CUBIC SIMILARITY IN DIMENSION FIVE

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Dedicated to Gary Meisters

Abstract. In this paper we classify all Drużkowski maps $F = X + (AX)^{*3}$ from $\mathbb{C}^5$ to $\mathbb{C}^5$ for which $J((AX)^{*3})$ is nilpotent. With this classification of maps we obtain the complete set of representatives of Meisters' cubic similarity relation in dimension five. This paper is a summary of the very large paper [3].

1 Introduction

The first time I got interested in the subject of cubic similarity was back in 1993. It was in the middle of June, a few days before I would go on a four week holiday to Moscow. In order to get credits for a class on polynomial mappings by Arno van den Essen, I had been working with two fellow students on linear cubic homogeneous maps. When we handed in our paper with the final results, Arno reminded us that the next day some American would give a talk about cubic linear maps. Naturally I went there and I listened to a very nice talk by Gary Meisters. One of the most impressive points in this talk was the point where he was showing some slides containing a list of matrices which turned out to be the representatives of the cubic similarity relation in dimension three, four and five. Though Gary already showed 19 matrices in dimension five, he was pretty sure that this list was not complete yet ...

Right after I returned from Moscow I started working on my Master's Thesis on cubic homogeneous maps. During the work done for this thesis I found that Gary's list in dimension four was complete.

After I started as a PhD-student in Nijmegen, I got back to this subject in the spring of 1996. And this time I was able to solve the dimension five case.

The reason that two years had passed after finishing my Master's Thesis and starting with the final research to the five dimensional case, was the complexity of this case. It was only in 1996 that we rediscovered the paper [1] by Drużkowski. In conjuction with a theorem from my Master's Thesis [2], we now were able to reduce the general case to the triangular cubic linear case.

2 Reduction to triangular matrices

We start with a few basic definitions.

Definition 1 Let $a = (a_1, \ldots, a_n) \in \mathbb{C}^n$. Then $a^{*3} := (a_1^3, \ldots, a_n^3)$.

Definition 2 Let A be a linear matrix over $\mathbb{C}$. Then the map $F = X + (AX)^{*3}$ is called *cubic-linear* or *in Drużkowski form*.

Note that in some other papers such a cubic linear map F is called Drużkowski map only if $\det(JF) = 1$.

Definition 3 Let $F = X + (AX)^{*3}$ and $G = X + (BX)^{*3}$ be two polynomial automorphisms in Drużkowski form. Then the matrices $A, B \in \mathrm{Mat}_{n,n}(\mathbb{C})$ are called *cubic similar* ($A \overset{3}{\sim} B$) if there exists a linear invertible polynomial map T with $T^{-1}FT = G$.

The idea behind this definition is that it is rather special that if T is a linear invertible map and F is a Drużkowski form one has that $T^{-1}FT$ is again on Drużkowski form and therefore this property deserves a name.

Definition 3 is in terms of maps. For computational use however it is often preferable to work in terms of matrices.

Lemma 1 *Let $F = X + (AX)^{*3}$ and $G = X + (BX)^{*3}$ be two polynomial maps on Drużkowski form. Then $A \overset{3}{\sim} B$ if and only if there exists $T \in \mathrm{Gl}_n(\mathbb{C})$ with $(ATX)^{*3} = T(BX)^{*3}$.*

Proof. The following statements can be read from top to bottom or the other way round. In either case each statement is equivalent with the next one in the sequence.

- $A \overset{3}{\sim} B$.
- There exists an invertible map T with $T^{-1}FT = G$.
- There exists an invertible map T with $T^{-1}(TX + (ATX)^{*3}) = X + (BX)^{*3}$.
- There exists an invertible map T with $X + T^{-1}(ATX)^{*3} = X + (BX)^{*3}$.

- There exists an invertible map T with $T^{-1}(ATX)^{*3} = (BX)^{*3}$.
- There exists an invertible matrix T with $T^{-1}(ATX)^{*3} = (BX)^{*3}$.
- There exists an invertible matrix T with $(ATX)^{*3} = T(BX)^{*3}$.

This proves the lemma. $\square$

From [2] we know that

Theorem 1 *Let $r \in \mathbb{N}$. If the Jacobian Conjecture holds for every polynomial map $F : \mathbb{C}^r \to \mathbb{C}^r$ where F has the special form*

$$F = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_r \end{pmatrix} + \begin{pmatrix} H_1(x_1, \ldots, x_r) \\ H_2(x_1, \ldots, x_r) \\ \vdots \\ H_r(x_1, \ldots, x_r) \end{pmatrix}$$

with $H_i = 0$ or $\deg(H_i) = 3$ (H_i homogeneous for all $i \in \{1, \ldots, r\}$) then for all $n \geq r$ and all $A \in \mathrm{Mat}_{n,n}(\mathbb{C})$ the Jacobian Conjecture holds for all Drużkowski forms

$$G = X + (AX)^{*3}$$

with $\mathrm{rank}(A) = r$ and $X = (x_1, \ldots, x_n)$.

Before we present our main reduction theorem we show a few lemmas, which we will need for the proof of this main theorem. The proofs can be found in [3]. The first two lemmas are proved purely theoretically. For the third and the fourth lemma we had to do some computations to solve the corresponding systems of equations.

