NORTH-HOLLAND

Polynomial Maps With Strongly Nilpotent Jacobian Matrix and the Jacobian Conjecture

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ABSTRACT

Let $H : k^n \to k^n$ be a polynomial map. It is shown that the Jacobian matrix JH is strongly nilpotent if and only if JH is linearly triangularizable if and only if the polynomial map $F = X + H$ is linearly triangularizable. Furthermore it is shown that for such maps F, sF is linearizable for almost all $s \in k$ (except a finite number of roots of unity).

INTRODUCTION

In [1] Bass, Connell, and Wright and in [7] Yagzhev showed that it suffices to prove the Jacobian conjecture for polynomial maps $F : \mathbb{C}^n \to \mathbb{C}^n$ of the form $F = X + H$, where $H = (H_1, \dots, H_n)$ is a cubic homogeneous polynomial map, i.e., each H_i is either zero or homogeneous of degree three. Since $\det(JF) \in \mathbb{C}^\times$ is equivalent to JH nilpotent (cf. [1, Lemma 4.1]), it follows that the Jacobian conjecture is equivalent to the following: if $F = X + H$ with JH nilpotent, then F is invertible. Hence it is clear that understanding nilpotent Jacobian matrices is crucial for the study of the Jacobian conjecture.

In [6], in an attempt to understand quadratic homogeneous polynomial maps, Meisters and Olech introduced the strongly nilpotent Jacobian matrices: a Jacobian matrix JH is strongly nilpotent if $JH(x_1) \cdots JH(x_n) = 0$ for all vectors $x_1, \dots, x_n \in \mathbb{C}^n$. They showed in [6] that for quadratic homogeneous polynomial maps JH is strongly nilpotent if and only if JH is nilpotent, if $n \leqslant 4$. However, for $n \geqslant 5$ there are counterexamples (cf. [4] and [6]).

LINEAR ALGEBRA AND ITS APPLICATIONS 247:121–132 (1996)

0024-3795/96/$15.00
SSDI 0024-3795(95)00095-9

On the other hand the obvious question whether the Jacobian conjecture is true for arbitrary polynomial maps $F = X + H$ with JH is strongly nilpotent has remained open.

In this paper we give an affirmative answer to this question. In fact we obtain a much stronger result: in Theorem 1.6 we show that the Jacobian matrix JH is strongly nilpotent if and only if JH is linearly triangularizable if and only if the polynomial map $F = X + H$ is linearly triangularizable. Furthermore we show that for such maps F the map sF is linearizable for almost all $s \in \mathbb{C}$ (except a finite number of roots of unity). So for such F the linearization conjecture of Meisters is true (it turned out to be false in general, as was shown in [3]).

1. DEFINITIONS AND FORMULATION OF THE FIRST MAIN RESULT

Throughout this paper k is a field with $\operatorname{char} k = 0$ and $k[X] := k[X_1, \dots, X_n]$ denotes the polynomial ring in n variables over k. Let $H = (H_1, \dots, H_n): k^n \to k^n$ be a polynomial map, i.e., $H_i \in k[X]$ for all i. By JH or $JH(X)$ we denote its Jacobian matrix. So $JH(X) \in M_n(k[X])$.

Now let $Y_{(1)} = (Y_{(1)1}, \dots, Y_{(1)n}), \dots, Y_{(n)} = (Y_{(n)1}, \dots, Y_{(n)n})$ be n sets of n new variables. So for each i, $JH(Y_{(i)})$ belongs to the ring of $n \times n$ matrices with entries in the n^2 variable polynomial ring $k[Y_{(i)j}; 1 \leqslant i, j \leqslant n]$.

DEFINITION 1.1. The Jacobian matrix JH is called *strongly nilpotent* if and only if the matrix $JH(Y_{(1)}) \cdots JH(Y_{(n)})$ is the zero matrix.

EXAMPLE 1.2. If JH is upper triangular with zeros on the main diagonal, then one readily verifies that JH is strongly nilpotent. In fact the main result of this paper (Theorem 1.6 below) asserts that a matrix JH is strongly nilpotent if and only if it is upper triangular with zeros on the main diagonal after a suitable linear change of coordinates.

