JOURNAL OF ALGEBRA **192,** 460–475 (1997)
ARTICLE NO. JA966947

$\mathsf{D}_n(A)$ for a Class of Polynomial Automorphisms and Stably Tameness

Arno van den Essen and Engelbert Hubbers

Dept. Wiskunde, Katholicke Universiteit Nijmegen, Postbus 9010, Nijmegen 6500 GL, The Netherlands

Communicated by Wilberd van der Kallen

Received September 12, 1996

In this paper we introduce a set, denoted by $\mathsf{D}_n(A)$, for every commutative ring A and every positive integer n. It is shown that the elements of this set can be used to give an explicit description of the class $\mathsf{H}_n(A)$ introduced in van den Essen and Hubbers [*J. Algebra* **187** (1997), 214–226]. We deduce that each polynomial map of the form $F = X + H$ with $H \in \mathsf{H}_n(A)$ can be written as a finite product of automorphisms of the form $\exp(D)$, where each D is a locally nilpotent derivation satisfying $D^2(X_i) = 0$ for all i. Furthermore we deduce that all such Fs are stably tame.

1. NOTATION, DEFINITIONS, AND AN EXPLICIT DESCRIPTION OF THE CLASS $\mathsf{H}_n(A)$

1.1. *Notation*

Throughout this paper A denotes an arbitrary commutative ring and $A[X] := A[X_1, \ldots, X_n]$ denotes the polynomial ring in n variables over A. Furthermore if $G = (G_1, \ldots, G_n) \in A[X]^n$ and $S = (S_{ij}(X)) \in M_{p,q}(A[X])$ then $S(G)$ or $S_{|G}$ denotes the $p \times q$ matrix $(S_{ij}(G_1, \ldots, G_n))_{i,j}$. In particular if $F \in A[X]^n$ $(= M_{n,1}(A[X]))$ then the composition of the polynomial maps F and G, denoted $F \circ G$, is equal to $F(G)$.

Matrix multiplication will be denoted by the symbol '$*$.' So if S, $T \in M_n(A[X])$ then the matrix product of S and T is denoted by $S * T$. By X we denote the column vector $(X_1, \ldots, X_n)^t$. In the sequel we also need another multiplication in $M_n(A[X])$, which we denote by $\triangle$. This multipli-

0021-8693/97 $25.00

cation is defined as follows:

$$S \triangle T := S(T * X) * T$$

for all $S, T \in M_n(A[X])$.

One easily verifies that this multiplication is associative, so it makes sense to write

$$S_1 \triangle S_2 \triangle \cdots \triangle S_n$$

for each n-tuple $S_1, \ldots, S_n$ in $M_n(A[X])$. Sometimes we need to extend a vector of length $1 \le p \le n-1$ or a $p \times p$ matrix to, respectively, a vector of length n, or an $n \times n$ matrix. This is done as follows: let $1 \le p \le n-1$, $c \in A[X]^p$, and $T \in M_p(A[X])$. Then $\tilde{c}^n$ denotes the vector

$$\tilde{c}^n = \begin{pmatrix} c \\ 0 \\ \vdots \\ 0 \end{pmatrix} \in A[X]^n,$$

obtained by extending c by $n-p$ zeros, and $\tilde{T}^n$ denotes the matrix

$$\tilde{T}^n = \begin{pmatrix} T & 0 \\ 0 & I_{n-p} \end{pmatrix} \in M_n(A[X]),$$

obtained by extending T with the $(n-p) \times (n-p)$ identity matrix. To simplify the notation we drop the superscript n and write $\tilde{c}$ and $\tilde{T}$, even sometimes when it is clear from the context that we mean $\tilde{c}^{n-1}$, respectively, $\tilde{T}^{n-1}$ instead of $\tilde{C}^n$, respectively, $\tilde{T}^n$.

Finally the adjoint of a matrix T is denoted by $\mathrm{Adj}(T)$ and if $a_1, \ldots, a_p$ are elements of a (nonnecessary commutative) ring then $\prod_{i=1}^{p} a_i$ denotes the element $a_1 \cdots a_p$.

1.2. $\mathsf{D}_n(A)$ *and the class* $\mathsf{H}_n(A)$

In [6] we introduced a new class of polynomial maps, denoted by $\mathsf{H}_n(A)$, and showed that for each $H \in \mathsf{H}_n(A)$ the Jacobian matrix JH is nilpotent and that the polynomial map $F = X + H$ is invertible over A with $\det(JF) = 1$.

Let us recall the definition of $\mathsf{H}_n(A)$.

DEFINITION 1.1. First if $n = 1$ we define $\mathsf{H}_1(A) = A$. If $n \ge 2$ we define $\mathsf{H}_n(A)$ inductively as follows: Let $H \in A[X]^n$. Then $H \in \mathsf{H}_n(A)$ if and only if there exist $T \in M_n(A)$, $c \in A^n$, and $H_\star \in \mathsf{H}_{n-1}(A[X_n])$ such that

$$H = \mathrm{Adj}(T) * \begin{pmatrix} H_* \\ 0 \end{pmatrix}_{|T * X} + c. \tag{1}$$

The main aim of this section is to give an explicit description of the elements of $\mathsf{H}_n(A)$. Therefore we introduce some useful objects.

DEFINITION 1.2. Let $n \geq 2$. Then $\mathsf{D}_n(A)$ is the set of $(2n-1)$-tuples

$$(T, c) := (T_2, \ldots, T_n, c_1, \ldots, c_n),$$

where $T_n \in M_n(A)$, $T_i \in M_i(A[X_{i+1}, \ldots, X_n])$ for all $2 \leq i \leq n-1$, $c_n \in A^n$ $(= M_{n,1}(A))$ and $c_i \in M_{i,1}(A[X_{i+1}, \ldots, X_n])$ for all $1 \leq i \leq n-1$.