Lemma 2 *Let $F = X + (AX)^{*3}$ with $A \in \mathrm{Mat}_{5,5}(\mathbb{C})$ and $J((AX)^{*3})$ is nilpotent. Then there exists linear invertible T such that $T^{-1}FT = X + (BX)^{*3}$ where the last row of B is a null row.*

Lemma 3 *Assume $\mathrm{rank}(A) = 2$. By lemma 2 we have that the last row is equal to zero. Now if we write*

$$A = \begin{pmatrix} a_1 & a_2 & a_3 & a_4 & a_5 \\ b_1 & b_2 & b_3 & b_4 & b_5 \\ c_1 & c_2 & c_3 & c_4 & c_5 \\ d_1 & d_2 & d_3 & d_4 & d_5 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} = \left(\begin{array}{cccc|c} & & & & a_5 \\ & & A' & & b_5 \\ & & & & c_5 \\ & & & & d_5 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

and we consider the Drużkowski form $X' + (A'X')^{*3}$ (where $X' = (x_1, \ldots, x_4)$) we may assume that A' equals

$$\begin{pmatrix} a_1 & a_2 & a_3 & a_4 \\ \lambda_2 a_1 & \lambda_2 a_2 & \lambda_2 a_3 & \lambda_2 a_4 \\ \lambda_3 a_1 & \lambda_3 a_2 & \lambda_3 a_3 & \lambda_3 a_4 \\ \lambda_4 a_1 & \lambda_4 a_2 & \lambda_4 a_3 & \lambda_4 a_4 \end{pmatrix} \text{ or } \begin{pmatrix} a_1 & a_2 & a_3 & a_4 \\ b_1 & b_2 & b_3 & b_4 \\ \lambda a_1 + \mu b_1 & \lambda a_2 + \mu b_2 & \lambda a_3 + \mu b_3 & \lambda a_4 + \mu b_4 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

and $d_5 = 0$.

Lemma 4 *Let A and A' be as in lemma 3. Assume*

$$A' = \begin{pmatrix} a_1 & a_2 & a_3 & a_4 \\ \lambda_2 a_1 & \lambda_2 a_2 & \lambda_2 a_3 & \lambda_2 a_4 \\ \lambda_3 a_1 & \lambda_3 a_2 & \lambda_3 a_3 & \lambda_3 a_4 \\ \lambda_4 a_1 & \lambda_4 a_2 & \lambda_4 a_3 & \lambda_4 a_4 \end{pmatrix}$$

Then there exists a linear invertible map $T \in \mathbb{C}[X]$ and $B \in \mathrm{Gl}_5(k)$ such that $T^{-1} \circ (X + (AX)^{*3}) \circ T = X + (BX)^{*3}$ with B is upper triangular with null diagonal.

Lemma 5 *Let A and A' be as in lemma 3. Assume*

$$A' = \begin{pmatrix} a_1 & a_2 & a_3 & a_4 \\ b_1 & b_2 & b_3 & b_4 \\ \lambda a_1 + \mu b_1 & \lambda a_2 + \mu b_2 & \lambda a_3 + \mu b_3 & \lambda a_4 + \mu b_4 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Then there exists a linear invertible map $T \in \mathbb{C}[X]$ and $B \in \mathrm{Gl}_5(k)$ such that $T^{-1} \circ (X + (AX)^{*3}) \circ T = X + (BX)^{*3}$ with B is upper triangular with null diagonal.

After these technical lemmas we can finally give the main reduction theorem, which is an improvement of [1, Theorem 2.1] for the case $n = 5$.

Theorem 2 *If a polynomial map $F = X + (AX)^{*3} : \mathbb{C}^5 \to \mathbb{C}^5$ has $\det(JF) = 1$ and $\mathrm{rank}(A) < 3$ or $\mathrm{corank}(A) < 3$, then there exists an invertible linear map L such that $L \circ F \circ L^{-1} = X + (BX)^{*3}$, with B is upper triangular with null diagonal.*

Proof. Though the original theorem in Drużkowski's paper [1] only claims that F is a tame automorphism, we can almost copy the proof as it is presented in that paper. Simply because in three of the four cases it is shown that LFL^{-1} has the desired form (and hence F is tame).

- $\mathrm{rank}(A) = 1$. The proof is exactly the same as in [1].
- $\mathrm{corank}(A) = 1$. From lemma 2 it follows that we are always in case (i) of Drużkowski's paper.
- $\mathrm{corank}(A) = 2$. From lemma 2 it now follows that we are always in case (iii) of Drużkowski's paper.
- $\mathrm{rank}(A) = 2$. This is the only part where Drużkowski doesn't show that F can be transformed to the desired form. To prove this case we use the lemmas 3, 4 and 5.

□

Since we are working in dimension five, we have that either $\mathrm{rank}(A) < 3$ or $\mathrm{corank}(A) < 3$ and hence:

Corollary 1 *Let $F = X + (AX)^{*3} : \mathbb{C}^5 \to \mathbb{C}^5$ such that $\det(JF) = 1$ then there exists an invertible linear map L such that $L \circ F \circ L^{-1} = X + (BX)^{*3}$, with B is upper triangular with null diagonal.*

3 Meisters' representatives

In [5] Meisters presents a list of seventeen mutually inequivalent matrices with respect to the cubic similarity relation in dimension five. The names of these matrices are based on the following notions.

- A J indicates that the matrix is on Jordan normal form.
- An N indicates that it is a nilpotent matrix which is not on Jordan normal form, but does not need parameters in it.
- A P indicates that it is a nilpotent matrix which contains parameters which cannot be reduced to a single complex number.
- The first number is the rank of the matrix.
- The second number is the nilpotence index of $J((AX)^{*3}$, where A is the matrix.
- The small letters at the end are used as an index.
- For some P matrices an extra integer is appended to show the number of parameters in it.

