convention for comprehension categories (cf. 4.1.2) leads us to denote the domain of $\{X\}$ by $\mathcal{Q}_0(X)$. Informally, $\mathcal{Q}_0(X) = \dot{\bigcup}_{a\in A}. X_a$ and for $(u, f)\colon X \to Y$ one has $f = \{f_a\colon X_a \to Y_{u(a)}\}_{a\in A}$.

The functor $\mathcal{Q}\colon \mathcal{F}(\mathbf{B}) \to \mathbf{B}^{\to}$ yields an equivalence $\mathcal{F}(\mathbf{B}) \simeq \mathbf{B}^{\to}$ over $\mathbf{B}$: for $f\colon C \to A$ let $f'\colon A \times C \to \Omega$ be the character of the the monic $\langle f, id\rangle$. Then $\mathcal{Q}(f') \cong f$ in $\mathbf{B}/A$.

This equivalence induces a CCompC-structure for $\mathcal{Q}\colon \mathcal{F}(\mathbf{B}) \to \mathbf{B}^{\to}$. But since we want $\mathcal{Q}$ to be a *split* CCompC, the unit, product and sum have to be constructed explicitly. Unit and sum are straightforward, but products are rather involved.

The unit is simply obtained by $A \mapsto [\top \circ \pi' : A \times t \to \Omega]$.

As to sums, for objects $X : A \times A_1 \to \Omega$ and $Y : \mathcal{Q}_0(X) \times A_2 \to \Omega$ one obtains $\Sigma_X.Y : A \times (A_1 \times A_2) \to \Omega$ as the character below.

$$\begin{tikzcd}
\mathcal{Q}_0(Y) \arrow[r, tail, "\{Y\}"] \arrow[d] & \mathcal{Q}_0(X) \times A_2 \arrow[r, tail, "\{X\} \times id"] & (A \times A_1) \times A_2 \cong A \times (A_1 \times A_2) \arrow[d, "\Sigma_X.Y"] \\
t \arrow[rr, "\top"] & & \Omega
\end{tikzcd}$$

In order to define products, we recall that the topos $\mathbf{B}$ has for every object $A \in \mathbf{B}$ a *partial map classifier* $\eta_A : A \rightarrowtail \tilde{A}$ with the property that for every monic $B' \rightarrowtail B$ and map $f\colon B' \to A$ there is a unique $\tilde{f}\colon B \to \tilde{A}$ forming a pullback as follows.

$$\begin{tikzcd} B' \arrow[r, tail] \arrow[d, "f"'] \arrow[dr, phantom, "\lrcorner", very near start] & B \arrow[d, dashed, "\tilde{f}"] \\ A \arrow[r, tail, "\eta_A"] & \widetilde{A} \end{tikzcd}$$

This construction is used to form the following arrows (where X, Y are as above).

$$\begin{tikzcd} \mathcal{Q}_0(X) \arrow[r, tail, "\{X\}"] \arrow[d, "id"'] & A \times A_1 \arrow[d, "id_{\widetilde{\mathcal{Q}_0(X)}}"] \\ \mathcal{Q}_0(X) \arrow[r, tail, "\eta"] & \widetilde{\mathcal{Q}_0(X)} \end{tikzcd} \qquad \begin{tikzcd} \mathcal{Q}_0(Y) \arrow[r, tail, "\eta"] \arrow[d, "\mathcal{Q}Y"'] & \widetilde{\mathcal{Q}_0(Y)} \arrow[d, "\widetilde{\mathcal{Q}Y}"] \\ \mathcal{Q}_0(X) \arrow[r, tail, "\eta"] & \widetilde{\mathcal{Q}_0(X)} \end{tikzcd}$$

$$\begin{tikzcd} \mathcal{Q}_0(X) \arrow[r, tail, "\eta"] \arrow[d, "\pi' \circ \{X\}"'] & \widetilde{\mathcal{Q}_0(X)} \arrow[d, "\underline{X}"] \\ A_1 \arrow[r, tail, "\eta"] & \widetilde{A_1} \end{tikzcd} \qquad \begin{tikzcd} \mathcal{Q}_0(Y) \arrow[r, tail, "\{X\} \times id \circ \{Y\}"] \arrow[d, "id"'] & (A \times A_1) \times \widetilde{A_2} \arrow[d, "\overline{Y}"] \\ \mathcal{Q}_0(Y) \arrow[r, tail, "\eta"] & \widetilde{\mathcal{Q}_0(Y)} \end{tikzcd}$$

Finally, we put

$$\begin{array}{rcll} \alpha & = & \langle \pi \times id,\ ev \circ \pi' \times id \rangle & : (A \times (A_1 \Rightarrow \widetilde{A_2})) \times A_1 \longrightarrow (A \times A_1) \times \widetilde{A_2} \\ f_1 & = & \Lambda(\underline{X} \circ id_{\widetilde{\mathcal{Q}_0(X)}}) \circ \pi & : A \times (A_1 \Rightarrow \widetilde{A_2}) \longrightarrow (A_1 \Rightarrow \widetilde{A_1}) \\ f_2 & = & \Lambda(\underline{X} \circ \widetilde{\mathcal{Q}Y} \circ \overline{Y} \circ \alpha) & : A \times (A_1 \Rightarrow \widetilde{A_2}) \longrightarrow (A_1 \Rightarrow \widetilde{A_1}). \end{array}$$

One can form the equalizer $\bullet \rightarrowtail A \times (A_1 \Rightarrow \widetilde{A_2})$ of f_1, f_2 and $\Pi_X.Y : A \times (A_1 \Rightarrow \widetilde{A_2}) \to \Omega$ as its characteristic. Informally one has $f_1(a, \phi) = \boldsymbol{\lambda} a_1 \in A_1.\ a_1 \in X_a$ and $f_2(a, \phi) = \boldsymbol{\lambda} a_1 \in A_1.\ \phi(a_1) \in Y_{(a,a_1)}$. The construction is so involved because in order to get "Beck-Chevalley" on-the-nose, the dependence on X, Y may only occur in the maps f_1, f_2.

(ii) A second CCompC $\mathcal{P} : \mathcal{L}(\mathbf{B}) \to \mathbf{B}^{\to}$ can be described more easily. The category $\mathcal{L}(\mathbf{B})$ — containing the logic of $\mathbf{B}$ — has maps $\varphi : A \to \Omega$ as objects. A map from $\varphi : A \to \Omega$ to $\psi : B \to \Omega$ in $\mathcal{L}(\mathbf{B})$ is a pair (u, f) making

$$\begin{tikzcd} \bullet \arrow[r, "f"] \arrow[d, tail, "\{\varphi\}"'] & \bullet \arrow[d, tail, "\{\psi\}"] \\ A \arrow[r, "u"] & B \end{tikzcd}$$

commute. This yields a split full comprehension category $\mathcal{P} : \mathcal{L}(\mathbf{B}) \to \mathbf{B}^{\to}$ by $\varphi \mapsto \{\varphi\}$. Notice that the splitting $u^*(\psi) = \psi \circ u$ is again obtained by composition. Using 4.1.2, we now write $\mathcal{P}_0(\varphi)$ for the domain of $\{\varphi\}$. The fibre categories are partial orders: one has a vertical map $\varphi \to \psi$ iff $\{\varphi\} \subseteq \{\psi\}$ iff $\varphi \Rightarrow \psi = \top$.

Obviously, a unit for $\mathcal{P}$ is given by $A \mapsto [\top_A = \top \circ ! : A \to \Omega]$.

For $\varphi : A \to \Omega$ and $\psi : \mathcal{P}_0(\varphi) \to \Omega$ one obtains $\exists_\varphi.\psi : A \to \Omega$ in

$$\begin{tikzcd} \mathcal{P}_0(\psi) \arrow[r, tail, "\{\psi\}"] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathcal{P}_0(\varphi) \arrow[r, tail, "\{\varphi\}"] & A \arrow[d, "\exists_\varphi.\psi"] \\ t \arrow[rr, "\top"] & & \Omega \end{tikzcd}$$

For the product $\forall_\varphi.\psi : A \to \Omega$ we need the following standard maps $\forall_C, \delta_C$.

$$\begin{tikzcd} t \arrow[r, tail, "\Lambda(\top \circ !)"] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \Omega^C \arrow[d, "\forall_C"] \\ t \arrow[r, "\top"] & \Omega \end{tikzcd} \qquad \begin{tikzcd} C \arrow[r, tail, "\langle id, id \rangle"] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & C \times C \arrow[d, "\delta_C"] \\ t \arrow[r, "\top"] & \Omega \end{tikzcd}$$

We then put $\forall_\varphi.\psi = \forall_{\mathcal{P}_0(\varphi)} \circ \Lambda(\hat{\psi})$ where $\hat{\psi} : A \times \mathcal{P}_0(\varphi) \to \Omega$ is described by $\hat{\psi} = (\delta_A \circ \langle \pi,\ \{\varphi\} \circ \pi' \rangle) \Rightarrow (\psi \circ \pi')$. This completes the example.

We proceed by investigating properties of closed comprehension categories. The first two results are about change-of-base and localization described in 4.1.11 (ii) and (iv).

4.3.6. Proposition. (i) *$\mathcal{P}$ is a CCompC $\Rightarrow$ $r^*(\mathcal{P})$ is a CCompC.*

(ii) *Given a morphism $\mathcal{P} \to \mathcal{P}'$ of CCompC's; a morphism of fibrations $r \to r'$ determines a morphism of CCompC's $r^*(\mathcal{P}) \to r'^*(\mathcal{P}')$.*

Proof. (i) Let $\mathcal{P}$ be a full comprehension category with unit, products and strong sums. $r^*(\mathcal{P})$ is again full and has a unit as remarked in 4.1.11 (ii); it admits products and strong sums by 4.2.12 (i).

(ii) By 4.2.12 (ii). □

4.3.7. Proposition. *Let $\mathcal{P}$ be a CCompC.*

(i) *For every object A in the basis, $\mathcal{P}[A]$ is a CCompC.*

(ii) *For every morphism $B \to A$ in the basis, there is a morphism $\mathcal{P}[A] \to \mathcal{P}[B]$ of CCompC's.*

Proof. Straightforward but laborious. □

4.3.8. Lemma. *Let $\mathcal{P} : \mathbf{E} \to \mathbf{B}^{\to}$ be a CCompC. Then considered as a functor, $\mathcal{P}$ preserves units, sums and products.*

Proof. Units are preserved by lemma 4.1.10 (iii) and sums are preserved because they are strong: $\mathcal{P}(\Sigma_E.E') \cong \mathcal{P}E \circ \mathcal{P}E' = \Sigma_{\mathcal{P}E}.\mathcal{P}E'$ in $\mathbf{B}/pE$. As to products we obtain for $u : A \to pE$ in $\mathbf{B}$,

$$\begin{array}{rcll}
\mathbf{B}/pE\big(u,\ \mathcal{P}(\Pi_E.E')\big) & \cong & \mathbf{E}_A\big(1A,\ u^*(\Pi_E.E')\big) & \\
& \cong & \mathbf{E}_A\big(1A,\ \Pi_{u^*(E)}.(\mathcal{P}E^{\#}(u))^*(E')\big), & \text{by Beck-Chevalley} \\
& \cong & \mathbf{E}_{\mathcal{P}_0u^*(E)}\big((\mathcal{P}u^*(E))^*(1A),\ (\mathcal{P}E^{\#}(u))^*(E')\big) & \\
& \cong & \mathbf{E}_{\mathcal{P}_0u^*(E)}\big(1\mathcal{P}u^*(E),\ (\mathcal{P}E^{\#}(u))^*(E')\big) & \\
& \cong & \mathbf{B}/\mathcal{P}_0E\big(\mathcal{P}E^{\#}(u),\ \mathcal{P}E'\big) &
\end{array}$$

in which the pullback functor $\mathcal{P}E^{\#}$ comes from 4.1.7. The first and last step hold by 4.1.10 (ii). □

The above lemma shows how the CCompC-structure defined "on the top level" in terms of (fibred) adjunctions shows up in the basis for display maps. In this way one can avoid rather cumbersome formulations of unit, product and sum for display maps.

4.3.9. Lemma. (i) *Let $\mathcal{P}$ be a CCompC; the fibration involved $p = cod \circ \mathcal{P}$ is a fibred CCC.*

(ii) *A morphism of CCompC's induces a morphism between the corresponding fibred CCC's.*

Proof. (i) Cartesian products are given by $E \times E' = \Sigma_E.\mathcal{P}E^*(E')$ and exponents by $E \Rightarrow E' = \Pi_E.\mathcal{P}E^*(E')$. In fact, strongness of the sums is not needed to obtain this, see lemma 2.2.8.

(ii) Straightforward. □

By looking at the fibre above the terminal object, one obtains from the previous result, a forgetful functor from closed comprehension categories to CCC's. This observation is the basis for the next result.

4.3.10. Theorem. *Let $\mathbf{B}$ be a CCC; $Cons_{\mathbf{B}} : \overline{\mathbf{B}} \to \mathbf{B}^{\to}$ is then the free CCompC generated by $\mathbf{B}$. The unit here is an isomorphism.*

Proof. The unit $\eta_{\mathbf{B}} : \mathbf{B} \to \overline{\mathbf{B}}_t$ is given by $B \mapsto (t, B)$ and $u \mapsto (id, u \circ \pi')$. Suppose $\mathcal{Q} : \mathbf{D} \to \mathbf{A}^{\to}$ is a CCompC with terminal $t' \in \mathbf{A}$ and $H : \mathbf{B} \to \mathbf{D}_{t'}$ is a functor which

preserves the CCC-structure. We construct

$$\begin{tikzcd}
\overline{\mathbf{B}} \arrow[rr, "L"] \arrow[d, shift right=3ex] \arrow[d, shift left=3ex] \arrow[d, phantom, "\overset{Cons_{\mathbf{B}}}{\Longrightarrow}"] & & \mathbf{D} \arrow[d, shift right=3ex] \arrow[d, shift left=3ex] \arrow[d, phantom, "\overset{\mathcal{Q}}{\Longrightarrow}"] \\
\mathbf{B} \arrow[rr, "K"] & & \mathbf{A}
\end{tikzcd}
\qquad \gamma : \mathcal{Q}_0 L \xrightarrow{\sim} K(Cons_{\mathbf{B}})_0$$

forming an (up-to-isomorphism) unique morphism of CCompC's with $L \restriction t \circ \eta_{\mathbf{B}} \cong H$. In fact, we can only take $KB = \mathcal{Q}_0(HB)$ — since $KB \cong K(t \times B) = K(Cons_{\mathbf{B}})_0(t, B) \cong \mathcal{Q}_0\, L\, \eta_{\mathbf{B}}(B) \cong \mathcal{Q}_0(HB)$. Similarly, we have to take $L(B, B') = !^*_{KB}(HB')$, because $L(B, B') = L(!^*_B(t, B')) \cong\, !^*_{KB}\, L\, \eta_{\mathbf{B}}(B') \cong\, !^*_{KB}(HB')$. Then indeed $L \restriction t \circ \eta_{\mathbf{B}} \cong H$. One easily verifies that $\mathcal{Q}_0 : \mathbf{D}_{t'} \to \mathbf{A}$ preserves finite products; hence we obtain $K(B \times B') \cong KB \times KB' \cong^* \mathcal{Q}_0(!^*_{KB}(HB')) = \mathcal{Q}_0 L(B, B')$; the isomorphism $\cong^*$ is there because the product $KB \times KB'$ can be obtained as a pullback. We finish by showing that L preserves products.

$$\begin{array}{rcl}
L(\Pi_{(B,B')}.(B \times B', B'')) & = & L(B, B' \Rightarrow B'') \\
& = & !^*_{KB}\,(H(B' \Rightarrow B'')) \\
& \cong & !^*_{KB}\,(HB') \Rightarrow\, !^*_{KB}\,(HB'') \\
& \cong & \Pi_{L(B,B')}.\ \mathcal{Q}(L(B, B'))^*\, !^*_{KB}(HB'') \\
& \cong & \Pi_{L(B,B')}.\ !^*_{\mathcal{Q}_0 L(B,B')}\,(HB'') \\
& \cong & \Pi_{L(B,B')}.\ \gamma^*_{(B,B')}\, !^*_{K(B \times B')}\,(HB'') \\
& \cong & \Pi_{L(B,B')}.\ \gamma^*_{(B,B')}\, L(B \times B', B''). \qquad \square
\end{array}$$

At the end of the next section we shall be able to establish two similar free constructions. We close this section with the description of categorical versions of the type systems based on the "Propositions as Types" setting.

4.3.11. Definition. (i) A λP1-category is a CCompC.

(ii) A λPi-category is a CCompC with fibred equalizers. A *morphism of λPi-categories* is a morphism of CCompC's which preserves the fibred equalizers.

(iii) A $\lambda*$-category is a CCompC $\mathcal{P} : \mathbf{E} \to \mathbf{B}^{\to}$ provided with an object $\Omega \in \mathbf{E}$ such that $p\Omega \in \mathbf{B}$ is terminal and $p = cod \circ \mathcal{P}$ has a generic object above $\mathcal{P}_0\Omega \in \mathbf{B}$.

Examples of λPi-categories are 4.3.2 (i), (iv) and 4.3.4. Also the term model of the calculus λPi yields an example, analogously to 4.3.2 (vi) using lemma 2.2.14. These λPi-categories are categorical versions of Martin-Löf's type theory.

The closure model in 4.3.3 is a $\lambda*$-category. Such categories are logically inconsistent in the sense that every proposition is inhabited. This result is known as *Girard's paradox*, see Girard [1972]. In Barendregt [199?] one can find a proof using only Π's and in Troelstra & van Dalen [1990] or Jacobs [1989] a proof which makes use of strong Σ's. Pitts & Taylor [1989] contains a similar inconsistency result which is obtained with identity types. It applies to λPi-categories with a generic object.

4.4 Category theory over a fibration

In the first chapter it was explained how a fibration forms a category fibred over a base category. Now we go one step up and consider fibrations as bases. This is not as bad as it may seem, since it turns out that one can reduce matters to the previous level. Lemma 1.1.5 lies at the basis of all this.

The first part of this section (up to 4.4.13) is a slightly extended version of the fifth section in Jacobs, Moggi & Streicher [1991].

4.4.1. A fibration as a basis. Suppose a cartesian functor p between fibrations q, r is given as in the following diagram.

$$\begin{tikzcd}
\mathbf{E} \arrow[rr, "p"] \arrow[dr, "q"'] & & \mathbf{B} \arrow[dl, "r"] \\
& \mathbf{A} &
\end{tikzcd}$$

Every object $A \in \mathbf{A}$ determines a "fibrewise" functor $p \upharpoonright A : \mathbf{E}_A \to \mathbf{B}_A$ by restriction. One calls p a *fibration over* r if all these fibrewise functors are fibrations and reindexing functors preserve the relevant cartesian structure (similarly to e.g. fibred cartesian products). More explicilty, p is a fibration over r if both

- for every $A \in \mathbf{A}$, $p \upharpoonright A$ is a fibration;
- for every $u : A \to A'$ in $\mathbf{A}$ and every reindexing functor $u^* : \mathbf{B}_{A'} \to \mathbf{B}_A$, there is a reindexing functor $u^\# : \mathbf{E}_{A'} \to \mathbf{E}_A$ forming a morphism of fibrations:

$$\begin{tikzcd}
\mathbf{E}_{A'} \arrow[r, "u^\#"] \arrow[d, "p \upharpoonright A'"'] & \mathbf{E}_A \arrow[d, "p \upharpoonright A"] \\
\mathbf{B}_{A'} \arrow[r, "u^*"] & \mathbf{B}_A
\end{tikzcd}$$

This rather complicated notion is equivalent to a more simple one; namely

$$p \text{ is a fibration over } r \quad \Leftrightarrow \quad p \text{ is a fibration itself.}$$

To verify the implication ($\Leftarrow$), notice that $p \upharpoonright A$ can be obtained from p by change-of-base. This yields that f in $\mathbf{E}_A$ is $p \upharpoonright A$-cartesian iff f is p-cartesian. The rest is not difficult. As to the implication ($\Rightarrow$), observe that if p is a fibration over r, then f in $\mathbf{E}$ is p-cartesian iff f can be written as $g \circ \alpha$ where g is q-cartesian and α is $p \upharpoonright A$-cartesian — with $A = q(dom\, f)$.

Next, consider a diagram,

$$\begin{tikzcd}
\mathbf{E} \arrow[rr, "F"] \arrow[dr, "p"] \arrow[ddr, "q"'] & & \mathbf{E}' \arrow[dl, "p'"'] \arrow[ddl, "q'"] \\
& \mathbf{B} \arrow[d, "r"] & \\
& \mathbf{A} &
\end{tikzcd}$$

in which r, q, q', p and p' are fibrations with $q = rp$, $q' = rp'$ and F is a cartesian functor from q to q'. One calls F a *cartesian functor from p to p' over r* if both

- for every $A \in \mathbf{A}$, $F \upharpoonright A$ is cartesian form $p \upharpoonright A$ to $p \upharpoonright A'$;
- $p' \circ F = p$.