If $n \geq 3$ we get a natural map $\pi\colon \mathsf{D}_n(A) \to \mathsf{D}_{n-1}(A[X_n])$ defined by

$$\pi\big((T_2, \ldots, T_n\, c_1, \ldots, c_n)\big) = (T_2, \ldots, T_{n-1}, c_1, \ldots, c_{n-1}).$$

Instead of $\pi((T, c))$ we often write (T', c').

DEFINITION 1.3. Let $n \geq 2$ and $0 \leq p \leq n-2$. Then

$$E_{n,p}\colon \mathsf{D}_n(A) \to A[X]^n$$

is given by

1. $E_{n,0}((T, c)) := \mathrm{Adj}(T_n) * \tilde{c}_{n-1\,|\,T_n * X}$ for all $(T, c) \in \mathsf{D}_n(A)$.
2. If $n \geq 3$ and $1 \leq p \leq n-2$, then inductively (with respect to n)

$$E_{n,p}((T, c)) := \mathrm{Adj}(T_n) * \begin{pmatrix} E_{n-1,p-1}((T', c')) \\ 0 \end{pmatrix}_{|T_n * X}$$

Instead of $E_{n,p}((T, c))$ we simply write $E_{n,p}(T, c)$.

Now we are able to give the main result of this section.

PROPOSITION 1.4. *Let $n \geq 2$ and $H \in A[X]^n$. Then $H \in \mathsf{H}_n(A)$ if and only if there exists $(T, c) \in \mathsf{D}_n(A)$ such that*

$$H = \sum_{p=0}^{n-2} E_{n,p}(T, c) + c_n.$$

Proof. The proof is by induction on n. The case $n = 2$ is obvious, so let $n \geq 3$. Then

$$H = \mathrm{Adj}(T_n) * \begin{pmatrix} H_\star \\ 0 \end{pmatrix}_{|T_n * X} + c_n,$$

where $T_n \in M_n(A)$, $c_n \in A^n$, and $H_\star \in \mathsf{H}_{n-1}(A[X_n])$. So by the induction hypothesis we have

$$H_\star = \sum_{p=0}^{n-3} E_{n-1,p}(T^*, c^*) + c^*_{n-1}$$

for some $(T^*, c^*) \in \mathsf{D}_{n-1}(A[X_n])$. Put $(T, c) := (T^*, T_n, c^*, c_n)$ and observe that $(T, c) \in \mathsf{D}_n(A)$ and $(T', c') = (T^*, c^*)$. So

$$\begin{aligned} H &= \sum_{p=0}^{n-3} \operatorname{Adj}(T_n) * E_{n-1,p}(T', c')_{|T_n * X} + \operatorname{Adj}(T_n) * \begin{pmatrix} c^*_{n-1} \\ 0 \end{pmatrix}_{|T_n * X} + c_n \\ &= \sum_{p=1}^{n-2} E_{n,p}(T, c) + E_{n,0}(T, c) + c_n \\ &= \sum_{p=0}^{n-2} E_{n,p}(T, c) + c_n. \end{aligned}$$

∎

PROPOSITION 1.5. *Let $n \geq 2$, $0 \leq p \leq n-2$, and $(T, c) \in \mathsf{D}_n(A)$. Then*

$$E_{n,p}(T, c) = \operatorname{Adj}\Big(\tilde{T}_{n-p} \vartriangle \cdots \vartriangle \tilde{T}_{n-1} \vartriangle T_n\Big) * \tilde{c}_{n-p-1 \,|\, (\tilde{T}_{n-p} \vartriangle \cdots \vartriangle \tilde{T}_{n-1} \vartriangle T_n) * X}.$$

Proof. The proof is by induction on p. The case $p = 0$ is obvious. So let $p \geq 1$. Then

$$\begin{aligned} E_{n,p}(T, c) &= \operatorname{Adj}(T_n) * \begin{pmatrix} E_{n-1,p-1}(T', c') \\ 0 \end{pmatrix}_{|T_n * X} \\ &= \operatorname{Adj}(T_n) * \Big[\operatorname{Adj}\Big(\tilde{T}_{n-p} \vartriangle \cdots \vartriangle \tilde{T}_{n-1}\Big)_{|T_n * X} \\ &\qquad * \Big(\tilde{c}_{n-p-1 \,|\, (\tilde{T}_{n-p} \vartriangle \cdots \vartriangle \tilde{T}_{n-1}) * X}\Big)_{|T_n * X}\Big] \\ &\qquad\qquad \text{(by the induction hypothesis)} \\ &= \operatorname{Adj}\Big(\Big(\tilde{T}_{n-p} \vartriangle \cdots \vartriangle \tilde{T}_{n-1}\Big)_{|T_n * X} * T_n\Big) \\ &\quad * \tilde{c}_{n-p-1}\Big(\Big(\tilde{T}_{n-p} \vartriangle \cdots \vartriangle \tilde{T}_{n-1}\Big)_{|T_n * X}\Big) * T_n * X \\ &= \operatorname{Adj}\Big(\tilde{T}_{n-p} \vartriangle \cdots \vartriangle \tilde{T}_{n-1} \vartriangle T_n\Big) * \tilde{c}_{n-p-1 \,|\, (\tilde{T}_{n-p} \vartriangle \cdots \vartriangle \tilde{T}_{n-1} \vartriangle T_n) * X}. \end{aligned}$$

∎

EXAMPLE 1.6. Consider the polynomial map $F := X + H \colon \mathbb{C}^4 \to \mathbb{C}^4$, where H equals

$$\begin{pmatrix} -X_2X_4^2 - e_4X_3^2X_4 - 2\dfrac{m_4}{g_4}X_2X_3X_4 - g_4X_1X_3X_4 - k_4X_3^3 - \dfrac{m_4^2}{g_4^2}X_2X_3^2 - m_4X_1X_3^2 \\ -X_3X_4^2 - e_3X_3^2X_4 + g_4X_2X_3X_4 - k_3X_3^3 + m_4X_2X_3^2 + g_4^2X_1X_3^2 \\ -\dfrac{1}{3}X_4^3 \\ 0 \end{pmatrix}$$

and $e_3, k_3, e_4, g_4, k_4, m_4 \in \mathbb{C}$ and $g_4 \neq 0$. This F is invertible. In fact if we take $P = P^{-1} = (X_4, X_3, X_2, X_1)$, we have that PFP is one of the eight representatives of the cubic homogeneous maps in dimension 4 as given by Hubbers [7] and also published in [4, Theorem 2.10].