In [4] it is shown that the rank and the nilpotence index as mentioned above are *invariants* with respect to the cubic similarity relation. Therefore it makes sense to use these figures to assign proper names to the matrices. Note that the nilpotence index of the matrix A itself is *not* an invariant. In fact A does not need to be nilpotent at all.

$$\underset{J(1,2)}{\begin{pmatrix} 0&1&0&0&0\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0 \end{pmatrix}} \quad \underset{J(2,2)}{\begin{pmatrix} 0&1&0&0&0\\0&0&0&0&0\\0&0&0&1&0\\0&0&0&0&0\\0&0&0&0&0 \end{pmatrix}} \quad \underset{J(2,3)}{\begin{pmatrix} 0&1&0&0&0\\0&0&1&0&0\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0 \end{pmatrix}}$$

$$\underset{N(2,3a)}{\begin{pmatrix} 0&0&1&0&0\\0&0&1&1&0\\0&0&0&1&0\\0&0&0&0&0\\0&0&0&0&0 \end{pmatrix}} \quad \underset{J(3,3)}{\begin{pmatrix} 0&1&0&0&0\\0&0&1&0&0\\0&0&0&0&0\\0&0&0&0&1\\0&0&0&0&0 \end{pmatrix}} \quad \underset{J(3,4)}{\begin{pmatrix} 0&1&0&0&0\\0&0&1&0&0\\0&0&0&1&0\\0&0&0&0&0\\0&0&0&0&0 \end{pmatrix}}$$

$$\underset{N(3,3a)}{\begin{pmatrix} 0&0&1&0&0\\0&0&1&1&0\\0&0&0&1&1\\0&0&0&0&0\\0&0&0&0&0 \end{pmatrix}} \quad \underset{N(3,4a)}{\begin{pmatrix} 0&1&1&0&0\\0&0&1&0&0\\0&0&0&1&0\\0&0&0&0&0\\0&0&0&0&0 \end{pmatrix}} \quad \underset{N(3,4b)}{\begin{pmatrix} 0&0&1&0&0\\0&0&1&1&0\\0&0&0&1&0\\0&0&0&0&1\\0&0&0&0&0 \end{pmatrix}}$$

$$\underset{N(3,4c)}{\begin{pmatrix} 0&0&1&0&0\\0&0&1&1&0\\0&0&0&1&1\\0&0&0&0&1\\0&0&0&0&0 \end{pmatrix}} \quad \underset{J(4,5)}{\begin{pmatrix} 0&1&0&0&0\\0&0&1&0&0\\0&0&0&1&0\\0&0&0&0&1\\0&0&0&0&0 \end{pmatrix}} \quad \underset{N(4,5a)}{\begin{pmatrix} 0&1&1&0&0\\0&0&1&0&0\\0&0&0&1&0\\0&0&0&0&1\\0&0&0&0&0 \end{pmatrix}}$$

$$\underset{N(4,5b)}{\begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}} \quad \underset{N(4,5c)}{\begin{pmatrix} 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}} \quad \underset{N(4,5d)}{\begin{pmatrix} 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}}$$

$$\underset{P(4,5c)}{\begin{pmatrix} 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & a & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}} \quad \underset{P(4,5c2)}{\begin{pmatrix} 0 & 1 & 1 & b & 0 \\ 0 & 0 & 1 & a & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}}$$

Remark 1 Note the following points:

- $P(4,5c)$ is not called $P(4,5a)$, which should be natural if one uses the small letter just as an index as with the N-matrices. However in this case the c is used because $P(4,5c)_{|a=1} = N(4,5c)$, where $P(4,5c)_{|a=1}$ means substitute $a = 1$ in $P(4,5c)$.
- Note also that $P(4,5c)_{|a=0} = N(4,5a)$. Hence we add the restriction that $a \notin \{0,1\}$ for $P(4,5c)$.
- $P(4,5c)_{|a=a_1} \overset{3}{\not\sim} P(4,5c)_{|a=a_2}$ if $a_1 \neq a_2$.
- $P(4,5c2)_{|b=0} = P(4,5c)$, hence we add the restriction $b \neq 0$ for $P(4,5c2)$. Note that there are no restrictions on the a in $P(4,5c2)$.

4 Classification of Drużkowski maps

Theorem 2 gives us the reduction we need. It means that the most general Drużkowski map $X + (AX)^{*3}$ in dimension five is given by the matrix A:

$$\begin{pmatrix} 0 & a_2 & a_3 & a_4 & a_5 \\ 0 & 0 & b_3 & b_4 & b_5 \\ 0 & 0 & 0 & c_4 & c_5 \\ 0 & 0 & 0 & 0 & d_5 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \tag{4.1}$$

Independent of the ten parameters in it this matrix is strong nilpotent. So this matrix is in fact on its own a description of all Drużkowski maps with $J((AX)^{*3})$ is nilpotent. However since our final goal is finding representatives with respect to the cubic similarity relation, it makes sense to split the general case into the five possible values for the nilpotence index of the associated Jacobian matrix. As was noted before, this nilpotence index is invariant under cubic similarity.

Using this observation we compute $J((AX)^{*3})^n$ for $n = 1, \ldots, 5$ and assume that the resulting matrix is the null matrix. Obviously for $n = 1$ this means that A equals the null matrix, which gives the identity map $I_5 : \mathbb{C}^5 \to \mathbb{C}^5$. Therefore we only consider the cases with nilpotence index ≥ 2.

4.1 Nilpotence index two

Assuming $J((AX)^{*3})^2 = 0$ gives a system of 119 equations in the ten parameters. Solving this system gives fifteen solutions. In figure 1 we show the tree along which we found these solutions. One starts at the top with the complete system. One solves a few simple equations. Normally this gives a few possible partial solutions. Each arrow presents such a solution. And each solution may imply some assumptions on the parameters. After substituting these partial solutions one gets new reduced systems of equations. And at this point the process is repeated. Hence each arrow represents some assumptions; these are listed at the bottom. Furthermore, the boxed numbers in the tree correspond to the numbered matrices given below.