As before, one can show that

$$F \text{ is cartesian } p \to p' \text{ over } r \quad \Leftrightarrow \quad F \text{ is cartesian } p \to p' \text{ in } \mathit{Fib}(\mathbf{B}).$$

In this way, one obtains a category $\mathit{Fib}(r)$ of fibrations and cartesian functors over r. As shown, one has $\mathit{Fib}(r) = \mathit{Fib}(\mathbf{B})$. It is left to the reader to formulate what natural transformations over r are and that the previous identification also concerns the 2-structure. Hence adjunctions over $r : \mathbf{B} \to \mathbf{A}$ are adjunctions over $\mathbf{B}$ (i.e. in the 2-category $\mathit{Fib}(\mathbf{B})$). In order to get an even better picture, the reader may want to verify that for $F : p \to p'$ in $\mathit{Fib}(\mathbf{B})$ as above and $G : p' \to p$ one has that $F \dashv G$ is an adjunction over r iff both

- for every $A \in \mathbf{A}$, there is fibred adjunction $F \upharpoonright A \dashv G \upharpoonright A$ in $\mathit{Fib}(\mathbf{B}_A)$;
- for every morphism $A \to A'$ in $\mathbf{A}$ there is a "pseudo map" of adjunctions from $F \upharpoonright A' \dashv G \upharpoonright A'$ to $F \upharpoonright A \dashv G \upharpoonright A$ (see Jacobs [1990] for the definition).

As a consequence we have for example that $p : \mathbf{E} \to \mathbf{B}$ is a "CCC over r" iff p is a fibred CCC iff every $p \upharpoonright A$ is a fibred CCC and reindexing functors form maps between these.

The above exposition is based on work of J. Bénabou; see also Pavlović [1990].

Next, we proceed to describe (closed) comprehension categories over a fibration. The intention is to obtain this structure fibrewise, preserved by reindexing functors.

4.4.2. Definition. Let $r:\mathbf{B}\to\mathbf{A}$ be a fibration. A functor $\mathcal{P}:\mathbf{E}\to\mathbf{B}^{\to}$ is a *comprehension category over* r if $\mathcal{P}$ is a comprehension category which restricts to a cartesian functor in

$$\begin{tikzcd} \mathbf{E} \arrow[rr, "\mathcal{P}"] \arrow[dr, "rp"'] & & V(\mathbf{B}) \arrow[dl, "r^{\to}"] \\ & \mathbf{A} & \end{tikzcd}$$

where $r^{\to}$ is the "arrow fibration" described at the end of 1.1.2.

4.4.3. Lemma. *Let* $\mathbf{E}\xrightarrow{p}\mathbf{B}\xrightarrow{r}\mathbf{A}$ *be fibrations and* $\mathcal{P}:\mathcal{P}_0\stackrel{\cdot}{\longrightarrow}p:rp\to r$ *a 2-cell in* $Fib(\mathbf{A})$. *Then* $\mathcal{P}$ *is a comprehension category over* r *iff both*

- *for every* $A\in\mathbf{A}$, $\mathcal{P}\restriction A:\mathbf{E}_A\to(\mathbf{B}_A)^{\to}$ *is a comprehension category;*
- *for every* $u:A\to A'$ *and* $u^*:\mathbf{B}_{A'}\to\mathbf{B}_A$, *there is a* $u^{\#}:\mathbf{E}_{A'}\to\mathbf{E}_A$ *forming a morphism of comprehension categories* $\mathcal{P}\restriction A'\to\mathcal{P}\restriction A$.

Moreover, $\mathcal{P}$ *is a full comprehension category iff all* $\mathcal{P}\restriction A$*'s are full.*

Proof. By the observations about cartesian arrows in 1.1.5 and 4.4.1 and the fact that $\mathcal{P}$ is a cartesian functor. □

4.4.4. Examples. (i) Constant comprehension categories as in 4.1.3 can also be described over a fibration $p:\mathbf{E}\to\mathbf{B}$ with fibred finite products. Let $T\subseteq Obj(\mathbf{E})$ be such that for every cartesian $f:E'\to E$ one has $E\in T\ \Rightarrow\ E'\in T$. The full subcategory of $\mathbf{E}$ determined by T then yields a full "subfibration" of p. Let's write $\mathbf{E}//T$ for the category with objects (E,X) where $E\in\mathbf{E}$ and $X\in T$ satisfy $pE=pX$. Morphisms $(f,g):(E,X)\to(E',X')$ in $\mathbf{E}//T$ are maps $f:E\to E'$ and $g:E\times X\to X'$ in $\mathbf{E}$ with $pf=pg$. A (full) comprehension category $Cons_T:\mathbf{E}//T\to\mathbf{E}^{\to}$ over p is obtained by $(E,X)\mapsto\pi:E\times X\to E$.

In the special case that $T=Obj(\mathbf{E})$, we have written $\overline{\mathbf{E}}$ for $\mathbf{E}//T$ in 1.2.7. The fibration was denoted there by $\overline{p}:\overline{\mathbf{E}}\to\mathbf{E}$. Let's write in this special case $\overline{\mathcal{P}}:\overline{\mathbf{E}}\to\mathbf{E}^{\to}$ for the comprehension category defined above (i.e. $\overline{\mathcal{P}}(E,E')=\pi:E\times E'\to E$).

If $\mathbf{B}$ is a category with finite products, the construction above applied to the fibration $\mathbf{B}\to\mathbf{1}$ (the terminal category) conincides with the one given in 4.1.3.

(ii) Let $\mathbf{B}$ be a category with pullbacks. The (obvious) functor $cod^{\to}:\mathbf{B}^{\to\to}\to\mathbf{B}^{\to}$ forms a fibration over $cod:\mathbf{B}^{\to}\to\mathbf{B}$. One obtains a full comprehension category $\mathbf{B}^{\to\to}\longrightarrow\mathbf{B}^{\to\to}$ over cod by $[\xrightarrow{v}\xrightarrow{u}]\mapsto[(id,v):u\circ v\to u]$ in $\mathbf{B}^{\to}$.

(iii) Let $p:\mathbf{E}\to\mathbf{B}$ be a LEX fibration (i.e. a fibration with fibred finite limits). As already mentioned at the end of 1.2.7, the functor $cod:V(\mathbf{E})\to\mathbf{E}$ is a fibration with $(f,g):\alpha\to\beta$ cartesian in $V(\mathbf{E})$ iff (f,g) is a pullback in $\mathbf{E}$. Hence we obtain a full comprehension category over p:

$$\begin{tikzcd} V(\mathbf{E}) \arrow[rr, shift left, "dom"] \arrow[rr, shift right, "cod"'] \arrow[rr, phantom, "\Downarrow Id"] \arrow[dr, "p^{\to}=p\circ cod"'] & & \mathbf{E} \arrow[dl, "p"] \\ & \mathbf{B} & \end{tikzcd}$$

Notice that $V(\mathbf{B}^{\to})\cong\mathbf{B}^{\to\to}$. This example generalizes the previous one.

4.4.5. Definition. Let $p:\mathbf{E}\to\mathbf{B}$ and $r:\mathbf{B}\to\mathbf{A}$ be fibrations; p forms a *comprehension category with unit over* r if there is

- a terminal object functor $1:\mathbf{B}\to\mathbf{E}$ for p in $Fib(\mathbf{B})$;
- a fibred right adjoint $\mathcal{P}_0$ of $1:r\to rp$ in $Fib(\mathbf{A})$.

4.4.6. Definition. A *closed comprehension category over a fibration* r is a full comprehension category with unit $\mathcal{P}$ over r which admits $\mathcal{P}$-products and strong $\mathcal{P}$-sums; moreover, r is required to have a fibred terminal object.

The next notion covers a special case.

4.4.7. Definition. Let $p:\mathbf{E}\to\mathbf{B}$ be a LEX fibration; p will be called a *fibred LCCC* if the comprehension category $V(\mathbf{E})\hookrightarrow\mathbf{E}^{\to}$ over p is closed.

In that case every fibre category $\mathbf{E}_A$ is an LCCC and reindexing functors preserve the LCCC-structure, see lemma 4.4.3.

4.4.8. Examples. (i) The first example from 4.4.4 is of interest again; it gives rise to a generalization of the bi-implication obtained in 4.3.2 (ii). For a fibration $p:\mathbf{E}\to\mathbf{B}$ with finite products one has

$$\overline{\mathcal{P}}:\overline{\mathbf{E}}\to\mathbf{E}^{\to}\text{ is a CCompC over }p\quad\Leftrightarrow\quad p\text{ is a fibred CCC.}$$

The implication $(\Rightarrow)$ goes as follows. $\overline{\mathcal{P}}$ is a CCompC over p $\Rightarrow$ $\overline{\mathcal{P}}$ is a CCompC $\Rightarrow$ $\overline{p}=cod\circ\overline{\mathcal{P}}$ is a fibred CCC $\Rightarrow$ p is a fibred CCC, using the change-of-base situation $p\to\overline{p}$ from 1.2.7. As to the reverse implication, one defines $\Sigma_{(E,E')}.(E\times E',E'')=(E,E'\times E'')$ and $\Pi_{(E,E')}.(E\times E',E'')=(E,E'\Rightarrow E'')$.

(ii) One easily verifies that a category $\mathbf{B}$ is an LCCC if and only if $cod:\mathbf{B}^{\to}\to\mathbf{B}$ is a fibred LCCC. This follows from the fact that $\mathbf{B}$ is an LCCC iff all its slice categories $\mathbf{B}/A$ are LCCC's.

(iii) The family fibration satisfies

$$\mathit{Fam}(\mathbf{C}) \to \mathbf{Sets} \text{ is a fibred LCCC} \quad \Leftrightarrow \quad \mathbf{C} \text{ is an LCCC.}$$

The implication ($\Leftarrow$) follows from a pointwise construction. The reverse implication follows from the fact that $\mathbf{C}$ is isomorphic to the fibre above the terminal object.

(iv) Streicher [1990] investigates an LCCC $\mathbf{B}$ and a collection of "display maps" $\mathcal{D}$ satisfying the conditions (Term), (Pb) and (Sub-lcc), see loc. cit. This precisely means that the fibration $cod : \mathbf{B}^{\to}(\mathcal{D}) \to \mathbf{B}$ is a fibred LCCC, where $\mathbf{B}^{\to}(\mathcal{D})$ is the full subcategory of $\mathbf{B}^{\to}$ with maps $f \in \mathcal{D}$ as objects.

(v) The fibration $\mathit{Fam}_{\mathrm{eff}}(\mathbf{M}) \to \omega\text{-}\mathbf{Set}$ from 1.2.12 is a fibred LCCC. Since $\mathbf{M}$ is an LCCC itself, this result follows from a pointwise construction.

A fibration is an LCCC if and only if the all fibres are LCCC's and reindexing preserves the LCCC structure. Equivalently, if all slices of the fibres are CCC's and reindexing preserves fibred finite limits and local exponentials. The next result contains another characterization; it is based on a suggestion by I. Moerdijk.

4.4.9. Proposition. *Let $p : \mathbf{E} \to \mathbf{B}$ be a fibration. For every object $E \in \mathbf{E}$ above $B \in \mathbf{B}$ one obtains a "slice fibration" $p/E : \mathbf{E}/E \to \mathbf{B}/B$. Then*

$$p \text{ is a fibred LCCC} \quad \Leftrightarrow \quad \text{every } p/E \text{ is a fibred CCC.}$$

Proof. Because for $E \in \mathbf{E}$ and $u : A \to pE$ in $\mathbf{B}$ one has an isomorphism (natural in u) between the fibre of the slice $(\mathbf{E}/E)_u$ and the slice of the fibre $(\mathbf{E}_A)/u^*(E)$. Considering CCC-structure preserved by reindexing yields the desired result. □

In the next construction, a generalization of $\overline{p}$ from 1.2.7 is obtained by using strong sums instead of cartesian products.

4.4.10. Proposition. *Let $\mathcal{P} : \mathbf{E} \to \mathbf{B}^{\to}$ be a closed comprehension category. By change-of-base, we form the fibration $\tilde{p} : \widetilde{\mathbf{E}} \to \mathbf{E}$.*

$$\begin{tikzcd}
\widetilde{\mathbf{E}} = \mathbf{E} \underset{\mathcal{P}_0, p}{\times} \mathbf{E} \arrow[r] \arrow[d, "\tilde{p}"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathbf{E} \arrow[d, "p"] \\
\mathbf{E} \arrow[r, "\mathcal{P}_0"] & \mathbf{B}
\end{tikzcd}$$

Then

(i) *$\tilde{p} : \widetilde{\mathbf{E}} \to \mathbf{E}$ forms part of a CCompC $\widetilde{\mathcal{P}}$ over p;*

(ii) *there is a "pseudo" change-of-base situation (in which 1 is terminal object functor),*

$$\begin{tikzcd}
\mathbf{E} \arrow[r] \arrow[d, "p"'] \arrow[dr, phantom, "\lrcorner", very near start] & \widetilde{\mathbf{E}} \arrow[d, "\tilde{p}"] \\
\mathbf{B} \arrow[r, "1"] & \mathbf{E}
\end{tikzcd}$$

By "pseudo" we mean that the fibration obtained by performing change-of-base on $\tilde{p}$ along 1 yields a fibration which is equivalent instead of isomorphic to p.

Proof. (i) One defines $\widetilde{\mathcal{P}} : \widetilde{\mathbf{E}} \to \mathbf{E}^{\to}$ by $(E, E') \mapsto$ [the projection $\Sigma_E.E' \to E$]; it is the unique vertical map f with $\mathcal{P}_0 f = \mathcal{P}E' \circ \mathcal{P}_0(in_{E,E'})^{-1}$, using the morphism described in lemma 4.2.9 (i) and the fact that $\mathcal{P}$ is full. One uses that these in-morphisms are cocartesian in order to define $\widetilde{\mathcal{P}}$ on morphisms. The rest is laborious but straightforward.

(ii) Easy. □

The constructions $\overline{p}$ and $\tilde{p}$ provide two ways to obtain closed comprehension categories over p. Later on in this section we shall see that both can be understood as free constructions. First we show that quantification for the base fibration p can be lifted to $\overline{p}$ and $\tilde{p}$. One gets strongness of the lifted sums for free.

4.4.11. Lemma. *Let $p : \mathbf{E} \to \mathbf{B}$ be a fibred CCC and $\mathcal{Q} : \mathbf{D} \to \mathbf{B}^{\to}$ a comprehension category. Then*

$$p \text{ admits } \mathcal{Q}\text{-products/sums} \quad \Rightarrow \quad \overline{\mathcal{P}} \text{ admits } p^*(\mathcal{Q})\text{-products/strong sums.}$$

Proof. Assume adjunctions $\Sigma_D \dashv \mathcal{Q}D^* \dashv \Pi_D$ in $\mathbf{E}$; we seek $\exists_{(E,D)} \dashv p^*(\mathcal{Q})(E,D)^* \dashv \forall_{(E,D)}$ in $\overline{\mathbf{E}}$. The product functor $\forall_{(E,D)} : \overline{\mathbf{E}}_{\mathcal{Q}D^*(E)} \to \overline{\mathbf{E}}_E$ defined by $(\mathcal{Q}D^*(E), E') \mapsto (E, \Pi_D.E')$ yields the desired result. The analogous definition $\exists_{(E,D)}(\mathcal{Q}D^*(E), E') = (E, \Sigma_D.E')$ does not work immediately; one has to use the Frobenius isomorphism $\phi : \Sigma_D.(\mathcal{Q}D^*(E) \times E') \xrightarrow{\sim} E \times \Sigma_D.E'$ from lemma 4.2.9 (ii).

Strongness follows from appropriate use of this Frobenius isomorphism. First one verifies that $\overline{\mathcal{P}}_0(in_{(E,D),(\mathcal{Q}D^*(E),E')}) = \overline{\mathcal{Q}D}(E) \times in_{D,E'}$; then one can assume a commuting diagram of the form

$$\begin{tikzcd}
\mathcal{Q}^*(E) \times E' \arrow[r, "\overline{\mathcal{Q}D}(E) \times in_{D,E'}"] \arrow[d, "f"'] & E \times \Sigma_D.E' \arrow[d, "g"] \\
E_1 \times E_2 \arrow[r, "\pi"] & E_1.
\end{tikzcd}$$

Let $f' : \mathcal{Q}D^*(E) \times E' \longrightarrow \mathcal{Q}D^*(pg)^*(E_1 \times E_2)$ be the vertical part of f. It gives rise to a transpose $\widehat{f'} : \Sigma_D.(\mathcal{Q}D^*(E) \times E') \longrightarrow (pg)^*(E_1 \times E_2)$ and thus one obtains $h = \overline{pg}(E_1 \times E_2) \circ \widehat{f'} \circ \phi^{-1} : E \times \Sigma_D.E' \longrightarrow E_1 \times E_2$ with the required properties. $\square$

4.4.12. Lemma. *Let $\mathcal{P}:\mathbf{E} \to \mathbf{B}^{\to}$ be a closed comprehension category and $\mathcal{Q}:\mathbf{D} \to \mathbf{B}^{\to}$ an arbitrary comprehension category. Then*

$$p \text{ admits } \mathcal{Q}\text{-products/sums} \quad \Rightarrow \quad \widetilde{\mathcal{P}} \text{ admits } p^*(\mathcal{Q})\text{-products/strong sums.}$$

Proof. Let's assume adjunctions $\Sigma_E \dashv \mathcal{P}E^* \dashv \Pi_E$ and $\exists_D \dashv \mathcal{Q}D^* \dashv \forall_D$ in $\mathbf{E}$; we intend to construct $\widetilde{\exists}_{(E,D)} \dashv p^*(\mathcal{Q})(E,D)^* \dashv \widetilde{\forall}_{(E,D)}$ in $\widetilde{\mathbf{E}}$. This is established by $\widetilde{\forall}_{(E,D)}.(\mathcal{Q}D^*(E), E') = (E, \forall_{\mathcal{P}E^*(D)}.\alpha^*(E'))$, where α is a mediating isomorphism in $\mathbf{B}$. A Similar definition works for sums. Strongness is obtained as in the previous proof, this time using a "generalized Frobenius" isomorphism $\exists_D.\Sigma_{\mathcal{Q}D^*(E)}.E' \cong \Sigma_E.\exists_{\mathcal{P}E^*(D)}.\alpha^*(E')$. $\square$

Remember from lemma 4.3.9 that there is a forgetful functor from CCompC's to fibred CCC's. It is used in the next result.

4.4.13. Theorem. *Let $p:\mathbf{E} \to \mathbf{B}$ be a fibred CCC; the construction $\overline{\mathcal{P}}:\overline{\mathbf{E}} \to \mathbf{E}$ yields the free CCompC generated by p.*
($\overline{\mathcal{P}}$ is described in 4.4.4 (i) and 4.4.8 (i).)

Proof. A unit $p \to \overline{p} = cod \circ \overline{\mathcal{P}}$ is given by the change-of-base situation in 1.2.7. Let $\mathcal{Q}:\mathbf{D} \to \mathbf{A}^{\to}$ be a CCompC and $(K:\mathbf{B} \to \mathbf{A}, L:\mathbf{E} \to \mathbf{D})$ be a morphism of fibred CCC's from p to $q = cod \circ \mathcal{Q}$. We have to construct an (up-to-isomorphism) unique morphism of CCompC's:

$$\begin{tikzcd}
\overline{\mathbf{E}} \arrow[rrr, "H"'] \arrow[d, shift right=4] \arrow[d, phantom, "\overset{\overline{\mathcal{P}}}{\Longrightarrow}"] \arrow[d, shift left=4] & & & \mathbf{D} \arrow[d, shift right=4] \arrow[d, phantom, "\overset{\mathcal{Q}}{\Longrightarrow}"] \arrow[d, shift left=4] \\
\mathbf{E} \arrow[rrr, "G"] & & & \mathbf{A}
\end{tikzcd}
\qquad \gamma : \mathcal{Q}_0 H \xrightarrow{\sim} G\overline{\mathcal{P}}_0$$

As in the proof of 4.3.10 one is forced to take $GE = \mathcal{Q}_0(LE)$ and $H(E,E') = \mathcal{Q}(LE)^*(LE')$. The main ingredient of the remaining verifications is that $\mathcal{Q}:\mathbf{D} \to \mathbf{A}^{\to}$ preserves (fibred) cartesian products, which follows from lemma 4.1.10 (ii). $\square$

4.4.14. Proposition. *Let $\mathcal{P}:\mathbf{E} \to \mathbf{B}^{\to}$ be a λPi-category, i.e. a CCompC with fibred equalizers.*

(i) *The functor $\widetilde{\mathcal{P}}:\widetilde{\mathbf{E}} \to \mathbf{E}^{\to}$ described in 4.4.10 is a cartesian functor in*

$$\begin{tikzcd}
\widetilde{\mathbf{E}} \arrow[rr, "\widetilde{\mathcal{P}}"] \arrow[dr, "\tilde{p}"'] & & V(\mathbf{E}) \arrow[dl, "cod"] \\
& \mathbf{E} &
\end{tikzcd}$$

and forms a morphisms of comprehension categories from $\widetilde{\mathcal{P}}$ to the inclusion $V(\mathbf{E}) \hookrightarrow \mathbf{E}^{\to}$.

(ii) *$\widetilde{\mathcal{P}}$ is an equivalence in the above diagram.*

Proof. (i) Obvious, since all projections $\widetilde{\mathcal{P}}(E,E')$ are vertical.