Now consider the following element (T, c) of $D_4(\mathbb{C})$, where

$$T = \left(\begin{pmatrix} 1 & 0 \\ g_4^2 X_3 & g_4 X_4 + m_4 X_3 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \right)$$

and

$$c = \left(\begin{pmatrix} \frac{-1}{g_4^2} X_2 \end{pmatrix}, \begin{pmatrix} -X_3^2(e_4 X_4 + k_4 X_3) \\ -X_3 X_4^2 - e_3 X_4 X_3^2 - k_3 X_3^3 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ \frac{-1}{3} X_4^3 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \right).$$

Our claim is that

$$H = \sum_{p=0}^{2} E_{4,p}(T, c) + c_4.$$

To prove this we will compute $E_{4,0}$, $E_{4,1}$, and $E_{4,2}$ by the method of Proposition 1.5. Note that $c_4 = 0$. Since $T_4 = \tilde{T}_3 = I_4$, $E_{4,0}$ and $E_{4,1}$ are easy:

$$E_{4,0} = \mathrm{Adj}(T_4) * \tilde{c}_{3\,|\,T_4 * X} = \tilde{c}_3 = \begin{pmatrix} 0 \\ 0 \\ \frac{-1}{3} X_4^3 \\ 0 \end{pmatrix},$$

$$E_{4,1} = \mathrm{Adj}\big(\tilde{T}_3 \triangle T_4\big) * \tilde{c}_{2\,|\,(\tilde{T}_3 \triangle T_4) * X} = \tilde{c}_2 = \begin{pmatrix} -X_3^2(e_4 X_4 + k_4 X_3) \\ -X_3 X_4^2 - e_3 X_4 X_3^2 - k_3 X_3^3 \\ 0 \\ 0 \end{pmatrix}.$$

Before we compute $E_{4,2}$ we present the following identities:

$$\tilde{T}_2 \vartriangle \tilde{T}_3 \vartriangle T_4 = \tilde{T}_2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ g_4^2 X_3 & g_4 X_4 + m_4 X_3 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix};$$

$$\begin{aligned} &\operatorname{Adj}\left(\tilde{T}_2 \vartriangle \tilde{T}_3 \vartriangle T_4\right) \\ &= \begin{pmatrix} g_4 X_4 + m_4 X_3 & 0 & 0 & 0 \\ -g_4^2 X_3 & 1 & 0 & 0 \\ 0 & 0 & g_4 X_4 + m_4 X_3 & 0 \\ 0 & 0 & 0 & g_4 X_4 + m_4 X_3 \end{pmatrix}; \end{aligned}$$

$$\left(\tilde{T}_2 \vartriangle \tilde{T}_3 \vartriangle T_4\right) * X = \begin{pmatrix} X_1 \\ g_4^2 X_1 X_3 + g_4 X_2 X_4 + m_4 X_2 X_3 \\ X_3 \\ X_4 \end{pmatrix};$$

$$\tilde{c}_{1\,|\,(\tilde{T}_2 \vartriangle \tilde{T}_3 \vartriangle T_4) * X} = \begin{pmatrix} -X_1 X_3 - \dfrac{1}{g_4} X_2 X_4 - \dfrac{m_4}{g_4^2} X_2 X_3 \\ 0 \\ 0 \\ 0 \end{pmatrix};$$

and finally

$$\begin{aligned} E_{4,2} &= \operatorname{Adj}\left(\tilde{T}_2 \vartriangle \tilde{T}_3 \vartriangle T_4\right) * \tilde{c}_{1\,|\,(\tilde{T}_2 \vartriangle \tilde{T}_3 \vartriangle T_4) * X} \\ &= \begin{pmatrix} -\left(X_4 + \dfrac{m_4}{g_4} X_3\right)\left(g_4 X_1 X_3 + X_2 X_4 + \dfrac{m_4}{g_4} X_2 X_3\right) \\ X_3\left(g_4^2 X_1 X_3 + g_4 X_2 X_4 + m_4 X_2 X_3\right) \\ 0 \\ 0 \end{pmatrix}. \end{aligned}$$

It is easy to verify that $H = E_{4,0} + E_{4,1} + E_{4,2} + c_4$, which was our claim.

2. NICE DERIVATIONS

Let $B := A[x_1, \ldots, x_n]$ be a finitely generated A-algebra and D a subset of $\operatorname{Der}_A(B)$. By B^D we denote the set of all $b \in B$ such that $d(b) = 0$ for all $d \in D$.

DEFINITION 2.1. Let $D \subset \mathrm{Der}_A(B)$ be a finite subset and $\tau \in \mathrm{Der}_A(B)$.

1. We say that τ is *derived from D in at most one step* if τ is of the form $\tau = \sum_{d \in D} b_d d$, where $b_d \in B^D$ for all $d \in D$.

2. Let $m \geq 2$. We say that τ is *derived from D in at most m steps* if there exists a sequence of finite subsets

$$D = D_0, D_1, D_2, \ldots, D_m$$

of $\mathrm{Der}_A(B)$ such that $\tau \in D_m$ and all elements of D_i are derived from D_{i-1} in at most one step, for all $1 \leq i \leq m$. If furthermore the elements of D satisfy $d_1 d_2(x_i) = 0$ for all $d_1, d_2 \in D$ and all i, then τ is called *nice of order* $\leq m$, with respect to $x_1, \ldots, x_n$ and D.