The fifteen solutions are presented by their corresponding matrices. For each matrix the rank is listed together with the assumptions used in the process to find them as mentioned above. Naturally, assumptions of the form $b_3 = 0$ are not shown since they are already used in the matrix and hence b_3 does not appear in the matrix anymore.

1. $$\begin{pmatrix} 0 & 0 & 0 & 0 & a_5 \\ 0 & 0 & 0 & 0 & b_5 \\ 0 & 0 & 0 & 0 & c_5 \\ 0 & 0 & 0 & 0 & d_5 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$
rank 1.

2. $$\begin{pmatrix} 0 & a_2 & a_3 & a_4 & a_5 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$
rank 1, $a_2 \neq 0$.

3. $$\begin{pmatrix} 0 & 0 & 0 & a_4 & a_5 \\ 0 & 0 & 0 & 0 & b_5 \\ 0 & 0 & 0 & 0 & c_5 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$
rank 2, $a_4 \neq 0$.

4. $$\begin{pmatrix} 0 & 0 & a_3 & a_4 & a_5 \\ 0 & 0 & 0 & 0 & b_5 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$
rank 2, $a_3 \neq 0$.

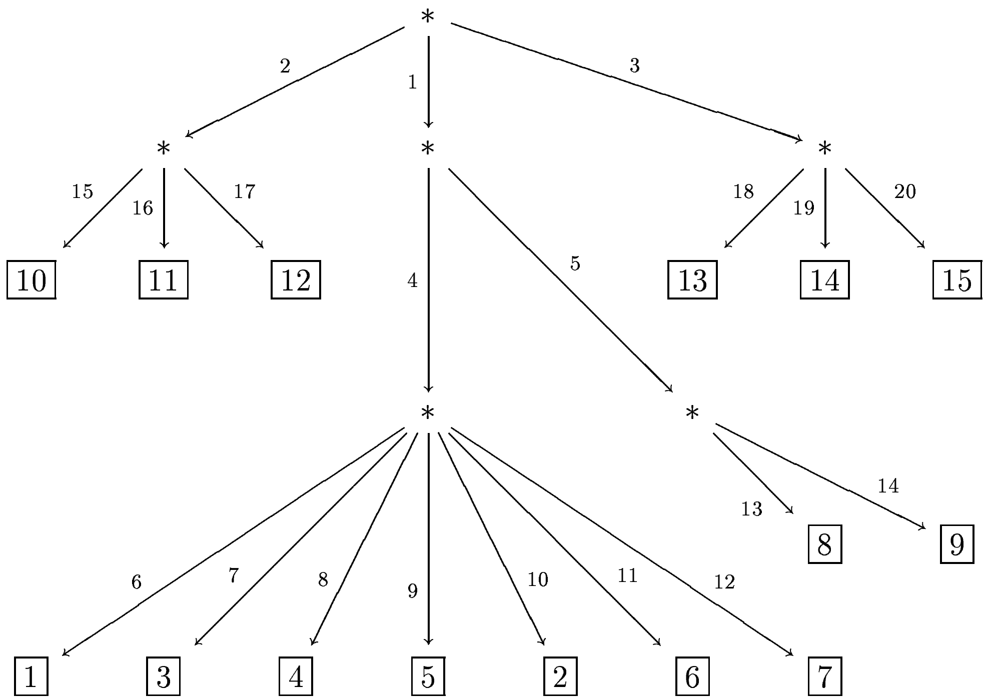
5. $$\begin{pmatrix} 0 & 0 & a_3 & -\frac{a_3 c_5^3}{d_5^3} & a_5 \\ 0 & 0 & 0 & 0 & b_5 \\ 0 & 0 & 0 & 0 & c_5 \\ 0 & 0 & 0 & 0 & d_5 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$
rank 2, $a_3 \neq 0$, $d_5 \neq 0$.

6. $$\begin{pmatrix} 0 & a_2 & -\frac{a_2 b_5^3}{c_5^3} & a_4 & a_5 \\ 0 & 0 & 0 & 0 & b_5 \\ 0 & 0 & 0 & 0 & c_5 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$
rank 2, $a_2 \neq 0$, $c_5 \neq 0$.

7. $$\begin{pmatrix} 0 & a_2 & a_3 & \frac{-a_2 b_5^3 - a_3 c_5^3}{d_5^3} & a_5 \\ 0 & 0 & 0 & 0 & b_5 \\ 0 & 0 & 0 & 0 & c_5 \\ 0 & 0 & 0 & 0 & d_5 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$
rank 2, $a_2 \neq 0$, $d_5 \neq 0$.

8. $$\begin{pmatrix} 0 & 0 & 0 & a_4 & a_5 \\ 0 & 0 & 0 & b_4 & b_5 \\ 0 & 0 & 0 & 0 & c_5 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$
rank 2, $b_4 \neq 0$.

9. $$\begin{pmatrix} 0 & 0 & a_3 & a_4 & a_5 \\ 0 & 0 & 0 & b_4 & b_5 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$
rank 2, $a_3 \neq 0$, $b_4 \neq 0$.



