

Chaotic Polynomial Automorphisms; counterexamples to several conjectures

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Abstract

We give a polynomial counterexample to a discrete version of the Markus-Yamabe Conjecture and a conjecture of Deng, Meisters and Zampieri, asserting that if $F : \mathbb{C}^n \rightarrow \mathbb{C}^n$ is a polynomial map with $\det(JF) \in \mathbb{C}^*$, then for all $\lambda \in \mathbb{R}$ large enough λF is global analytic linearizable. These counterexamples hold in any dimension ≥ 4 .

Introduction

In [4] a new approach to the Jacobian Conjecture is introduced. The authors conjecture that if $F : \mathbb{C}^n \rightarrow \mathbb{C}^n$ is a polynomial map with $F(0) = 0$ and $JF(0) = I$, then for all $\lambda > 1$, λ large enough there exists an analytic automorphism $\varphi_\lambda : \mathbb{C}^n \rightarrow \mathbb{C}^n$ such that $\varphi_\lambda^{-1} \circ \lambda F \circ \varphi_\lambda = \lambda I$ i.e. φ_λ conjugates λF to its linear part. We also say that λF is analytic linearisable to its linear part. We call this conjecture the *DMZ*-conjecture (after Deng, Meisters and Zampieri). Of course this conjecture, if true, would imply the Jacobian Conjecture since it follows readily that λF and hence F is injective. The local existence of φ_λ is guaranteed by the Poincaré-Siegel theorem (cf. [1, section 25, p. 193]) since if $\lambda > 1$ the eigenvalues of λI are non-resonant. Furthermore $\varphi_\lambda(0) = 0$ and φ_λ is unique if we assume that $J\varphi_\lambda(0) = I$, which we can do without loss of generality. It was shown in [4] that φ_λ^{-1} is entire, however the convergence of φ_λ could only be proved in some neighbourhood of 0. Meisters in [8] restricted the problem to polynomial maps of the form $F = X + H$ with H cubic homogeneous and $\det(JF) = 1$ (or equivalently JH nilpotent) and conjectured that for such maps λF can be conjugated to its linear part λI by means of polynomial automorphisms φ_λ , for almost all $\lambda \in \mathbb{C}$, except a finite number of roots of unity. In [5] the first author gave a

counterexample to this conjecture for any dimension ≥ 4 . On the other hand it was recently shown by Gorni and Zampieri in [7] that this example can be conjugated to its linear part for all λ with $|\lambda| \neq 1$ by means of an analytic automorphism φ_λ ! So the *DMZ*-conjecture remained open.

Another proof of the fact that the counterexample of [5] satisfies the *DMZ*-conjecture was even more recently given by Deng in [3]. In his very elegant and short paper he proves that an analytic map $F : \mathbb{C}^n \rightarrow \mathbb{C}^n$ with $F(0) = 0$ can be analytically conjugated to its linear part if and only if F is an analytic automorphism of \mathbb{C}^n and 0 is a global attractor of F (i.e. for every $x \in \mathbb{C}$ the sequence $x, F(x), F^2(x), \dots$ tends to 0). In the same paper he conjectured that if $F = X + H$ with H cubic homogeneous and JH nilpotent then 0 is a global attractor of $F \circ \lambda$ for all λ with $|\lambda| < 1$. (In fact in the argument he gave to motivate this conjecture he does not use that H is of degree 3.)

A similar kind of question was brought up independently by Cima, Gasull and Mañosas in [2]. They studied the problem that if $F : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is a polynomial map with $F(0) = 0$ and such that the eigenvalues of $JF(x)$ are smaller than 1 in absolute value for all $x \in \mathbb{R}^n$, then 0 is a global attractor of F . They call it the discrete Markus-Yamabe Question and show that this problem implies the Jacobian Conjecture and that it is true for triangular maps.

In this paper we give a counterexample to the *DMZ*-conjecture of the form $F = X + H$, where H is homogeneous of degree 5 in any dimension $n \geq 4$. Furthermore we show that if $0 < \lambda < 1$ λF is a counterexample to the discrete Markus-Yamabe Question.