PROPOSITION 2.2. *The notation is as in Definition* 2.1. *If* $d_1 d_2(x_i) = 0$ *for all* $d_1, d_2 \in D$ *and all* i, *then* $d_1 d_2(x_i) = 0$ *for all* $d_1, d_2 \in D_m$ *and all* i. *In particular* $d^2(x_i) = 0$ *for every nice derivation.*

Proof. We use induction on m. The case $m = 0$ is obvious since $D_0 = D$. Now let $m \geq 1$. Then $d_1 = \sum_{d \in D_{m-1}} b_d d$, $d_2 = \sum_{d' \in D_{m-1}} b'_{d'} d'$ with $b'_{d'}, b_{d'} \in B^{D_{m-1}}$. Then

$$d_1 d_2(x_i) = \sum_{d, d'} bd(b'_{d'}) d(x_i) + \sum_{d, d'} b_d b'_{d'}, dd'(x_i). \tag{2}$$

Now observe that $d(b'_{d'}) = 0$ since $b'_{d'} \in B^{D_{m-1}}$ and $d \in D_{m-1}$. Finally the induction hypothesis gives $dd'(x_i) = 0$ for all $d, d' \in D_{m-1}$ and all i, so (2) implies $d_1 d_2(x_i) = 0$. ∎

We demonstrate these aspects by the so-called Winkelmann derivation. See [11].

EXAMPLE 2.3. Let $\tau = (1 + X_4 X_2 - X_5 X_3)\partial_{X_1} + X_5 \partial_{X_2} + X_4 \partial_{X_3}$, a derivation on $B := A[X_1, X_2, X_3, X_4, X_5]$. Let $D = \{\partial_{X_1}, \partial_{X_2}, \partial_{X_3}\}$. Then τ is nice of order 2 with respect to X_1, X_2, X_3, X_4, X_5, and D. To show that this is true, we present a sequence of finite subsets of $\mathrm{Der}_A(B)$,

$$D = D_0, D_1, D_2.$$

Take $D_1 := \{\partial_{X_1}, X_5 \partial_{X_2} + X_4 \partial_{X_3}\}$ and $D_2 := \{\tau\}$. Note that in Definition

2.1 it is not demanded that the set D_i of this sequence is a subset of D_{i+1}. The only demand is that each D_i is a finite subset of $\mathrm{Der}_A(B)$. Since $X_4, X_5 \in B^D$ it follows immediately that ∂_{X_1} and $X_5\partial_{X_2} + X_4\partial_{X_3}$ are derived from D in one step. And from $1 + X_4X_2 - X_5X_3 \in B^{D_1}$ it follows that τ is derived from D_1 in one step. Obviously we have $d_1d_2(X_i) = 0$ for all $d_1, d_2 \in D$ and hence with Proposition 2.2 also $\tau^2(X_i) = 0$.

3. DERIVATIONS ASSOCIATED WITH POLYNOMIAL MAPS

The main aim of this section is to show that for each $0 \le p \le n-2$ the polynomial map $X + E_{n,p}(T,c)$ (where $(T,c) \in \mathbf{D}_n(A)$) is of the form $\exp(d)$, for some nice A-derivation d of $A[X]$. Observe that d is locally nilpotent if d is nice with respect to $X_1, \dots, X_n$ since $d^2(X_i) = 0$ for all i, by Proposition 2.2.

In order to prove this result (see Theorem 3.3), we need to generalise some of the notions of Sect. 1 to arbitrary finitely generated A-algebras. So let $B := A[x_1, \dots, x_n]$ be a finitely generated A-algebra, and let $\varphi\colon A[X_1, \dots, X_n] \to B$ be the A-ring homomorphism defined by $\varphi(X_i) = x_i$ for all i. For each $p, q \ge 1$ consider the natural extension

$$\varphi\colon M_{p,q}(A[X_1, \dots, X_n]) \to M_{p,q}(B).$$

Then for each $(T,c) \in \mathbf{D}_n(A)$ we define

$$E_{n,p}(T,c)(x) := \varphi\big(E_{n,p}(T,c)\big) \in B^n.$$

Now let $(\partial_1, \dots, \partial_n)$ be an n-tuple of A-derivations of B. With each vector $b = (b_1, \dots, b_n)^t \in B^n$ we associate the following A-derivation of B:

$$D(b; \partial_1, \dots, \partial_n) := b_1\partial_1 + \cdots + b_n\partial_n \qquad \left(= b^t * \begin{pmatrix} \partial_1 \\ \vdots \\ \partial_n \end{pmatrix} \right).$$

To formulate the next lemma we need some more notation: Let $(T,c) \in \mathbf{D}_n(A)$. Put

$$\begin{aligned}
(x_1', \dots, x_n')^t &:= T_n * (x_1, \dots, x_n)^t, \\
(\partial_1', \dots, \partial_n') &:= (\mathrm{Adj}(T_n))^t * (\partial_1, \dots, \partial_n)^t, \\
x'' &:= (x_1', \dots, x_{n-1}'), \\
(T'', c'') &:= (T'(X_n = x_n'), c'(X_n = x_n')) \in \mathbf{D}_{n-1}(A[x_n']).
\end{aligned}$$

LEMMA 3.1. *Let $n \geq 3$ and $1 \leq p \leq n-2$. Then*

$$D\big(E_{n,p}(T,c)(x);\, \partial_1, \ldots, \partial_n\big) = D\big(E_{n-1,p-1}(T'',c'')(x'');\, \partial'_1, \ldots, \partial'_{n-1}\big).$$

Proof.