1:$b_3=0,c_4=0$

2:$b_3=0,c_4\neq 0,d_5=0$

3:$b_3\neq 0,c_4=0,a_2=0$

4:$b_3=0,c_4=0,b_4=0$

5:$b_3=0,c_4=0,b_4\neq 0,a_2=0,d_5=0$

6:$b_3=0,c_4=0,b_4=0,a_2=0,a_3=0,a_4=0$

7:$b_3=0,c_4=0,b_4=0,a_2=0,a_3=0,a_4\neq 0,d_5=0$

8:$b_3=0,c_4=0,b_4=0,a_2=0,a_3\neq 0,d_5=0,c_5=0$

9:$b_3=0,c_4=0,b_4=0,a_2=0,a_3,d_5\neq 0$

10:$b_3=0,c_4=0,b_4=0,a_2\neq 0,c_5=0,b_5=0,d_5=0$

11:$b_3=0,c_4=0,b_4=0,a_2,c_5\neq 0,d_5=0$

12:$b_3=0,c_4=0,b_4=0,a_2,d_5\neq 0$

13:$b_3=0,c_4=0,b_4\neq 0,a_3=0,a_2=0,d_5=0$

14:$b_3=0,c_4=0,b_4,a_3\neq 0,a_2=0,d_5=0,c_5=0$

15:$b_3=0,c_4\neq 0,a_2=0,a_3=0,d_5=0$

16:$b_3=0,c_4,a_2\neq 0,a_3=0,b_5=0,b_4=0,d_5=0$

17:$b_3=0,c_4,a_2,b_4\neq 0,d_5=0$

18:$b_3\neq 0,d_5=0,c_5=0,c_4=0,a_2=0$

19:$b_3,d_5\neq 0,a_3=0,a_4=0,c_4=0,a_2=0$

20:$b_3,d_5,a_3\neq 0,c_4=0,a_2=0$

Figure 1. Solution tree for nilpotence index two

10. $\begin{pmatrix} 0 & 0 & 0 & a_4 & a_5 \\ 0 & 0 & 0 & b_4 & b_5 \\ 0 & 0 & 0 & c_4 & c_5 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$
rank 2, $c_4 \neq 0$.

11. $\begin{pmatrix} 0 & a_2 & 0 & a_4 & a_5 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & c_4 & c_5 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$
rank 2, $a_2 \neq 0$, $c_4 \neq 0$.

12. $\begin{pmatrix} 0 & a_2 & -\dfrac{a_2 b_4^3}{c_4^3} & a_4 & a_5 \\ 0 & 0 & 0 & b_4 & \dfrac{b_4 c_5}{c_4} \\ 0 & 0 & 0 & c_4 & c_5 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$
rank 2, $a_2 \neq 0$, $b_4 \neq 0$, $c_4 \neq 0$.

13. $\begin{pmatrix} 0 & 0 & a_3 & a_4 & a_5 \\ 0 & 0 & b_3 & b_4 & b_5 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$
rank 2, $b_3 \neq 0$.

14. $\begin{pmatrix} 0 & 0 & 0 & 0 & a_5 \\ 0 & 0 & b_3 & -\dfrac{b_3 c_5^3}{d_5^3} & b_5 \\ 0 & 0 & 0 & 0 & c_5 \\ 0 & 0 & 0 & 0 & d_5 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$
rank 2, $b_3 \neq 0$, $d_5 \neq 0$.

15. $\begin{pmatrix} 0 & 0 & a_3 & -\dfrac{a_3 c_5^3}{d_5^3} & a_5 \\ 0 & 0 & b_3 & -\dfrac{b_3 c_5^3}{d_5^3} & b_5 \\ 0 & 0 & 0 & 0 & c_5 \\ 0 & 0 & 0 & 0 & d_5 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$
rank 2, $a_3 \neq 0$, $b_3 \neq 0$, $d_5 \neq 0$.

4.2 Nilpotence index three

In this case we have a system of 123 equations. Solving this system gives ten solutions. Ordered by rank these solutions are:

16. $\begin{pmatrix} 0 & a_2 & a_3 & a_4 & a_5 \\ 0 & 0 & b_3 & b_4 & b_5 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$
rank 2, $a_2 \neq 0$, $b_3 \neq 0$.

17. $\begin{pmatrix} 0 & 0 & 0 & a_4 & a_5 \\ 0 & 0 & 0 & b_4 & b_5 \\ 0 & 0 & 0 & c_4 & c_5 \\ 0 & 0 & 0 & 0 & d_5 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$
rank 2, $d_5 \neq 0$.

18. $\begin{pmatrix} 0 & a_2 & a_3 & a_4 & a_5 \\ 0 & 0 & 0 & 0 & b_5 \\ 0 & 0 & 0 & 0 & c_5 \\ 0 & 0 & 0 & 0 & d_5 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$
rank 2, $a_3 \neq 0$, $d_5 \neq 0$.

19. $\begin{pmatrix} 0 & a_2 & a_3 & a_4 & a_5 \\ 0 & 0 & 0 & b_4 & b_5 \\ 0 & 0 & 0 & c_4 & c_5 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$
rank 3.

20. $\begin{pmatrix} 0 & a_2 & 0 & a_4 & a_5 \\ 0 & 0 & 0 & 0 & b_5 \\ 0 & 0 & 0 & c_4 & c_5 \\ 0 & 0 & 0 & 0 & d_5 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$
rank 3, $a_2 \neq 0$, $d_5 \neq 0$.

21. $\begin{pmatrix} 0 & 0 & a_3 & a_4 & a_5 \\ 0 & 0 & 0 & b_4 & b_5 \\ 0 & 0 & 0 & 0 & c_5 \\ 0 & 0 & 0 & 0 & d_5 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$
rank 3, $a_3 \neq 0$, $b_4 \neq 0$, $d_5 \neq 0$.

22. $\begin{pmatrix} 0 & -\dfrac{a_3 c_4^3}{b_4^3} & a_3 & a_4 & a_5 \\ 0 & 0 & 0 & b_4 & \dfrac{b_4 c_5}{c_4} \\ 0 & 0 & 0 & c_4 & c_5 \\ 0 & 0 & 0 & 0 & d_5 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$
rank 3, $a_3 \neq 0$, $b_4 \neq 0$, $c_4 \neq 0$, $d_5 \neq 0$.