(ii) Since $\widetilde{\mathcal{P}}$ is a CCompC one has that $\widetilde{\mathcal{P}}$ is a full and faithful functor. Hence it suffices to define for a vertical $\alpha:E' \to E$ in $\mathbf{E}$ an object $(E,E'') \in \widetilde{\mathbf{E}}$ with $\widetilde{\mathcal{P}}(E,E'') \cong \alpha$ vertically. This is done by a standard construction, see e.g. Seely [1984]. In informal type theoretical notation, we construct the type $\alpha^{-1}(x) = \Sigma y{:}E'.\, I_E(x, \alpha(y))$ depending on $x : E$. In category theoretical formulation, we form the following pullback in the fibre above $\mathcal{P}_0E$.

$$\begin{tikzcd}
& \alpha^{-1}(E) \arrow[dl] \arrow[dr] & & \\
1\mathcal{P}_0E \arrow[dr, "var^E"'] & & \mathcal{P}E^*(E') \arrow[r] \arrow[dl, "\mathcal{P}E^*(\alpha)"] & E' \arrow[d, "\alpha"] \\
& \mathcal{P}E^*(E) \arrow[rr] & & E
\end{tikzcd}$$

where var^E is the unique vertical map with $\overline{\mathcal{P}E}(E) \circ var^E = \varepsilon_E : 1\mathcal{P}_0E \to E$. One can then show that $\mathcal{P}(\alpha^{-1}(E)) \cong \mathcal{P}_0(\alpha)$ in $\mathbf{B}/\mathcal{P}_0E$. It follows readily that $\widetilde{\mathcal{P}}(E, \alpha^{-1}(E)) \cong \alpha$ in $\mathbf{E}_A/E$. $\square$

4.4.15. Corollary. *Let $\mathcal{P}$ be a λPi-category; $cod \circ \mathcal{P}$ is then a fibred LCCC.* $\square$

The functor $\mathit{Fam}(\mathbf{Sets}) \to \mathbf{Cat}^{\to}$ from 4.3.2 (iv) is a λPi-category. Indeed the fibre categories $\mathbf{Sets}^{\mathbf{C}}$ are LCCC's (even more, they are toposes). The term model of the calculus λPi (i.e. Martin-Löf's type theory) also forms a λPi-category. As all fibres, the one above the terminal object (i.e. the empty context) is an LCCC. Seely [1984], section 3, constructs only this fibre category as a term model.

From the corollary above one obtains a forgetful functor from λPi-categories to LCCC's by looking at the fibre above the terminal object. This forms the background for the next result.

4.4.16. Theorem. *Let $\mathbf{B}$ be an LCCC; the identity functor on $\mathbf{B}^{\rightarrow}$ is then the free λPi-category generated by $\mathbf{B}$. The unit here is an isomorphism.*

Proof. The unit $\eta_{\mathbf{B}} : \mathbf{B} \to \mathbf{B}/t$ is described by $B \mapsto !_B$ and $u \mapsto u$. A λPi-category $\mathcal{Q} : \mathbf{D} \to \mathbf{A}^{\rightarrow}$ together with a morphism $H : \mathbf{B} \to \mathbf{D}_{t'}$ of LCCC's gives rise to an (up-to-isomorphism) unique morphism of λPi-categories:

$$\begin{tikzcd}
\mathbf{B}^{\rightarrow} \arrow[r, "L"] \arrow[d, shift right=3] \arrow[d, shift left=3] \arrow[d, phantom, "\overset{Id}{\Longrightarrow}"] & \mathbf{D} \arrow[d, shift right=3] \arrow[d, shift left=3] \arrow[d, phantom, "\overset{\mathcal{Q}}{\Longrightarrow}"] \\
\mathbf{B} \arrow[r, "K"] & \mathbf{A}
\end{tikzcd} \qquad \gamma : \mathcal{Q}_0 H \xrightarrow{\sim} K\, dom$$

We put $KB = \mathcal{Q}_0(HB)$, since $KB = K\,dom(!_B) \cong \mathcal{Q}_0\,L\,(!_B) = \mathcal{Q}_0\,L\,\eta_{\mathbf{B}}\,(B) \cong \mathcal{Q}_0(HB)$. Furthermore, we take $L(f : B' \to B) = (Hf)^{-1}(HB)$, where $(-)^{-1}$ is determined in the proof of proposition 4.4.14 (ii). We are forced to proceed like this since $f \cong (\eta_{\mathbf{B}}(f))^{-1}(\eta_{\mathbf{B}}(B))$. $\square$

4.5 Locally small fibrations

In this last section of chapter 4 so-called "locally small" fibrations will be investigated. These are of interest in our research because of

- connections with comprehension categories, see 4.5.4 and 4.5.5;
- connections with small fibrations, see 4.5.8;
- their role in a fibred version of an adjoint functor theorem, see 4.5.11.

The results are used only in section 5.2. We stress that the material presented below is standard (except perhaps 4.5.4 and 4.5.10).

The notion to be introduced next comes from Bénabou [1975], see also Bénabou [1985]. A few different formulations are available. We start with the one below because it is clearly intrinsic.

4.5.1. Definition. A fibration $p : \mathbf{E} \to \mathbf{B}$ is *locally small* if for each $A \in \mathbf{B}$ and $E, E' \in \mathbf{E}_A$ one can find two morphisms $\xi : E_0 \to E$, $\xi' : E_0 \to E'$ in $\mathbf{E}$ with ξ cartesian over $p(\xi')$ such that for every pair $f : D \to E$, $f' : D \to E'$ with f cartesian over $p(f')$, there is a unique $\phi : D \to E_0$ with $\xi \circ \phi = f$ and $\xi' \circ \phi = f'$. In a

diagram,

$$\begin{tikzcd}
D \arrow[rr, dashed, "\phi"] \arrow[dr, "f"] \arrow[ddr, "f'"'] & & E_0 \arrow[dl, "\xi"'] \arrow[ddl, "\xi'"] \\
& E & \\
& E' &
\end{tikzcd}$$

This ϕ is then necessarily cartesian. A suggestive notation for the arrow $p(\xi) = p(\xi')$ in $\mathbf{B}$ is $\pi_0 : \underline{\mathrm{Hom}}_A(E, E') \to A$.

We immediately mention an equivalent formulation; it involves representability of the hom-sets in the fibres and thus explains the name "locally small". The proof is easy and left to the reader.

4.5.2. Lemma. *Let $p : \mathbf{E} \to \mathbf{B}$ be a cloven fibration; p is locally small if and only if for each $A \in \mathbf{B}$ and $E, E' \in \mathbf{E}_A$, the functor $(\mathbf{B}/A)^{op} \to \mathbf{Ens}$ given by*

$$B \xrightarrow{u} A \quad \mapsto \quad \mathbf{E}_B\,(\,u^*(E),\ u^*(E')\,)$$

is representable — where $\mathbf{Ens}$ is a suitably large universe.

More explicitly, a morphism $\pi_0 : \underline{\mathrm{Hom}}_A(E, E') \to A$ in $\mathbf{B}$ together with a vertical $\pi_1 : \pi_0^*(E) \to \pi_0^*(E')$ in $\mathbf{E}$ should exist such that for every $u : B \to A$ in $\mathbf{B}$ and vertical $f : u^*(E) \to u^*(E')$, there is a unique $v : B \to \underline{\mathrm{Hom}}_A(E, E')$ making the following two diagrams commute.

$$\begin{tikzcd}
B \arrow[rr, "v"] \arrow[dr, "u"'] & & \underline{\mathrm{Hom}}_A(E, E') \arrow[dl, "\pi_0"] \\
& A &
\end{tikzcd} \qquad \begin{tikzcd}
u^*(E) \arrow[r, dashed] \arrow[d, "f"'] & \pi_0^*(E) \arrow[d, "\pi_1"] \\
u^*(E') \arrow[r, dashed] & \pi_0^*(E')
\end{tikzcd}$$

where the dashed arrows are the unique ones over v. $\square$

4.5.3. Examples. (i) The fibration $Fam(\mathbf{C}) \to \mathbf{Sets}$ is locally small iff $\mathbf{C}$ is locally small (i.e. has small hom-sets). As to the if-part, for I-indexed collections $\{X_i\}$ and $\{X'_i\}$ one finds appropriate maps $\pi_0 : \dot{\bigcup}_{i \in I}.\mathbf{C}(X_i, X'_i) \to I$ in $\mathbf{Sets}$ and $\pi_1 : \{X_i\}_{(i,f)} \to \{X'_i\}_{(i,f)}$ in $Fam(\mathbf{C})$ over $\dot{\bigcup}_{i \in I}.\mathbf{C}(X_i, X'_i)$, the latter described by $\boldsymbol{\lambda}(i, f).\,f$.

The only-if-part is obtained by looking at the fibre above the terminal object $t = \{\emptyset\}$ in $\mathbf{Sets}$. For $X, X' \in \mathbf{C}$ one obtains a *set* A as domain of the π_0 belonging to $\{X\}, \{X'\}$ considered as objects of $Fam(\mathbf{C})_t$. It satisfies

$$A \;\cong\; \mathbf{Sets}/t\,(id_t, \pi_0) \;\cong\; Fam(\mathbf{C})_t(\{X\}, \{X'\}) \;\cong\; \mathbf{C}(X, X').$$

(ii) Let $\mathbf{B}$ be a category with finite limits. Then

$$cod : \mathbf{B}^{\rightarrow} \rightarrow \mathbf{B} \text{ is locally small } \Leftrightarrow \mathbf{B} \text{ is an LCCC.}$$

For $u : B \rightarrow A$ and $f, f' \in \mathbf{B}/A$ one has

$$\mathbf{B}/B\,(u^*(f),\ u^*(f')) \;\cong\; \mathbf{B}/A\,(\Sigma_u.u^*(f),\ f') \;\cong\; \mathbf{B}/A\,(u \times f,\ f').$$

Hence the LHS has a representing object iff the RHS has one, i.e. *cod* is locally small iff all slices $\mathbf{B}/A$ are CCC's.

(iii) Every small fibration is locally small. For a fibration of the form $\sum(\mathbf{C}) \rightarrow \mathbf{B}$ where $\mathbf{C}$ is internal in $\mathbf{B}$, one takes for $A \in \mathbf{B}$ and objects $X, X' : A \rightarrow C_0$ above A, the following pullback.

$$\begin{tikzcd}
\underline{\mathrm{Hom}}_A(X, X') \arrow[r, "\pi_1"] \arrow[d, "\pi_0"'] \arrow[dr, phantom, very near start, "\lrcorner"] & C_1 \arrow[d, "\langle \partial_0, \partial_1 \rangle"] \\
A \arrow[r, "\langle X, X' \rangle"] & C_0 \times C_0
\end{tikzcd}$$

We presuppose that all such pullbacks exist in the base category. It is left to the reader to check that being locally small is an essentially categorical property, i.e. one which is preserved under equivalence.

Part of the relevance of locally small fibrations lies in their relation to comprehension categories. The next result is as one would expect, given the idea behind comprehension categories.

4.5.4. Proposition. *Let $p : \mathbf{E} \rightarrow \mathbf{B}$ be a locally small fibration. There is then a "Hom"-comprehension category of the following form.*

$$\begin{tikzcd}
\bullet \arrow[rr, "\mathcal{P}"] \arrow[dr, "p^{op} \times p"'] & & \mathbf{B}^{\rightarrow} \arrow[dl, "cod"] \\
& \mathbf{B} &
\end{tikzcd}$$

See 1.1.11 for the opposite of a fibration.

Proof. The domain of $p^{op} \times p$ is the category with pairs E, E' where $pE = pE'$ as objects. One takes $\mathcal{P}(E, E') = \pi_0 : \underline{\mathrm{Hom}}_{pE}(E, E') \rightarrow pE$. Arrows $(E, E') \rightarrow (D, D')$

are pairs $[f_1, f_2] : E \rightarrow D$ in $\mathbf{E}^{(op)}$ and $g : E' \rightarrow D'$ in $\mathbf{E}$ with $pf_1 = pg = u$, say. We construct

$$\begin{tikzcd}
\pi_0^* u^*(D) \arrow[r] \arrow[d, "\pi_0^*(f_2)"'] & u^*(D) \arrow[r, "f_1"] \arrow[d, "f_2"'] & D \\
\pi_0^*(E) \arrow[r] \arrow[d, "\pi_1"'] & E \arrow[ur, "{[f_1, f_2]}"'] & \\
\pi_0^*(E') \arrow[r] \arrow[d, "\pi_0^*(g')"'] & E' \arrow[d, "g'"'] \arrow[dr, "g"] & \\
\pi_0^* u^*(D') \arrow[r] & u^*(D') \arrow[r] & D'
\end{tikzcd}$$

Hence one obtains a vertical arrow $(u \circ \pi_0)^*(D) \rightarrow (u \circ \pi_0)^*(D')$. It determines a unique map $v : \underline{\mathrm{Hom}}_{pE}(E, E') \rightarrow \underline{\mathrm{Hom}}_{pD}(D, D')$ with $\mathcal{P}(D, D') \circ v = u \circ \mathcal{P}(E, E')$. Hence we put $\mathcal{P}([f_1, f_2], g) = (u, v)$. $\square$

We recall from section 4.1 that a comprehension category has a unit if the fibrewise global sections functors are representable. In the presence of fibrewise exponents, one easily sees that this is equivalent to representability of fibred hom-sets. This is the content of the next result, see also Pavlović [1990].

4.5.5. Proposition. *Let $p : \mathbf{E} \rightarrow \mathbf{B}$ be a fibred CCC. Then*

$$\begin{array}{ccl} p \textit{ is locally small} & \Leftrightarrow & \textit{there is a comprehension category with unit} \\ & & \mathcal{P} : \mathbf{E} \rightarrow \mathbf{B}^{\rightarrow} \textit{ such that } p = cod \circ \mathcal{P} \end{array}$$

Proof. Let $1 : \mathbf{B} \rightarrow \mathbf{E}$ describe the fibred terminal object.
($\Rightarrow$) For $E \in \mathbf{E}$ above $A \in \mathbf{B}$, put $\mathcal{P}E = \pi_0 : \underline{\mathrm{Hom}}_A(1A, E) \rightarrow A$ in $\mathbf{B}$. Then

$$\begin{array}{rcll}
\mathbf{E}(1B, E) & \cong & \dot{\bigcup}_{u : B \rightarrow A}.\ \mathbf{E}_B\,(1B,\ u^*(E)\,) & \\
& \cong & \dot{\bigcup}_{u : B \rightarrow A}.\ \mathbf{E}_B\,(u^*(1A),\ u^*(E)\,) & \\
& \cong & \dot{\bigcup}_{u : B \rightarrow A}.\ \mathbf{B}/A\,(u,\ \mathcal{P}E) & \\
& \cong & \mathbf{B}(B,\ \mathcal{P}_0 E) & \text{where } \mathcal{P}_0 E = dom(\mathcal{P}E).
\end{array}$$

($\Leftarrow$) For $E, E' \in \mathbf{E}$ above $A \in \mathbf{B}$ one has for $u : B \rightarrow A$ in $\mathbf{B}$,

$$\begin{array}{rcll}
\mathbf{E}_B\,(u^*(E),\ u^*(E')\,) & \cong & \mathbf{E}_B\,(1B,\ u^*(E) \Rightarrow u^*(E')\,) & \\
& \cong & \mathbf{E}_B\,(1B,\ u^*(E \Rightarrow E')\,) & \\
& \cong & \mathbf{B}/A\,(u,\ \mathcal{P}(E \Rightarrow E')\,) & \text{see 4.1.10 (ii).}
\end{array}$$

Hence $\mathcal{P}(E \Rightarrow E')$ is an appropriate representing arrow. $\square$

In case one has pullbacks in the base category, another description of a locally small fibration can be given; it may be found e.g. in Johnstone [1977], A2.

4.5.6. Lemma. *Let $p : \mathbf{E} \to \mathbf{B}$ be a cloven fibration on a basis $\mathbf{B}$ with pullbacks. Then p is locally small if and only if for all $A, A' \in \mathbf{B}$ and $E \in \mathbf{E}_A$, $E' \in \mathbf{E}_{A'}$, the functor $(\mathbf{B}/A \times A')^{op} \to \mathbf{Ens}$ given by*

$$B \xrightarrow{u} A \times A' \quad \mapsto \quad \mathbf{E}_B\big((\pi \circ u)^*(E),\ (\pi' \circ u)^*(E')\big)$$

is representable (where π, π' are cartesian projections). $\square$

The next result is due to Penon [1974], see also Johnstone [1977], A6.

4.5.7. Theorem. *Let $p : \mathbf{E} \to \mathbf{B}$ be a locally small fibration on a basis $\mathbf{B}$ with pullbacks. Then*

(i) *Every object $E \in \mathbf{E}$ determines an internal category $\mathit{Full}(E)$ in $\mathbf{B}$ together with a full and faithful cartesian functor $\widehat{E}$ from the externalization $\sum(\mathit{Full}(E))$ to $\mathbf{E}$.*

(ii) *Let $\mathbf{C}$ be an internal category in $\mathbf{B}$. Every cartesian functor $F : \sum(\mathbf{C}) \to \mathbf{E}$ has an up-to-isomorphism unique factorization*

$$\begin{tikzcd} \sum(\mathbf{C}) \arrow[r, "{[G]}"] \arrow[dr] & \sum(\mathit{Full}(E)) \arrow[r, "\widehat{E}"] \arrow[d] & \mathbf{E} \arrow[dl, "p"] \\ & \mathbf{B} & \end{tikzcd}$$

where $G : \mathbf{C} \to \mathit{Full}(E)$ is an internal functor which is the identity on objects.

Proof. (i) Write $\Omega_0 = pE$ and $\langle \partial_0, \partial_1 \rangle : \Omega_1 \to \Omega_0 \times \Omega_0$ for the representing arrow obtained by the previous lemma from the pair E, E. The identity on E yields a map $i : \langle id, id \rangle \to \langle \partial_0, \partial_1 \rangle$ in $\mathbf{B}/\Omega_0 \times \Omega_0$. Similarly, one obtains internal composition.

The object-part of $\widehat{E} : \sum(\mathit{Full}(E)) \to \mathbf{E}$ is defined by $[X : A \to \Omega_0] \mapsto X^*(E)$. Then

$$\begin{array}{rcl} \sum(\mathit{Full}(E))\big(A \xrightarrow{X} \Omega_0,\ B \xrightarrow{Y} \Omega_0\big) & \cong & \dot{\bigcup}_{u : A \to B}.\ \sum(\mathit{Full}(E))_A\big(A \xrightarrow{X} \Omega_0,\ A \xrightarrow{Y \circ u} \Omega_0\big) \\ & = & \dot{\bigcup}_{u : A \to B}.\ \mathbf{B}/\Omega_0 \times \Omega_0\big(\langle X, Y \circ u \rangle,\ \langle \partial_0, \partial_1 \rangle\big) \\ & \cong & \dot{\bigcup}_{u : A \to B}.\ \mathbf{E}_A\big(X^*(E),\ u^*Y^*(E)\big) \\ & \cong & \mathbf{E}\big(\widehat{E}(X),\ \widehat{E}(Y)\big). \end{array}$$

Hence $\widehat{E}$ can be extended to a full and faithful functor.

(ii) Take $E = F(id_{C_0}) \in \mathbf{E}_{C_0}$ and $G_0 = id_{C_0} : C_0 \to pE = \Omega_0$. The map $G_1 : C_1 \to \Omega_1$ is obtained from $F(id_{C_1}) : F(\partial_0) \to F(\partial_1)$ using that

$$\begin{array}{rcll} \mathbf{E}_{C_0}\big(F(\partial_0),\ F(\partial_1)\big) & = & \mathbf{E}_{C_0}\big(F(\partial_0^*(id_{C_0})),\ F(\partial_1^*(id_{C_0}))\big) & \\ & \cong & \mathbf{E}_{C_0}\big(\partial_0^*(E),\ \partial_1^*(E)\big) & \text{since } F \text{ is cartesian} \\ & \cong & \mathbf{B}/\Omega_0 \times \Omega_0\big(\langle \partial_0, \partial_1 \rangle,\ \langle \partial_0^E, \partial_1^E \rangle\big) & \text{by definition of } \langle \partial_0^E, \partial_1^E \rangle. \end{array}$$

Then indeed,

$$\{\widehat{E} \circ [G]\}(A \xrightarrow{X} C_0) \ = \ X^*(E) \ = \ X^*(F(id_{C_0})) \cong F(X^*(id_{C_0})) \ = \ F(X).$$

If also $D \in \mathbf{E}_{C_0}$ yields a diagram as above, then

$$D \ \cong \ id_{C_0}^*(D) \ = \ \widehat{D}(id_{C_0}) \ = \ (\widehat{D} \circ [G])(id_{C_0}) \ \cong \ F(id_{C_0}) = E. \ \square$$

4.5.8. Corollary. *On a basis with pullbacks, one has*

a fibration is small iff it is locally small and has a generic object.

