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The Jacobian Conjecture

Cubic Homogeneous Maps in Dimension Four

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Abstract

This thesis presents some interesting results concerning the research on the Jacobian Conjecture.

In a short historical summary of the research on this topic in chapter 1 we show that the general Jacobian Conjecture can be reduced to the case where $F = X - H$ is a so-called cubic homogeneous map. Wright has solved this case for $n = 3$; we present his result in theorem 1.20.

In chapter 2 we generalize Wright's result to the case $n = 4$. Here the most important result is given in theorem 2.7, where we give a complete classification of all cubic homogeneous polynomial maps in four indeterminates, which have non-zero constant determinants of their Jacobian matrices. Furthermore with this classification we prove that all these cubic homogeneous polynomial maps are invertible in corollary 2.9.

In chapter 3 we present a general theorem (theorem 3.1) concerning the form of cubic homogeneous maps in two indeterminates and coefficients in a uniform factorization domain. It is proved that all such polynomial maps can be written in a very simple way. As an illustration of this theorem we write the explicit maps of theorem 2.7 in the form of theorem 3.1.

In chapter 4 we return to the four dimensional cubic homogeneous maps. Here we consider a subclass of the general class of chapter 2: we examine the so-called Drużkowski forms. The result of this research is presented in theorem 4.2 where we give a complete classification of these maps. With this classification of these particular maps we prove a conjecture of Meisters (see conjecture 4.5 and theorem 4.6) concerning representatives for the 4×4 power similarity relation. We prove that the six representatives Meisters described are a complete set of representatives for the power similarity relation. We also prove a very general theorem (theorem 4.7) concerning the validity of the Jacobian Conjecture for Drużkowski forms in dimension n with the $n \times n$ coefficient matrix A and with $\text{rank}(A) = r$, depending on the validity of the Jacobian Conjecture for cubic homogeneous maps in dimension r. So this theorem gives a criterion to determine whether a Drużkowski map in a higher dimension is invertible or not. In particular we prove in theorem 4.8 that the Jacobian Conjecture is true if the rank $r \leq 4$. Consequently we prove in theorem 4.9 that a polynomial map on Drużkowski form which satisfies the Jacobian hypothesis is invertible if $n \leq 7$.

Chapter 5 deals with some related problems. For instance we give an answer to the question whether cubic homogeneous maps which satisfy the Jacobian hypothesis are exponents of some locally nilpotent derivations (theorem 5.2 and theorem 5.7). Although there was one map that could not be written as $\exp(D)$, we found that all maps could be written as $\exp(D_1) \circ \exp(D_2)$. See theorem 5.8. Furthermore we give examples of homogeneous Jacobian matrices in four variables that are not strongly nilpotent (theorem 5.11) and we show how we can extend these examples to higher dimensions (example 5.15). We also prove that all homogeneous parts of the cubic homogeneous maps in dimension four vanish after a finite time of iterations (theorem 5.13).

Chapter 6 describes the results we found in dimension five. We studied the linear cubic homogeneous case: remarkably enough all maps we found in this class can be seen as maps in two dimensions (examples 6.2, 6.3, 6.4, 6.5 and 6.6). And with theorem 3.1 we know exactly how we can describe these maps. Finally we present some topics for future research in this area in section 6.3.

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Contents

Chapter 1

History

1.1 Introduction

The original aim of this thesis was to find a counterexample for the Jacobian Conjecture. This may seem a little bit opportunistic since the conjecture is open since 1939. But because we used a very systematic way of searching there were only three things that could happen:

1. We could find a counterexample as desired.
2. We could find that the system of equations could not be solved completely.
3. We couldn't find a counterexample but we could find a complete solution of the systems of equations, i.e. we could find a complete classification of all possible polynomial maps.

Only the second possibility is a bad result. In order to avoid this problem we had to restrict ourselves. The main problem in dimension four -or in any higher dimension- is the number of indeterminates, since this number is a measure for the number of monomials in F and thus for the number of variables in the system of equations. Since it is known from [Yagzhev 80] and [Bass et al. 82] that the validity of the Jacobian Conjecture rests on the so-called cubic homogeneous maps we first restricted ourselves to these maps. The second restriction was to examine dimension four. This was a natural choice since in [Wright 93] the situation in dimension three is completely described for cubic homogeneous maps. So if we wanted to find anything interesting we had to examine dimension four. However in that particular article Wright writes:

> 'Here it becomes useful to assume F is cubic homogeneous, since this limits the number of its monomials. The dimension four case may still be out of range even with this reduction, however; the number of monomials of degree three in four variables is 20, so the number of monomials for a cubic homogeneous map in dimension four is $20 \times 4 = 80$.'

Though this was not very encouraging we decided to give it a try anyway. Mainly because we could reduce the 80 variables to 64 by using some transformation matrix. With the current possibilities to solve a set of equations in a systematic, interactive way, we hoped that it would work out. And, fortunately, it did!

But before we present how we did succeed in avoiding the second -bad- possibility, we start with some history of the Jacobian Conjecture.

1.2 The Jacobian Conjecture

Throughout this chapter we shall mainly consider polynomial maps

$$F : \mathbb{C}^n \to \mathbb{C}^n$$

where $F = (F_1, \ldots, F_n)$ and each $F_i \in \mathbb{C}[x_1, \ldots, x_n]$. Throughout this paper we shall use the abbreviation $X = (x_1, x_2, \ldots, x_n)$. Most of the time $n = 4$. And it will be clear from the context if it is not.

Some of the main aims of the study of invertible polynomial maps concerns the following questions:

Question 1.1 *How can we determine whether a polynomial map $F : \mathbb{C}^n \to \mathbb{C}^n$ is invertible or not?*

and of course

Question 1.2 *Knowing that a certain polynomial map F is invertible, how can we determine its inverse?*

If we denote -as usual- the Jacobian matrix of map F by JF, i.e.

$$JF = \left(\frac{\partial}{\partial x_j} F_i\right)_{i,j=1}^{n}$$

we can deduce the first theorem that deals with these questions:

Theorem 1.3 *Let $F : \mathbb{C}^n \to \mathbb{C}^n$ be an invertible polynomial map. Then $\det(JF) \in \mathbb{C}^*$.*

Proof: Let G be the inverse of F. Then we have of course that

$$G(F(X)) = X.$$

By differentiation of this equation with the chain rule we get

$$(JG)(F(X)).JF(X) = I_n$$

so $\det((JG)(F(X)).JF(X)) = \det((JG)(F(X)).\det(JF(X)) = \det(I_n) = 1$ and it follows immediately from the degrees that $\det(JF) \in \mathbb{C}^*$. □

This result immediately leads to the following conjecture:

Conjecture 1.4 (The Jacobian Conjecture) *Let $F : \mathbb{C}^n \to \mathbb{C}^n$ be a polynomial map. Now if $\det(JF) \in \mathbb{C}^*$ then F is an invertible polynomial map.*

Though this conjecture was already posed in [Keller 39], it is still open for $n \geq 2$.

Actually his formulation of the conjecture was in terms of $\mathbb{Z}$ instead of $\mathbb{C}$. Naturally this observation gives rise to the next question.

Question 1.5 *Can $\mathbb{C}$ be replaced by any arbitrary field, or even by a ring like $\mathbb{Z}$?*

For instance let us take the field $\mathbb{F}_p$ for some prime p. Then we have a field with characteristic $p > 1$. Furthermore let $n = 1$. Take $F(x_1) = x_1 + x_1^p$. Then $JF = I_1$ and $\det(JF) = 1 \in \mathbb{F}_p^*$. However for every $G \in \mathbb{F}_p[x_1]$ with $\deg(G) \geq 1$ we have that $\deg(G(x_1 + x_1^p)) = p \deg(G) > 1$. So certainly there is no such G with $G(F(x_1)) = x_1$, thus F is not an invertible map. So the Jacobian Conjecture is not true in this case. So in particular we see that we cannot choose any arbitrary field instead of $\mathbb{C}$.