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1 A counterexample to the discrete Markus-Yamabe Question

Let $n \geq 4$ and consider the polynomial ring $\mathbb{R}[X] := \mathbb{R}[X_1, \dots, X_n]$. In $\mathbb{R}[X]$ define the element

$$d(X) := X_3X_1 + X_4X_2$$

Theorem 1.1 *Let $n \geq 4$ and $m \in \mathbb{N}, m \geq 1$. Define the polynomial automorphism*

$$F = (X_1 + X_4d(X)^2, X_2 - X_3d(X)^2, X_3 + X_4^m, X_4, \dots, X_n).$$

Then for each $0 < \lambda < 1$ λF is a counterexample to the discrete Markus-Yamabe Question. More precisely, if $0 < \lambda < 1$ and $a \in \mathbb{R}$ is such that $a\lambda > 1$ then the first component of $(\lambda F)^k(a, a, \dots, a)$ tends to infinity if k tends to infinity.

Definition 1.2 For each $\lambda > 0$ and $a > 0$ we put $(\lambda F)^k(a) := (\lambda F)^k(a, a, \dots, a)$ and denote the first component of this vector by $f_k(\lambda, a)$. So

$$f_k(\lambda, a) := ((\lambda F)^k(a))_1,$$

for all $k \geq 1$. Furthermore we put

$$d_k(\lambda, a) := d((\lambda F)^k(a)),$$

for all $k \geq 1$.

Lemma 1.3 *i). $d(\lambda F(X)) = \lambda^2[X_4^{m+1}d(X)^2 + d(X) + X_4^mX_1]$*

ii). $d_{k+1}(\lambda, a) \geq \lambda^2(\lambda^k a)^{m+1}(d_k(\lambda, a))^2$, for all $k \geq 1$.

iii). $f_{k+1}(\lambda, a) \geq \lambda^{k+1}a(d_k(\lambda, a))^2$, for all $k \geq 1$.

Proof. i) is easy to verify. Consequently, since all monomials in $d(\lambda F(X))$ have positive coefficients, we get

$$\begin{aligned} d_{k+1}(\lambda, a) &= d((\lambda F)(\lambda F)^k(a)) \\ &\geq \lambda^2((\lambda F)^k(a))_4^{m+1}d((\lambda F)^k(a))^2 \\ &= \lambda^2(\lambda^k a)^{m+1}(d_k(\lambda, a))^2 \end{aligned}$$

since the fourth component of $(\lambda F)^k(a)$ equals $\lambda^k a$. This proves ii). Finally

$$\begin{aligned} f_{k+1}(\lambda, a) &= (\lambda F)_1((\lambda F)^k(a)) \\ &\geq \lambda((\lambda F)^k(a))_4 d((\lambda F)^k(a)) \end{aligned}$$

(using that $(\lambda F)_1 = \lambda X_4 d(X)^2 + \lambda X_1$). So $f_{k+1}(\lambda, a) \geq \lambda^{k+1} a (d_k(\lambda, a))^2$, which proves iii). \square

Proposition 1.4 *We have:*

$$\begin{aligned} f_k(\lambda, a) &\geq \lambda^{p_k} a^{p_k + (2m+1)(k-1) + 4} \\ d_k(\lambda, a) &\geq \lambda^{p_k + m(k-1) + 1} a^{p_k + (2m+1)(k-1) + m + 4} \end{aligned}$$

for all $k \geq 1$, where $p_1 = 1$ and $p_{k+1} = 2p_k + (2m+1)(k-1) + 4$ for all $k \geq 1$.

Proof. Use induction on k . Details are left to the reader. \square

Proof of theorem 1.1. It follows immediately from the estimation of $f_k(\lambda, a)$ in proposition 1.4 that $\lim_{k \rightarrow \infty} f_k(\lambda, a) = \infty$ if $\lambda a > 1$. Furthermore one easily verifies that $\lambda F = \lambda X + H$ with JH nilpotent. So for all $x \in \mathbb{R}^n$ the eigenvalues of $JF(x)$ are equal to λ . \square

Corollary 1.5 *Let $m = 5$ and $0 < \lambda < 1$. Put $\tilde{F} := \lambda F \lambda^{-1}$. Then $\tilde{F} = X + H$ with H homogeneous of degree 5 and JH is nilpotent. However 0 is not a global attractor of $\tilde{F} \circ \lambda (= \lambda F)$.*

2 A counterexample to the DMZ-conjecture

Let $n \geq 4$ and consider the polynomial ring $\mathbb{C}[X] := \mathbb{C}[X_1, \dots, X_n]$. In $\mathbb{C}[X]$ define the element $d(X) := X_3 X_1 + X_4 X_2$.