Proof. The only-if-part follows from 4.5.3 (iii) and the remark following definition 1.4.4. Hence we only consider the if-part. Let $p : \mathbf{E} \to \mathbf{B}$ be a locally small fibration with a generic object $T \in \mathbf{E}$. One obtains an internal category $\mathit{Full}(T)$ in $\mathbf{B}$ provided with a full and faithful functor $\sum(\mathit{Full}(T)) \to \mathbf{E}$ by $u \mapsto u^*(T)$. The latter is essentially surjective on objects because T is a generic object, see definition 1.2.9. Hence it is a (weak) equivalence. $\square$

The above result comes from Bénabou [1975]. In somewhat different formulation, it also occurs in Paré and Schumacher [1978], II, theorem 3.11.1.

4.5.9. Remarks. (i) Close inspection of the constructions above reveals that one does not need the existence of all pullbacks in the base category or of all representing arrows. Hence the results can be obtained if enough of these are around. This is used in the proof of theorem 3.3.3.

(ii) As a special case of the construction in theorem 4.5.7 (i), one can start from an LCCC $\mathbf{B}$ (see 4.5.3 (ii)) and an arrow τ in $\mathbf{B}$, say with codomain Ω, see e.g. Johnstone [1977], 2.38 or Pitts [1987], 3.2. An internal category $\mathit{Full}(\tau)$ in $\mathbf{B}$ is obtained, where $\langle \partial_0, \partial_1 \rangle$ is the representing arrow corresponding to the pair $\pi^*(\tau)$, $\pi'^*(\tau) \in \mathbf{B}/\Omega \times \Omega$ obtained by pullbacks. Viewed a bit differently, $\langle \partial_0, \partial_1 \rangle$ is the "local exponential" $\pi^*(\tau) \Rightarrow \pi'^*(\tau)$, see 4.5.3 (ii). This $\mathit{Full}(\tau)$ is called a *full internal subcategory of* $\mathbf{B}$, because it comes equipped with a full and faithful functor $\sum(\mathit{Full}(\tau)) \to \mathbf{B}^{\to}$. The latter is of course a full comprehension category.

A bit more subtle, one can speak in the spirit of the first remark about a full internal subcategory of an arbitrary ambient category $\mathbf{B}$, provided there is enough structure around to perform the relevant constructions. In terms of comprehension categories, there is an alternative description.

4.5.10. Definition. (i) Assume $\mathbf{C} \in \mathit{Cat}(\mathbf{B})$; $\mathbf{C}$ will be called a *full internal subcategory of* $\mathbf{B}$ if there is a full comprehension category $\mathcal{P}$ of the following form,

$$\begin{tikzcd} \sum(\mathbf{C}) \arrow[rr, "\mathcal{P}"] \arrow[dr, "{[\mathbf{C}]}"'] & & \mathbf{B}^{\to} \arrow[dl, "cod"] \\ & \mathbf{B} & \end{tikzcd}$$

which preserves fibred terminal objects (if any).

(ii) A *full small* fibration is a fibration which is equivalent to the externalization $[\mathbf{C}]$ of a full internal subcategory $\mathbf{C}$.

Indeed, given such a full internal subcategory, the relevant pullbacks and local exponential (as in 4.5.9 (ii)) exist: put $\tau = \mathcal{P}(id_{C_0}) \in \mathbf{B}/C_0$. Then $\langle \partial_0, \partial_1 \rangle$ is the local exponential $\pi^*(\tau) \Rightarrow \pi'^*(\tau)$, since for $u\colon A \to C_0 \times C_0$ one has

$$\begin{array}{rcl} \mathbf{B}/C_0 \times C_0\big(\, u,\ \langle \partial_0, \partial_1 \rangle \,\big) & = & \sum(\mathbf{C})_A\big(\, \pi \circ u,\ \pi' \circ u \,\big) \\ & = & \sum(\mathbf{C})_A\big(\, u^*\pi^*(id_{C_0}),\ u^*\pi'^*(id_{C_0}) \,\big) \\ & \cong & \mathbf{B}/A\big(\, \mathcal{P}(u^*\pi^*(id_{C_0})),\ \mathcal{P}(u^*\pi'^*(id_{C_0}))\,\big) \\ & \cong & \mathbf{B}/A\big(\, u^*\pi^*(\mathcal{P}(id_{C_0})),\ u^*\pi'^*(\mathcal{P}(id_{C_0}))\,\big) \\ & \cong & \mathbf{B}/C_0 \times C_0\big(\, \Sigma_u.u^*\pi^*(\tau),\ \pi'^*(\tau) \,\big) \\ & \cong & \mathbf{B}/C_0 \times C_0\big(\, u \times \pi^*(\tau),\ \pi'^*(\tau) \,\big) \end{array}$$

Now suppose $\mathbf{C}$ has an internal terminal object. There is then a terminal object functor $1\colon \mathbf{B} \to \sum(\mathbf{C})$ which satisfies by assumption $\mathcal{P}1 \cong id_{(-)}$. Then $\mathcal{P}$ is a comprehension category with unit, since

$$\begin{array}{rcll} \sum(\mathbf{C})\,(1A,\ X) & \cong & \mathbf{B}^{\to}\,(\mathcal{P}1A,\ \mathcal{P}X) & \\ & \cong & \mathbf{B}^{\to}\,(id_A,\ \mathcal{P}X) & \\ & \cong & \mathbf{B}\,(A,\ \mathcal{P}_0X) & \qquad \text{using } id_{(-)} \dashv dom. \end{array}$$

As remarked after lemma 4.1.10, $\mathcal{P}$ is then a continuous functor. It is in fact the internal global sections functor, analogously to 4.1.6 (ii). Hyland [1989], 0.1 uses this description to define full internal subcategories.

Using the above terminology, one can say that if $p\colon \mathbf{E} \to \mathbf{B}$ is a $\lambda{\to}$-category, then the total category $\mathbf{E}$ contains a full internal subcategory, see 3.3.3. The comprehension category involved there is $\overline{\mathcal{P}}\colon \overline{\mathbf{E}} \to \mathbf{E}^{\to}$, see 4.4.4 (ii).

Finally, we mention without proof an adjoint functor theorem. It is basically a translation of theorem 1.9 in Paré and Schumacher [1978], IV.

4.5.11. Theorem. *Suppose G is a cartesian functor in*

$$\begin{tikzcd} \mathbf{E} \arrow[rr, "G"] \arrow[dr, "p"'] & & \mathbf{D} \arrow[dl, "q"] \\ & \mathbf{B} & \end{tikzcd}$$

where p and q are locally small fibrations. Further, suppose that p is complete and that G is continuous (i.e. preserves the fibred finite limits and products). Then

G has a fibred left adjoint iff

the following solution set condition is satisfied:

for every $A \in \mathbf{B}$ and $D \in \mathbf{D}_A$ there are objects $B \in \mathbf{B}$ and $E \in \mathbf{E}_B$ such that for every $E' \in \mathbf{E}_A$ and vertical $f\colon D \to GE'$, one can find

- *$u\colon A \to B$ in $\mathbf{B}$*
- *$\alpha\colon D \to G(u^*(E))$ in $\mathbf{D}_A$*
- *$g\colon u^*(E) \to E'$ in $\mathbf{E}_A$*

such that $G(g) \circ \alpha = f$. □

Chapter 5

Applications

In this final chapter the type theoretical and categorical lines come together. In the first section the main ideas of how to translate type theoretical settings and features into categorical ones are described. The subsequent three sections work out the details for the calculi CC, HML and λHOL together with λPRED. The part about HML is borrowed from Jacobs, Moggi & Streicher [1991].

The last section is about the untyped λ-calculus. It can be considered as spin-off: using that "untyped" can be understood as "typed with only one type" we are lead to use monoid constant comprehension categories for the semantics of the untyped λ-calculus. As main new result we obtain an adjunction between categorical and set theoretical λ-algebras (see theorem 5.5.10).

5.1 From type theory to category theory

The theory developed in the previous chapters allows us to construct for a given type theoretical setting a categorical one; it will consist of fibrations and comprehension categories suitably linked together. Next we can show how type theoretical features (on top of a certain setting) correspond to categorical ones (on top of the translated setting).

This section will consist of two parts: the first one about the translation of settings and the second one about the translation of features. The second part will be a bit shorter; more extensive expositions of a number of examples can be found in the other sections of this chapter.

Settings and features in type theory can be found in chapter 2.

5.1.1. Translation of settings. As stressed in chapters 2 and 3, a (type theoretical) setting determines the organization of contexts. In the translation below, we therefore loosely speak about objects of a certain category as "contexts". More precisely, we speak about $s_1, \ldots, s_n$-contexts when these contexts contain s_i-types (for all i). This makes sense, since one cannot always separate contexts into more simple ones consisting of types of a single sort. For example, in a setting with $s_1 \succ s_2$ and $s_2 \succ s_1$ contexts will consist of alternating sequences of s_1- and s_2-types.

The translation of settings follows the next four guidelines.

(1.1) Every sort s requires a separate full comprehension category $\mathcal{P}(s)$ with a terminal object in the basis. Objects of the base category are to be understood as contexts containing s'-types for $s \succ s'$. Morphisms between such contexts are "substitutions", i.e. sequences of terms. Objects in the total category are s-types. Morphisms in the fibres are then single terms between such s-types.

(1.2) If there is no s-type dependency, i.e. $s \not\succ s$, then $\mathcal{P}(s)$ is required to be *constant*, i.e. to be of the form $Cons_{T(s)}: \mathbf{B}//T(s) \to \mathbf{B}^{\to}$, where $T(s) \subseteq Obj(\mathbf{B})$ is the collection of s-types, see 4.1.3. The base category $\mathbf{B}$ is required to have finite products.

Implicitly, we require that both these points yield comprehension categories *over* appropriate fibrations. These fibrations are determined by the rest of the structure.

(2.1) If s_0, s are two different sorts with s not depending on s_0, i.e. $s \not\succ s_0$, then we require a fibration from a category of s, s_0-contexts to a category of s-contexts. Such a fibration should have a terminal object functor, which describes empty s_0-contexts.

(2.2) If $s \not\succ s_0$ as above, but also $s_0 \not\succ s$, then the fibration described before should be *constant*, i.e. of the form $Fst: \mathbf{A} \times \mathbf{B} \to \mathbf{A}$ (or $Snd: \mathbf{A} \times \mathbf{B} \to \mathbf{B}$, depending on how one starts).

In these latter two principles one may read for s also a sequence of sorts.

These points guide the constructions below: the examples should make clear how to apply them. For example, what to take as a base for the whole categorical setting can be discovered by inspection of the dependencies: one should start with the sort(s) which do not depend on any other. The pictures of the categories of contexts roughly follow the ordering $\succ$: if $s_1 \succ s_2$ then s_1-contexts will be fibred over s_2-contexts; $\succ$-cycles shrink to a single category. We hope that such details will become clear as we proceed.

The minimal setting $Sort = \{*\}$ with $\prec\, = \emptyset$.

Using (1.1) and (1.2) we obtain a constant comprehension category

$$\begin{tikzcd} \mathbf{B}//T(*) \arrow[r, shift left] \arrow[r, shift right] \arrow[r, phantom, "\Downarrow Cons_{T(*)}"] & \mathbf{B} \end{tikzcd}$$

where $\mathbf{B}$ is a category with finite products. Here one does not really need that this comprehension category is over a fibration (in a degenerate sense though, it can be understood as a comprehension category over the fibration from $\mathbf{B}$ to the terminal category).

In the beginning of chapter 3 (before comprehension categories were introduced) we said that such cartesian categories $\mathbf{B}$ formed the appropriate setting. The above picture is slightly more precise. A term model example of this setting may be found in the beginning of section 3.1. There a base category $\mathbf{B}$ of contexts is formed. For $T(*) \subseteq Obj(\mathbf{B})$ one takes the collection of $*$-types (where a type σ is identified with the singleton context $x{:}\,\sigma$).

The propositions as types setting $Sort = \{*\}$ with $* \succ *$.

An appropriate setting consists of a structure of the form

$$\begin{tikzcd} \mathbf{E} \arrow[r, shift left] \arrow[r, shift right] \arrow[r, phantom, "\Downarrow \mathcal{P}"] & \mathbf{B} \end{tikzcd}$$

where $\mathcal{P}$ is a full comprehension category with a terminal object in the base category $\mathbf{B}$, see (1.1). A term model example of this setting is described after 4.1.2.

The propositional setting $Sort = \{*, \Box\}$ with $* \succ \Box$.

The starting point is a base category $\mathbf{B}$ for the $\Box$-contexts. It forms by (1.2) the basis for a constant comprehension category $Cons_{T(\Box)}$. On top of $\mathbf{B}$ one has by (2.1) a fibration $p: \mathbf{E} \to \mathbf{B}$ from $\Box, *$-contexts to $\Box$-contexts. Finally, $\mathbf{E}$ forms the basis for a constant comprehension category $Cons_{T(*)}: \mathbf{E}//T(*) \to \mathbf{E}^{\to}$ over p, see 4.4.4 (i). In a diagram,

$$\begin{tikzcd} \mathbf{E}//T(*) \arrow[r, shift left] \arrow[r, shift right] \arrow[r, phantom, "\Downarrow Cons_{T(*)}"] & \mathbf{E} \arrow[d, shift right, "p"'] \\ \mathbf{B}//T(\Box) \arrow[r, shift left] \arrow[r, shift right] \arrow[r, phantom, "\Downarrow Cons_{T(\Box)}"] & \mathbf{B} \arrow[u, shift right, "1"'] \end{tikzcd}$$

Here, $\mathbf{B}$ is a category with finite products and $p: \mathbf{E} \to \mathbf{B}$ is a fibration with finite products. Only this part was presented as constituting the propositional setting in chapter 3. However there was a warning that it formed a simplified version, see 3.2.1. The above picture forms the appropriate refinement.

A term model example $p: \mathbf{E} \to \mathbf{B}$ for this setting has been described at the end of section 3.1. For $T(\Box) \subseteq Obj(\mathbf{B})$ and $T(*) \subseteq Obj(\mathbf{E})$ one takes the collection of $\Box$- resp. $*$-types.

The setting $Sort = \{*, \Box\}$ with $* \succ \Box$, $* \succ *$, $\Box \succ \Box$.

Basically the same analysis as before applies, except that the relevant comprehension categories are not constant. Hence one obtains

$$\begin{tikzcd}
\mathbf{E} \arrow[r, shift left] \arrow[r, shift right] \arrow[r, phantom, "\Downarrow \mathcal{P}(*)"] & \mathbf{B} \arrow[d, shift right, "r"'] \\
\mathbf{D} \arrow[r, shift left] \arrow[r, shift right] \arrow[r, phantom, "\Downarrow \mathcal{P}(\Box)"] & \mathbf{A} \arrow[u, shift right, "1"']
\end{tikzcd}$$

where $\mathcal{P}(*)$ is a comprehension category over r. A term model for this setting may be found in Jacobs, Moggi & Streicher [1991].

The setting $Sort = \{*, \Box\}$ with $* \succ \Box,\ * \succ *,\ \Box \succ \Box,\ \Box \succ *$.

Obviously an appropriate categorical setting will consist of two comprehension categories: one for $*$ and one for $\Box$. Since these sorts are mutually depending on each other, their contexts cannot be separated. Hence the picture looks like this.

$$\begin{tikzcd}
\mathbf{E} \arrow[r, shift left] \arrow[r, shift right] \arrow[r, phantom, "\Downarrow \mathcal{P}(*)"] & \mathbf{B} & \mathbf{D} \arrow[l, shift left] \arrow[l, shift right] \arrow[l, phantom, "\mathcal{P}(\Box) \Downarrow"]
\end{tikzcd}$$

where $\mathcal{P}(*)$ and $\mathcal{P}(\Box)$ are full comprehension categories and $\mathbf{B}$ is a category with a terminal object. In terms of display categories, one has a base category with two collections $\{\mathcal{P}(*)(E) \mid E \in \mathbf{E}\}$ and $\{\mathcal{P}(\Box)(D) \mid D \in \mathbf{D}\}$ of display maps.

The setting $Sort = \{*, \Box, \triangle\}$ with $* \succ \Box,\ \Box \succ \triangle,\ * \succ \triangle$.

One starts with a base category $\mathbf{A}$ of $\triangle$-contexts. On top, there should be two fibrations: one from $\triangle, \Box$-contexts to $\triangle$-contexts and one from $\triangle, \Box, *$-contexts to $\triangle, \Box$-contexts, using (2.1) twice. By composition one obtains a new fibration from $\triangle, \Box, *$-contexts to $\triangle$-contexts. The latter corresponds to the transitivity requirement imposed on the dependency relation in definition 2.1.1. Because there are no dependencies of the form $s \succ s$, the three comprehension categories involved are constant.

$$\begin{tikzcd}
\mathbf{E}//T(*) \arrow[r, shift left] \arrow[r, shift right] \arrow[r, phantom, "\Downarrow Cons_{T(*)}"] & \mathbf{E} \arrow[d, shift right, "p"'] \\
\mathbf{B}//T(\Box) \arrow[r, shift left] \arrow[r, shift right] \arrow[r, phantom, "\Downarrow Cons_{T(\Box)}"] & \mathbf{B} \arrow[u, shift right, "\top"'] \arrow[d, shift right, "r"'] \\
\mathbf{A}//T(\triangle) \arrow[r, shift left] \arrow[r, shift right] \arrow[r, phantom, "\Downarrow Cons_{T(\triangle)}"] & \mathbf{A} \arrow[u, shift right, "1"']
\end{tikzcd}$$

In this diagram $Cons_{T(\Box)}$ is a comprehension category over r and $Cons_{T(*)}$ is a comprehension category over p.

The setting $Sort = \{*, \triangle, \Box\}$ $* \succ \triangle,\ * \succ \Box$.

Since the two sorts $\triangle, \Box$ are mutually independent, they determine by (2.2) two constant fibrations $\mathbf{A} \xleftarrow{\mathrm{Fst}} \mathbf{A} \times \mathbf{B} \xrightarrow{\mathrm{Snd}} \mathbf{B}$, where $\mathbf{A}$ contains $\Box$-contexts and $\mathbf{B}$ the $\triangle$-contexts. On top of $\mathbf{A} \times \mathbf{B}$ one has a fibration from $\Box, \triangle, *$-contexts to $\Box, \triangle$-contexts. Finally one has three constant comprehension categories:

$$\begin{tikzcd}
 & & \mathbf{E}//T(*) \arrow[r, shift left] \arrow[r, shift right] \arrow[r, phantom, "\Downarrow Cons_{T(*)}"] & \mathbf{E} \arrow[d, shift right, "p"'] & \\
 & & & \mathbf{A} \times \mathbf{B} \arrow[u, shift right, "1"'] \arrow[dll, shift right, "Fst"'] \arrow[ddr, shift left, "Snd"] & \\
\mathbf{A}//T(\Box) \arrow[r, shift left] \arrow[r, shift right] \arrow[r, phantom, "\Downarrow Cons_{T(\Box)}"] & \mathbf{A} \arrow[urr, shift right] & & & \\
 & & & \mathbf{B}//T(\triangle) \arrow[r, shift left] \arrow[r, shift right] \arrow[r, phantom, "\Downarrow Cons_{T(\triangle)}"] & \mathbf{B} \arrow[uul, shift left]
\end{tikzcd}$$

Here, $Cons_{T(*)}$ is a comprehension category over p.

5.1.2. Translation of features. Type theoretical features require a certain setting as background. Similarly for categorical features. This makes it difficult to describe them uniformily. We mention the three main guidelines. Afterwards a few exemplaric cases are described.

(3.1) The feature (s_1, s_2)-quantification corresponds to the requirement that the comprehension category $\mathcal{P}(s_2)$ has both

- $\mathcal{P}(s_1)$-products and (strong) $\mathcal{P}(s_1)$-sums, in case $\mathcal{P}(s_1)$ and $\mathcal{P}(s_2)$ have the same base category. Else, one first has to perform change-of-base on $\mathcal{P}(s_1)$ along a suitable fibration r connecting the two base categories; the requirement then is that $\mathcal{P}(s_2)$ has $r^*(\mathcal{P}(s_1))$-products and (strong) $r^*(\mathcal{P}(s_1))$-sums. Hence in this case one first has to move $\mathcal{P}(s_1)$ "upwards"
  One requires *strong* sums if $s_2 \succ s_2$, see the first stipulation about quantification in the beginning of section 2.3.
- $\mathcal{P}(s_2)$ has a unit; this requirement is a result of the second stipulation about quantification in section 2.3.

(3.2) The axiom feature $s_1 : s_2$ is described by a generic object: there should be an object Ω in the total category of $\mathcal{P}(s_2)$ above the terminal such that — in case the base categories are the same — the fibration $cod \circ \mathcal{P}(s_1)$ has a generic object above $\mathcal{P}(s_1)_0(\Omega)$. In case the base categories don't match, one first has to perform change-of-base on the fibration $cod \circ \mathcal{P}(s_1)$ along a suitable terminal object functor connecting the base categories. Hence in this case one first has to move $cod \circ \mathcal{P}(s_1)$ "upwards".

(3.3) The feature s-closure corresponds to the requirement that

- $Cons_{T(s)}$ is a CCompC, if $s \not\succ s$.
- $cod \circ \mathcal{P}(s)$ is a fibred CCC, if $s \succ s$.

The second point is of minor relevance: it seems a bit strange to require only s-closure in a setting with $s \succ s$.