In particular we see that the characteristic has to be zero. So if we assure this by taking R as a subring of some $\mathbb{Q}$-algebra and if $F_1, \ldots, F_n \in R[X]$ we can formulate a generalized Jacobian Conjecture. Before we do this, we present a useful invertibility criterion.

Lemma 1.6 *Let F be a polynomial map $\mathbb{C}^n \to \mathbb{C}^n$. Then we have F is invertible if and only if $\mathbb{C}[F] = \mathbb{C}[X]$.*

Proof: Assume that F is invertible with inverse $G = (G_1, \ldots, G_n)$. Then for each i we have $x_i = G_i(F_1, \ldots, F_n)$ and thus $x_i \in \mathbb{C}[F]$ where $\mathbb{C}[F] = \mathbb{C}[F_1, \ldots, F_n]$. Naturally we also have that for each i $F_i \in \mathbb{C}[X]$, so the assumption that F is invertible implies that $\mathbb{C}[F] = \mathbb{C}[X]$, which proves one side of the lemma. Now if we assume that $\mathbb{C}[F] = \mathbb{C}[X]$ then for each i we have $x_i \in \mathbb{C}[F]$. But this implies that for each i there exists some $G_i \in \mathbb{C}[X]$ with $x_i = G_i(F_1, \ldots, F_n)$. If we combine these G_i's we get that there exists some $G = (G_1, \ldots, G_n) : \mathbb{C}^n \to \mathbb{C}^n$ with $G(F(X)) = X$. From, for instance, [Adjamagbo et al. 88, Theorem 1.11] it now follows that also $F(G(X)) = X$. But then we have that G is the inverse of F and thus F is invertible which completes the proof of the lemma. □

Conjecture 1.7 (Generalized Jacobian Conjecture ($JC_n(R)$)) *Let R be a subring of a $\mathbb{Q}$-algebra. Let $F : R^n \to R^n$ be a polynomial map. If $\det(JF) \in R[X]^*$ then $R[F] = R[X]$.*

We see that both Keller's problem and the usual conjecture are instances of this generalized Jacobian Conjecture with respectively $R = \mathbb{Z}$ and $R = \mathbb{C}$. But a more interesting observation deals with the following question:

Question 1.8 *Is the generalized Jacobian Conjecture really more general than the usual Jacobian Conjecture?*

The answer to this question is no. It is proved in [Essen 91] that, at the cost of enlarging the number of indeterminates, $JC_n(R)$ is true for all n if and only if $JC_n(\mathbb{C})$ is true for all n. From this it follows that it suffices to examine the usual Jacobian Conjecture. This is especially practical since one can use methods of complex analysis if one regards $JC_n(\mathbb{C})$.

Over the years some other equivalent conjectures have been stated. For instance by Magnus in 1955. In [Magnus 55] he considers so-called volume preserving transformations of complex planes given by analytic functions.

Theorem 1.9 *Let $f, g \in \mathbb{C}[x_1, x_2]$ with $\deg(f) = n$ and $\deg(g) = m$. Consider the map:*

$$(x_1, x_2) \mapsto (f(x_1, x_2), g(x_1, x_2)) \text{ with } \det(J(f, g)) = 1.$$

Then if m or n is a prime then (f, g) is invertible.

Proof: See [Magnus 55]. □

This result has been improved several times. In [Nakai et al. 77] to

Theorem 1.10 *Assume* $\det(J(f,g)) = 1$. *Then the map* (f,g) *is invertible if*

- n *or* m *is prime or*
- n *or* m *equals 4 or*
- $m > n$, $m = 2p$ *for some odd prime* p.

And in [Appelgate et al. 85] and [Nagata 88] to

Theorem 1.11 *If* n *or* m *has at most two prime factors then the map* (f,g) *is invertible.*

Another formulation of the conjecture is the so-called 'Rolle' formulation (see for instance [Essen 91]):

Conjecture 1.12 *If* $F(a) = F(b)$ *with* $a \neq b$ *in* $\mathbb{C}^n$, *then there exists* $\xi \in \mathbb{C}^n$ *such that* $F'(\xi) = 0$.

The equivalence between this formulation and the usual formulation follows from the following lemma:

Lemma 1.13 *Let* $F : \mathbb{C}^n \to \mathbb{C}^n$ *be a polynomial. If* F *is injective, then* F *is invertible.*

Proof: See [Rosenlicht et al. 62]. □

This lemma forms also the base for an important theorem in [Wang 80]:

Theorem 1.14 *Let* $\det(JF) \in \mathbb{C}^*$ *and* $\deg(F_i) \leq 2$ *for all* i. *Then* F *is invertible.*

Proof: From lemma 1.13 it follows that we only have to prove the injectivity of F. So suppose we have some $a \neq b \in \mathbb{C}^n$ with $F(a) = F(b)$. If we define $G(X) := F(X+a) - F(a)$, then we have that $\deg(G) \leq 2$ and $G(0) = 0$. If we take $c = b - a$ then we have that $c \neq 0$ and furthermore $G(c) = 0$. From this it follows that we can take $b = 0$. From the equation

$$(JG)(X) = (JF)(X+a)$$

it follows immediately that also $\det(JG) \in \mathbb{C}^*$. Since $\deg(G) \leq 2$ we can write

$$G = G_{(1)} + G_{(2)}$$

where $G_{(i)}$ is G's homogeneous component of degree i. From this it follows that for all $t \in \mathbb{C}$

$$\begin{aligned} G(tc) &= G_{(1)}(tc) + G_{(2)}(tc) \\ &= tG_{(1)}(c) + t^2 G_{(2)}(c) \end{aligned}$$

By differentiation we get that for all $t \in \mathbb{C}$

$$G_{(1)}(c) + 2tG_{(2)}(c) = \frac{d}{dt} G(tc) = (JG)(tc) \neq 0.$$

But if we substitute $t = \frac{1}{2}$ we get $G(c) \neq 0$ which is a contradiction to the previous derived $G(c) = 0$. So we have proved that F is indeed injective and thus invertible. □

It may seem that this theorem is just a very special case of the general case. However this is not really true. Independently Bass, Connell and Wright on one side and Yagzhev on the other side proved the following theorem.

Theorem 1.15 *If the Jacobian Conjecture holds for all $n \geq 2$ and all polynomial maps F with $\deg(F_i) \leq 3$ for all i, then the Jacobian Conjecture holds.*

Proof: See [Bass et al. 82] or [Yagzhev 80]. □

From this theorem it immediately follows that the special case of theorem 1.14 is nearly the general case.

Also in [Bass et al. 82] the result of theorem 1.15 has been improved.

Theorem 1.16 *If the Jacobian Conjecture holds for all $n \geq 2$ and all polynomial maps F of the form $F = X - H$ where H is homogeneous of degree three, then the Jacobian Conjecture holds.*

However even this theorem could be improved as was shown in [Drużkowski 83]:

Theorem 1.17 *If the Jacobian Conjecture holds for all $n \geq 2$ and all polynomial maps F of the form*

$$F = \left(x_1 - \left(\sum_{j=1}^{n} c_{j1} x_j \right)^3, \ldots, x_n - \left(\sum_{j=1}^{n} c_{jn} x_j \right)^3 \right)$$

where $c_{ji} \in \mathbb{C}$ then the Jacobian Conjecture holds.