REMARK 1.3. One easily verifies that if k is an infinite field, then Definition 1.1 is equivalent to $JH(x_1) \cdots JH(x_n) = 0$ for all $x_1, \dots, x_n \in k^n$. So for $k = \mathbb{R}$ and H homogeneous of degree two we obtain the strong nilpotence properly introduced by Meisters and Olech in [6]. See also [4].

To formulate the first main result of this paper we need more definition.

DEFINITION 1.4.

(i) Let $F = X + H$ be a polynomial map. We say that F is in (*upper*) *triangular form* if $H_i \in k[X_{i+1}, \dots, X_n]$ for all $1 \leqslant i \leqslant n-1$ and $H_n \in k$.

(ii) We say that F is *linearly triangularizable* if there exists $T \in \mathrm{GL}_n(k)$ such that $T^{-1}FT$ is in upper triangular form.

One easily verifies the following lemma:

LEMMA 1.5. *Let $F = X + H$ be a polynomial map. Then F is in upper triangular form if and only if JH is upper triangular with zeros on the main diagonal.*

Now we are ready to formulate the first main result of this paper:

THEOREM 1.6. *Let $H = (H_1, \dots, H_n): k^n \to k^n$ be a polynomial map. Then there is equivalence between the following statements*:

(i) *JH is strongly nilpotent.*

(ii) *There exists $T \in \mathrm{GL}_n(k)$ such that $J(T^{-1}HT)$ is upper triangular with zeros on the main diagonal.*

(iii) *$F := X + H$ is linearly triangularizable.*

From this theorem it immediately follows that:

COROLLARY 1.7. *If $F = X + H$ with JH strongly nilpotent, then F is invertible.*

2. THE PROOF OF THEOREM 1.6

The proof of Theorem 1.6 is based on the following two results.

LEMMA 2.1. *Let $JH = \sum_{|\alpha| \leqslant d} A_\alpha X^\alpha$, where $d = \max_i \deg H_i - 1$ and $A_\alpha \in M_n(k)$ for all α. Then JH is strongly nilpotent if and only if $A_{\alpha_{(1)}} \cdots A_{\alpha_{(n)}} = 0$ for all multiindices $\alpha_{(i)}$ with $|\alpha_{(i)}| \leqslant d$.*

Proof. By Definition 1.1 we obtain

$$\left(\sum_{|\alpha_{(1)}| \leqslant d} A_{\alpha_{(1)}} Y_{(1)}^{\alpha_{(1)}} \right) \cdots \left(\sum_{|\alpha_{(n)}| \leqslant d} A_{\alpha_{(n)}} U_{(n)}^{\alpha_{(n)}} \right) = 0.$$

The result then follows by looking at the coefficients of $Y_{(1)}^{\alpha_{(1)}} \cdots Y_{(n)}^{\alpha_{(n)}}$. ∎

PROPOSITION 2.2. *Let V be a finite dimensional k-vector-space, and $l_1, \ldots, l_p$ k-linear maps from V to V. Let $r \in \mathbb{N}$, $r \geqslant 1$. If $l_{i_1} \circ \cdots \circ l_{i_r} = 0$ for each r-tuple $l_{i_1}, \ldots, l_{i_r}$ with $1 \leqslant i_1, \ldots, i_r \leqslant p$, then there exists a basis (v) of V such that $\mathrm{Mat}(l_i, (v)) = D_i$, where D_i is an upper triangular matrix with zeros on the main diagonal.*

Proof. Let $d := \dim V$. We use induction on d. First let $d = 1$. Then the hypothesis implies that $l_i^r = 0$ for each i. So $l_i = 0$ for each i, and we are done. So let $d > 1$, and assume that the assertion is proved for all $d - 1$ dimensional vector spaces. Now we (also) use induction on r. If $r = 1$ then each $l_i = 0$. So let $r \geqslant 2$. Then for each $(r-1)$-tuple $l_{i_2}, \ldots, l_{i_r}$ with $1 \leqslant i_2, \ldots, i_r \leqslant p$ we have

$$l_1 l_{i_2} \cdots l_{i_r} = 0, \ldots, \qquad l_p l_{i_2} \cdots l_{i_r} = 0. \tag{2.1}$$