23. $\begin{pmatrix} 0 & 0 & a_3 & a_4 & a_5 \\ 0 & 0 & b_3 & b_4 & b_5 \\ 0 & 0 & 0 & 0 & c_5 \\ 0 & 0 & 0 & 0 & d_5 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$
rank 3, $b_3 \neq 0$.

24. $\begin{pmatrix} 0 & a_2 & a_3 & a_4 & a_5 \\ 0 & 0 & b_3 & -\frac{b_3 c_5^3}{d_5^3} & b_5 \\ 0 & 0 & 0 & 0 & c_5 \\ 0 & 0 & 0 & 0 & d_5 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$
rank 3, $a_2 \neq 0$, $d_5 \neq 0$.

25. $\begin{pmatrix} 0 & 0 & a_3 & a_4 & a_5 \\ 0 & 0 & b_3 & b_4 & b_5 \\ 0 & 0 & 0 & c_4 & c_5 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$
rank 3, $b_3 \neq 0$, $c_4 \neq 0$.

In [3] one can find the solution tree corresponding to these solutions.

4.3 Nilpotence index four

Here we have a system of 56 equations. There are only four solutions:

26. $\begin{pmatrix} 0 & 0 & a_3 & a_4 & a_5 \\ 0 & 0 & b_3 & b_4 & b_5 \\ 0 & 0 & 0 & c_4 & c_5 \\ 0 & 0 & 0 & 0 & d_5 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$
rank 3.

27. $\begin{pmatrix} 0 & a_2 & a_3 & a_4 & a_5 \\ 0 & 0 & 0 & b_4 & b_5 \\ 0 & 0 & 0 & c_4 & c_5 \\ 0 & 0 & 0 & 0 & d_5 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$
rank 3, $a_2 \neq 0$.

28. $\begin{pmatrix} 0 & a_2 & a_3 & a_4 & a_5 \\ 0 & 0 & b_3 & b_4 & b_5 \\ 0 & 0 & 0 & 0 & c_5 \\ 0 & 0 & 0 & 0 & d_5 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$
rank 3, $a_2 \neq 0$, $b_3 \neq 0$.

29. $\begin{pmatrix} 0 & a_2 & a_3 & a_4 & a_5 \\ 0 & 0 & b_3 & b_4 & b_5 \\ 0 & 0 & 0 & c_4 & c_5 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$
rank 3, $a_2 \neq 0$, $b_3 \neq 0$, $c_4 \neq 0$.

The solution tree is quite simple. It can be found in [3].

4.4 Nilpotence index five

Finally the last case gives one solution since all matrices of the form (4.1) are nilpotent.

30. $\begin{pmatrix} 0 & a_2 & a_3 & a_4 & a_5 \\ 0 & 0 & b_3 & b_4 & b_5 \\ 0 & 0 & 0 & c_4 & c_5 \\ 0 & 0 & 0 & 0 & d_5 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$, rank 4, $a_2 \neq 0$, $b_3 \neq 0$, $c_4 \neq 0$, $d_5 \neq 0$.

5 Cubic similarity reduction

The basic result of the previous section is that we can reformulate corollary 1 to:

Corollary 2 *Let $F = X + (AX)^{*3} : \mathbb{C}^5 \to \mathbb{C}^5$ such that $\det(JF) = 1$ then there exists an invertible linear map L such that $L \circ F \circ L^{-1} = X + (BX)^{*3}$, where B is the null matrix or B is one of the thirty matrices presented in section 4.*

The next thing we have to do is check whether these maps are cubic similar to the matrices of section 3. In order to find these relations we use the fact that the rank is an invariant of this matrix. At this point it is more practical to use the rank as an invariant than the nilpotence index of the corresponding jacobian. This is because we have to make some assumptions on the parameters still appearing in the matrices of the previous section, and mostly the effect on the rank of these assumptions are more clearly than the effects on the nilpotence index.

The basic approach taken is:

1. Try to reduce A to cases already known by use of permutaion matrices.
2. Take a general linear map T containing parameters.
3. Compute B where B is given by $X + (BX)^{*3} = T^{-1} \circ (X + (AX)^{*3}) \circ T$.
4. Compare B with the already known representatives.
5. Guess which one of those can be identified with B. (Call this matrix M.)
6. Solve $B = M$ in the variables of T.
7. If this system has no solution:
    - Guess another M.
    - If all representatives have been tried, one probably has found a matrix which is not equivalent to the known representatives.
    - Reduce A as much as possible to M', i.e. solve $B_{i,j} = 0$ or $B_{i,j} = 1$ for as many entries $B_{i,j}$ as possible.
    - Prove that the new M' is indeed not cubic similar to all the old representatives of the same rank.
8. If this system has at least one solution:
    - Try to simplify the solution(s) by setting free parameters equal to zero or to one in case they cannot be set to zero.
    - Check if this T implies some new assumptions on the original parameters in the matrices in order to have that T is invertible.
        - If it does not, you have found that $A \overset{3}{\sim} M$ in general.
        - If it does, assume these assumptions don't hold and apply this information to reduce A to A' and repeat the complete process on A'.

In [3] this process is described for each of the thirty matrices. Here we will show one example.