More recently Drużkowski combined some of his own results from [Drużkowski 83] and [Drużkowski 85] and proved the theorem:

Theorem 1.18 *Let F be a polynomial map of the form in theorem 1.17. Let A be the matrix of the coefficients of F, i.e. $A = (c_{ij})$. Then the Jacobian Conjecture is true if* $\operatorname{rank}(A) \leq 2$ *or* $\operatorname{corank}(A) \leq 2$.

Proof: See [Drużkowski 93]. □

An immediate consequence of this theorem is:

Corollary 1.19 *Let F be a polynomial map of the form in theorem 1.17. Then the Jacobian Conjecture holds for this F if $n \leq 5$.*

Proof: Let $A = (c_{ij})$. If $\operatorname{rank}(A) \leq 2$ then the corollary is true by theorem 1.18. If $\operatorname{rank}(A) > 2$ then $\operatorname{corank}(A) \leq 2$ and the corollary is again true by theorem 1.18. □

However this result has been improved, based on the following theorem by Wright.

Theorem 1.20 *Suppose $F = (F_1, F_2, F_3)$ is cubic homogeneous and satisfies the Jacobian hypothesis. Then F is linearly triangularizable, i.e., there exists a linear (homogeneous) automorphism A of K^3 such that AFA^{-1} is triangular. In particular, F is an automorphism.*

Proof: See [Wright 93]. □

If we now look ahead and use theorem 4.7 we can improve corollary 1.19 to:

Corollary 1.21 *Let F be a polynomial map of the form in theorem 1.17. Then the Jacobian Conjecture holds for this F if $n \leq 6$.*

Proof: Let $A = (c_{ij})$. If $\text{rank}(A) \leq 3$ then the corollary is true by theorem 1.20 and theorem 4.7. If $\text{rank}(A) > 3$ then $\text{corank}(A) \leq 2$ and the corollary is true by theorem 1.18. □

Probably this is the best result so far, thus -as we have stated before- the Jacobian Conjecture in general is still open.

1.3 Invertibility criteria

As we have seen in the previous section, one of the main interests concerns invertibility criteria. Since Keller formulated his conjecture in 1939 a lot of research has taken place on this subject. In lemma 1.6 and lemma 1.13 we have already given some criteria. Since for our research we needed a good invertibility criterion we shall present some more in this section.

Already in 1974 Gurjar and Abhyankar found the so-called *Abhyankar's inversion formula.*

Theorem 1.22 *Let K be a field with characteristic zero. Let $F : K^n \to K^n$ be a polynomial map with $F_i = x_i + H_i$ where all terms in H_i have a degree greater than one. If F is invertible then the inverse $G = (G_1, \ldots, G_n)$ is given by*

$$G_i = \sum_{p_1,\ldots,p_n \geq 0} \frac{1}{p_1! \ldots p_n!} \partial_1^{p_1} \ldots \partial_n^{p_n} (x_i H_1^{p_1} \cdots H_n^{p_n}).$$

The problem with this theorem lies in the fact that you don't know how far you have to go with the summation in order to find G_i. Fortunately Gabber proved a theorem that provided an upperbound for the degree of G depending on the degree of F. If we define $\deg(F) = \max \deg(F_i)$ we get:

Theorem 1.23 *If K is a field and $F : K^n \to K^n$ is an invertible polynomial map with inverse G, then $\deg(G) \leq (\deg(F)^{n-1})$.*

Proof: See for instance [Rusek 89]. □

However even with this upperbound for the degrees, still Abhyankar's inversion formula is not very practical. It took until 1986 before there was some progress at this point. With the use of Gröbner bases for ideals in polynomial rings, Van den Essen gave some nice answers to questions 1.1 and 1.2.

Here we can even use an arbitrary field K; characteristic zero is no longer needed. Now let $F : K^n \to K^n$ be a polynomial map and $F_1, \ldots, F_n \in K[X]$. Introduce n new variables $y_1, \ldots, y_n$ and regard the ideal I generated by the elements $y_1 - F_1(X), \ldots, y_n - F_n(X)$. Then I is an ideal in $K[X, Y] = K[x_1, \ldots, x_n, y_1, \ldots, y_n]$. Now choose some admissible $<$ such that any power product in $y_1, \ldots, y_n$ is smaller than any power product in $x_1, \ldots, x_n$. Now we have:

Theorem 1.24 *Let G be the reduced Gröbner basis of I. Then F is invertible if and only if $G = \{x_1 - G_1(Y), \ldots, x_n - G_n(Y)\}$ for some $G_i \in K[Y]$. Furthermore, if F is invertible then the inverse is given by $G = (G_1, \ldots, G_n)$.*

Proof: See [Essen 90]. □

Though this algorithm is a great improvement from Abhyankar's inversion formula, it is still very slow due to the computation of the Gröbner bases, which can be -even for small examples- very slow.

In order to decrease the computation time one needed bigger steps in the reduction process. By looking at the Gröbner basis G one sees that I contains an element of the form $x_1 - G_1(Y)$. In case of two indeterminates x_1 and x_2, this means that there has to be an element in I from which x_2 is eliminated. The main tool in elimination theory is the resultant.

Definition 1.25 *Let A be a commutative ring without zero divisors, K its quotient field and $A[T]$ the polynomial ring in the indeterminate T with coefficients in A. Furthermore let*

$$\begin{aligned} f &= f_n T^n + f_{n-1} T^{n-1} + \cdots + f_0 \\ g &= g_m T^m + g_{m-1} T^{m-1} + \cdots + g_0 \end{aligned}$$

with f_n and $g_m \neq 0$. Then the resultant of f and g is defined by

1. *if $n, m \geq 1$:*

$$R_T(f,g) = \begin{pmatrix} f_n & \cdots & f_0 & & \\ & \ddots & & \ddots & \\ & & f_n & \cdots & f_0 \\ g_m & \cdots & g_0 & & \\ & \ddots & & \ddots & \\ & & g_m & \cdots & g_0 \end{pmatrix} \begin{matrix} \left.\vphantom{\begin{matrix}a\\b\\c\end{matrix}}\right\} m \\ \left.\vphantom{\begin{matrix}a\\b\\c\end{matrix}}\right\} n \end{matrix}$$

2. *if $m = 0$: $R_T(f,g) = g_0^n$*
3. *if $n = 0$: $R_T(f,g) = f_0^m$*

With this resultant in 1988 Adjamagbo and Van den Essen proved that:

Theorem 1.26 *Let K be an arbitrary field and $F = (F_1, F_2) : K^2 \to K^2$ a polynomial map. Then there is equivalence between:*

1. *F is invertible*
2. *There exists $\lambda_1, \lambda_2 \in K^*$ and $G_1, G_2 \in K[y_1, y_2]$ such that*

$$\begin{aligned} R_{x_2}(F_1 - y_1, F_2 - y_2) &= \lambda_1(x_1 - G_1) \\ R_{x_1}(F_1 - y_1, F_2 - y_2) &= \lambda_2(x_2 - G_2) \end{aligned}$$

Furthermore, if F is invertible then $G = (G_1, G_2)$ is the inverse of F.

Proof: See [Adjamagbo et al. 88]. □

If we substitute $x_1 = 0$ respectively $x_2 = 0$ in the resultants of theorem 1.26 we see easily that G is completely determined by the four so-called face polynomials $F_1(0, x_2), F_1(x_1, 0), F_2(0, x_2)$ and $F_2(x_1, 0)$. But from this it follows that F is completely determined by its face polynomials, if F is invertible. A result also obtained by McKay and Wang in [McKay et al. 86]. The idea is: if F is invertible with inverse G, then G is completely determined by the face polynomials of F. But F is completely determined by the face polynomials of G, which were determined by the face polynomials of F. In 1988 McKay and Wang gave a generalization of this result for higher dimensions.