Theorem 2.1 *Let $n \geq 4$ and $m \geq 3$, m odd. Define the polynomial automorphism*

$$F = (X_1 + X_4 d(X)^2, X_2 - X_3 d(X)^2, X_3 + X_4^m, X_4, \dots, X_n).$$

Then F is a counterexample to the DMZ-conjecture. More precisely, for every $\lambda > 0$, $\lambda \neq 1$, λF is not global analytic linearisable to λX .

The proof of this theorem is based on the following observation which is due to Bo Deng (cf [3]).

Lemma 2.2 *Let $F : \mathbb{C}^n \rightarrow \mathbb{C}^n$ be an analytic map with $F(0) = 0$. Put $A := JF(0)$ and suppose that the eigenvalues of A are smaller than 1 in absolute value. If F is global analytic linearisable to its linear part A then 0 is a global attractor of F .*

Proof. Let $x \in \mathbb{C}^n$ and let $\varphi : \mathbb{C}^n \rightarrow \mathbb{C}^n$ be the analytic automorphism of \mathbb{C}^n such that $\varphi^{-1}F\varphi = A$. Then $F = \varphi A \varphi^{-1}$ and hence $F^k(x) = \varphi A^k \varphi^{-1}(x)$, for all $k \geq 1$. By the hypothesis on the eigenvalues of A it follows that $A^k \varphi^{-1}(x) \rightarrow 0$ if $k \rightarrow \infty$. Consequently $F^k(x) = \varphi(A^k \varphi^{-1}(x)) \rightarrow 0$ if $k \rightarrow \infty$. \square

Proof of theorem 2.1. i). From lemma 2.2 and theorem 1.1 it follows that λF is not analytic linearisable if $0 < \lambda < 1$.

ii). Now let $\lambda > 1$. Suppose that λF is analytic linearisable. We derive a contradiction. Then $(\lambda F)^{-1} = F^{-1} \circ \lambda^{-1}$ is also analytic linearisable. Put $\mu := \lambda^{-1}$ and $G := F^{-1}$. So $G \circ \mu$ is analytic linearisable. One easily verifies that

$$G = (X_1 - X_4 \tilde{d}(X)^2, X_2 + (X_3 - X_4^m) \tilde{d}(X)^2, X_3 - X_4^m, X_4, \dots, X_n) \quad (1)$$

where

$$\tilde{d}(X) := d(X) - X_4^m X_1. \quad (2)$$

Since $0 < \mu < 1$ it follows from lemma 2.2 that 0 is a global attractor of $G \circ \mu$. However we will show below (corollary 2.6) that for every $0 < \mu < 1$ 0 is not a global attractor of $G \circ \mu$. Hence we have derived a contradiction. \square

So it remains to show that 0 is not a global attractor of $G \circ \mu$. First we show that 0 is not a global attractor of μG if $0 < \mu < 1$. To prove this we need some lemmas. So let G and $\tilde{d}(X)$ be as in (1) resp. (2).

For each $a > 0$ let $a^* := (a, -a, a, -a, a, \dots, a) \in \mathbb{R}^n$. Then we define for each $a > 0$ and $\mu > 0$:

$$\begin{aligned} g_k(\mu, a) &:= ((\mu G)^k(a^*))_1 \\ \tilde{d}_k(\mu, a) &:= \tilde{d}((\mu G)^k(a^*)) \end{aligned}$$

for all $k \geq 1$.