In (3.1) one can see the advantage of the double role that comprehension categories play: at one time as a "model" and at another time as domain of quantification. It enables this high level description of the quantification rules.

Probably a good example to start with is the structure

$$\begin{tikzcd}
\mathbf{E} \arrow[r, "\mathcal{P}(*)"] \arrow[dr, "p(*)"'] & \mathbf{B}^{\to} \arrow[d, "cod"] & \mathbf{D} \arrow[l, "\mathcal{P}(\Box)"'] \arrow[dl, "p(\Box)"] \\
& \mathbf{B} &
\end{tikzcd}$$

where $\mathcal{P}(*)$ and $\mathcal{P}(\Box)$ are full comprehension categories and $\mathbf{B}$ is a category with a terminal. As argued above, it forms a categorical version of the setting $Sort = \{*, \Box\}$ with $* \succ \Box$, $* \succ *$, $\Box \succ \Box$, $\Box \succ *$. Let's consider some relevant features, see section 2.3.

The (s_1, s_2)-quantification rule simply corresponds to the comprehension category $\mathcal{P}(s_2)$ having a unit and $\mathcal{P}(s_1)$-products and strong $\mathcal{P}(s_1)$-sums — where $s_1, s_2 \in \{*, \Box\}$.

Axioms are described by generic objects. In this case, $* : \Box$ corresponds to having an object $\Omega \in \mathbf{D}$ such that

- $p(\Box)(\Omega) \in \mathbf{B}$ is terminal;
- there is a generic object for $p(*)$ above $\mathcal{P}(\Box)_0(\Omega) \in \mathbf{B}$.

The inclusion (s_1, s_2) corresponds to the presence of a full and faithful functor $\mathcal{I} : \mathbf{E} \to \mathbf{D}$ forming a morphism of comprehension categories. Such structures with inclusion will be considered more closely in the next section.

A categorical version of the propositions as types setting $Sort = \{*\}$ with $* \succ *$ consists as we have seen several times now of a full comprehension category $\mathcal{P}(*) : \mathbf{E} \to \mathbf{B}^{\to}$ with a terminal object in the basis $\mathbf{B}$. By (3.1), $(*, *)$-quantification corresponds to $\mathcal{P}(*)$ being a closed comprehension category. The axiom $* : *$ corresponds to having an object $\Omega \in \mathbf{E}$ above the terminal together with a generic object above $\mathcal{P}(*)_0(\Omega)$. The feature $(*, *)$-identity types correspond to fibred equalizers (in the presence of strong sums, see lemma 2.2.14). In this way we find the notions of a λP1, $\lambda*$ and λPi-category as defined at the end of section 4.3.

In the minimal setting $Sort = \{*\}$ with $\prec = \emptyset$ one can only have the feature $*$-closure. This amounts to the requirement that the corresponding constant comprehension category of the form $Cons_{T(*)} : \mathbf{B}/\!/T(*) \to \mathbf{B}^{\to}$ is closed. Essentially this means that the collection of $*$-types $T(*)$ contains a unit and is closed under cartesian products and exponents, see 4.2.5 (iv).

The remaining systems will be considered in the next three sections. In 5.3 and 5.4 one may find examples where the change-of-base described in (3.1) and (3.2) is necessary.

5.2 CC-categories

In the previous section we already looked in some detail at the setting $Sort = \{*, \Box\}$ with $* \succ \Box$, $* \succ *$, $\Box \succ \Box$, $\Box \succ *$. It was argued that a corresponding categorical setting consists of two comprehension categories with the same basis. Categorical studies in the literature of such structures all assume the feature $(*, \Box)$-inclusion, motivated both by concrete examples and by the presence of this feature in early formulations of the calculus of constructions. Both on the type theoretical side and on the categorical side one has that the $(*, \Box)$-inclusion "transports" features: it enables more economical formulations since certain features result from others. The underlying categorical structure will now be depicted as

$$\begin{tikzcd}
\mathbf{E} \arrow[r, "\mathcal{I}"] \arrow[dr] & \mathbf{D} \arrow[r, "\mathcal{Q}"] \arrow[d] & \mathbf{B}^{\to} \arrow[dl, "cod"] \\
& \mathbf{B} &
\end{tikzcd}$$

where $\mathcal{I} : \mathbf{E} \to \mathbf{D}$ is the — full and faithful — $(*, \Box)$-inclusion functor and $\mathcal{Q}$ is a full comprehension category. As a result, $\mathcal{QI} : \mathbf{E} \to \mathbf{B}^{\to}$ is a full comprehension category again. Such a diagram underlies the work in Hyland and Pitts [1989].

The first definition below concerns *weak* CC-categories which have weak $(*, *)$ and $(\Box, *)$-sums, see section 2.3. Subsequent definitions will deal with ramifications.

After some examples and constructions we take a brief look at the role of small complete fibrations (cf. 4.2.4 and 4.2.5 (ii)). These have received much attention, especially in Hyland [1989]. Our own contribution in this section concerns the split topos model in example 5.2.6 (i) and a systematic presentation in terms of comprehension categories.

5.2.1. Definition. A *weak CC-category* is a structure $\mathbf{E} \xrightarrow{\mathcal{I}} \mathbf{D} \xrightarrow{\mathcal{Q}} \mathbf{B}^{\to}$ as above where

- $\mathcal{Q}$ is a CCompC, i.e. a closed comprehension category;
- $p = cod \circ \mathcal{QI} : \mathbf{E} \to \mathbf{B}$ is a fibration and $\mathcal{I}$ is a full and faithful cartesian functor (from p to $q = cod \circ \mathcal{Q}$) which has a fibred left adjoint;
- there is an object $\Omega \in \mathbf{D}$ such that $q\Omega \in \mathbf{B}$ is terminal and p has a generic object above $\mathcal{Q}_0\Omega \in \mathbf{B}$

This definition is quite compact and needs some unravelling; therefore we use lemma 4.2.13. By the reflection $\mathbf{E} \leftrightarrows \mathbf{D}$, the fibration p has a terminal object which is preserved by $\mathcal{I}$. Then $\mathcal{P} = \mathcal{QI} : \mathbf{E} \to \mathbf{B}^{\to}$ is a full comprehension category with unit. Again by the reflection, $\mathcal{P}$ has $\mathcal{Q}$-products and weak sums. Especially, $\mathcal{P}$ has ($\mathcal{P}$-) products and weak sums. Thus, the reflection yields all the structure of the calculus "weak CC", see section 2.3. The notion of a (weak) CC-category is essentially due to Hyland and Pitts [1989].

The first ramification we mention concerns strengthening the weak $(*, *)$ and $(\Box, *)$-sums. This cannot be done separately, see section 2.2, especially 2.2.10 and 2.2.11 (ii). In view of our stipulation to treat strong sums as the "normal" situation, we speak simply of a "CC-category" instead of a "strong CC-category". Again we use a compact formulation: only strong $(*, *)$-sums are required. By lemma 2.2.10 one obtains strongness of the $(\Box, *)$-sums as a result.

5.2.2. Definition. A *CC-category* is a weak CC-category $\mathbf{E} \xrightarrow{\mathcal{I}} \mathbf{D} \xrightarrow{\mathcal{Q}} \mathbf{B}^{\to}$ in which $\mathcal{P} = \mathcal{QI}$ is a CCompC.

5.2.3. Definition. A (weak) CC-category will be called *split* if the fibrations involved are split and all units, products and sums as well as the generic object are split.

In case one is willing to view a logic as a type theory in which propositions have at most one proof-object, the name introduced below makes sense.

5.2.4. Definition. Consider a (weak) CC-category $\mathbf{E} \xrightarrow{\mathcal{I}} \mathbf{D} \xrightarrow{\mathcal{Q}} \mathbf{B}^{\to}$ as defined above. It will be called *logical* if the "fibration of propositions" $p = cod \circ \mathcal{QI} : \mathbf{E} \to \mathbf{B}$ is a preorder (i.e. has preorder categories as fibres).

Notice that if in a weak CC-category $\mathbf{E} \xrightarrow{\mathcal{I}} \mathbf{D} \xrightarrow{\mathcal{Q}} \mathbf{B}^{\to}$ the functor $\mathcal{I}$ is an equivalence, $\mathcal{Q}$ becomes a $\lambda*$-category, i.e. a CCompC with a suitable generic object yielding a type of all types. The next notion covers the case when the other functor $\mathcal{Q}$ is an equivalence. It goes back to Ehrhard [1989].

5.2.5. Definition. A *dictos* is a weak CC-category $\mathbf{E} \xrightarrow{\mathcal{I}} \mathbf{D} \xrightarrow{\mathcal{Q}} \mathbf{B}^{\to}$ in which $\mathcal{Q}$ is an equivalence. The base category $\mathbf{B}$ is then an LCCC. In the sequel, we shall loosely speak about "a dictos $\mathbf{E} \to \mathbf{B}^{\to}$".

Later in this section the notion of a dictos will be investigated further.

5.2.6. Examples. (i) An easy example is obtained from a *topos* $\mathbf{B}$. As in 1.2.10 (ii) we write $Sub(\mathbf{B})$ for the full subcategory of $\mathbf{B}^{\to}$ consisting of monic arrows. One obtains a *logical CC-category* $Sub(\mathbf{B}) \hookrightarrow \mathbf{B}^{\to} \xrightarrow{Id} \mathbf{B}^{\to}$. The reflection comes from the fact that every morphism in a topos has a unique epi-mono factorization, see e.g. Johnstone [1977], 1.52.

Using the split CCompC's $\mathcal{Q} : \mathcal{F}(\mathbf{B}) \to \mathbf{B}^{\to}$ and $\mathcal{P} : \mathcal{L}(\mathbf{B}) \to \mathbf{B}^{\to}$ described in 4.3.5, one can improve the above topos example a bit by describing it as a *split* CC-category. The only thing left to verify is that the fibration $p : \mathcal{L}(\mathbf{B}) \to \mathbf{B}$ has *split* $\mathcal{Q}$-products and strong sums. This will be done below; the notation is as in 4.3.5.

For $X : A \times A' \to \Omega$ in $\mathcal{F}(\mathbf{B})$ and $\varphi : \mathcal{Q}_0(X) \to \Omega$ one defines two maps $\varphi_1, \varphi_2 : A \times \mathcal{Q}_0(X) \to \Omega$ by

$$\begin{aligned} \varphi_1 &= (\delta_A \circ \langle \pi,\ \pi \circ \{X\} \circ \pi' \rangle) \Rightarrow (\varphi \circ \pi') \\ \varphi_2 &= (\delta_A \circ \langle \pi,\ \pi \circ \{X\} \circ \pi' \rangle) \,\&\, (\varphi \circ \pi') \end{aligned}$$

Then one takes $\forall_X.\varphi = \forall_{\mathcal{Q}_0(X)} \circ \Lambda(\varphi_1)$ and $\exists_X.\varphi = \exists_{\mathcal{Q}_0(X)} \circ \Lambda(\varphi_2)$ — where for $C \in \mathbf{B}$, $\exists_C : \Omega^C \to \Omega$ is the standard map obtained as character of the monic part of $\pi \circ \epsilon_C : \bullet \rightarrowtail \Omega^C \times C \to \Omega^C$.

(ii) A second (split) CC-category is obtained as follows.

$$\begin{tikzcd} Fam_{\mathrm{eff}}(\mathbf{M}) \arrow[r, "\mathcal{I}"] \arrow[dr] & Fam_{\mathrm{eff}}(\omega\text{-}\mathbf{Set}) \arrow[r, "\mathcal{Q}", "\sim"'] \arrow[d] & \omega\text{-}\mathbf{Set}^{\to} \arrow[dl, "cod"] \\ & \omega\text{-}\mathbf{Set} & \end{tikzcd}$$

The equivalence $\mathcal{Q}$, the reflection $\mathcal{I}$ and the generic object are described in 1.2.12. $\mathcal{QI}$ is a CCompC as mentioned in 4.3.2 (iii).

(iii) The above two examples are dictoses. Here is another one. Let $\mathbf{C}$ be a complete Heyting (pre-) algebra, considered as a (small complete) category. Since $\mathbf{C}$ has infinite coproducts, the fibration $\mathit{Fam}(\mathbf{C}) \to \mathbf{Sets}$ from 4.1.6 (ii) has sums, see example 4.2.5 (i). It yields by lemma 4.2.13 a fibred left adjoint to $\mathit{Fam}(\mathbf{C}) \to \mathbf{Sets}^{\to}$. Thus one obtains a *logical dictos*.

(iv) Term models of the calculi CC and weak CC as described in section 2.3 also yield appropriate examples. The construction is by now familiar so we only give a sketch. A (weak) CC-cateogry $\mathbf{E} \xrightarrow{\mathcal{I}} \mathbf{D} \xrightarrow{\mathcal{Q}} \mathbf{B}^{\to}$ is obtained as follows. Objects of $\mathbf{B}$ are (equivalence classes of) contexts $[\Gamma]$ with sequences of terms ("substitutions") between them. Objects of $\mathbf{E}$ are $[\Gamma \vdash \sigma : *]$ and objects of $\mathbf{D}$ are $[\Gamma \vdash A : \Box]$. The functor $\mathcal{I}$ is then given by $[\Gamma \vdash \sigma : *] \mapsto [\Gamma \vdash \mathit{In}(\sigma) : \Box]$. Finally, $\mathcal{Q}([\Gamma \vdash A : \Box])$ is the usual projection $[\Gamma, \alpha : A] \longrightarrow [\Gamma]$.

The next two results go further in unravelling the structure of a weak CC-category.

5.2.7. Proposition. *Let* $\mathbf{E} \xrightarrow{\mathcal{I}} \mathbf{D} \xrightarrow{\mathcal{Q}} \mathbf{B}^{\to}$ *be a weak CC-category. We write* $\mathcal{P} = \mathcal{QI}$ *and* $p = \mathit{cod} \circ \mathcal{P} : \mathbf{E} \to \mathbf{B}$ *for the "fibration of propositions". Then*

(i) *p is a fibred CCC;*

(ii) *p is a full small fibration. (cf. definition 4.5.10 (ii))*

Proof. (i) Analogously to lemma 4.3.9 (i), using that $\mathcal{P}$ is a full comprehension category with unit, products and weak sums. As remarked in the proof there, the result does not require that the sums are strong.

(ii) By proposition 4.5.5 ($\Leftarrow$) and (i) above, one obtains that p is locally small. Since p has a generic object, corollary 4.5.8 tells us that p is small, provided the relevant constructions can be performed, see remark 4.5.9 (i). We check these details.

Let $T \in \mathbf{E}$ above $\mathcal{Q}_0\Omega \in \mathbf{B}$ be generic for p, where $\Omega \in \mathbf{D}$ is above the terminal $t \in \mathbf{B}$. Let's write $C_0 = \mathcal{Q}_0\Omega$. The following pullback in $\mathbf{B}$ yields the product $C_0 \times C_0$.

$$\begin{tikzcd}
C_0 \times C_0 \arrow[r, "\pi'"] \arrow[d, "\pi"'] \arrow[dr, phantom, "\lrcorner", very near start] & C_0 \arrow[d, "\mathcal{Q}\Omega"] \\
C_0 \arrow[r, "\mathcal{Q}\Omega"'] & t
\end{tikzcd}$$

Notice that π is obtained as $\mathcal{Q}(\mathcal{Q}\Omega^*(\Omega))$. The pair $\langle \partial_0, \partial_1 \rangle = \mathcal{P}(\pi^*(T) \Rightarrow \pi'^*(T))$ in $\mathbf{B}/C_0 \times C_0$ is obtained as in the proofs of 4.5.7 (i) and 4.5.5 ($\Leftarrow$). We have to check that the pullbacks of composable tuples and triples C_2 and C_3 (described in 1.4.1) can be formed. But these are both obtained by pulling back $\partial_0 = \pi \circ \langle \partial_0, \partial_1 \rangle$. Since

the latter is a composition of $\mathcal{P}$- and $\mathcal{Q}$-projections, this can always be done, see lemma 4.1.7.

We conclude that p is a small fibration. Since $\mathcal{P}$ is a full comprehension category with unit, p is a full small fibration. $\Box$

5.2.8. Theorem (From weak CC to $\lambda\omega$). *Let* $\mathbf{E} \xrightarrow{\mathcal{I}} \mathbf{D} \xrightarrow{\mathcal{Q}} \mathbf{B}^{\to}$ *be a weak CC-category. Let $t \in \mathbf{B}$ be the terminal and* $p = \mathit{cod} \circ \mathcal{QI} : \mathbf{E} \to \mathbf{B}$*. By change-of-base we form*

$$\begin{tikzcd}
\mathbf{E}'' \arrow[r] \arrow[d, "p''"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathbf{E}' \arrow[r] \arrow[d, "p'"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathbf{E} \arrow[d, "p"] \\
\mathbf{D}_t \arrow[r, hook] & \mathbf{D} \arrow[r, "\mathcal{Q}_0"] & \mathbf{B}
\end{tikzcd}$$

Then p'' is a $\lambda\omega$-category. (see definition 3.2.2)

Proof. The base category $\mathbf{D}_t$ of p'' is a CCC since $\mathit{cod} \circ \mathcal{Q} : \mathbf{D} \to \mathbf{B}$ is a fibred CCC, see lemma 4.3.9 (i). p' is a fibred CCC since it is obtained by change-of-base from a fibred CCC p, see (i) in the previous proposition. p'' has a generic object because p has a generic object $T \in \mathbf{E}$ above $\mathcal{Q}_0\Omega \in \mathbf{B}$ with $\Omega \in \mathbf{D}_t$. Finally we have to find products and sums for p'' along cartesian projections. For $D, D' \in \mathbf{D}_t$, we have $D \times D' = \Sigma_D . \mathcal{Q}D^*(D')$. The first projection $\pi : D \times D' \to D$ is $\widetilde{\mathcal{Q}}(D, \mathcal{Q}D^*(D'))$, where $\widetilde{\mathcal{Q}} : \widetilde{\mathbf{D}} \to \mathbf{D}^{\to}$ is the CCompC defined in 4.4.10. Analogously to lemma 4.2.12 (i) one can verify that p' in the above diagram has $\widetilde{\mathcal{Q}}$-products and sums. Hence p'' has products and sums along cartesian projections. $\Box$

In Jacobs, Moggi & Streicher [1991] one may find how — in the other direction — every $\lambda\omega$-category can be turned into a CC-category.

The content of the next result goes back to Hyland [1989], 3.1, proposition 2 and to Ehrhard [1989], corollary 1. We made some changes in the formulation.

5.2.9. Theorem. *Let* $p : \mathbf{E} \to \mathbf{B}$ *be a fibration where $\mathbf{B}$ is an LCCC. Then*

$$p \text{ is full small complete} \quad \Leftrightarrow \quad \text{there is a dictos } \mathcal{P} : \mathbf{E} \to \mathbf{B}^{\to} \text{ with } p = \mathit{cod} \circ \mathcal{P}$$

For the relevant notions, see 4.5.10 (ii), 4.2.4 (ii) and 5.2.5.

Proof. ($\Leftarrow$) By 5.2.7 (ii) one has that p is a full small fibration. By the reflection $\mathbf{E} \leftrightarrows \mathbf{B}^{\to}$ it follows that p inherits completeness from $\mathit{cod} : \mathbf{B}^{\to} \to \mathbf{B}$.

($\Rightarrow$) By definition 4.5.10 there is a full comprehension category (with unit) $\mathcal{P} : \mathbf{E} \to \mathbf{B}^{\to}$ such that $p = \mathit{cod} \circ \mathcal{P}$. A left adjoint to $\mathcal{P}$ is obtained from the adjoint functor theorem 4.5.11. Indeed p and cod are locally small and complete fibrations and $\mathcal{P}$ is a continuous functor (see the argumentation after definition 4.5.10). It can be shown that for every $u : A' \to A$ in $\mathbf{B}/A$ there is an object $E \in \mathbf{E}_A$ such that for every $E' \in \mathbf{E}_A$ and $f \in \mathbf{B}/A(u, \mathcal{P}E')$ one can find $\alpha \in \mathbf{B}/A(u, \mathcal{P}E)$ and $g : E \to E'$ in $\mathbf{E}_A$ with $\mathcal{P}_0 g \circ \alpha = f$. This yields the solution set condition mentioned in theorem 4.5.11. One takes $E = \Sigma_u.1A'$, where Σ_u denotes the "sum" obtained by a higher order definition in terms of products (which are available). In informal type theoretical formulation: $\Sigma_u.D = \Pi_{x:*}.(\Pi_u.(D \to x)) \to x$. For the solution mentioned above, one takes for $y : u$ the term $\alpha(y) = \lambda x : *. \lambda z : \Pi_u.(1A' \to x). zy\langle\rangle$ and for $w : \Sigma_u.1A'$ the term $g(w) = f(wu(\lambda y : u. \lambda z : 1A'. y))$. Then indeed $g(\alpha(y)) = f(y)$. □

As an application of this theorem it can be shown that there are no non-logical models of the calculus of constructions with families of sets as types and set-indexed collections as propositions.