Theorem 1.27 *Let $\phi = (F_1, \ldots, F_n)$ define a K-automorphism of $K[X]$. Then ϕ is completely determined by its face polynomials $F_i(x_1, \ldots, x_{j-1}, 0, x_{j+1}, \ldots, x_n)$.*

Proof: See [McKay et al. 88]. □

The only problem here is that for a period of three years it was not known how to reconstruct an invertible polynomial map $F : K^n \to K^n$ if $n \geq 3$. However this problem was solved in 1991 by Kwieciński and Van den Essen. The algorithm uses Gröbner bases and theorem 1.24. Furthermore it uses the next lemma.

Lemma 1.28 *Let $F : K^n \to K^n$ be an invertible polynomial map with inverse $G = (G_1, \ldots, G_n)$. Define for each $j \in \{1, \ldots, n\}$ the ideal*

$$I_j = (y_1 - F_1(x_j = 0), \ldots, y_n - F_n(x_j = 0)).$$

Then for each j we have $I_j \subset K[x_1, \ldots, \hat{x_j}, \ldots, x_n, y_1, \ldots, y_n]$. Furthermore we have

$$I_j \cap K[y_1, \ldots, y_n] = (G_j).$$

Proof: From theorem 1.24 it follows that

$$I_j = (x_1 - G_1(Y), \ldots, x_{j-1} - G_{j-1}(Y), -G_j(Y), x_{j+1} - G_{j+1}(Y), \ldots, x_n - G_n(Y)).$$

So it is obvious that $(G_j) \subset I_j \cap K[Y]$. Conversely let $g(Y) \in I_j$. Then

$$g(Y) = \sum_{p \neq j} a_p(X, Y)(x_p - G_p(Y)) + b(X, Y)G_j(Y)$$

with $a_p, b \in K[x_1, \ldots, \hat{x_j}, \ldots, x_n, y_1, \ldots, y_n]$. But if $g(Y) \in K[Y]$ we must have that for each p

$$a_p(X, Y)(x_p - G_p(Y)) \in K[Y],$$

so $a_p = 0$ for all p, since $\{x_i - G_i(Y)\}_i^n$ is a reduced Gröbner basis. But then $g(Y) = b(X, Y)G_j(Y)$. And thus $I_j \cap K[Y] \subset (G_j)$, which completes the proof. □

Now we can reconstruct the invertible map F from its face polynomials

$$\{F_i(x_j = 0)\}_{i,j}^n$$

in a few steps.

1. Choose some order $<$ with all power products with $y_1, \ldots, y_n$ smaller than all power products with $x_1, \ldots, x_n$.
2. Compute with respect to this order the reduced Gröbner basis B_j of the ideal I_j for all j, where I_j is defined as in lemma 1.28.
3. Then $B_j \cap K[Y]$ is the reduced Gröbner basis of $I_j \cap K[Y] = (G_j)$.
4. So $B_j \cap K[Y] = \{\tilde{G}_j\}$ for some $\tilde{G}_j \in K[Y]$ and $\tilde{G}_j = \lambda_j G_j$ for some $\lambda_j \in K^*$.
5. Since $G_j(F(e_j)) = 1$ (e_j is the j^{th} unit vector) we have $\tilde{G}_j(F(e_j)) = \lambda_j$.
6. From this we can derive:
$$G_j = \frac{\tilde{G}_j}{\tilde{G}_j(F(e_j))}.$$
7. Let $G = (G_1, \ldots, G_n)$. Then we can compute F -the inverse of G- by theorem 1.24.

Some of the invertibility criteria described in this section are already implemented at the University of Nijmegen in the so-called Jacobian package for Maple. So we were able to verify very easily whether a polynomial map which we had found was invertible or not.

1.4 Linear triangularization in dimension three

The idea for our systematic search in dimension four was handed to us by means of the paper[1] 'The Jacobian Conjecture: Linear Triangularization For Cubics in Dimension Three' by David Wright. In this paper a proof is given for the Jacobian Conjecture in case of polynomial endomorphisms, which are cubic homogeneous, of three space. Before we can introduce the main result of Wright's paper, we have to give a few definitions first:

Definition 1.29 *An n-dimensional polynomial map F is called* cubic homogeneous *if it has the form*

$$F_i = x_i - H_i$$

where H_i is homogeneous of degree three or $H_i = 0$, for $i = 1, \ldots, n$.

And more generally:

Definition 1.30 *An n-dimensional polynomial map F is called* d-homogeneous *if it has the form*

$$F_i = x_i - H_i$$

where H_i is homogeneous of degree d or $H_i = 0$, for $i = 1, \ldots, n$. We can write

$$H_i = \sum_{|\alpha|=d} a_i^\alpha X^\alpha$$

for $i = 1, \ldots, n$, where α is the ordered n-tuple $(\alpha_1, \ldots, \alpha_n)$ and $|\alpha| = \alpha_1 + \cdots + \alpha_n$. Furthermore:

$$X^\alpha = x_1^{\alpha_1} \ldots x_n^{\alpha_n}.$$

Definition 1.31 *A polynomial map F is said to* satisfy the Jacobian hypothesis *if the Jacobian determinant is a non-zero constant.*

Definition 1.32 *A polynomial map F is called* triangular *if it has the form $(F_1, \ldots, F_n)$ with $F_i - \lambda_i x_i \in K[x_1, \ldots, x_{i-1}]$ and $\lambda_i \in K^*$, for $i = 1, \ldots, n$.*

The main theorem in [Wright 93] is now:

Theorem 1.33 *Suppose $F = (F_1, F_2, F_3)$ is cubic homogeneous and satisfies the Jacobian hypothesis. Then F is linearly triangularizable, i.e., there exists a linear (homogeneous) automorphism A of K^3 such that AFA^{-1} is triangular. In particular, F is an automorphism.*

Proof: This is the same theorem as theorem 1.20. For the proof see [Wright 93]. □

The main method Wright used, was to assume that the most general cubic homogeneous map in dimension three satisfied the Jacobian hypothesis. This implied that the Jacobian matrix JH was nilpotent and from that observation he computed some system of equations on the coefficients of the general map.

The important step in this method is how to construct the system of equations from the property that JH is nilpotent. Before we can give this construction we take a look at the following lemma, concerning general $n \times n$ nilpotent matrices.

[1] See [Wright 93].

Lemma 1.34 *The following two statements are equivalent:*

1. *The matrix $U = (u_{ij})$ is an $n \times n$ nilpotent matrix.*
2. *For $r = 1, \ldots, n$ we have $P_r(U) = 0$, where*

$$P_r(U) = (-1)^r \sum_{1 \le i_1 < \cdots < i_r \le n} \begin{vmatrix} u_{i_1 i_1} & \ldots & u_{i_1 i_r} \\ \vdots & & \vdots \\ u_{i_r i_1} & \ldots & u_{i_r i_r} \end{vmatrix}.$$

Proof: See [Wright 87]. □

From this lemma it follows that F satisfies the Jacobian hypothesis is equivalent to $P_r(JH) = 0$ for all r with $1 \le r \le n$. Since H is homogeneous of degree d, we have that JH is homogeneous of degree $d-1$. From this observation it follows that the polynomial $P_r(JH)$ is a homogeneous polynomial of degree $r(d-1)$. If we use the notation from definition 1.30 to denote a homogeneous polynomial map, we can write

$$P_r(JH) = \sum_{|\beta| = r(d-1)} w_r^\beta X^\beta.$$

In order to calculate the concrete coefficients w_r^β Wright proves the following proposition, which is in fact even more general than we need, since H doesn't need to be homogeneous.