$$\begin{aligned}
&D\big(E_{n,p}(T,c)(x);\, \partial_1, \ldots, \partial_n\big) \\
&\quad = \big(E_{n,p}(T,c)(x)\big)^t * \begin{pmatrix} \partial_1 \\ \vdots \\ \partial_n \end{pmatrix} \\
&\quad = \Big(\big(E_{n-1,p-1}(T',c')_{|T_n * x}\big)^t \quad 0\Big) * \big(\mathrm{Adj}(T_n)\big)^t * \begin{pmatrix} \partial_1 \\ \vdots \\ \partial_n \end{pmatrix} \\
&\quad = \Big(\big(E_{n-1,p-1}(T'',c'')(x'')\big)^t \quad 0\Big) * \begin{pmatrix} \partial'_1 \\ \vdots \\ \partial'_n \end{pmatrix} \\
&\quad = D\big(E_{n-1,p-1}(T'',c'')(x'');\, \partial'_1, \ldots, \partial'_{n-1}\big). \qquad \blacksquare
\end{aligned}$$

LEMMA 3.2. *The notation is as above. Let $a \in A$ and let $\partial_1, \ldots, \partial_n$ be A-derivations of B such that $\partial_i(x_j) = a\delta_{ij}$ for all i, j. Then*

$$\partial'_i(x'_j) = a \det(T_n)\, \delta_{ij}$$

for all i, j.

Proof. Denote the ith column of $\mathrm{Adj}(T_n)$ by $(t^*_{1i}, \ldots, t^*_{ni})^t$ and the jth row of T_n by $(t_{j1}, \ldots, t_{jn})$. Then

$$\begin{aligned}
\partial'_i(x'_j) &= \left(\sum_{s=1}^{n} t^*_{si}\partial_s\right)\left(\sum_{s=1}^{n} t_{js}x_s\right) \\
&= \sum_{s=1}^{n} a t^*_{si} t_{js} \\
&= a\big(T_n * \mathrm{Adj}(T_n)\big)_{ji} \\
&= a \det(T_n)\, \delta_{ij}. \qquad \blacksquare
\end{aligned}$$

Now we are able to prove:

THEOREM 3.3. *Let $\partial_1, \dots, \partial_n$ be A-derivations on $A[x_1, \dots, x_n]$ such that there exists an element $a \in A$ such that $\partial_i(x_j) = a\,\delta_{ij}$ for all i, j. Let $(T, c) \in \mathsf{D}_n(A)$. Then the A-derivation $d := D(E_{n,p}(T, c)(x); \partial_1, \dots, \partial_n)$ is nice with respect to $x_1, \dots, x_n$ and $D_0 := \{\partial_1, \dots, \partial_n\}$, for all $n \geq 2$ and all $0 \leq p \leq n - 2$.*

Proof. 1. The hypotheses on the ∂_i imply that $dd'(x_i) = 0$ for all $d, d' \in D_0$ and all i.

2. First we consider the case $p = 0$. Then

$$E_{n,0}(T, c) = \operatorname{Adj}(T_n) * \tilde{c}_{n-1\,|\,T_n * X}.$$

So

$$d = \left(\tilde{c}_{n-1\,|\,T_n * X}\right)^t * \left(\operatorname{Adj}(T_n)\right)^t * \begin{pmatrix} \partial_1 \\ \vdots \\ \partial_n \end{pmatrix}.$$

Write $\tilde{c}^t_{n-1} = (\gamma_1(X_n), \dots, \gamma_{n-1}(X_n), 0)$. Then the definition of x'_n and the ∂'_j imply that

$$d = \left(\gamma_1(x'_n), \dots, \gamma_{n-1}(x'_n), 0\right) * \left(\partial'_1, \dots, \partial'_n\right)^t = \sum_{i=1}^{n-1} \gamma_i(x'_n)\,\partial'_i. \tag{3}$$

Put $D_1 := \{\partial'_1, \dots, \partial'_{n-1}\}$ and observe that $D_1 \subset \operatorname{Der}_A(B)$ and that each element of D_1 is derived from D_0 in at most one step. Finally since $\partial'_i(x'_n) = 0$ for all $1 \leq i \leq n - 1$ (by Lemma 3.2) we get that $\gamma_i(x'_n) \in B^{D_1}$ for all $1 \leq i \leq n - 1$. So (3) implies that d is derived from D_1 in at most one step. Consequently d is derived from D_0 in at most two steps. So d is nice with respect to $x_1, \dots, x_n$ and D_0 by case 1.

3. Now we prove the theorem by induction on n. If $n = 2$, then $p = 0$ and we are in case 2. So let $n \geq 3$. By case 2 we may assume that $p \geq 1$. Then by Lemma 3.1 we have

$$d = D\left(E_{n-1,\,p-1}(T'', c'')(x''); \partial'_1, \dots, \partial'_{n-1}\right)$$

with $(T'', c'') \in D_{n-1}(A[x'_n])$. By Lemma 3.2 we can apply the induction hypothesis to the ring $A[x'_n]$ and the $(n-1)$-tuple of $A[x'_n]$-derivations $\partial'_1, \dots, \partial'_{n-1}$ on the $A[x'_n]$-algebra $B' := A[x'_n][x'_1, \dots, x'_{n-1}]$. So the $A[x'_n]$-derivation d on B' is nice with respect to $D'_0 := \{\partial'_1, \dots, \partial'_{n-1}\}$ and $x'_1, \dots, x'_{n-1}$. So there exists a sequence

$$D'_0, D'_1, \dots, D'_m$$

of finite subsets of $\mathrm{Der}_{A[x'_n]}(B')$ such that $d \in D'_m$ and D'_i is derived from D'_{i-1} in at most one step for all $1 \le i \le m$. Now observe that $D'_0 \subset \mathrm{Der}_A(B)$ and that $B' \subset B$ since by definition obviously $x'_i \in B$ for all i. Consequently if d' is an $A[x'_n]$-derivation of B' derived from D'_0 in at most one step, then $d' \in \mathrm{Der}_A(B)$. Hence $D'_1 \subset \mathrm{Der}_A(B)$. Arguing in a similar way we conclude by induction on i that $D'_i \subset \mathrm{Der}_A(B)$ for all $0 \le i \le m$. Since as remarked in case 2 above, all elements of D'_0 ($= D_1$ in case 2) are derived from D_0 in at most one step we deduce that d is derived from D_0 in at most $m+1$ steps. Just define $D_i := D'_{i-1}$ for all $1 \le i \le m+1$. Hence d is nice with respect to $x_1, \ldots, x_n$ and D_0 by 1. ∎