If $l_{i_2} \cdots l_{i_r} = 0$ for each such $(r-1)$-tuple, we are done by the induction hypothesis on r. So we may assume that for some $(r-1)$-tuple $l_{i_2}, \ldots, l_{i_r}$ the map $l_{i_2} \cdots l_{i_r} \neq 0$. So there exists $v \neq 0$, $v \in V$ with $v_1 := l_{i_2} \cdots l_{i_r} v \neq 0$. From (2.1) we deduce that $l_i v_1 = 0$ for all i. Then consider $\overline{V} := V/kv_1$. Since $l_i v_1 = 0$ for all i, we get induced k-linear maps $\bar{l}_i : \overline{V} \to \overline{V}$. Since $\dim \overline{V} = d - 1$, the induction hypothesis implies that there exist $v_2, \ldots, v_r$ in V such that $(\bar{v}_2, \ldots, \bar{v}_r)$ is a k-basis of $\overline{V}$ and $\mathrm{Mat}(\bar{l}_i, (\bar{v}_2, \ldots, \bar{v}_r))$ is in upper triangular form. Then $(v) = (v_1, v_2, \ldots, v_r)$ is as desired. ∎

COROLLARY 2.3. *Let $A_1, \ldots, A_p \in M_n(k)$. Let $r \in \mathbb{N}$, $r \geqslant 1$. If $A_{i_1} \cdots A_{i_r} = 0$ for each r-tuple $A_{i_1}, \ldots, A_{i_r}$ with $1 \leqslant i_1, \ldots, i_r \leqslant p$, then there exists $T \in \mathrm{GL}_n(k)$ such that $T^{-1} A_i T = D_i$, where each D_i is an upper triangular matrix with zeros on the main diagonal.*

Now we are able to present the proof of Theorem 1.6.

Proof. (ii) → (iii) follows from Lemma 1.5. So let's prove (iii) → (i). If $F = X + H$ is linearly triangularizable, then by Lemma 1.5 $J(T^{-1}HT)$ is an upper triangular matrix with zeros on the main diagonal. As remarked in Example 1.2, this implies that $J(T^{-1}HT)$ is strongly nilpotent. Finally observe that $J(T^{-1}HT) = T^{-1}JH(TX)T$. So the strong nilpotency of $J(T^{-1}HT)$ implies that $JH(TY_{(1)}) \cdots JH(TY_{(n)}) = 0$, which implies in turn that JH is strongly nilpotent.

Finally we prove (i) → (ii). So let JH be strongly nilpotent. Now if we write $JH = \sum_{|\alpha| \leqslant d} A_\alpha X^\alpha$, then by Lemma 2.1 $A_{\alpha_{(1)}} \cdots A_{\alpha_{(n)}} = 0$ for all n-tuples with $|\alpha_{(i)}| \leqslant d$. So by Corollary 2.3 there exists $T \in \mathrm{GL}_n(k)$ such that $T^{-1}A_\alpha T = D_\alpha$ for all α with $|\alpha| \leqslant d$, where D_α is an upper triangular matrix with zeros on the main diagonal. Consequently so is $T^{-1}JH(X)T$ $(= \sum T^{-1}A_\alpha T X^\alpha)$, and hence so is $J(T^{-1}HT) = T^{-1}JH(TX)T$, which is obtained by replacing X by TX in $T^{-1}JH(X)T$. ∎

3. STRONGLY NILPOTENT JACOBIAN MATRICES AND MEISTERS LINEARIZATION CONJECTURE

In [2] Deng, Meisters, and Zampieri studied dilations of polynomial maps with $\det(JF) \in \mathbb{C}^\times$. They were able to prove that for large enough $s \in \mathbb{C}$ the map sF is locally linearizable to $sJF(0)X$ by means of an analytic map φ_s, the so-called Schröder map, whose inverse is an entire function and satisfies some nice properties.

Their original aim was to show that φ_s is entire analytic, which would imply that sF and hence F is injective, which in turn would imply the Jacobian conjecture. Although they were not able to prove the entireness of φ_s, calculations of many examples of polynomial maps of the form $X + H$ with H cubic homogeneous showed that in all these cases the Schröder map was even much better than expected, namely, it was a polynomial automorphism (cf. [5]). This led Meisters to the following conjecture:

CONJECTURE 3.1 (Linearization conjecture, Meisters [5]). *Let $F = X + H$ be a cubic homogeneous polynomial map with JH nilpotent. Then for almost all $s \in \mathbb{C}$ (except a finite number of roots of unity) there exists a polynomial automorphism φ_s such that $\varphi_s^{-1} sF\varphi = sX$.*

Recently in [3] it was shown by the first author that the conjecture is false if $n \geqslant 5$ and true if $n \leqslant 4$.