Example 1 Consider $F = X + (AX)^{*3}$ where

$$A = \begin{pmatrix} 0 & 0 & 0 & a_4 & a_5 \\ 0 & 0 & 0 & b_4 & b_5 \\ 0 & 0 & 0 & 0 & c_5 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

We already know that $b_4 \neq 0$. If we compute $T_1^{-1} F T_1 = X + (BX)^{*3}$ for a general map T_1 and try to solve the cases $B = J(2,2)$, $B = J(2,3)$ and $B = N(2,3a)$ we don't get any solution at all. So most probably we have found a new representative. If we try to reduce this B, we see that we can find T_1 such that $B_{1,4} = 1$, $B_{2,4} = 1$, $B_{2,5} = 1$ and $B_{3,5} = 1$ and all other $B_{i,j} = 0$. We call this M'. Looking carefully at the definition of cubic similarity shows that this M' is indeed not cubic similar to the known representatives with rank two. We call this new representative $N(2,2a)$. The T_1 we have used is

$$\left(\frac{(b_5\, a_4 - b_4\, a_5)^3\; x_1}{{b_4}^3}, \frac{(b_5\, a_4 - b_4\, a_5)^3\; x_2}{{a_4}^3}, {c_5}^3\, x_3\;, \frac{(b_5\, a_4 - b_4\, a_5)\; x_4}{b_4\, a_4} - \frac{a_5\, x_5}{a_4}, x_5 \right)$$

If we look at this T_1 we see that it is invertible only if $a_4 \neq 0$, $c_5 \neq 0$ and $b_5 a_4 - b_4 a_5 \neq 0$. (We already know that $b_4 \neq 0$.)

Now assume that $a_4 \neq 0$ and $c_5 \neq 0$ but $b_5 a_4 - b_4 a_5 = 0$ and start the process again. After taking a new T_2 and compute $T_2^{-1} F T_2$, we get a matrix B that can be identified with $J(2,2)$. Solving this system yields that T_2 is

$$\left(x_5 + {a_4}^3\, x_3, {b_4}^3\, x_3, x_1, x_4 - \frac{b_5\, x_2}{b_4\, c_5}, \frac{x_2}{c_5} \right)$$

Looking at T_2 we note that we don't need any new assumptions. From T_1 it already follows that we have to look at the cases where $a_4 = 0$ and $c_5 = 0$.

Now assume $a_4 \neq 0$ and $b_5 a_4 - b_4 a_5 \neq 0$ but $c_5 = 0$. In this case the map T_3 gives $T_3^{-1} F T_3$ which is cubic similar to $J(2,2)$ where T_3 is given by

$$\left(-\frac{(b_5\, a_4 - b_4\, a_5)^3\; x_1}{{b_4}^3}, -\frac{(b_5\, a_4 - b_4\, a_5)^3\; x_3}{{a_4}^3}, x_5, \frac{a_5\, x_4}{a_4} - \frac{x_2\, b_5}{b_4}, x_2 - x_4 \right)$$

Note that this map T_3 does not imply any new assumptions.

Now assume $a_4 \neq 0$ but $b_5 a_4 - b_4 a_5 = 0$ and $c_5 = 0$. We can immediately skip this case since it gives a matrix A with $\operatorname{rank}(A) = 1$.

So the next case is $a_4 = 0$. In order to remain in a rank two case we must have that either $a_5 \neq 0$ or $c_5 \neq 0$. We may assume $c_5 \neq 0$ since a simple permutation $P = (x_3, x_2, x_1, x_4, x_5)$ swaps the first and third row. So now we can use T_4 is

$$\left(x_5 + {a_5}^3\, x_3, {b_4}^3\, x_1, {c_5}^3\, x_3, x_2 - \frac{b_5\, x_4}{b_4}, x_4 \right)$$

to get that $T_4^{-1} F T_4$ is cubic similar to $J(2,2)$.

And with this last case we have solved the case for this matrix completely, since T_4 does not imply any new assumptions.

6 New representatives

Examining all thirty matrices from section 4 in a similar way as in example 1 completely classifies the Drużkowski maps in dimension five with respect to the cubic similarity relation. This tedious process gives the following new representatives:

$$\underset{N(2,2a)}{\begin{pmatrix} 0&0&0&1&0\\ 0&0&0&1&1\\ 0&0&0&0&1\\ 0&0&0&0&0\\ 0&0&0&0&0 \end{pmatrix}} \quad \underset{N(2,3b)}{\begin{pmatrix} 0&0&0&1&0\\ 0&0&0&1&1\\ 0&0&0&1&-1\\ 0&0&0&0&1\\ 0&0&0&0&0 \end{pmatrix}} \quad \underset{N(3,3b)}{\begin{pmatrix} 0&1&1&0&0\\ 0&0&0&1&0\\ 0&0&0&0&1\\ 0&0&0&0&0\\ 0&0&0&0&0 \end{pmatrix}}$$

$$\underset{N(3,4e)}{\begin{pmatrix} 0&0&1&0&1\\ 0&0&1&0&0\\ 0&0&0&1&0\\ 0&0&0&0&1\\ 0&0&0&0&0 \end{pmatrix}} \quad \underset{N(3,4f)}{\begin{pmatrix} 0&0&1&0&0\\ 0&0&0&1&1\\ 0&0&0&1&0\\ 0&0&0&0&1\\ 0&0&0&0&0 \end{pmatrix}} \quad \underset{N(3,4g)}{\begin{pmatrix} 0&0&1&1&0\\ 0&0&0&1&1\\ 0&0&0&1&0\\ 0&0&0&0&1\\ 0&0&0&0&0 \end{pmatrix}}$$

$$\underset{N(3,4h)}{\begin{pmatrix} 0&0&1&1&1\\ 0&0&1&0&0\\ 0&0&0&1&0\\ 0&0&0&0&1\\ 0&0&0&0&0 \end{pmatrix}} \quad \underset{N(3,4i)}{\begin{pmatrix} 0&1&1&0&0\\ 0&0&0&1&1\\ 0&0&0&1&0\\ 0&0&0&0&1\\ 0&0&0&0&0 \end{pmatrix}} \quad \underset{N(3,4j)}{\begin{pmatrix} 0&1&0&1&0\\ 0&0&0&1&1\\ 0&0&0&1&0\\ 0&0&0&0&1\\ 0&0&0&0&0 \end{pmatrix}}$$