COROLLARY 3.4. *Let $(T, c) \in \mathsf{D}_n(A)$ and $0 \le p \le n-2$. Put*

$$D := D\left(E_{n,p}(T,c); \frac{\partial}{\partial X_1}, \ldots, \frac{\partial}{\partial X_n}\right).$$

Then D is nice with respect to $X_1, \ldots, X_n$ and $\{\partial/\partial X_1, \ldots, \partial/\partial X_n\}$. Furthermore we have $\exp(D) = X + E_{n,p}(T,c)$ and the inverse map is given by $\exp(-D) = X - E_{n,p}(T,c)$.

Proof. The first part is an immediate consequence of Theorem 3.3. Furthermore $D^2(X_i) = 0$ by Proposition 2.2. So $\exp(D)(X) = X + E_{n,p}(T,c)$ and the inverse map is given by $\exp(-D)(X) = X - E_{n,p}(T,c)$. ∎

4. THE MAIN THEOREM

In this section we show that for every $H \in \mathsf{H}_n(A)$ the polynomial map $F = X + H$ is a product of n polynomial automorphisms of the form $\exp(D)$, where each D is a nice derivation on $A[X]$. More precisely

THEOREM 4.1. *Let $F = X + H$, where $H = \sum_{p=0}^{n-2} E_{n,p}(T,c) + c_n$, for some $(T, c) \in \mathsf{D}_n(A)$. Then*

$$F = \exp\left(D\left(c_n; \frac{\partial}{\partial X_1}, \ldots, \frac{\partial}{\partial X_n}\right)\right) \times \prod_{p=0}^{n-2} \exp\left(D\left(E_{n,p}(T,c); \frac{\partial}{\partial X_1}, \ldots, \frac{\partial}{\partial X_n}\right)\right).$$

Proof. Observe that

$$\exp\left(-D\left(c_n; \frac{\partial}{\partial X_1}, \dots, \frac{\partial}{\partial X_n}\right)\right) \circ F = \sum_{p=0}^{n-2} E_{n,p}(T, c).$$

So the case $n = 2$ follows from Corollary 3.4. Hence we may assume that $n \geq 3$. Now Theorem 4.1 follows directly from Proposition 4.2 below and Corollary 3.4. ∎

PROPOSITION 4.2. *Let $n \geq 3$, $0 \leq p \leq n-3$, and $(T, c) \in \mathsf{D}_n(A)$. Then*

$$\exp\big(-D\big(E_{n,p}(T, c)\big)\big) \circ \left(X + \sum_{q=p}^{n-2} E_{n,q}(T, c)\right) = X + \sum_{q=p+1}^{n-2} E_{n,q}(T, c).$$

Proof. Put $G := \exp(-D(E_{n,p}(T, c)))$. So $G = X - E_{n,p}(T, c)$ (by Corollary 3.4). Hence if we put

$$U := \tilde{T}_{n-p} \triangle \cdots \triangle \tilde{T}_{n-1} \triangle T_n$$

then by Proposition 1.4 we get

$$G = X - \operatorname{Adj}(U) * \tilde{c}_{n-p-1 \mid U * X}.$$

So if we put

$$f := X + \sum_{q=p}^{n-2} E_{n,q}(T, c)$$

then

$$G \circ f = f - \operatorname{Adj}(U(f)) * \tilde{c}_{n-p-1 \mid U(f) * f}.$$

Since $U(f) = f$ (by Corollary 4.4 below, with $j = 0$) we get

$$G \circ f = f - \operatorname{Adj}(U) * \tilde{c}_{n-p-1 \mid U * f}.$$

Now observe that each component of $\tilde{c}_{n-p-1}$ belongs to $A[X_{n-p}, \dots, X_n]$ and that for each $i \geq n-p$ $(U * f)_i = (U * X)_i$ (by Lemma 4.3 below). So $\tilde{c}_{n-p-1 \mid U * f} = \tilde{c}_{n-p-1 \mid U * X}$ and hence

$$\begin{aligned} G \circ f &= f - \operatorname{Adj}(U) * \tilde{c}_{n-p-1 \mid U * X} \\ &= f - E_{n,p}(T, c) \end{aligned}$$

(by Proposition 1.4) ∎

LEMMA 4.3. *Let $n \geq 3$, $0 \leq p \leq n-2$, $0 \leq j \leq p$, and $(T, c) \in \mathsf{D}_n(A)$. Put $f := X + \sum_{q=p}^{n-2} E_{n,q}(T, c)$. Then*

$$\left[\left(\tilde{T}_{n-p+j} \triangle \cdots \triangle \tilde{T}_{n-1} \triangle T_n\right) * f\right]_i = \left[\left(\tilde{T}_{n-p+j} \triangle \cdots \triangle \tilde{T}_{n-1} \triangle T_n\right) * X\right]_i$$

for all $i \geq n-p+j$.

Proof. Put $U := \tilde{T}_{n-p+j} \triangle \cdots \triangle \tilde{T}_{n-1} \triangle T_n$. It suffices to show that for each $q \geq p$

$$\left[U * E_{n,q}(T, c)\right]_i = 0 \tag{4}$$

for all $i \geq n-p+j$. So let $q \geq p$. Then $q \geq p-j$.