In this section we show that Meisters linearization conjecture is true for all $n \geqslant 1$ if we replace "JH is nilpotent" by "JH is strongly nilpotent". In fact we don't even need the assumption that this H is cubic homogeneous. More precisely we have:

THEOREM 3.2. *Let k be a field, $k(s)$ the field of rational functions in one variable, and $F : k^n \to k^n$ a polynomial map of the form $F = X + H$ with $F(0) = 0$ and JH strongly nilpotent. Then there exists a polynomial automorphism $\varphi_s \in \operatorname{Aut}_{k(s)}(k(s)[X])$, linearly triangularizable over k, such that*

$$\varphi_s^{-1} sF\varphi_s = sJF(0)X.$$

Furthermore; the zeros of the denominators of the coefficients of the X-monomials appearing in φ_s are roots of unity.

Before we can prove this result we need one definition and some lemmas.

DEFINITION 3.3. We say that $X_1^{i_1} \cdots X_n^{i_n} > X_1^{i'_1} \cdots X_n^{i'_n}$ if and only if $\sum_{j=1}^n i_j > \sum_{j=1}^n i'_j$ or if $\sum_{j=1}^n i_j = \sum_{j=1}^n i'_j$ and there exists some $l \in \{1, 2, \dots, n\}$ such that $i_j = i'_j$ for all $j < l$ and $i_l > i'_l$.

Furthermore we say that the *rank* of the monomial $M := X_1^{i_1} \cdots X_n^{i_n}$ is the index of this monomial in the ascending ordered list of all monomials M' in $X_1, \dots, X_n$ with $\deg M' \leqslant \deg M$ (total degree).

EXAMPLE 3.4. The rank of $X_1 X_2 X_3$ is 15, since the ascending ordered list of all monomials in X_1, X_2, and X_3 of total degree at most three is

$$X_3, X_2, X_1,$$

$$X_3^2, X_2 X_3, X_2^2, X_1 X_3, X_1 X_2, X_1^2,$$

$$X_3^3, X_2 X_3^2, X_2^2 X_3, X_2^3, X_1 X_3^2, X_1 X_2 X_3, X_1 X_2^2, X_1^2 X_3, X_1^2 X_2, X_1^3.$$

LEMMA 3.5. *For each $2 \leqslant j \leqslant n-1$ let $l_j(X_{j+1}, \dots, X_n)$ be a linear form in $X_{j+1}, \dots, X_n$ and let $\mu \in k$. Then the leading monomial with respect* *to the order of Definition* 3.3 *in the expansion of*

$$\mu \prod_{j=2}^{n} \left[sX_j + sl_j(X_{j+1}, \ldots, X_n) \right]^{i_j} \tag{3.1}$$

is

$$\mu s^{i_2 + \cdots + i_n} X_2^{i_2} \cdots X_n^{i_n}.$$

Proof. It is obvious that the monomial $\mu s^{i_2 + \cdots + i_n} X_2^{i_2} \cdots X_n^{i_n}$ appears in the expansion of (3.1). Now we have to show that this is really the leading monomial. Note that all monomials in the expansion have the same (total) degree: $i_2 + \cdots + i_n$. For each $j = 2, \ldots, n$ we get a contribution of $[sX_j + sl_j(X_{j+1}, \ldots, X_n)]^{i_j}$ that is of the form

$$\sum_{k=0}^{i_j} \binom{i_j}{k} X_j^k \left[l_j(X_{j+1}, \ldots, X_n) \right]^{i_j - k},$$

and since l_j is a linear term that does not contain X_j it is obvious that we get the highest order monomial if we take $k = i_j$. So if we start with $j = 2$, we see that the highest X_2 power is i_2. And if we apply this result to $j = 3$, we see that the leading power product must begin with $X_2^{i_2} X_3^{i_3}$. If we do this for all j, it is obvious that the leading monomial is $\mu s^{i_2 + \cdots + i_n} X_2^{i_2} \cdots X_n^{i_n}$. ∎