5.2.10. Proposition. *The "family model" from 4.1.6 (ii) satisfies*

$$Fam(\mathbf{C}) \to \mathbf{Sets}^{\to} \text{ is a dictos} \quad \Leftrightarrow \quad \mathbf{C} \text{ is a complete Heyting pre-algebra.}$$

Proof. The implication ($\Leftarrow$) is example 5.2.6 (ii). As to ($\Rightarrow$) one has

$$\begin{aligned} Fam(\mathbf{C}) \to \mathbf{Sets}^{\to} \text{ is a dictos} \quad &\Rightarrow \quad Fam(\mathbf{C}) \to \mathbf{Sets} \text{ is small complete} \\ &\Rightarrow \quad \mathbf{C} \text{ is equivalent to a small complete category} \\ &\Rightarrow \quad \mathbf{C} \text{ is a complete Heyting pre-algebra.} \end{aligned}$$

The latter implication is based on a result of P. Freyd, see e.g. Mac Lane [1971], V.2, proposition 3. □

5.3 HML-categories

As shown in 5.1.1 a categorical version of the setting $Sort = \{*, \square\}$ with $* \succ \square$, $* \succ *$, $\square \succ \square$ looks like this

$$\begin{tikzcd} \mathbf{E} \arrow[r, shift left] \arrow[r, shift right] \arrow[r, phantom, "\Downarrow \mathcal{P}"] & \mathbf{B} \arrow[d, shift right, "r"'] \\ \mathbf{D} \arrow[r, shift left] \arrow[r, shift right] \arrow[r, phantom, "\Downarrow \mathcal{Q}"] & \mathbf{A} \arrow[u, shift right, "1"'] \end{tikzcd}$$

where $\mathcal{P}$ is a full comprehension category over r. By dressing this setting up with appropriate features one obtains the notion of a HML-category. We don't give any concrete examples but show instead how $\lambda\omega$-categories and CC-categories can be transformed into HML-categories — which indirectly yields examples. At the end of this section, we reconsider features for the propositional setting — which is a special case of the one above with $\mathcal{P}$ and $\mathcal{Q}$ constant comprehension categories.

The next definition and the subsequent two theorems are borrowed from Jacobs, Moggi & Streicher [1991]. Remember from section 2.3 that the features for HML are $(\square, \square)$, $(*, *)$ and $(\square, *)$-quantification and an $* : \square$-axiom. The following categorical description follows the guidelines (3.1) and (3.2) in 5.1.2. Notice that the change-of-base as described there is used twice.

5.3.1. Definition. An HML-category is given by a setting as above in which

- $\mathcal{Q}$ is a CCompC;
- $\mathcal{P}$ is a CCompC over r;
- $\mathcal{P}$ admits $r^*(\mathcal{Q})$-products and strong sums;
- there is an object $\Omega \in \mathbf{D}$ such that $q\Omega \in \mathbf{A}$ is terminal; further, the fibration p' obtained by change-of-base as below has a generic object above $\mathcal{Q}_0\Omega \in \mathbf{A}$.

$$\begin{tikzcd} \mathbf{E}' \arrow[r] \arrow[d, "p'"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathbf{E} \arrow[d, "p"] \\ \mathbf{A} \arrow[r, "1"] & \mathbf{B} \end{tikzcd}$$

5.3.2. Theorem. (i) *Every $\lambda\omega$-category can be transformed into an HML-category.*
(ii) *Every HML-category can be transformed into a $\lambda\omega$-category.*
(iii) *The output of first applying (i) and then (ii) is isomorphic to the input.*

Proof. (i) Let $p : \mathbf{E} \to \mathbf{B}$ be a $\lambda\omega$-category, i.e. a fibred CCC on a CCC $\mathbf{B}$, with a generic object and $Cons_{\mathbf{B}}$-products and sums. One forms

$$\begin{tikzcd} \overline{\mathbf{E}} \arrow[r, shift left] \arrow[r, shift right] \arrow[r, phantom, "\Downarrow \overline{\mathcal{P}}"] & \mathbf{E} \arrow[d, shift right, "p"'] \\ \overline{\mathbf{B}} \arrow[r, shift left] \arrow[r, shift right] \arrow[r, phantom, "\Downarrow Cons_{\mathbf{B}}"] & \mathbf{B} \arrow[u, shift right, "1"'] \end{tikzcd}$$

This structure forms an HML-category since

- $Cons_{\mathbf{B}}$ is a CCompC, see example 4.3.2 (ii).
- $\overline{\mathcal{P}} : \overline{\mathbf{E}} \to \mathbf{E}^{\to}$ is a CCompC over p, see example 4.4.8 (i); moreover, it has $p^*(Cons_{\mathbf{B}})$-products and strong sums by lemma 4.4.11.
- The generic object for p also works here, by the change-of-base situation $p \to \overline{p}$ described in 1.2.7.

(ii) Suppose an HML-category as describe above is given. We form the fibration p'' by change-of-base

$$\begin{tikzcd} \mathbf{E}'' \arrow[rr] \arrow[d, "p''"'] \arrow[dr, phantom, "\lrcorner", very near start] & & \mathbf{E}' \arrow[r] \arrow[d, "p'"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathbf{E} \arrow[d, "p"'] \\ \mathbf{D}_t \arrow[r, hook] & \mathbf{D} \arrow[r, "\mathcal{Q}_0"] & \mathbf{A} \arrow[r, "1"] & \mathbf{B} \end{tikzcd}$$

where $t \in \mathbf{A}$ is terminal object. Then

- $\mathbf{D}_t$ is CCC, since $q = cod \circ \mathcal{Q}$ is a fibred CCC, see 4.3.9 (i).
- p'' is a fibred CCC, since fibred CCC's are preserved by change-of-base.
- The generic object T for p' above $\mathcal{Q}_0\Omega \in \mathbf{A}$ where $\Omega \in \mathbf{D}_t$ yields a generic object for p'': for every $E \in \mathbf{E}$ and $D \in \mathbf{D}_t$ with $pE = 1\mathcal{Q}_0 D$, there is a morphism $u : \mathcal{Q}_0 D \to \mathcal{Q}_0\Omega$ in $\mathbf{A}$ with $u^*(T) \cong E$ in $\mathbf{E}'$. Since $\mathcal{Q}$ is a *full* comprehension category there is a (unique) $f : D \to \Omega$ in $\mathbf{D}_t$ with $\mathcal{Q}_0 f = u$. But then we are done.
- p'' has products and sums along cartesian projections, by an argument similar to the one in the proof of theorem 5.2.8.

(iii) By the change-of-base situation $p \to \overline{p}$ from 1.2.7 and the fact that $\overline{\mathbf{B}}_t \cong \mathbf{B}$. □

5.3.3. Theorem. (i) *Every CC-category can be transformed into an HML-category.*

(ii) *Doing CC $\longrightarrow$ HML $\longrightarrow$ $\lambda\omega$ and CC $\longrightarrow$ $\lambda\omega$ yields equivalent results.*

(The transformation CC $\longrightarrow$ $\lambda\omega$ is described in theorem 5.2.8.)

Proof. (i) Asume we have a CC-category as in definition 5.2.2. One forms

$$\begin{tikzcd} \widetilde{\mathbf{E}} \arrow[r, shift left=2] \arrow[r, shift right=2] \arrow[r, phantom, "\Downarrow \widetilde{\mathcal{P}}"] & \mathbf{E} \arrow[d, shift right, "p"'] \\ \mathbf{D} \arrow[r, shift left=2] \arrow[r, shift right=2] \arrow[r, phantom, "\Downarrow \mathcal{Q}"] & \mathbf{B} \arrow[u, shift right, "1"'] \end{tikzcd}$$

where $\mathcal{P} = \mathcal{Q}\mathcal{I}$ is a CCompC. Hence $\widetilde{\mathcal{P}}$ is a CCompC over p by 4.4.10, admitting $p^*(\mathcal{Q})$-products and strong sums by lemma 4.4.12. The generic object of the CC-category also works here, because of the "pseudo" change-of-base situation $\widetilde{p} \to p$ from 4.4.10.

(ii) Again by the "pseudo" change-of-base situation $\widetilde{p} \to p$. □

Finally we take a brief look at the features for the refined propositional setting

$$\begin{tikzcd} \mathbf{E}//T(*) \arrow[r, shift left=2] \arrow[r, shift right=2] \arrow[r, phantom, "\Downarrow Cons_{T(*)}"] & \mathbf{E} \arrow[d, shift right, "p"'] \\ \mathbf{B}//T(\Box) \arrow[r, shift left=2] \arrow[r, shift right=2] \arrow[r, phantom, "\Downarrow Cons_{T(\Box)}"] & \mathbf{B}. \arrow[u, shift right, "1"'] \end{tikzcd}$$

as described in section 5.1. Following (3.1) – (3.3) in 5.1.2 we obtain the following features.

5.3.4. Redefinition. The above setting will be called

(i) a $\lambda{\rightarrow}$-category if $Cons_{T(*)}$ is a CCompC over p; further, if there is an object $(t, \Omega) \in \mathbf{B}//T(\Box)$ above the terminal such that above $\Omega \in \mathbf{B}$ there is a generic object for the fibration q obtained by change-of-base:

$$\begin{tikzcd} \bullet \arrow[r] \arrow[d, "q"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathbf{E}//T(*) \arrow[d, "cod \circ Cons_{T(*)}"] \\ \mathbf{B} \arrow[r, "1"] & \mathbf{E} \end{tikzcd}$$

(ii) a $\lambda\underline{\omega}$-category if it is a $\lambda{\rightarrow}$-category in which $Cons_{T(\Box)}$ is a CCompC.

(iii) a $\lambda 2$-category if it is a $\lambda{\rightarrow}$-category in which $Cons_{T(*)}$ has $p^*(Cons_{T(\Box)})$-products and sums.

(iv) a $\lambda\omega$-category if it is both a $\lambda\underline{\omega}$-category and a $\lambda 2$-category.

The notions introduced earlier in definition 3.2.2 are special cases in the above redefinition: one can take $T(*) = Obj(\mathbf{E})$ — we then write $\overline{\mathcal{P}}$ for $Cons_{T(*)}$, see 4.4.8 (i). Furthermore, for a $\lambda{\rightarrow}$-category and a $\lambda 2$-category one takes $T(\Box) = \{\Omega\}$ and $T(\Box) = Obj(\mathbf{B})$ for a $\lambda\underline{\omega}$-category and a $\lambda\omega$-category.

One might ask about the motivation for these refined descriptions of the minimal and propositional settings and their features. We mention two points.

- This refined description comes out as a result of a general method of translation. As such, it has more value than the somewhat ad hoc notions introduced in definition 3.2.2.
- In case one is interested in modelling calculi having exponent-types but no (cartesian) product-types (as used e.g. in Barendregt [1991], [199?]), only the refined framework can be used, see the discussion in example 4.2.6.

5.4 λHOL-categories and λPRED-categories

This section follows the same pattern as the previous one: λHOL- and λPRED-categories are defined by dressing up the correspond settings from section 5.1 with appropriate features following 5.1.2. No concrete examples are given, but it is shown how to obtain these from $\lambda\omega$-categories (as defined in 3.2.2).

Remember the features for λHOL are $*$- and $\Box$-closure, $(\Box, *)$-quantification and $* : \Box, \Box : \triangle$ axioms.

5.4.1. Definition. We consider the categorical setting described in 5.1.1 for the setting $\mathit{Sort} = \{*, \Box, \triangle\}$ with $* \succ \Box$, $\Box \succ \triangle$, $* \succ \triangle$. It is called a *λHOL-category* if

- $\mathit{Cons}_{T(*)}$ is a CCompC over p;
- $\mathit{Cons}_{T(\Box)}$ is a CCompC over r;
- $\mathit{Cons}_{T(*)}$ has $p^*(\mathit{Cons}_{T(\Box)})$-products and sums;
- there is an object $(t, \Omega_*) \in \mathbf{B}//T(\Box)$ such that the following fibration obtained by change-of-base has a generic object above Ω_*;

$$\begin{tikzcd} \bullet \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathbf{E}//T(*) \arrow[d, "cod \circ \mathit{Cons}_{T(*)}"] \\ \mathbf{B} \arrow[r, "\top"] & \mathbf{E} \end{tikzcd}$$

- there is an object $(t, \Omega_\Box) \in \mathbf{A}//T(\triangle)$ such that the following fibration obtained by change-of-base has a generic object above $\Omega_\Box$.

$$\begin{tikzcd} \bullet \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathbf{B}//T(*) \arrow[d, "cod \circ \mathit{Cons}_{T(\Box)}"] \\ \mathbf{A} \arrow[r, "1"] & \mathbf{B} \end{tikzcd}$$

5.4.2. Theorem. *Every $\lambda\omega$-category on a small base category can be transformed into a λHOL-category.*

Proof. Let $p : \mathbf{E} \to \mathbf{B}$ be a fibred CCC on a small CCC with a generic object and $\mathit{Cons}_{\mathbf{B}}$-products and sums. We lift it to a λHOL-category by the family construction described in 1.1.2. The functor $\mathit{Fam}(p) : \mathit{Fam}(\mathbf{E}) \to \mathit{Fam}(\mathbf{B})$ given by $\{E_i\}_{i \in I} \mapsto$

$\{pE_i\}_{i \in I}$ is a fibration over $\mathit{Fam}(\mathbf{B}) \to \mathbf{Sets}$; let's write r for the latter fibration. One easily verifies that $\mathit{Fam}(p)$ is a fibred CCC again. Hence we consider

$$\begin{tikzcd} \overline{\mathit{Fam}(\mathbf{E})} \arrow[r, shift left] \arrow[r, shift right] \arrow[r, phantom, "\Downarrow"] & \mathit{Fam}(\mathbf{E}) \arrow[d, shift right, "\mathit{Fam}(p)"'] \\ \overline{\mathit{Fam}(\mathbf{B})} \arrow[r, shift left] \arrow[r, shift right] \arrow[r, phantom, "\Downarrow \overline{\mathcal{R}}"] & \mathit{Fam}(\mathbf{B}) \arrow[u, shift right, "\mathit{Fam}(1)"'] \arrow[d, shift right, "r"'] \\ \mathbf{Sets}//T(\triangle) \arrow[r, shift left] \arrow[r, shift right] \arrow[r, phantom, "\Downarrow"] & \mathbf{Sets} \arrow[u, shift right, "\{t\}_-"'] \end{tikzcd}$$

For $T(\triangle)$ we take $\{\mathit{Obj}(\mathbf{B})\}$ using that $\mathbf{B}$ is small. It yields a generic object for r. Since $\mathbf{B}$ is a CCC, $r : \mathit{Fam}(\mathbf{B}) \to \mathbf{Sets}$ is a fibred CCC. Hence one obtains a constant CCompC $\overline{\mathcal{R}}$ over r, see 4.4.8 (i). One has $\overline{\mathcal{R}}(\{A_i\}_I, \{B_i\}_I) = \{\pi_{A_i,B_i} : A_i \times B_i \to A_i\}_I$.

Similarly, using that $\mathit{Fam}(p)$ is a fibred CCC over r, one obtains a constant CCompC $\overline{\mathit{Fam}(\mathbf{E})} \to \mathit{Fam}(\mathbf{E})^{\to}$ over r. The axiom $* : \Box$ and the $(\Box, *)$-quantification follow from a pointwise construction. $\Box$

We turn to λPRED-categories. Remember that the features are $*, \triangle$-closure, $(\triangle, *)$-quantification and an $* : \Box$ axiom.

5.4.3. Definition. The categorical setting described in 5.1.1 for $\mathit{Sort} = \{*, \triangle, \Box\}$ with $* \succ \triangle$, $* \succ \Box$ will be called a *λPRED-category* if

- $\mathit{Cons}_{T(*)}$ is a CCompC over p;
- $\mathit{Cons}_{T(\triangle)}$ is a CCompC;
- $\mathit{Cons}_{T(*)}$ has $p^*\mathit{Snd}^*(\mathit{Cons}_{T(\triangle)})$-products and sums;
- there is an object $(t, \Omega) \in \mathbf{A}//T(\Box)$ such that the following fibration obtained by change-of-base has a generic object above Ω.

$$\begin{tikzcd} \bullet \arrow[rr] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & & \mathbf{E}//T(\triangle) \arrow[d, "cod \circ \mathit{Cons}_{T(*)}"] \\ \mathbf{A} \arrow[r, "- \times t"] & \mathbf{A} \times \mathbf{B} \arrow[r, "1"] & \mathbf{E} \end{tikzcd}$$

where $- \times t$ is the terminal object functor for the fibration $\mathit{Fst} : \mathbf{A} \times \mathbf{B} \to \mathbf{A}$.

5.4.4. Theorem. *Every $\lambda\omega$-category can be transformed into a λPRED-category.*

Proof. Let $p : \mathbf{E} \to \mathbf{B}$ be a $\lambda\omega$-category. We are going to use the base category $\mathbf{B}$ to model both $\Box$- and $\triangle$-contexts. Therefore, we first form the fibration p' by change-of-base in

$$\begin{tikzcd} \mathbf{E}' \arrow[r] \arrow[d, "p'"'] & \mathbf{E} \arrow[d, "p"] \\ \mathbf{B} \times \mathbf{B} \arrow[r, "Prod"] & \mathbf{B} \end{tikzcd}$$

The rest is then straightforward: p' is a fibred CCC and thus one obtains a constant CCompC over p'. Let $T \in \mathbf{E}$ be generic for p and put $\Omega = pT \in \mathbf{B}$. Then $Cons_\Omega$ is used to model $\Box$. The constant CCompC $Cons_{\mathbf{B}}$ is used to model $\triangle$. $\Box$

5.5 The untyped lambda calculus revisited

D. Scott often stressed that the untyped λ-calculus should be considered as a special form of typed λ-calculus, viz. as a calculus with one type (satisfying e.g. $\Omega = \Omega \to \Omega$). Following this view we obtain a new notion of model for the untyped λ-calculus by considering "monoid" constant comprehension categories which have a single type. We include non-extensional abstraction in our investigation via S. Hayashi's "semi-adjunctions". At the end of this section we compare our new notion to the one consisting of a "CCC with a reflexive object" as introduced by Scott and further developed by Koymans, see Scott [1980], Koymans [1982], [1984] and Barendregt [1984].

The categorical concepts used in this section will all be described "on-the-nose", i.e. without mediating isomorphisms. We first recall the notion of a semi-adjunction from Hayashi [1985]. The subsequent lemma comes from Jacobs [1991].

A *semi-functor* $F : \mathbf{C} \to \mathbf{D}$ is a 'functor' except that it needs not preserve identities. Another semi-functor $G : \mathbf{D} \to \mathbf{C}$ is a right *semi-adjoint* of F — notation $F \dashv_s G$ — if there are collections $\{\alpha_{X,Y}, \beta_{X,Y}\}_{X \in \mathbf{C}, Y \in \mathbf{D}}$ such that the big four squares in the following diagram commute (for all f, g).

$$\begin{tikzcd} Y \arrow[d, "f"'] & \mathbf{D}(FX, Y) \arrow[r, shift left, "\alpha_{X,Y}"] \arrow[d, "f \circ - \circ Fg"'] & \mathbf{C}(X, GY) \arrow[l, shift left, "\beta_{X,Y}"] \arrow[d, "Gf \circ - \circ g"] & X \\ Y' & \mathbf{D}(FX', Y') \arrow[r, shift left, "\alpha_{X',Y'}"] & \mathbf{C}(X', GY') \arrow[l, shift left, "\beta_{X',Y'}"] & X' \arrow[u, "g"'] \end{tikzcd}$$

5.5.1. Lemma. *Suppose $F \dashv_s G$ as described above, but with F an ordinary functor; then — omitting indices — one has*

(i) $\beta \circ \alpha = id$, *i.e.* $\mathbf{D}(FX, Y)$ *is a retract of* $\mathbf{C}(X, GY)$.
(ii) $\alpha(u \circ Fv) = \alpha(u) \circ v$ $\quad Gu \circ \alpha(v) = \alpha(u \circ v)$.
(iii) $\beta(u) \circ Fv = \beta(u \circ v)$ $\quad \beta(Gu \circ v) = u \circ \beta(v)$.

Proof. (i) $(\beta \circ \alpha)(u) = id \circ \beta(\alpha(u)) \circ F(id) = \beta(G(id) \circ \alpha(u) \circ id) = id \circ u \circ F(id) = u$.
(ii) & (iii) Similarly. $\Box$

Further, two more notions are needed. A *morphism of semi-adjunctions* from $\langle F, G, \{\alpha, \beta\}\rangle : \mathbf{C} \to \mathbf{D}$ to $\langle F', G', \{\alpha', \beta'\}\rangle : \mathbf{C}' \to \mathbf{D}'$ consists of a pair of functors $\langle K : \mathbf{C} \to \mathbf{C}', L : \mathbf{D} \to \mathbf{D}'\rangle$ such that

$$\begin{array}{rclcrcl} LF & = & F'K & \text{and} & G'L & = & KG \\ K\alpha_{X,Y} & = & \alpha'_{KX,LY}L & \text{and} & L\beta_{X,Y} & = & \beta'_{KX,LY}K. \end{array}$$

Finally, a *semi-CCC* is a category provided with *semi*-adjunctions for semi-terminal, product and exponent. In equantional presentation, it is a 'CCC' except that one does not have $!_t = id_t$, $\langle \pi, \pi' \rangle = id$ and $\Lambda(ev) = id$, see Hayashi [1985] for more details.