$$\underset{P(3,4a)}{\begin{pmatrix} 0&0&1&a&0\\ 0&0&1&1&0\\ 0&0&0&1&0\\ 0&0&0&0&1\\ 0&0&0&0&0 \end{pmatrix}} \quad \underset{P(3,4c)}{\begin{pmatrix} 0&0&1&0&a\\ 0&0&1&1&0\\ 0&0&0&1&0\\ 0&0&0&0&1\\ 0&0&0&0&0 \end{pmatrix}} \quad \underset{P(3,4g)}{\begin{pmatrix} 0&0&1&1&0\\ 0&0&0&1&a\\ 0&0&0&1&0\\ 0&0&0&0&1\\ 0&0&0&0&0 \end{pmatrix}}$$

$$\underset{P(3,4h)}{\begin{pmatrix} 0&0&1&1&a\\ 0&0&1&0&0\\ 0&0&0&1&0\\ 0&0&0&0&1\\ 0&0&0&0&0 \end{pmatrix}} \quad \underset{P(3,4i)}{\begin{pmatrix} 0&1&a&0&0\\ 0&0&0&1&1\\ 0&0&0&1&0\\ 0&0&0&0&1\\ 0&0&0&0&0 \end{pmatrix}} \quad \underset{P(3,4j)}{\begin{pmatrix} 0&1&0&1&0\\ 0&0&0&1&a\\ 0&0&0&1&0\\ 0&0&0&0&1\\ 0&0&0&0&0 \end{pmatrix}}$$

$$\underset{P(3,4a2)}{\begin{pmatrix} 0&0&1&a&b\\ 0&0&1&1&0\\ 0&0&0&1&0\\ 0&0&0&0&1\\ 0&0&0&0&0 \end{pmatrix}} \quad \underset{P(3,4j2)}{\begin{pmatrix} 0&1&a&1&0\\ 0&0&0&1&b\\ 0&0&0&1&0\\ 0&0&0&0&1\\ 0&0&0&0&0 \end{pmatrix}} \quad \underset{N(4,5e)}{\begin{pmatrix} 0&1&0&1&0\\ 0&0&1&1&0\\ 0&0&0&1&0\\ 0&0&0&0&1\\ 0&0&0&0&0 \end{pmatrix}}$$

$$\underset{P(4,5e)}{\begin{pmatrix} 0&1&0&1&0\\ 0&0&1&a&0\\ 0&0&0&1&0\\ 0&0&0&0&1\\ 0&0&0&0&0 \end{pmatrix}}$$

Remark 2 Similar to remark 1 we note the following:

- In $N(2,3b)$ the -1 seems a bit strange: why isn't it $P(2,3a)$ with a parameter a on the place of the -1? The answer is in fact pretty simple. As long as $a \notin \{0,1\}$, $P(2,3a) \overset{3}{\sim} N(2,3b)$. Furthermore $P(2,3a)_{|a=0} \overset{3}{\sim} P(2,3a)_{|a=1} \overset{3}{\sim} N(2,3a)$. So independent of the value of the parameter a, $P(2,3a)$ can be reduced to a matrix with no parameters left in it. So there's no need to add a P-matrix.
- $P(3,4a)_{|a=1} \overset{3}{\sim} N(3,4a)$ and $P(3,4a)_{|a=0} = N(3,4b)$.
- $P(3,4c)_{|a=1} \overset{3}{\sim} N(3,4c)$ and $P(3,4c)_{|a=0} = N(3,4b)$.
- $P(3,4g)_{|a=1} = N(3,4g)$ and $P(3,4g)_{|a=0} \overset{3}{\sim} N(3,4a)$.
- $P(3,4h)_{|a=1} = N(3,4h)$ and $P(3,4h)_{|a=0} \overset{3}{\sim} N(3,4b)$.
- $P(3,4i)_{|a=1} = N(3,4i)$ and $P(3,4i)_{|a=0} \overset{3}{\sim} N(3,4a)$.
- $P(3,4j)_{|a=1} = N(3,4j)$ and $P(3,4j)_{|a=0} \overset{3}{\sim} N(3,4a)$.
- $P(3,4a2)_{|a=0} = P(3,4c)$ and $P(3,4a2)_{|b=0} = P(3,4a)$, hence $P(3,4c2)$ would have been a correct name also.
- $P(3,4j2)_{|a=0} = P(3,4j)$. Furthermore we have $P(3,4j2)_{|b=0,a=-1} \overset{3}{\sim} N(3,3a)$ and $P(3,4j2)_{|b=0,a\neq 0,a\neq -1} \overset{3}{\sim} N(3,4a)$.
- $P(4,5e)_{|a=1} = N(4,5e)$ and $P(4,5e)_{|a=0} = N(4,5d)$.
- So we add for $P(3,4a)$, $P(3,4c)$, $P(3,4g)$, $P(3,4h)$, $P(3,4i)$, $P(3,4j)$ and $P(4,5e)$ the restriction that $a \notin \{0,1\}$. For $P(3,4a2)$ and $P(3,4j2)$ we add $a, b \neq 0$.

The final claim in this paper is that the seventeen matrices by Meisters in section 3 together with the nineteen matrices in this section give a complete family of inequivalent matrices with respect to Meisters' cubic similarity relation in dimension five. Unfortunately in dimension five the amount of work compared to the work in dimension four has increased enormously. Therefore it doesn't look very promising to start with research on the dimension six case. Especially if one bares in mind that the five-dimensional case only worked out because of the strong reduction theorem 2 and the fact that we don't have such a theorem in dimension six. The problem for this theorem is that we now have the possibility that $\text{rank}(A) = \text{corank}(A) = 3$ and we cannot use Drużkowski theorem anymore. But nevertheless, even with an equivalent reduction theorem in dimension six, it would most probably still be too complex to compute.

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