LEMMA 3.6. *Let F be a polynomial map of the form*

$$F = \begin{pmatrix} X_1 + a(X_2, \ldots, X_n) + l_1(X_2, \ldots, X_n) \\ X_2 + l_2(X_3, \ldots, X_n) \\ \vdots \\ X_{n-1} + l_{n-1}(X_n) \\ X_n \end{pmatrix},$$

where $a(X_2, \ldots, X_n)$ is a polynomial with leading monomial (with respect to the order of Definition 3.3) *$\lambda X_2^{i_2} \cdots X_n^{i_n}$ and $i_2 + \cdots + i_n \geqslant 2$. Furthermore $l_i(X_{i+1}, \ldots, X_n)$ are some linear forms. Then there exists a polynomial map φ*

on triangular form such that

$$\varphi^{-1} sF\varphi = s\begin{pmatrix} X_1 + \tilde{a}(X_2,\ldots,X_n) + l_1(X_2,\ldots,X_n) \\ X_2 + l_2(X_3,\ldots,X_n) \\ \vdots \\ X_{n-1} + l_{n-1}(X_n) \\ X_n \end{pmatrix}, \tag{3.2}$$

where the leading monomial of $\tilde{a}(X_2,\ldots,X_n)$, *say* $\tilde{\lambda} X_2^{j_2} \cdots X_n^{j_n}$, *is of strictly lower order than the leading monomial of* $a(X_2,\ldots,X_n)$, *i.e.,*

$$X_2^{j_2} \cdots X_n^{j_n} < X_2^{i_2} \cdots X_n^{i_n}.$$

Proof. Let

$$\varphi = \begin{pmatrix} X_1 + \mu X_2^{i_2} \cdots X_n^{i_n} \\ X_2 \\ \vdots \\ X_n \end{pmatrix}$$

for some $\mu \in k$. It is obvious that φ is of triangular form. Proving that the equation (3.2) is valid is equivalent with showing that

$$sF\varphi = \varphi\left(s\begin{pmatrix} X_1 + \tilde{a}(X_2,\ldots,X_n) + l_1(X_2,\ldots,X_n) \\ X_2 + l_2(X_3,\ldots,X_n) \\ \vdots \\ X_{n-1} + l_{n-1}(X_n) \\ X_n \end{pmatrix}\right) \tag{3.3}$$

is valid. We do this by looking at the n components. For $i \geqslant 2$ it is easy to see that the ith component of the left hand side of (3.3) equals that of the right hand side of (3.3). Hence our only concern is the first component. Put $\hat{a}(X_2,\ldots,X_n) := a(X_2,\ldots,X_n) - \lambda X_2^{i_2} \cdots X_n^{i_n}$. On the left hand side we have

$$sF\varphi|_1 = sX_1 + s\mu X_2^{i_2} \cdots X_n^{i_n} + s\lambda X_2^{i_2} \cdots X_n^{i_n} + s\hat{a}(X_2, \ldots, X_n) + sl_1(X_2, \ldots, X_n), \tag{3.4}$$

and on the right hand side:

$$\left.\varphi\left(s\begin{pmatrix} X_1 + \tilde{a}(X_2, \ldots, X_n) + l_1(X_2, \ldots, X_n) \\ X_2 + l_2(X_3, \ldots, X_n) \\ \vdots \\ X_{n-1} + l_{n-1}(X_n) \\ X_n \end{pmatrix}\right)\right|_1$$

$$= sX_1 + s\tilde{a}(X_2, \ldots, X_n) + sl_1(X_2, \ldots, X_n) + \mu \prod_{j=2}^{n} \left[sX_j + sl_j(X_{j+1}, \ldots, X_n) \right]^{i_j}. \tag{3.5}$$

By subtracting Equation (3.5) from Equation (3.4) under the assumption that Equation (3.3) holds, we get

$$s(\mu + \lambda) X_2^{i_2} \cdots X_n^{i_n} + s\hat{\hat{a}}(X_2, \ldots, X_n) = \mu \prod_{j=2}^{n} \left[sX_j + sl_j(X_{j+1}, \ldots, X_n) \right]^{i_j}, \tag{3.6}$$

where $\hat{\hat{a}} = \hat{a} - \tilde{a}$. Now we have to derive a relation for μ to achieve that Equation (3.3) indeed holds. We can do this by restricting Equation (3.6) to the coefficients of $X_2^{i_2} \cdots X_n^{i_n}$. With Lemma 3.5 we see that the restriction of the right hand side of (3.6) to $X_2^{i_2} \cdots X_n^{i_n}$ gives $\mu s^{i_2 + \cdots + i_n}$, so we get

$$s\mu + s\lambda = s^{i_2 + \cdots + i_n}\mu,$$

and from this equation we can compute μ:

$$\mu = \frac{\lambda}{s^{i_2 + \cdots + i_n - 1} - 1}.$$

Note that we have assumed that $i_2 + \cdots + i_n \geqslant 2$, so $s^{i_2+\cdots+i_n-1} - 1 \neq 0$; hence μ is well defined. ■

Now we are able to give the proof of Theorem 3.2.