Next we describe semi-products and sums for split comprehension categories. It is a straightforward generalization of ordinary products and sums as described in section 5.2 (except that we now require everything "up-to-equality").

5.5.2. Definition. Let $\mathcal{P} : \mathbf{E} \to \mathbf{B}^{\to}$ be a split comprehension category.
(i) $\mathcal{P}$ has *semi-products* (resp. *semi-sums*) if both

- for each $E \in \mathbf{E}$, the weakening functor $\mathcal{P}E^*$ has a right semi-adjoint Π_E (resp. a left semi-adjoint Σ_E);
- for each cartesian $f : E' \to E$ in $\mathbf{E}$ the pair $\langle (pf)^*, (\mathcal{P}_0 f)^* \rangle$ is a morphism of semi-adjunctions $\mathcal{P}E^* \dashv_s \Pi_E \longrightarrow \mathcal{P}E'^* \dashv_s \Pi_{E'}$ (resp. $\langle (\mathcal{P}_0 f)^*, (pf)^* \rangle$ is a morphism $\Sigma_E \dashv_s \mathcal{P}E^* \longrightarrow \Sigma_{E'} \dashv_s \mathcal{P}E'^*$).

(ii) A *morphism* $(\mathcal{P} : \mathbf{E} \to \mathbf{B}^{\to}) \longrightarrow (\mathcal{P}' : \mathbf{E}' \to \mathbf{B}'^{\to})$ *of comprehension categories with semi-products* consists of a pair of functors $K : \mathbf{B} \to \mathbf{B}'$ and $L : \mathbf{E} \to \mathbf{E}'$ such that

- $\langle K, L \rangle$ is a morphism of split fibrations $p \to p'$;
- $\langle K, L, id \rangle$ is a morphism of comprehension cateogries (see 4.1.4);
- for each $E \in \mathbf{E}$, the pair $\langle L \restriction pE, L \restriction \mathcal{P}_0 E \rangle$ is a morphism of semi-adjunctions $\mathcal{P}E^* \dashv_s \Pi_E \longrightarrow \mathcal{P}'(LE)^* \dashv_s \Pi'_{LE}$.

Similarly for semi-sums.

5.5.3. Definition. Let $\mathbf{B}$ be a category with terminal object t and let $\Omega \in \mathbf{B}$. One says that
(i) Ω is *non-empty* if $\mathbf{B}(t, \Omega)$ is non-empty;

(ii) Ω has *enough points* if for all $f, g : \Omega \to A$ in $\mathbf{B}$,

$$\forall x : t \to \Omega.\, f \circ x = g \circ x \quad \Rightarrow \quad f = g.$$

Constant comprehension categories are described in 4.1.3. For details about the semantics of the untyped λ-calculus, we refer to Barendregt [1984], especially chapter 5.

5.5.4. Definition. (i) A *categorical λ-algebra* is given by a base category $\mathbf{B}$ with finite products containing a non-empty object Ω such that the constant comprehension category $\mathit{Cons}_\Omega : \mathbf{B}/\!/\Omega \to \mathbf{B}^\to$ has semi-products.

(ii) A *morphism of categorical λ-algebras* $(\mathbf{B}, \Omega) \to (\mathbf{B}', \Omega')$ is a functor $K : \mathbf{B} \to \mathbf{B}'$ such that

- $K\Omega = \Omega'$ and $Kt = t'$, the terminal object in $\mathbf{B}'$; moreover, $K(!_A) = !_{KA}$;
- for every $A \in \mathbf{B}$ one has $K(A \times \Omega) = (KA) \times \Omega'$ with $K(\pi_{A,\Omega}) = \pi_{KA,\Omega'}$ and $K(\pi'_{A,\Omega}) = \pi'_{KA,\Omega'}$;
- the pair (K, K') is a morphism $\mathit{Cons}_\Omega \to \mathit{Cons}_{\Omega'}$ of comprehension categories with semi-products.
  (The functor $K' : \mathbf{B}/\!/\Omega \to \mathbf{B}'/\!/\Omega'$ is defined in 4.1.6 (i).)

This yields a category **Cat-$\boldsymbol{\lambda}$-Alg**.

One might wonder why we don't simply require that K preserves all cartesian products (on-the-nose) in the second point in (ii) above. But that would be too strong: the counit functor ε in the proof of 5.5.10 below satisfies $\varepsilon(n+m) = \Omega^{n+m} \cong \Omega^n \times \Omega^m = \varepsilon(n) \times \varepsilon(m)$. In the domain of ε, $+$ is $\times$, 1 is Ω and $\varepsilon(n) = \Omega^n$. As it stands, the second requirement above says precisely that (K, K', id) is a morphism of comprehension categories, see the second point in 5.5.2 (ii).

5.5.5. Definition. Let $(\mathbf{B}, \Omega)$ be a categorical λ-algebra. It will be called
(i) a *categorical λ-model* if Ω has enough points;
(ii) a *categorical $\lambda\eta$-algebra* if Cons_Ω has ordinary products;
(iii) a *categorical $\lambda\eta$-model* if it is both a categorical λ-model and a $\lambda\eta$-algebra.

Let $(\mathbf{B}, \Omega)$ be a categorical λ-algebra as described above. Recall from 4.1.3 that the fibre categories $(\mathbf{B}/\!/\Omega)_A$ are monoids, i.e. categories with only one object, viz. (A, Ω). Morphisms in $(\mathbf{B}/\!/\Omega)_A$ are arrows $f : A \times \Omega \to \Omega$ in $\mathbf{B}$. Composition in $(\mathbf{B}/\!/\Omega)_A$ is given by $g \bullet f = g \circ \langle \pi, f \rangle$; the projection $\pi' : A \times \Omega \to \Omega$ serves as identity. Reindexing along $u : B \to A$ is done by $u^*(f) = f \circ u \times id$.

The product semi-adjunctions are described by maps

$$\mathbf{B}(\,(A \times \Omega) \times \Omega,\ \ \Omega\,) \underset{\beta(A)}{\overset{\alpha(A)}{\rightleftarrows}} \mathbf{B}(\,A \times \Omega,\ \ \Omega\,)$$

A map $\alpha(A)(f) : A \times \Omega \to \Omega$ should be understood as the result of abstraction in the underlined Ω in $f : (A \times \underline{\Omega}) \times \Omega \to \Omega$. This follows from the fact that $\alpha(A)(f) \bullet h = \alpha(f \bullet \mathit{Cons}_\Omega(A, \Omega)^*(h))$, see lemma 5.5.1 (ii). More explicitly, it gives the following naturality condition

$$\alpha(A)(f) \circ \langle \pi, h \rangle \quad = \quad \alpha(A)(f \circ \langle \pi,\, h \circ \pi \times id \rangle)$$

Because one abstracts in the underlined Ω a form of "twisting" is often necessary.

A deeper analysis of categorical λ-algebras may be found after the following examples.

5.5.6. Examples. (i) Let D be a reflexive cpo via maps $F : D \to [D \to D]$ and $G : [D \to D] \to D$ with $F \circ G = id$, see Barendregt [1984], 5.4. As usual we write $a \cdot b = F(a)(b)$ and $\lambda x.- = G(\boldsymbol{\lambda} x.-)$.

A base category $\mathbf{D}$ is formed with $n \in \mathbb{N}$ as objects; n can be considered as the context containing the first n variables from an enumeration $\{x_n \mid n \in \mathbb{N}\}$. Morphism $n \to m$ are sequences $(f_1, \ldots, f_m)$ where each f_i is a continuous function $D^n \to D$, i.e. $f_i \in [D^n \to D]$. Composition in $\mathbf{D}$ is done in the obvious way and identities are sequences of projections. The object $0 \in \mathbf{D}$ is terminal and $n + m$ is a product. Thus $\mathbf{D}$ is an algebraic theory. As distinguished object ("Ω") we take $1 \in \mathbf{D}$. Notice that 1 is a non-empty object iff the cpo D is non-empty.

The product semi-functors $\Pi_n : (\mathbf{D}/\!/1)_{n+1} \to (\mathbf{D}/\!/1)_n$ are given by $(n{+}1, 1) \mapsto (n, 1)$ and $f \mapsto \boldsymbol{\lambda}\vec{x}, z \in D^{n+1}.\, \lambda y.\, f(\vec{x}, y, z \cdot y)$. The α's and β's as described above are given by

$$\begin{array}{rcl} \alpha(n)(f) & = & \boldsymbol{\lambda}\vec{x}, z \in D^{n+1}.\ \lambda y.\, f(\vec{x}, y, z) \\ \beta(n)(g) & = & \boldsymbol{\lambda}\vec{x}, y, z \in D^{n+2}.\ g(\vec{x}, z) \cdot y. \end{array}$$

One easily verifies that $(\mathbf{D}, 1)$ is a categorical λ-model. In case $G \circ F = id$ — i.e. $D \cong [D \to D]$ — it becomes a categorical $\lambda\eta$-model.

(ii) Let $\mathsf{M} = \langle D, \cdot, \boldsymbol{K}, \boldsymbol{S} \rangle$ be a λ-algebra, see Barendregt [1984], 5.2. One writes $\mathbf{1}_n = \lambda x_0 \ldots x_n.\, x_0 \ldots x_n$; inductively, one can define $\mathbf{1}_0 = \boldsymbol{I} = \boldsymbol{SKK}$ and $\mathbf{1}_{n+1} = \boldsymbol{S}(\boldsymbol{K}\mathbf{1}_n)$, see loc. cit. 5.6. Let's put $(D^n \to D) = \{a \in D \mid \mathbf{1}_n \cdot a = a\}$. Then $(D^0 \to D) = D$; we write $\mathbf{1}$ for $\mathbf{1}_1$ and $(D \to D)$ for $(D^1 \to D)$.

Let $\mathbf{D}$ be a base category, once again with $n \in \mathbb{N}$ as objects, but with m tuples $(a_1, \ldots, a_m)$ with $a_i \in (D^n \to D)$ as morphisms $n \to m$. Then $(b_1, \ldots, b_k) \circ (a_1, \ldots, a_m) = (c_1, \ldots, c_k)$ where $c_i = \lambda x_1 \ldots x_n.\, b_i(a_1 x_1 \ldots x_n) \ldots (a_m x_1 \ldots x_n)$. The identity on n is $(\lambda x_1 \ldots x_n.\, x_0, \ldots, \lambda x_1 \ldots x_n.\, x_n)$. The category $\mathbf{D}$ has terminal 0 and products $n + m$ as before. Hence it is an algebraic theory again. We take $1 \in \mathbf{D}$ as distinguished object.

The comprehension category $\mathit{Cons}_1 : \mathbf{D}/\!/1 \to \mathbf{D}^\to$ has semi-products: for morphisms $a \in (D^{n+2} \to D)$ in $(\mathbf{D}/\!/1)_{n+1}$ and $b \in (D^{n+1} \to D)$ in $(\mathbf{D}/\!/1)_n$ one takes

$$\begin{array}{rcl} \alpha(n)(a) & = & \lambda x_1 \ldots x_n z y.\ a x_1 \ldots x_n y z \\ \beta(n)(b) & = & \lambda x_1 \ldots x_n y z.\ b x_1 \ldots x_n z y. \end{array}$$

Then $\beta(n)(\alpha(n)(a)) = \mathbf{1}_{n+2} \cdot a = a$.

In case M is a λ-model, i.e. $\forall x \in D.\ a \cdot x = b \cdot x \Rightarrow \mathbf{1} \cdot a = \mathbf{1} \cdot b$, one obtains a categorical λ-model: suppose morphisms $(a_1, \ldots, a_m), (b_1, \ldots, b_m) : 1 \to m$ in $\mathbf{D}$ are given with $\forall x : 0 \to 1.\ (a_1, \ldots, a_m) \circ x = (b_1, \ldots, b_m) \circ x$. Then $a_i, b_i \in (D \to D)$ satisfy $\forall x \in D.\ a_i \cdot x = b_i \cdot x$. Hence $a_i = \mathbf{1} \cdot a_i = \mathbf{1} \cdot b_i = b_i$. Thus the object $1 \in \mathbf{D}$ has enough points.

The next result describes the structure given by the fibred semi-products of a categorical λ-algebra in a down-to-earth way.

5.5.7. Lemma. *Let $\mathbf{B}$ be a category with finite products and $\Omega \in \mathbf{B}$ be a non-empty object. Then*

(i) *$(\mathbf{B}, \Omega)$ is a categorical λ-algebra if and only if there is a map*

$$app \ : \ \Omega \times \Omega \ \to \ \Omega$$

together with an operation

$$\lambda(-) \ : \ \mathbf{B}(A \times \Omega, \Omega) \ \to \ \mathbf{B}(A, \Omega)$$

such that

$$\begin{aligned} app \circ \lambda(f) \times id &= f \\ \lambda(f \circ g \times id) &= \lambda(f) \circ g. \end{aligned}$$

(ii) *$(\mathbf{B}, \Omega)$ is a categorical $\lambda\eta$-algebra if and only if there are app and λ as in (i) which additionally satisfy*

$$\lambda(app) \ = \ id.$$

Proof. For the (if)-part of (i) and (ii), one defines

$$\begin{aligned} \alpha(A)(f) &= \lambda(f \circ \langle \pi \times id,\ \pi' \circ \pi \rangle) \\ \beta(A)(g) &= app \circ \langle g \circ \pi \times id,\ \pi' \circ \pi \rangle. \end{aligned}$$

In order to prove the (only if)-part, we first unravel the structure given by the semi-products. Let $\alpha(A), \beta(A)$ be as described before the examples. The naturality conditions following from lemma 5.5.1 (ii),(iii) are

$$\begin{aligned} \alpha(A)(f) \circ \langle \pi, h \rangle &= \alpha(A)(f \circ \langle \pi,\ h \circ \pi \times id \rangle) \\ \beta(A)(g) \circ \langle \pi,\ h \circ \pi \times id \rangle &= \beta(A)(g \circ \langle \pi, h \rangle). \end{aligned}$$

The "Beck-Chevalley" condition — the second point in 5.5.2 (i) — implies that for $u : B \to A$ in $\mathbf{B}$ one has

$$\begin{aligned} \alpha(A)(f) \circ u \times id &= \alpha(B)(f \circ (u \times id) \times id) \\ \beta(A)(g) \circ (u \times id) \times id &= \beta(B)(g \circ u \times id). \end{aligned}$$

Applying $\beta(t)$ to $\pi' : t \times \Omega \to \Omega$ yields a map $\beta(t)(\pi') : (t \times \Omega) \times \Omega \to \Omega$. By arranging the input appropriately, one obtains

$$app \ = \ \beta(t)(\pi') \circ \langle \langle !, \pi' \rangle, \pi \rangle \ : \ \Omega \times \Omega \ \to \ \Omega.$$

For an arrow $f : A \times \Omega \to \Omega$ in $\mathbf{B}$ one has $f \circ \pi : (A \times \Omega) \times \Omega \to \Omega$ by introducing an extra "dummy" variable. It enables us to apply $\alpha(A)$ which yields an arrow $A \times \Omega \to \Omega$. Finally, we remove the first "dummy" Ω by substituting an arbitrary element $c_0 : t \to \Omega$ — which exists because Ω is non-empty. Hence we have

$$\lambda(f) \ = \ \alpha(A)(f \circ \pi) \circ \langle id, c_0 \circ ! \rangle \ : \ A \ \to \ \Omega.$$

An easy argument shows that the definition of $\lambda(f)$ does not depend on a choice for c_0: if we would have taken $c_1 : t \to \Omega$ then $\varphi = c_1 \circ !_\Omega : \Omega \to \Omega$ satisfies $\varphi \circ c_0 = c_1$ and thus

$$\begin{aligned} \alpha(A)(f \circ \pi) \circ \langle id, c_1 \circ ! \rangle &= \alpha(A)(f \circ \pi) \circ \langle \pi, \varphi \circ \pi' \rangle \circ \langle id, c_0 \circ ! \rangle \\ &= \alpha(A)(f \circ \pi) \circ \langle id, c_0 \circ ! \rangle, \end{aligned}$$

the latter by naturality of $\alpha(A)$. We compute

$$\begin{aligned} app \circ \lambda(f) \times id &= \beta(t)(\pi') \circ \langle \langle !_{\Omega \times \Omega}, \pi' \rangle, \pi \rangle \circ \langle \lambda(f) \circ \pi, \pi' \rangle \\ &= \beta(t)(\pi') \circ \langle \langle !_{A \times \Omega}, \pi' \rangle, \lambda(f) \circ \pi \rangle \\ &= \beta(t)(\pi') \circ (!_A \times id) \times id \circ \langle id, \lambda(f) \circ \pi \rangle \\ &= \beta(A)(\pi' \circ !_A \times id) \circ \langle \pi,\ \alpha(A)(f \circ \pi) \circ \pi \times id \rangle \circ \langle id,\ c_0 \circ ! \rangle \\ &\qquad \text{by Beck-Chevalley for } \beta \\ &= \beta(A)(\pi' \circ \langle \pi,\ \alpha(A)(f \circ \pi) \rangle) \circ \langle id,\ c_0 \circ ! \rangle \\ &\qquad \text{by naturality of } \beta(A) \\ &= f \circ \pi \circ \langle id,\ c_0 \circ ! \rangle \\ &\qquad \text{by lemma 5.5.1 (i)} \\ &= f. \end{aligned}$$

Assuming $g : B \to A$ one obtains

$$\begin{aligned} \lambda(f \circ g \times id) &= \alpha(B)(f \circ g \times id \circ \pi) \circ \langle id,\ c_0 \circ !_B \rangle \\ &= \alpha(B)(f \circ \pi \circ (g \times id) \times id)) \circ \langle id,\ c_0 \circ !_B \rangle \\ &= \alpha(A)(f \circ \pi) \circ g \times id \circ \langle id,\ c_0 \circ !_B \rangle \\ &\qquad \text{by Beck-Chevalley for } \alpha \\ &= \alpha(A)(f \circ \pi) \circ \langle id,\ c_0 \circ !_A \rangle \circ g \\ &= \lambda(f) \circ g. \end{aligned}$$

In case $(\mathbf{B}, \Omega)$ is a $\lambda\eta$-algebra, one has $\alpha(A) \circ \beta(A) = id$. In order to prove $\lambda(app) = id$, we first notice that $app \circ \pi = \beta(\Omega)(\pi) : (\Omega \times \Omega) \times \Omega \to \Omega$. Indeed,

$$\begin{aligned} app \circ \pi &= \beta(t)(\pi') \circ \langle\langle !_{\Omega\times\Omega}, \pi'\rangle, \pi\rangle \circ \pi \\ &= \beta(t)(\pi') \circ (!_\Omega \times id) \times id \circ \langle\pi, \pi \circ \pi\rangle \\ &= \beta(\Omega)(\pi') \circ \langle\pi, \pi \circ \pi \times id\rangle \\ &\qquad \text{by Beck-Chevalley, as before} \\ &= \beta(\Omega)(\pi' \circ \langle\pi, \pi\rangle) \\ &\qquad \text{by naturality} \\ &= \beta(\Omega)(\pi). \end{aligned}$$

Hence one obtains

$$\begin{aligned} \lambda(app) &= \alpha(\Omega)(app \circ \pi) \circ \langle id, c_0 \circ !\rangle \\ &= \alpha(\Omega)(\beta(\Omega)(\pi)) \circ \langle id, c_0 \circ !\rangle \\ &= \pi \circ \langle id, c_0 \circ !\rangle \\ &= id. \qquad \square \end{aligned}$$

5.5.8. Examples. (i) Suppose $\mathbf{B}$ is a CCC which has a reflexive object Ω. The latter means that there are maps $F : \Omega \to \Omega^\Omega$ and $G : \Omega^\Omega \to \Omega$ with $F \circ G = id$. Such structures are used by Scott and Koymans for the semantics of the untyped λ-calculus. Using the above lemma one easily obtains a λ-category $(\mathbf{B}, \Omega)$; one defines

$$\begin{aligned} app &= ev \circ F \times id \\ \lambda(f) &= G \circ \Lambda(f). \end{aligned}$$

This yields the required equations.

$$\begin{aligned} app \circ \lambda(f) \times id &= ev \circ F \times id \circ (G \circ \Lambda(f)) \times id \\ &= ev \circ \Lambda(f) \times id \\ &= f. \\ \lambda(f \circ g \times id) &= G \circ \Lambda(f \circ g \times id) \\ &= G \circ \Lambda(f) \circ g \\ &= \lambda(f) \circ g. \end{aligned}$$

Moreover, in case $(\mathbf{B}, \Omega)$ is extensional in the sense of Scott and Koymans — which means that $G \circ F = id$ and thus $\Omega^\Omega \cong \Omega$ — then

$$\begin{aligned} \lambda(app) &= G \circ \Lambda(ev \circ F \times id) \\ &= G \circ F \\ &= id. \end{aligned}$$

Notice that a categorical λ-algebra as it is used here is "more economical" than the structure used by Scott and Koymans: in our case the base category $\mathbf{B}$ need not have exponents (see the discussion at the end of this section).