1. We first treat the case that $q = p-j$. Then $j = 0$ and $q = p$. Consequently $U = \tilde{T}_{n-p} \triangle \cdots \triangle \tilde{T}_{n-1} \triangle T_n$, $E_{n,q}(T, c) = E_{n,p}(T, c)$, and hence by Proposition 1.4

$$\begin{aligned} U * E_{n,q}(T, c) &= U * \operatorname{Adj}(U) * \tilde{c}_{n-p-1 \mid U * X} \\ &= \det(U) * \tilde{c}_{n-p-1 \mid U * X}. \end{aligned}$$

Since the last $p+1$ coordinates of $\tilde{c}_{n-p-1}$ are zero, we obtain that

$$\left[U * E_{n,q}(T, c)\right]_i = 0$$

for all $i \geq n-p$, which proves the case that $q = p-j$.

2. Now assume that $q \geq p-j+1$. So $n-q \leq n-p+j-1$. Put $V := \tilde{T}_{n-q} \triangle \cdots \triangle \tilde{T}_{n-p+j-1}$. Then by Proposition 1.4 we can write

$$\begin{aligned} E_{n,q}(T, c) &= \operatorname{Adj}(V \triangle U) * \tilde{c}_{n-q-1 \mid (V \triangle U) * X} \\ &= \operatorname{Adj}\left(V_{\mid U * X} * U\right) * \tilde{c}_{n-q-1 \mid (V \triangle U) * X} \\ &= \operatorname{Adj}(U) * \operatorname{Adj}\left(V_{\mid U * X}\right) * \tilde{c}_{n-q-1 \mid (V \triangle U) * X}. \end{aligned}$$

Consequently

$$U * E_{n,q}(T, c) = \det(U) * \operatorname{Adj}\left(V_{\mid U * X}\right) * \tilde{c}_{n-q-1 \mid (V \triangle U) * X}. \tag{5}$$

Note that V, and hence $V_{\mid U * X}$, is of the form $\tilde{B}$ for some $B \in M_{n-p+j-1}(A[X])$. Furthermore $(\tilde{c}_{n-q-1})_i = 0$ if $i \geq n-q$, which implies that $(\tilde{c}_{n-q-1 \mid (V \triangle U) * X})_i = 0$ if $i \geq n-p+j$ (since $n-p+j > n-q$). Now the desired result (4) follows from (5). ∎

COROLLARY 4.4. *The notation is as in Lemma* 4.3. *Then*

$$\left(\tilde{T}_{n-p+j} \triangle \cdots \triangle \tilde{T}_{n-1} \triangle T_n\right)(f) = \tilde{T}_{n-p+j} \triangle \cdots \triangle \tilde{T}_{n-1} \triangle T_n.$$

Proof. The proof is by induction on $N := p - j$. If $N = 0$ the result is obvious. So let $N \geq 1$. Then

$$\begin{aligned}
&\left(\tilde{T}_{n-p+j} \triangle \cdots \triangle \tilde{T}_{n-1} \triangle T_n\right)(f) \\
&\quad = \tilde{T}_{n-p+j\,|\,(\tilde{T}_{n-p+j+1} \triangle \cdots \triangle \tilde{T}_{n-1} \triangle T_n)(f) * f} * \left(\tilde{T}_{n-p+j+1} \triangle \cdots \triangle \tilde{T}_{n-1} \triangle T_n\right)(f) \\
&\quad = \tilde{T}_{n-p+j\,|\,(\tilde{T}_{n-p+j+1} \triangle \cdots \triangle \tilde{T}_{n-1} \triangle T_n) * f} * \left(\tilde{T}_{n-p+j+1} \triangle \cdots \triangle \tilde{T}_{n-1} \triangle T_n\right)(f)
\end{aligned}$$

by the induction hypothesis. Finally observe that the matrix elements of $\tilde{T}_{n-p+j}$ depend only on $X_{n-p+j+1}, \ldots, X_n$. The result follows immediately from Lemma 4.3 (with $j + 1$ instead of j). ∎

5. STABLY TAMENESS

With Theorem 4.1 we are now able to prove the stably tame generators conjecture for all maps in our class $\mathsf{H}_n(A)$, and, we will also show that this result is "sharp": We give an example of an element of our class which is not tame, so in general we cannot get a better result than this stable tameness.

First let us recall the conjecture (it has already been mentioned in [1–4] and [8]):

CONJECTURE 5.1. *For every invertible polynomial map $F: k^n \to k^n$ over a field k there exist $t_1, \ldots, t_m$ such that*

$$F^{[m]} = (F, t_1, \ldots, t_m): k^{n+m} \to k^{n+m}$$

is tame, i.e., F is stably tame.

THEOREM 5.2. *Let $F = X + H$ with $H \in \mathsf{H}_n(A)$. Then F is stably tame.*

To do this we use the following result due to Martha Smith [10]:

PROPOSITION 5.3. *Let D be a locally nilpotent derivation of $A[X]$. Let $a \in \ker(D)$. Extend D to $A[X][t]$ by setting $D(t) = 0$. Note that tD is locally nilpotent. Define $\rho \in \mathrm{Aut}_A\, A[X][t]$ by $\rho(X_i) = X_i$, $i = 1, \ldots, n$, and $\rho(t) = t + a$. Then*

$$(\exp(aD), t) = \rho^{-1} \exp(-tD)\, \rho \exp(tD).$$

COROLLARY 5.4. *Let D and a be as in Proposition* 5.3 *If D is conjugate by a tame automorphism to a triangular derivation, then* $(\exp(aD, t))$ *is tame.*

LEMMA 5.5. *Let τ be a nice derivation of order m with respect to $X_1, \ldots, X_n$ and $D := \{\partial/\partial X_1, \ldots, \partial/\partial X_n\}$ on $A[X]$. Then* $\exp(a\tau)$ *is stably tame for all $a \in \ker(\tau)$.*

Proof. We use induction on m. Consider the case that $m = 1$. Then $\tau = \sum_{d \in D} b_d d$ with $b_d \in A[X]^D = \bigcap_{d \in D} \ker(d) = A$. Hence $\tau(X_i) \in A$ and clearly τ is on triangular form. So now we can apply Corollary 5.4 and find that $\exp(a\tau)$ is stably tame.