Proof. By Theorem 1.6 we may assume that $F = (F_1, \ldots, F_n)$ is of triangular form. We use induction on n. If $n = 1$, F degenerates to the identical map X_1 and the theorem follows immediately.

If $n = 2$, we can write

$$F = \begin{pmatrix} X_1 + a(X_2) + l_1(X_2) \\ X_2 \end{pmatrix},$$

where $a = \sum_{i=1}^{m} a_i X_2^i$ and $l_1 = aX_2$, the linear part. In particular we have that the leading monomial of a is $a_m X_2^m$. So with Lemma 3.6 we know that there exists a map φ_m of triangular form such that

$$\varphi_m^{-1} sF\varphi_m = \begin{pmatrix} sX_1 + \tilde{a}(X_2) + sl_1(X_2) \\ sX_2 \end{pmatrix}.$$

where $\deg(\tilde{a}) < m$. By applying the same lemma m times (if necessary we can use φ_j as the identity) we find a sequence $\varphi_1, \ldots, \varphi_m$ such that

$$\varphi_1^{-1} \cdots \varphi_m^{-1} sF\varphi_m \cdots \varphi_1 = s\begin{pmatrix} X_1 + l_1(X_2) \\ X_2 \end{pmatrix}$$

So $\varphi_s := \varphi_m \circ \cdots \circ \varphi_1$ is as desired. Now consider $F = (F_1, F_2, \ldots, F_n)$. Put $\tilde{F} := (F_2, \ldots, F_n)$ and $\tilde{X} := (X_2, \ldots, X_n)$. Then by the induction hypothesis we know that there exists an invertible polynomial map $\tilde{\varphi}_s$ such that

$$\tilde{\varphi}_s^{-1} s\tilde{F}\tilde{\varphi}_s = sJ_{\tilde{X}}\tilde{F}(0).$$

So with $\chi = (X_1, \tilde{\varphi}_s)$ and with the notation

$$F = \big(X_1 + a(X_2, \ldots, X_n) + l_1(X_2, \ldots, X_n), \tilde{F}\big)$$

we get

$$\chi^{-1} sF\chi = s \begin{pmatrix} X_1 + \tilde{a}(X_2, \ldots, X_n) + l_1(X_2, \ldots, X_n) \\ X_2 + l_2(X_3, \ldots, X_n) \\ \vdots \\ X_{n-1} + l_{n-1}(X_n) \\ X_n \end{pmatrix}.$$

Now we only have to make the first component linear. Let r be the rank of the leading monomial in $\tilde{a}(X_2, \ldots, X_n)$. With Lemma 3.6 we know that there exists a φ_r such that

$$\varphi_r^{-1} \chi^{-1} sF\chi\varphi_r = s \begin{pmatrix} X_1 + \tilde{a}_r(X_2, \ldots, X_n) + l_1(X_2, \ldots, X_n) \\ X_2 + l_2(X_3, \ldots, X_n) \\ \vdots \\ X_{n-1} + l_{n-1}(X_n) \\ X_n \end{pmatrix},$$

where the rank of the leading monomial of $\tilde{a}_r(X_2, \ldots, X_n)$ is less than r. So after r applications of Lemma 3.6 we have obtained a sequence $\varphi_1, \ldots, \varphi_r$ such that

$$\varphi_1^{-1} \cdots \varphi_r^{-1} \chi sF\chi\varphi_r \cdots \varphi_1 = s \begin{pmatrix} X_1 + l_1(X_2, \ldots, X_n) \\ X_2 + l_2(X_3, \ldots, X_n) \\ \vdots \\ X_{n-1} + l_{n-1}(X_n) \\ X_n \end{pmatrix},$$

which proves the theorem. ■

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Received 22 November 1994; final manuscript accepted 15 January 1995