(ii) We investigate what app and λ are in the examples in 5.5.6. In the first case one has $app : 1 + 1 \to 1$ as a continuous function $D \times D \to D$ described by $(x, y) \mapsto x \cdot y$. For $f : n + 1 \to 1$ in $\mathbf{D}$ one has $\lambda(f) = \boldsymbol{\lambda}\vec{x}.\lambda y.\, f(\vec{x}, y)$. This is as one would expect.

In the second case one starts from a (set-theoretical) λ-algebra. One has $app = \lambda xy.\, xy \in (D^2 \to D)$. If $a \in (D^{n+1} \to D)$ then $\lambda(a) = \lambda x_1 \ldots x_n.\, \lambda y.\, a x_1 \cdots x_n y = \mathbf{1}_n \cdot a$, which, indeed is in $(D^n \to D)$.

The formulation obtained in lemma 5.5.7 in terms of app and λ is quite practical. It will be extended to morphisms.

5.5.9. Lemma. *Let $(\mathbf{B}, \Omega)$ and $(\mathbf{B}', \Omega')$ be categorical λ-algebras. A functor $K : \mathbf{B} \to \mathbf{B}'$ is a morphism of categorical λ-algebras if and only if*

- $K\Omega = \Omega'$ *and* $K(!_A) = !_{KA}$;
- $K(\pi_{A,\Omega}) = \pi_{KA,\Omega'}$ *and* $K(\pi'_{A,\Omega}) = \pi'_{KA,\Omega'}$;
- $K(app) = app'$ *and* $K(\lambda(f)) = \lambda'(Kf)$.

Proof. We have to show that the third requirement above is equivalent to the third requirement in definition 5.5.4 (ii); the latter boils down to $K\alpha(A) = \alpha'(KA)K$ and $K\beta(A) = \beta'(KA)K$. Thus, using the definitions of $\alpha(A)$ and $\beta(A)$ from the proof of 5.5.7, the (if)-part is easily established.

In the reverse direction, one obtains $K(app) = app'$ and $K(\lambda(f)) = \lambda'(Kf)$ for the description of app and λ in the same proof. One has to use that $\lambda(f)$ does not depend on the constant c_0 occurring in the definition of $\lambda(f)$. $\square$

Let $(\mathbf{B}, \Omega)$ be a categorical λ-algebra. For $a, b \in \mathbf{B}(A, \Omega)$ put $a \cdot b = app \circ \langle a, b\rangle$. We write $\|\Omega\|$ for the (non-empty) collection $\mathbf{B}(t, \Omega)$ and claim that $\langle\|\Omega\|, \cdot\rangle$ is a λ-algebra as described in Barendregt [1984]. Abstraction is done as follows. For a term $a(x) : t \times \Omega \to \Omega$ containing a free variable x one takes

$$\lambda x.\, a(x) = \lambda(a(x)) : t \to \Omega.$$

Then

$$\begin{aligned} \big(\lambda x.\, a(x)\big) \cdot b &= app \circ \langle\lambda(a(x)), b\rangle \\ &= app \circ \lambda(a(x)) \times id \circ \langle id, b\rangle \\ &= a(x) \circ \langle id, b\rangle \\ &= a(b). \end{aligned}$$

Let's write

$$\pi_i^n : t \times \underbrace{\Omega \times \cdots \times \Omega}_{n\text{times}} \to \Omega$$

for the i-the projection. One has

$$\begin{array}{rcl} \boldsymbol{K} & = & \lambda(\lambda(\pi_1^2)) \\ \boldsymbol{S} & = & \lambda(\lambda(\lambda(\,(\pi_1^3\cdot\pi_3^3)\;\cdot\;(\pi_2^3\cdot\pi_3^3)\,))) \\ \boldsymbol{I} & = & \lambda(\pi_1^1) \\ \mathbf{1} & = & \lambda(\lambda(\,\pi_1^2\cdot\pi_2^2\,)) \end{array}$$

which yields essentially de Bruijn's nameless notation.

Notice that for $a \in \|\Omega\|$ one has $\mathbf{1}\cdot a = \lambda y.\, a\cdot y = \lambda(app \circ a \times id)$. Hence if $(\mathbf{B}, \Omega)$ is a categorical λ-model, one obtains the (ξ)-rule.

$$\begin{array}{ll} \multicolumn{2}{l}{\forall x \in \|\Omega\|.\;\; a\cdot x = b\cdot x} \\ \Rightarrow & \forall x : t \to \Omega.\;\; app \circ a \times id \circ \langle id,\, x\rangle = app \circ b \times id \circ \langle id,\, x\rangle \\ \Rightarrow & app \circ a \times id = app \circ b \times id, \qquad \text{since } t \times \Omega \cong \Omega \text{ has enough points} \\ \Rightarrow & \mathbf{1}\cdot a = \mathbf{1}\cdot b. \end{array}$$

And if $(\mathbf{B}, \Omega)$ is a categorical $\lambda\eta$-algebra, then (η) holds.

$$\begin{array}{rcl} \lambda y.\, a\cdot y & = & \lambda(app \circ a \times id) \\ & = & \lambda(app)\;\circ\; a \\ & = & a. \end{array}$$

Let's write $\boldsymbol{\lambda}$**-Alg** for the category with (set theoretical) λ-algebras $\langle D,\,\cdot,\,\boldsymbol{K},\,\boldsymbol{S}\rangle$ as objects; we allow D to be a collection of arbitrary size. Morphisms are maps between the underlying collections preserving application and $\boldsymbol{K}, \boldsymbol{S}$, see Barendregt [1984], 5.2.2 (ii).

The assignment $(\mathbf{B}, \Omega) \mapsto \langle \|\Omega\|, \cdot\,\rangle$ forms the object-part of a "forgetful" functor $U : \textbf{Cat-}\boldsymbol{\lambda}\textbf{-Alg} \longrightarrow \boldsymbol{\lambda}\textbf{-Alg}$: for a morphism $K : (\mathbf{B}, \Omega) \to (\mathbf{B}', \Omega')$ of categorical λ-algebras, one has $UK : \|\Omega\| \to \|\Omega'\|$ defined by $a \mapsto Ka$. By lemma 5.5.9, K preserves app and λ on-the-nose; hence UK is a morphism of λ-algebras.

5.5.10. Theorem. *The forgetful functor U : $\textbf{Cat-}\boldsymbol{\lambda}\textbf{-Alg} \longrightarrow \boldsymbol{\lambda}\textbf{-Alg}$ has a left adjoint; the unit of the adjunction is an identity.*

Proof. The object-part of a functor $F : \boldsymbol{\lambda}\textbf{-Alg} \longrightarrow \textbf{Cat-}\boldsymbol{\lambda}\textbf{-Alg}$ is described in example 5.5.6 (ii). For a morphism of λ-algebras $h : \langle D, \cdot\,\rangle \longrightarrow \langle D', \cdot'\,\rangle$ one defines $Fh : (\mathbf{D}, 1) \to (\mathbf{D}', 1)$ by $n \mapsto n$ and $(a_1, \ldots, a_m) \mapsto (h(a_1), \ldots, h(a_m))$. By lemma 5.5.9 and proposition 5.1.14 (i) from Barendregt [1984], h preserves the relevant structure. Notice that the underlying collection of $UF(\langle D, \cdot\,\rangle)$ is $\|1\| = \mathbf{D}(0,\,1) = (D^0 \to D) = D$. One obtains $UF = id$.

A counit $\varepsilon : FU(\mathbf{B}, \Omega) \to (\mathbf{B}, \Omega)$ is defined on objects by $n \mapsto \Omega^n$. To define it on morphisms, we need some notation. For an element $a \in \|\Omega\|$ we define $a^{(n)} : \Omega^n \to \Omega$ by $a^{(n)} = (a \circ !_{\Omega^n}) \cdot \pi_1^n \cdot \ldots \cdot \pi_n^n$ where $\pi_i^n : \Omega^n \to \Omega$ is the i-th projection.

On a morphism $(a_1, \ldots, a_m) : n \to m$ in $FU(\mathbf{B}, \Omega)$ — where $a_i \in (\|\Omega\|^n \to \|\Omega\|)$ — we put $\varepsilon(a_1, \ldots, a_m) = \langle a_1^{(n)}, \ldots, a_m^{(n)}\rangle : \Omega^n \to \Omega^m$. One has $\varepsilon(\lambda x_1 \ldots x_n.\, x_i) = ((\lambda x_1 \ldots x_n.\, x_i) \circ !)\cdot\pi_1^n\cdot\ldots\cdot\pi_n^n = \pi_i^n$. Hence ε preserves identities and the projections $n \leftarrow n+1 \to 1$. Composition is preserved since

$$\varepsilon(\lambda x_1 \ldots x_n.\, b_i(a_1 x_1 \ldots x_n) \ldots (a_m x_1 \ldots x_n)) \;\;=\;\; b_i^{(m)} \circ \langle a_1^{(n)}, \ldots, a_m^{(n)}\rangle$$

In order to show that ε is a morphism of categorical λ-algebras it suffices by lemma 5.5.9 to check

$$\begin{array}{rcll} \varepsilon(app) & = & (\lambda xy.\, xy)^{(2)} & \text{see 5.5.6 (ii)} \\ & = & \pi_1^2\;\cdot\;\pi_2^2 & \\ & = & app\;\circ\;\langle \pi, \pi'\rangle & \\ & = & app. & \end{array}$$

and for $a \in (\|\Omega\|^{n+1} \to \|\Omega\|)$,

$$\begin{array}{rcll} \varepsilon(\lambda(a)) & = & \varepsilon(\mathbf{1}_n \cdot a) & \text{see 5.5.6 (ii)} \\ & = & (a \circ !)\cdot\pi_1^n\cdot\ldots\cdot\pi_n^n & \\ & = & \lambda x.\,(a \circ !)\cdot\pi_1^n\cdot\ldots\cdot\pi_n^n\cdot x & \text{since } \mathbf{1}_{n+1}\cdot a = a \\ & = & \lambda(app \circ ((a \circ !)\cdot\pi_1^n\cdot\ldots\cdot\pi_n^n) \times id) & \\ & = & \lambda((a \circ !)\cdot\pi_1^{n+1}\cdot\ldots\cdot\pi_{n+1}^{n+1}) & \\ & = & \lambda(\varepsilon(a)). & \end{array}$$

Finally, the triangular identities boil down to

$$\varepsilon F \;=\; id \qquad \text{and} \qquad U\varepsilon \;=\; id.$$

These are easily verified. $\square$

The pattern obtained here is the same as established in Jacobs [1991], 7.4.3 for the second order λ-calculus $\lambda 2$: the functor from categorical to set theoretical models has a left-adjoint-right-inverse.

The next two theorems deal with some categorical properties of categorical λ-algebras.

5.5.11. Theorem. *Let $(\mathbf{B}, \Omega)$ be a categorical λ-algebra. By definition $Cons_\Omega$ has semi-products; it also has semi-sums.*

Proof. The standard (non-surjective) pairing from λ-calculus yields "combinators" $fst, snd : \Omega \to \Omega$ and $pair : \Omega \times \Omega \to \Omega$ satisfying $fst \circ pair = \pi$ and $snd \circ pair = \pi'$. In λ-calculus notation, $fst(z) = z\boldsymbol{K}, snd(z) = z\boldsymbol{K}'$ — where $\boldsymbol{K}' = \lambda xy.\, y$ — and $pair(x, y) = \lambda z.\, zxy$. A bit more categorically, $fst = id_\Omega \cdot (\boldsymbol{K} \circ !_\Omega), snd = id_\Omega \cdot (\boldsymbol{K}' \circ !_\Omega)$ and $pair = \lambda(\pi' \cdot (\pi \circ \pi) \cdot (\pi' \circ \pi))$.

For the semi-adjunctions $\Sigma_{(A,\Omega)} \dashv_s Cons_\Omega(A,\Omega)^*$, maps

$$\mathbf{B}(\,A\times\Omega,\;\;\Omega\,) \underset{\beta(A)}{\overset{\alpha(A)}{\rightleftarrows}} \mathbf{B}(\,(A\times\Omega)\times\Omega,\;\;\Omega\,)$$

are required. One takes

$$\begin{aligned}\alpha(A)(f) &= f\circ\langle\pi\circ\pi,\ pair\circ\langle\pi'\circ\pi,\ \pi'\rangle\rangle\\ \beta(A)(g) &= g\circ\langle\langle\pi,\ fst\circ\pi'\rangle,\ snd\circ\pi'\rangle.\end{aligned}$$

Then $\alpha(A)\circ\beta(A) = id$. $\square$

5.5.12. Theorem. *Let* $(\mathbf{B},\Omega)$ *be a categorical* λ*-algebra. We write* $Fst = cod \circ Cons_\Omega : \mathbf{B}/\!/\Omega \to \mathbf{B}$ *for the fibration involved. Then*

(i) *Fst is a fibred monoid, i.e. all fibre categories are monoids;*

(ii) *Fst is a fibred semi-CCC, i.e. all fibre categories are semi-CCC's and reindexing preserves this structure.*

Proof. (i) Obvious, since one starts from a single type Ω.

(ii) Remember (from 4.1.3) that composition in the fibre categories $(\mathbf{B}/\!/\Omega)_A$ is described by $g\bullet f = g\circ\langle\pi,\ f\rangle$. We define the semi-CCC structure, see Hayashi [1985].

(1) $!' = c_0\circ !_{A\times\Omega} : A\times\Omega\to\Omega$, where $c_0 : t\to\Omega$ is an arbitrary constant; then $!'\bullet f = c_0\circ !_{A\times\Omega}\circ\langle\pi,\ f\rangle = c_0\circ !_{A\times\Omega} = !'$.

(2) $\pi_0 = fst\circ\pi'$, $\pi_1 = snd\circ\pi' : A\times\Omega\to\Omega$. Further, for $f,g : A\times\Omega\to\Omega$ one takes $\langle\langle f,g\rangle\rangle = pair\circ\langle f,g\rangle$, see the proof of the previous result for the combinators fst, snd and $pair$. One has $\pi_0\bullet\langle\langle f,g\rangle\rangle = f$, $\pi_1\bullet\langle\langle f,g\rangle\rangle = g$ and $\langle\langle f,g\rangle\rangle\bullet h = \langle\langle f\bullet h,\ g\bullet h\rangle\rangle$.

(3) $ev = app\circ\langle fst,\ snd\rangle\circ\pi' : A\times\Omega\to\Omega$. For $f : A\times\Omega\to\Omega$ one takes $\Lambda(f) = \lambda(f\circ\langle\pi\circ\pi,\ pair\circ\pi\times id\rangle)$. Then $ev\bullet\langle\langle\Lambda(f)\bullet g,\ h\rangle\rangle = \Lambda(f)\bullet\langle\langle g,h\rangle\rangle$, $\Lambda(f\bullet\langle\langle g\bullet\pi_0,\ \pi_1\rangle\rangle) = \Lambda(f)\bullet g$ and $ev\bullet\langle\langle\pi_0,\ \pi_1\rangle\rangle = ev$. $\square$

The previous theorem indicates how to obtain a "CCC with reflexive object" from a categorical λ-algebra. The next two facts should be used.

- Taking the Karoubi envelope of a semi-CCC $\mathbf{C}$ yields a CCC $K(\mathbf{C})$, see Hayashi [1985].
- If $\Omega\cong\Omega^\Omega$ in a semi-CCC $\mathbf{C}$, then id_Ω is a reflexive object in $K(\mathbf{C})$. The latter is easily verified.

Obviously, the object, say Ω, of a monoid semi-CCC satisfies $\Omega\cong\Omega^\Omega$. Hence taking the Karoubi envelope of one of the fibre categories of a categorical λ-algebra yields a CCC with a reflexive object.

Finally we are in a position to compare our new notion of "monoid constant comprehension category with semi-products" with the "CCC with reflexive object" as used by Scott and Koymans. We mention the advantages of our approach.

- It captures "untyped" as the monoid-case in a "typed world". Explicitly: constant comprehension categories describe simply typed λ-calculi, i.e. calculi on the minimal setting. *Monoid* constant comprehension categories describe the untyped λ-calculus. This follows a general categorical understanding of "untyped".
- It describes the β- (plus naturality-) rules by semi-adjunctions and the additional η-rule by ordinary adjunctions. This also fits into a general categorical pattern, see e.g. Hayashi [1985], Jacobs [1991].
- It gives rise to the adjointness in theorem 5.5.10 between categorical and set theoretical models. In the Scott-Koymans approach, turning a CCC with reflexive object first into a λ-algebra and then again into a category yields incomparable results. This is due to the fact that the Karoubi envelope introduces unnecessary junk, see Koymans [1984], Barendregt [1984].
- It enables a direct and uniform presentation of concrete examples, see 5.5.6 (i),(ii). In order to present (ii) as CCC with a reflexive object, one first has to take the Karoubi envelope.

However, we have to concede that the notion of a CCC with a reflexive object is more elementary.

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Index

Samenvatting

Het onderhavige proefschrift is opgebouwd uit vijf hoofdstukken. Het eerste gaat over indicering van categorieën. De typentheoretische motivatie ligt in het feit dat een context een index vormt voor de categorie van typen en termen die afleidbaar zijn in die context. Het centrale begrip is 'vezeling' (fibration, in het Engels) zoals geïntroduceerd door Grothendieck. Een aantal elementaire definities en resultaten wordt besproken. Zijdelings worden twee alternatieve vormen van indicering beschreven: 'geïndiceerde categorieën' en 'interne categorieën'.

In het tweede hoofdstuk komt typentheorie aan de orde. Gebaseerd op een categorische intuïtie wordt het typentheoretische begrip 'achtergrond' (in het Engels, setting) ingevoerd. Een achtergrond bestaat uit een verzameling soorten voorzien van een transitieve relatie die beschrijft wat afhankelijk mag zijn van wat. Een achtergrond kan bijvoorbeeld bepalen dat een propositie af mag hangen van een type, dat wil zeggen, dat een propositie een variabele van een type mag bevatten. Een achtergrond bepaalt tevens welke 'aspecten' (features, in het Engels) toelaatbaar zijn. Voorbeelden van aspecten zijn exponenten, producten, sommen en identiteiten. Om bijvoorbeeld afhankelijke producten te kunnen vormen moet de achtergrond waartegen men werkt betreffende afhankelijkheid bevatten. Aldus wordt een typensysteem begrepen als een achtergrond plus een aantal daardoor toegestane aspecten. Verschillende bekende systemen worden zo opnieuw beschreven. Dit vergemakkelijkt de overgang naar een categorische beschrijvingswijze.

Een achtergrond kent typenafhankelijkheid indien er een soort is die van zichzelf afhangt. Achtergronden zonder deze eigenschap zijn categorisch eenvoudig: contexten kunnen simpelweg als cartesische producten beschreven worden. De systemen $\lambda\!\rightarrow$, $\lambda 2$, $\lambda\underline{\omega}$ en $\lambda\omega$ die het linkervlak van Barendregt's cubus vormen hebben eenzelfde achtergrond zonder typenafhankelijkheid. Beschrijving van de bijbehorende categorieën vindt men in hoofdstuk drie.

Achtergronden met typenafhankelijkheid zijn iets minder eenvoudig te beschrijven. In hoofdstuk vier wordt de benodigde theorie ontwikkeld. Het centrale begrip hier is 'comprehensie categorie'. Zo'n structuur bescrijft de organisatie van contexten, die nu niet meer als cartesische producten begrepen kunnen worden: vanwege de afhankelijkheid is een vorm van disjuncte vereniging vereist. Een comprehensie categorie geeft een passende categorische beschrijving van zulke disjuncte verenigingen en de bijbehorende projecties. Verder wordt een algemeen begrip van quantificatie voor vezelingen beschreven in termen van comprehensie categorieën.

Deze twee ingrediënten worden aan een gedetailleerd onderzoek onderworpen. De resulterende inzichten worden vervolgens in het vijfde hoofdstuk aangewend: eerst om een algemene schets te geven van de omzetting van typentheoretische achtergronden en aspecten in overeenkomstige categorische; daarna om enkele individuele typensystemen categorisch te beschrijven; tenslotte om de categorische semantiek van de ongetypeerde lambda calculus te herzien. Zogenaamde 'constante' comprehensie categorieën met één type geven een adequate beschrijving.

Curriculum Vitae

De auteur van dit proefschrift is geboren op 2 augustus 1963 te Nuenen (N. Br.). In dezelfde plaats doorliep hij de lagere school. In Eindhoven volgde hij van 1975 tot 1981 de gymnasium-β opleiding aan het Augustinianum.

Als vervolgstudies werd gekozen voor wiskunde in combinatie met wijsbegeerte aan de Katholieke Universiteit Nijmegen. Het candidaats- en vervolgens het doctoraal-examen wiskunde zijn behaald op 1 september 1983 en 27 augustus 1987; het laatste cum laude. Bepalende docenten waren de Iongh, Veldman en Barendregt. In de wijsbegeerte is het candidaatsexamen afgelegd op 30 november 1984 en het doctoraalexamen op 10 juni 1988. Vermeldenswaard zijn hier de docenten Sundholm en Boukema.

Inmiddels was de schrijver per 1 september 1987 als toegevoegd onderzoeker in dienst getreden bij de afdeling Grondslagen van de Informatica van de KUN — in eerste instantie zonder volledige aanstelling. Onderbroken door detachering aan de universiteiten van Pisa en Cambridge is tot aan de zomer van 1991 in Nijmegen de promotie voorbereid.