Proposition 1.35 *Let $H = (H_1, \ldots, H_n)$ be a polynomial map $K^n \to K^n$, with $H_i = \sum a_i^\alpha X^\alpha$. Then $P_r(JH) = \sum_r^\beta X^\beta$ where*

$$w_r^\beta = (-1)^r \sum_{1 \le t_1 < \cdots < t_r \le n} \sum_{\substack{\{\alpha^{(1)}, \ldots, \alpha^{(r)}\} \\ \alpha^{(1)} + \cdots + \alpha^{(r)} = \beta + \varepsilon_{t_1} + \cdots + \varepsilon_{t_r}}} \left|(\alpha_{t_i}^{(j)})\right| \left|(a_{t_i}^{\alpha^{(j)}})\right|$$

where $\varepsilon_t = (0, \ldots, 0, 1, 0, \ldots, 0)$ with the '1' on the t^{th} position, $\alpha^{(j)}$ is an n-tuple of non-negative integers $(\alpha_1^{(j)}, \ldots, \alpha_n^{(j)})$ and i, j index the rows and the columns (respectively) of the matrices that appear behind the summation sign.

Proof: See [Wright 93]. □

In fact all possible β-s together form a representation system for all monomials that can appear in the cubic homogeneous part, e.g. $\beta = (2, 0, 1)$ stands for the monomial $x_1^2 x_2^0 x_3^1$.

In his paper Wright proved theorem 1.33 by hand. In dimension four it is to complex to do the calculations by hand. So we wrote some Maple procedures which computed the system of equations. In order to verify the credibility of our procedures we also checked the dimension three case.

1.4.1 The algorithm

The basic algorithm[2] is very simple. First we compute the most general $F = X - H$ that is cubic homogeneous and we put the coefficients of the monomials that appear in H in a matrix M. The order of the monomials is important. The order used here

[2]See appendix A for the actual implementation.

is the pure lexicographically as defined in [Geddes et al. 92], in descending way so the first column of M stands for x_1^3 and the last column stands for x_n^3. Using this matrix, we begin with computing all w_r^β-s, following the scheme of proposition 1.35, and add them all to a system of equations. When we have drawn up the complete system, we start with solving it. Finally we substitute the solutions -if any- in the original matrix M, to see whether the solutions represent triangularizable maps.

The interesting part is the solving of the system. You have to do it in a clever way; it won't work if you use the standard Maple procedure `solve` on the complete system. The first step was suggested by Wright: use an invertible linear map A such that $A^{-1}HA$ is of a nice form. Here 'a nice form' means as much zeroes as possible in the matrix $J(A^{-1}HA)(e_1)$. The second step was implementing the way one normally solves systems of equations: start with the easy equations and substitute their solutions in the remaining part of the system. Repeat this as long as possible. Here 'easy equations' are equations that have only one solution, which is very short, i.e. one doesn't substitute a complex term in a single variable since this would increase the complexity of the system very fast. A consequence of solving only very easy equations at a time is that one can be sure that one doesn't miss a solution, which could happen if Maple tried to solve the complete system at once.

With this strategy we checked the validity of theorem 1.33.

At first it turned out that some of the computations in the original paper -done by hand- were wrong. For instance on [Wright 93, page 5] it is stated that

$$w_3^{(4,2,0)} = -e_1f_2 + e_2f_1 - 2e_1f_3 + 2e_3f_1 - \tfrac{1}{3}h_1.$$

But according to our calculations, this should have been:

$$\begin{aligned} w_3^{(4,2,0)} &= (-1)^3 \left(\begin{vmatrix} 3 & 2 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{vmatrix} \begin{vmatrix} 0 & 0 & h_1 \\ \frac{1}{3} & 0 & h_2 \\ 0 & 1 & h_3 \end{vmatrix} + \begin{vmatrix} 3 & 2 & 0 \\ 0 & 0 & 3 \\ 0 & 1 & 0 \end{vmatrix} \begin{vmatrix} 0 & 0 & g_1 \\ \frac{1}{3} & 0 & g_2 \\ 0 & 0 & g_3 \end{vmatrix} + \right. \\ &\qquad \left. \begin{vmatrix} 3 & 1 & 1 \\ 0 & 2 & 1 \\ 0 & 0 & 1 \end{vmatrix} \begin{vmatrix} 0 & d_1 & e_1 \\ \frac{1}{3} & d_2 & e_2 \\ 0 & d_3 & e_3 \end{vmatrix} + \begin{vmatrix} 2 & 2 & 1 \\ 1 & 0 & 2 \\ 0 & 1 & 0 \end{vmatrix} \begin{vmatrix} 0 & 0 & d_1 \\ 0 & 0 & d_2 \\ 1 & 0 & d_3 \end{vmatrix} \right) \\ &= (-1)^3 (h_1 + 0 + 2e_1d_3 - 2d_1e_3 + 0) \\ &= 2d_1e_3 - 2e_1d_3 - h_1 \end{aligned}$$

And at this point of his calculations, Wright doesn't know anything about d_1 or e_3. So instead of claiming $h_1 = 0$, he should have claimed $h_1 = 2d_1e_3$. Fortunately, with some other equations it turned out that both d_1 and e_3 are equal to 0, so also $h_1 = 0$. So this small error could be fixed. However, although the derivation may have been changed a bit by this mistake, the final result of our computations was exactly the same as the result claimed in [Wright 93]. So in particular we concluded that indeed theorem 1.33 was valid.

In fact the dimension three case was so easy that we could automate the process completely. If we put the coefficients of the most general cubic homogeneous polynomial map in a matrix M we get

$$M = \begin{pmatrix} a_1 & b_1 & c_1 & d_1 & e_1 & f_1 & g_1 & h_1 & i_1 & j_1 \\ a_2 & b_2 & c_2 & d_2 & e_2 & f_2 & g_2 & h_2 & i_2 & j_2 \\ a_3 & b_3 & c_3 & d_3 & e_3 & f_3 & g_3 & h_3 & i_3 & j_3 \end{pmatrix}$$

and if we substitute the matrix

$$\begin{pmatrix} 0 & 0 & 0 \\ \frac{1}{3} & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \text{ for the matrix } \begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{pmatrix}$$

so that we get

$$M = \begin{pmatrix} 0 & 0 & 0 & d_1 & e_1 & f_1 & g_1 & h_1 & i_1 & j_1 \\ \frac{1}{3} & 0 & 0 & d_2 & e_2 & f_2 & g_2 & h_2 & i_2 & j_2 \\ 0 & 1 & 0 & d_3 & e_3 & f_3 & g_3 & h_3 & i_3 & j_3 \end{pmatrix}$$

we can use the single command **simplifyM(M,3)** to compute the one and only solution in the case where $JH^2.X \neq 0$. The case where $JH^2.X = 0$ can be handled with the same procedure call, but now with

$$M = \begin{pmatrix} 0 & 0 & 1 & d_1 & e_1 & f_1 & g_1 & h_1 & i_1 & j_1 \\ \frac{1}{3} & 0 & 0 & d_2 & e_2 & f_2 & g_2 & h_2 & i_2 & j_2 \\ 0 & 0 & 0 & d_3 & e_3 & f_3 & g_3 & h_3 & i_3 & j_3 \end{pmatrix}.$$

This leads to a contradiction, as was shown also in [Wright 93].