Now consider the case $m > 1$. We may assume that for all nice derivations $\sigma \in \mathrm{Der}_A(A[X])$ of order $m - 1$ with respect to D and $X_1, \ldots, X_n$ and for any commutative ring A we have that $\exp(a\sigma)$ is stably tame for all $a \in \ker(\sigma)$. Let τ be nice of order m. Define ρ and extend τ to $A[X][t]$ as in Proposition 5.3 (in fact we extend all derivations of D_i to $A[X][t]$ in this way). Now from

$$(\exp(a\tau), t) = \rho^{-1} \exp(-t\tau)\, \rho \exp(t\tau)$$

it follows that it suffices to see that $\exp(t\tau)$ is stably tame. Now we see that $t\tau = \sum_{d \in D_{m-1}} tb_d d$ with $tb_d \in A[X][t]^{D_{m-1}}$. But from this it follows that

$$\begin{aligned} \exp(t\tau) &= \exp\left(\sum_{d \in D_{m-1}} tb_d d \right) \\ &= \prod_{d \in D_{m-1}} \exp(tb_d d). \end{aligned}$$

This last equation follows from Proposition 1.5. Obviously it suffices to prove that each $\exp(tb_d d)$ is stably tame to conclude that $\exp(t\tau)$ is stably tame. But d is a nice derivation of order $m - 1$, $tb_d \in \ker(d)$, and hence we can apply the induction hypothesis to the ring $A[t]$ and find that $\exp(t\tau)$ is stably tame and hence $\exp(a\tau)$ is stably tame. ∎

Proof of Theorem 5.2. Now if we look at Theorem 4.1 we see that each $F = X + H$ with $H \in \mathsf{H}_n(A)$ can be written as the product of a finite number of $\exp(a_i D_i)$s, where each D_i is a nice derivation with respect to $X_1, \ldots, X_n$ and $\{\partial/\partial X_1, \ldots, \partial/\partial X_n\}$ and $a_i \in \ker(D_i)$. Applying Lemma 5.5 n times gives us the desired result: F is stably tame. ∎

Remark 5.6. Note that we do not give an indication of the value of m in Conjecture 5.1. As can be seen from the proof above, this m can be very high. At the highest level we have n $\exp(a_i D_i)$s, but each of these factors can give rise to a great number of extra variables, depending on the "order of niceness" of each D_i.

To conclude this paper we show that in general the automorphisms $F = X + H$ with $H \in \mathsf{H}_2(A)$ need not be tame. Actually, this idea was already presented by Nagata [9].

EXAMPLE 5.7. Let A be a domain, but not a principle ideal domain. Let $a, b \in A$ such that $Aa + Ab$ is not a principal ideal. Let $f(T) \in A[T]$ with $\deg(f) \geq 2$ and let $F = X + H$ with

$$H = \begin{pmatrix} bf(aX_1 + bX_2) \\ -af(aX_1 + bX_2) \end{pmatrix}$$

Since $H \in \mathsf{H}_2(A)$ F is an automorphism of $A[X_1, X_2]$. However, it is shown in [9] that F is not tame.

REFERENCES

1. H. Bass, A non-triangular action of G_a on A^3, *J. Pure Appl. Algebra* **33** (1984), 1–5.
2. L. M. Drużkowski, The Jacobian conjecture, Preprint 492, Institute of Mathematics, Polish Academy of Sciences, IMPAN, Sniadeckich 8, P.O. Box 137, 00-950 Warsaw, Poland, 1991.
3. A. R. P. van den Essen, Locally finite and locally nilpotent derivations with applications to polynomial flows and polynomial morphisms, *Proc. Amer. Math. Soc.* **116**(3) (1992), 861–871.
4. A. R. P. van den Essen, Seven lectures on polynomial automorphisms, *in* "Automorphisms of Affine Spaces, Proceedings of the Conference 'Invertible Polynomial Maps'" (A. R. P. van den Essen, Ed.), pp. 3–40, Kluwer Academic, Dordrecht/Norwell, MA, 1995.
5. A. R. P. van den Essen (Ed.), "Automorphisms of Affine Spaces, Proceedings of the Conference 'Invertible Polynomial Maps,' July 4–8, 1994, Curaçao," Caribbean Mathematics Foundation, Kluwer Academic, Dordrecht/Norwell, MA, 1995.
6. A. R. P. van den Essen and E.-M. G. M. Hubbers, "A New Class of Invertible Polynomial Maps," *J. Algebra* **187** (1997), 214–226.
7. E.-M. G. M. Hubbers, "The Jacobian Conjecture: Cubic Homogeneous Maps in Dimension Four," Master's thesis, University of Nijmegen, Toernooiveld, 6525 ED Nijmegen, The Netherlands, February 17, 1994, directed by A. R. P. van den Essen.
8. A. Joseph, A wild automorphism of $U(sl(2))$, *Math. Proc. Cambridge Philos. Soc.* **80** (1976), 61–65.
9. M. Nagata, "On the automorphism group of $k[X, Y]$," Kyoto University Lectures in Mathematics, vol. 5, Kyoto University, Kinokuniya, Tokyo, 1972.
10. M. K. Smith, Stably tame automorphisms, *J. Pure Appl. Algebra* **58** (1989), 209–212.
11. J. Winkelmann, On free holomorphic **C**-actions on $\mathbf{C}^n$ and homogeneous Stein manifolds, *Math. Ann.* **286** (1990), 593–612.