Lemma 2.3 *i).* $d(G(X)) = \tilde{d}(X)$.

$$ii). \tilde{d}((\mu G)(X)) = \mu^2 \tilde{d}(X) - \mu^{m+1} X_4^m X_1 + \mu^{m+1} X_4^{m+1} \tilde{d}(X)^2.$$

$$iii). \tilde{d}_{k+1}(\mu, a) = (\mu^{k+1} a)^{m+1} (\tilde{d}_k(\mu, a))^2 + \mu^2 \tilde{d}_k(\mu, a) + \mu (\mu^{k+1} a)^m g_k(\mu, a) \text{ for all } k \geq 1.$$

$$iv). g_{k+1}(\mu, a) = \mu^{k+1} a (\tilde{d}_k(\mu, a))^2 + \mu g_k(\mu, a) \text{ for all } k \geq 1.$$

Proof. The proofs of i) and ii) are straightforward and left to the reader. From ii) we deduce that

$$\begin{aligned} \tilde{d}_{k+1}(\mu, a) &= \tilde{d}((\mu G)^{k+1}(a^*)) \\ &= \tilde{d}((\mu G)((\mu G)^k(a^*))) \\ &= \mu^2 \tilde{d}((\mu G)^k(a^*)) - \mu^{m+1} (((\mu G)^k(a^*))_4)^m ((\mu G)^k(a^*))_1 \\ &\quad + \mu^{m+1} (((\mu G)^k(a^*))_4)^{m+1} \tilde{d}((\mu G)^k(a^*))^2 \end{aligned}$$

Now observe that $((\mu G)^k(a^*))_4 = \mu^k(-a)$, hence since m is odd $((\mu G)^k(a^*))_4^m = -(\mu^k a)^m$. So we get

$$\begin{aligned} \tilde{d}_{k+1}(\mu, a) &= \mu^2 \tilde{d}_k(\mu, a) + \mu^{m+1} (\mu^k a)^m g_k(\mu, a) + \mu^{m+1} (\mu^k a)^{m+1} (\tilde{d}_k(\mu, a))^2 \\ &= (\mu^{k+1} a)^{m+1} (\tilde{d}_k(\mu, a))^2 + \mu (\mu^{k+1} a)^m g_k(\mu, a) + \mu^2 \tilde{d}_k(\mu, a) \end{aligned}$$

which proves iii). Finally

$$\begin{aligned} g_{k+1}(\mu, a) &= ((\mu G)^{k+1}(a^*))_1 \\ &= (\mu G)_1((\mu G)^k(a^*)) \\ &= \mu((\mu G)^k(a^*))_1 - \mu((\mu G)^k(a^*))_4 (\tilde{d}((\mu G)^k(a^*)))^2 \\ &= \mu g_k(\mu, a) - \mu \cdot \mu^k(-a) (\tilde{d}_k(\mu, a))^2 \\ &= \mu g_k(\mu, a) + \mu^{k+1} a (\tilde{d}_k(\mu, a))^2 \end{aligned}$$

which proves iv). □

Corollary 2.4 *i).* $\tilde{d}_{k+1}(\mu, a) \geq (\mu^{k+1} a)^{m+1} (\tilde{d}_k(\mu, a))^2$ for all $k \geq 1$.

$$ii). g_{k+1}(\mu, a) \geq \mu^{k+1} a (\tilde{d}_k(\mu, a))^2 \text{ for all } k \geq 1.$$

Proof. By induction on k one readily verifies that for all $k \geq 1$ both $\tilde{d}_k(\mu, a)$ and $g_k(\mu, a)$ are polynomials in μ and a with coefficients in \mathbb{N} . Then the result follows from lemma 2.3 iii) and iv). □

Proposition 2.5 *We have:*

$$\begin{aligned} g_k(\mu, a) &\geq \mu^{q_k(m+1)+k} a^{(q_k+2k)(m+1)+1} \\ \tilde{d}_k(\mu, a) &\geq \mu^{(q_k+k)(m+1)} a^{(q_k+2k+1)(m+1)} \end{aligned}$$

for all $k \geq 1$, where $q_1 = 0$ and $q_{k+1} = 2q_k + 2k$ for all $k \geq 1$.

Proof. Use induction on k . □

Corollary 2.6 *If $\mu a > 1$ and $a > 1$ then $\lim_{k \rightarrow \infty} ((G \circ \mu)^k(G(a^*)))_1 = \infty$. So 0 is not a global attractor of $G \circ \mu$.*

Proof. Observe that $(G\mu)^k(G(a^*)) = \mu^{-1}(\mu G)^{k+1}(a^*)$. Then apply proposition 2.5. □

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