1.5 Linear triangularization in dimension four

In [Wright 93] a counterexample of theorem 1.33 in dimension four is given:

$$F = \begin{pmatrix} x_1 \\ x_2 \\ (1 - x_1x_2)x_3 - x_2^2x_4 \\ x_1^2x_3 + (1 + x_1x_2)x_4 \end{pmatrix}.$$

If we write this polynomial map in the form

$$F = X - H = \begin{pmatrix} x_1 \\ x_2 \\ x_3 - x_1x_2x_3 - x_2^2x_4 \\ x_4 + x_1^2x_3 + x_1x_2x_4 \end{pmatrix}$$

we see that F is of the desired cubic homogeneous form. The Jacobian matrix of H is given by the matrix:

$$JH = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ x_2x_3 & x_1x_3 + 2x_2x_4 & x_1x_2 & x_2^2 \\ -2x_1x_3 - x_2x_4 & -x_1x_4 & -x_1^2 & -x_1x_2 \end{pmatrix}$$

It is obvious that JH is of rank two. Furthermore we have:

$$JH^2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -x_1x_2^2x_3 - x_2^3x_4 & x_1^2x_2x_3 + x_1x_2^2x_4 & 0 & 0 \\ x_1^2x_2x_3 + x_1x_2^2x_4 & -x_1^3x_3 - x_1^2x_2x_4 & 0 & 0 \end{pmatrix} \text{ and } JH^2.X = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Of course since JH is nilpotent of rank two, $JH^3 = 0$ also. Furthermore it is easy to verify that the determinant of the Jacobian matrix of F is equal to $(1 - x_1x_2)(1 + x_1x_2) + x_1^2x_2^2 = 1$, so this polynomial map F satisfies the Jacobian hypothesis. In [Wright 93] it is shown that this map is not triangularizable. In fact in chapter 2 we shall present a whole class containing this particular example of non-triangularizable maps.

However if we use our procedure **simplifyM** on this map we find that all w_r^β are equal to 0. This leads to the conclusion that in the four dimensional case, the demand that all w_r^β are equal to 0 is not strong enough to force triangularizability.

But although this polynomial map may not be triangularizable, it is invertible. Its inverse is given by:

$$F^{-1} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 + x_2^2 x_4 + x_3 x_1 x_2 \\ x_4 - x_4 x_1 x_2 - x_1^2 x_3 \end{pmatrix}.$$

So this example is not a counterexample of the Jacobian Conjecture.

Chapter 2

Classification in dimension four

2.1 Introduction

The aim of this chapter is to describe the general forms of four dimensional polynomial maps that satisfy the Jacobian hypothesis. As we have seen in chapter 1 it suffices to describe the so-called cubic homogeneous polynomial maps.

Definition 2.1 *The most general cubic homogeneous polynomial map in dimension four ($F : K^4 \to K^4$) is given by:*

$$F = X - H = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} - \begin{pmatrix} H_1 \\ H_2 \\ H_3 \\ H_4 \end{pmatrix}$$

Where for each u with $1 \leq u \leq 4$ H_u is of the form

$$\begin{aligned} H_u \;=\; & a_u x_1^3 + b_u x_1^2 x_2 + c_u x_1^2 x_3 + d_u x_1^2 x_4 + e_u x_1 x_2 x_3 + f_u x_1 x_2 x_4 + g_u x_1 x_3 x_4 + \\ & h_u x_1 x_3^2 + i_u x_1 x_3 x_4 + j_u x_1 x_4^2 + k_u x_2^3 + l_u x_2^2 x_3 + m_u x_2^2 x_4 + n_u x_2 x_3^2 + \\ & o_u x_2 x_3 x_4 + p_u x_2 x_4^2 + q_u x_3^3 + r_u x_3^2 x_4 + s_u x_3 x_4^2 + t_u x_4^3, \end{aligned}$$

and K is an arbitrary field of characteristic zero.

We use the pure lexicographically order of the monomials as described in for instance [Geddes et al. 92]. This order of the monomials is important. We use it explicitly when we translate the map F to the matrix of the coefficients of F. We need this matrix to compute the system of equations that must hold if F satisfies the Jacobian hypothesis, as is described in [Wright 93].

Furthermore we have that 'F satisfies the Jacobian hypothesis' is equivalent to 'the Jacobian matrix JH of the homogeneous part H of F is nilpotent'. This holds because of the (cubic) homogeneity of F.

So the matrix JH plays a crucial role in this chapter. Since F is a four dimensional polynomial map, JH is a 4×4 matrix. And assuming that F satisfies the Jacobian hypothesis, we know that this JH is nilpotent. So it is of rank zero, one, two or three. Of course rank zero means that

$$JH.X = 0$$

for all X. So with Euler's formula we have that

$$3.H = 0$$

which means that $H = 0$ and F is the identity. And therefore rank zero is not interesting. Also rank one is not interesting (anymore) to examine. From [Bass et al. 82, Theorem 6.2(c)] it is known that F is triangularizable in any dimension and therefore in particular F is invertible.

So the only two interesting cases are JH has rank two and JH has rank three.

2.2 JH has rank two

Since JH is a nilpotent matrix of rank two, we must have that

$$JH^3 = 0.$$

Furthermore from the following proposition it follows that $JH^3 = 0$ implies that JH is of rank zero, one or two, and in particular not of rank three.

Proposition 2.2 *A nilpotent 4×4 matrix N with $N^3 = 0$ is not of rank three.*

Proof: A nilpotent 4×4 matrix N can be of rank zero, one, two or three. With an invertible linear transformation A we can reduce this N to the Jordan form. Since there is only one 4×4 Jordan form of rank three, say

$$J = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

then we must have that $A^{-1}NA = J$, if N is of rank three. But this gives the following contradiction:

$$0 = (A^{-1}NA)^3 = J^3 \neq 0.$$

So N has rank zero, one or two. □

By using definition 2.1 we computed the matrices JH, JH^2 and JH^3. By collecting the coefficients of the monomials in the sixteen entries of JH^3 we got a system of equations, since all coefficients of the monomials had to be equal to zero.

Furthermore we divided this case into two subcases: $JH^2.X = 0$ and $JH^2.X \neq 0$.

2.2.1 $JH^2.X = 0$

In this case we added the equations we obtained by computing $JH^2.X$ and collecting the coefficients of the monomials in the four resulting polynomials to the system of equations we obtained from $JH^3 = 0$.

Before we started to solve this system, we chose some matrices to reduce the complexity of the system. This was done in such a way that we didn't loose any generality.

To find these matrices we can do the following. First we choose a vector $x \in K^4$ with $JH(x).x \neq 0$. Since $H \neq 0$ such an x exists. Now define $y = JH(x).x$. We can extend these two independent vectors to a basis of K^4 by choosing some z and w. Define $e_i = (0, \ldots, 0, 1, 0, \ldots, 0)$ as the standard basis. Now using a linear invertible map A with $A.e_1 = x, A.e_2 = y, A.e_3 = z$ and $A.e_4 = w$ and replacing H by $A^{-1}HA$ we got the following equations:

$$\begin{aligned} JH(e_1).e_1 &= e_2 \\ JH(e_1).e_2 &= JH(e_1)^2.e_1 = 0 \end{aligned}$$

Furthermore we have

$$JH(e_1) = \begin{pmatrix} 3a_1 & b_1 & c_1 & d_1 \\ 3a_2 & b_2 & c_2 & d_2 \\ 3a_3 & b_3 & c_3 & d_3 \\ 3a_1 & b_4 & c_4 & d_4 \end{pmatrix}$$

and solving these equations gives us: $a_2 = \frac{1}{3}$ and $a_1, a_3, a_4, b_1, b_2, b_3$ and b_4 are all equal to zero.

If we look at the Jordan form of JH we see that there are two possibilities for this matrix:

$$N_1 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \text{ and } N_2 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

In both situations we see that there is a two dimensional kernel. So we can assume all c_i or all d_i equal to zero. Substituting this in $JH(e_1)$ we get the matrix:

$$\begin{pmatrix} 0 & 0 & c_1 & 0 \\ 1 & 0 & c_2 & 0 \\ 0 & 0 & c_3 & 0 \\ 0 & 0 & c_4 & 0 \end{pmatrix}.$$

From the trace of this matrix and the notion that it is nilpotent it follows that $c_3 = 0$. Furthermore by replacing z by $z - c_2 x$, we get the matrix:

$$\begin{pmatrix} 0 & 0 & c_1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & c_4 & 0 \end{pmatrix}.$$

If we rename the current basis into (f_1, f_2, f_3, f_4) we see that we can transform this to $(f_1, f_2, \frac{1}{c_4} f_3, f_4)$, assuming $c_4 \neq 0$. This gives the matrix:

$$\begin{pmatrix} 0 & 0 & c_1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

And if we assume that $c_4 = 0$ we can transform to the basis $(f_1, f_2, \frac{1}{c_1} f_3, f_4)$. Since the matrix must be of rank two, it is clear that $c_1 \neq 0$ in this case. This results in the following matrix:

$$\begin{pmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

These two matrices imply the following two substitution matrices

$$\begin{pmatrix} 0 & 0 & c_1 & 0 \\ \frac{1}{3} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \text{ and } \begin{pmatrix} 0 & 0 & 1 & 0 \\ \frac{1}{3} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

to substitute in the matrix

$$\begin{pmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \\ a_4 & b_4 & c_4 & d_4 \end{pmatrix}.$$

After we made these substitutions in our system of equations we were able to find all solutions of the system. The substitutions of the first matrix resulted in two solutions. The distinction between these two was the choice of $c_1 = 0$ or $c_1 \neq 0$. The solutions were respectively:

1. $$\begin{pmatrix} x_1 \\ x_2 - \frac{1}{3}x_1^3 - h_2 x_1 x_3^2 - q_2 x_3^3 \\ x_3 \\ x_4 - x_1^2 x_3 - h_4 x_1 x_3^2 - q_4 x_3^3 \end{pmatrix}$$

2. $$\begin{pmatrix} x_1 \; - c_1 x_1^2 x_3 + \frac{1}{4} r_4 x_1 x_3^2 - c_1^2 x_1 x_3 x_4 + 9c_1^2 x_2 x_3^2 - c_1 q_4 x_3^3 \\ \quad - r_4 c_1 x_3^2 x_4 + 2c_1^3 x_3 x_4^2 \\ x_2 \; - \frac{1}{3}x_1^3 + 3c_1 x_1 x_2 x_3 - \frac{16 q_4 c_1^2 - r_4^2}{48 c_1^2} x_1 x_3^2 - \frac{1}{2} r_4 x_1 x_3 x_4 \\ \quad + c_1^2 x_1 x_4^2 + \frac{3}{4} r_4 x_2 x_3^2 - 3c_1^2 x_2 x_3 x_4 - \frac{r_4 q_4}{12 c_1} x_3^3 \\ \quad + \frac{4 q_4 c_1^2 - r_4^2}{12 c_1} x_3^2 x_4 + \frac{1}{2} r_4 c_1 x_3 x_4^2 - \frac{2}{3} c_1^3 x_4^3 \\ x_3 \\ x_4 \; - x_1^2 x_3 + \frac{r_4}{4c_1} x_1 x_3^2 - c_1 x_1 x_3 x_4 + 9c_1 x_2 x_3^2 - q_4 x_3^3 \\ \quad - r_4 x_3^2 x_4 + 2c_1^2 x_3 x_4^2 \end{pmatrix}$$

The second substitution matrix lead to a contradiction so it didn't give any new solutions.

If we look at the first solution, we note that this map trivially satisfies $JH^2.X = 0$ since we already have $JH^2 = 0$. For the second solution this doesn't hold. That this situation occurs is not a big surprise. If we look at the Jordan forms of 4×4 nilpotent matrices of rank two, we have seen already (on page 17) that there are two non-equivalent matrices N_1 and N_2. The difference between these two matrices is that $N_2^2 = 0$ whereas $N_1^2 \neq 0$.

Example 2.3 *As we have mentioned already in section 1.5, the example from [Wright 93], where*

$$F = \begin{pmatrix} x_1 \\ x_2 \\ x_3 - x_1 x_2 x_3 - x_2^2 x_4 \\ x_4 + x_1^2 x_3 + x_1 x_2 x_4 \end{pmatrix}$$

is a special case of this class. If we use

$$A = \begin{pmatrix} x_1 - \frac{1}{3}x_4 \\ x_3 \\ x_1 + \frac{2}{3}x_4 \\ -3x_2 \end{pmatrix} \quad \textit{then } A^{-1} = \begin{pmatrix} \frac{2}{3}x_1 + \frac{1}{3}x_3 \\ -\frac{1}{3}x_4 \\ x_2 \\ -x_1 + x_3 \end{pmatrix}$$

and $A^{-1}FA$ is a special case of

$$\begin{pmatrix} x_1 & - c_1x_1^2x_3 + \frac{1}{4}r_4x_1x_3^2 - c_1^2x_1x_3x_4 + 9c_1^2x_2x_3^2 - c_1q_4x_3^3 \\ & - r_4c_1x_3^2x_4 + 2c_1^3x_3x_4^2 \\ x_2 & - \frac{1}{3}x_1^3 + 3c_1x_1x_2x_3 - \frac{16q_4c_1^2-r_4^2}{48c_1^2}x_1x_3^2 - \frac{1}{2}r_4x_1x_3x_4 \\ & + c_1^2x_1x_4^2 + \frac{3}{4}r_4x_2x_3^2 - 3c_1^2x_2x_3x_4 - \frac{r_4q_4}{12c_1}x_3^3 \\ & + \frac{4q_4c_1^2-r_4^2}{12c_1}x_3^2x_4 + \frac{1}{2}r_4c_1x_3x_4^2 - \frac{2}{3}c_1^3x_4^3 \\ x_3 & \\ x_4 & - x_1^2x_3 + \frac{r_4}{4c_1}x_1x_3^2 - c_1x_1x_3x_4 + 9c_1x_2x_3^2 - q_4x_3^3 \\ & - r_4x_3^2x_4 + 2c_1^2x_3x_4^2 \end{pmatrix}.$$

Just substitute:

$$c_1 = \frac{1}{3}, q_4 = 0, r_4 = 0.$$

If we take another look at the second solution, we see that H_1 equals c_1H_4. So we can reduce this map even further by looking at $(F_1 - c_1F_4, F_2, F_3, F_4)$. This gives the map

$$\begin{pmatrix} x_1 - c_1x_4 & \\ x_2 & - \frac{1}{3}x_1^3 + 3c_1x_1x_2x_3 - \frac{16q_4c_1^2-r_4^2}{48c_1^2}x_1x_3^2 - \frac{1}{2}r_4x_1x_3x_4 \\ & + c_1^2x_1x_4^2 + \frac{3}{4}r_4x_2x_3^2 - 3c_1^2x_2x_3x_4 - \frac{r_4q_4}{12c_1}x_3^3 \\ & + \frac{4q_4c_1^2-r_4^2}{12c_1}x_3^2x_4 + \frac{1}{2}r_4c_1x_3x_4^2 - \frac{2}{3}c_1^3x_4^3 \\ x_3 & \\ x_4 & - x_1^2x_3 + \frac{r_4}{4c_1}x_1x_3^2 - c_1x_1x_3x_4 + 9c_1x_2x_3^2 - q_4x_3^3 \\ & - r_4x_3^2x_4 + 2c_1^2x_3x_4^2 \end{pmatrix}$$

By making the transformation of coordinates by substituting x_1 by $x_1 + c_1x_4$ and collecting the coefficients of the monomials we get the following polynomial map:

$$\begin{pmatrix} x_1 & \\ x_2 & - \frac{1}{3}x_1^3 - c_1x_1^2x_4 + 3c_1x_1x_2x_3 - \frac{16q_4c_1^2-r_4^2}{48c_1^2}x_1x_3^2 - \frac{1}{2}r_4x_1x_3x_4 \\ & + \frac{3}{4}r_4x_2x_3^2 - \frac{r_4q_4}{12c_1}x_3^3 - \frac{r_4^2}{16c_1}x_3^2x_4 \\ x_3 & \\ x_4 & - x_1^2x_3 + \frac{r_4}{4c_1}x_1x_3^2 - 3c_1x_1x_3x_4 + 9c_1x_2x_3^2 - q_4x_3^3 - \frac{3}{4}r_4x_3^2x_4 \end{pmatrix}$$

We should have found this polynomial map already if we had used the substitution matrix

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ \frac{1}{3} & 0 & 0 & c_1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \text{ instead of } \begin{pmatrix} 0 & 0 & c_1 & 0 \\ \frac{1}{3} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

to begin with.

2.2.2 $JH^2.X \neq 0$

In this case we had no other equations we could add to the system coming from $JH^3 = 0$. So we only had to choose a good matrix to start with.

To find this matrix we first choose a vector $x \in K^4$ with $JH(x)^2.x \neq 0$. Since we assumed $JH^2.X \neq 0$, such an x must clearly exist. Now define $y = JH(x).x$ and $z = JH(x)^2.x$. We extend these three independent vectors with the vector w. Again there exists a linear invertible map A that maps e_1, e_2, e_3 and e_4 to x, y, z and w respectively. Replacement of H by $A^{-1}HA$ leads to the following equations:

$$\begin{aligned} JH(e_1).e_1 &= e_2 \\ JH(e_1).e_2 &= JH(e_1)^2.e_1 = e_3 \\ JH(e_1).e_3 &= JH(e_1)^2.e_2 = JH(e_1)^3.e_1 = 0 \end{aligned}$$

Solving these equations result in a matrix:

$$\begin{pmatrix} 0 & 0 & 0 & d_1 \\ 1 & 0 & 0 & d_2 \\ 0 & 1 & 0 & d_3 \\ 0 & 0 & 0 & d_4 \end{pmatrix}.$$

Again since the trace of this matrix must be equal to zero, $d_4 = 0$. If we rename the basis to (f_1, f_2, f_3, f_4) and apply this matrix several times on the fourth basis vector f_4 we get:

$$\begin{aligned} JH(e_1).f_4 &= d_1 f_1 + d_2 f_2 + d_3 f_3 \\ JH(e_1)^2.f_4 &= d_1 f_2 + d_2 f_3 \\ JH(e_1)^3.f_4 &= d_1 f_3. \end{aligned}$$

But since $JH^3 = 0$ we must have $d_1 = 0$. Furthermore if we transform to the basis $(f_1, f_2, f_3, f_4 - d_2 f_1 - d_3 f_3)$ we get the matrix:

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

This matrix leads to the substitution matrix

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ \frac{1}{3} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \text{ for } \begin{pmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \\ a_4 & b_4 & c_4 & d_4 \end{pmatrix}$$

After we made this substitution in our system of equations we were able to solve the remaining system. We found four solutions. We had to make some assumptions to find these solutions.

1. After assuming that $e_4 \neq 0$ we found the solution:
$$\begin{pmatrix} x_1 \\ x_2 - \frac{1}{3}x_1^3 \\ x_3 - x_1^2 x_2 - e_3 x_1 x_2^2 - k_3 x_2^3 \\ x_4 - e_4 x_1 x_2^2 - k_4 x_2^3 \end{pmatrix}$$

2. With the assumption $e_4 = 0$ and $i_3 \neq 0$ we found:
$$\begin{pmatrix} x_1 \\ x_2 - \frac{1}{3}x_1^3 + i_3 x_1 x_2 x_4 - j_2 x_1 x_4^2 + s_3 x_2 x_4^2 + i_3^2 x_3 x_4^2 - t_2 x_4^3 \\ x_3 \; - x_1^2 x_2 - \frac{2s_3}{i_3} x_1 x_2 x_4 - i_3 x_1 x_3 x_4 - j_3 x_1 x_4^2 - \frac{s_3^2}{i_3^2} x_2 x_4^2 \\ \quad - s_3 x_3 x_4^2 - t_3 x_4^3 \\ x_4 \end{pmatrix}$$

3. And with $e_4 = 0, i_3 = 0$ and $k_4 \neq 0$:
$$\begin{pmatrix} x_1 \\ x_2 - \frac{1}{3}x_1^3 \\ x_3 - x_1^2 x_2 - e_3 x_1 x_2^2 - k_3 x_2^3 \\ x_4 - k_4 x_2^3 \end{pmatrix}$$

4. And finally with $e_4 = 0, i_3 = 0$ and $k_4 = 0$:
$$\begin{pmatrix} x_1 \\ x_2 - \frac{1}{3}x_1^3 - j_2x_1x_4^2 - t_2x_4^3 \\ x_3 - x_1^2x_2 - e_3x_1x_2^2 - g_3x_1x_2x_4 - j_3x_1x_4^2 - k_3x_2^3 - m_3x_2^2x_4 \\ \quad - p_3x_2x_4^2 - t_3x_4^3 \\ x_4 \end{pmatrix}$$

Note that in the first and the third solution the assumptions are not really important. They were important to find these solutions, since we needed them to divide through e_4 respectively k_4. But the final solutions found are still solutions if we substitute $e_4 = 0$ and $k_4 = 0$ in respectively the first and the third solution. The first substitution yields exactly the third solution. And the second substitution yields a special case of the fourth solution. And since we note further that the results of these substitutions are still members of the class with $\text{rank}(JH) = 2$ and $JH^2.X \neq 0$ we can reduce these four solutions by omitting the third one to three general solutions where the only restriction is that $i_3 \neq 0$. All other appearing variables are completely free.

2.3 JH has rank three

In this case we have that JH is a nilpotent matrix of rank three. This implies that $JH^4 = 0$. So at first we tried to compute the system of equations that follows from this observation like we did in the rank two case. However on our computer it wasn't possible to compute JH^4 completely in one run and collect the coefficients of the monomials afterwards. We were only able to compute the entries of one row of JH^4 at a time and extract the equations coming from this row before we could compute the entries of the other rows. Afterwards we were able to union all equations, but we couldn't solve the remaining system of equations. It was so large that the system operator killed the job since it required about 160 megabytes and therefore it made work for other users impossible on our 128 megabytes computer. So we had to take a different approach.

Instead of computing JH^4 we used the method described in [Wright 93]. In this paper a method is given that can be used to construct a system of equations that must hold if the given polynomial map satisfies the Jacobian hypothesis. After we computed this system of equations we made a similar division in subcases as in the rank two case: $JH^3.X = 0$ and $JH^3.X \neq 0$.

2.3.1 $JH^3.X = 0$

In this case we added the extra equations coming from $JH^3.X = 0$ to the original system we computed with Wright's method. Before we started to solve the system we substituted as initial values the matrix

$$\begin{pmatrix} 0 & 0 & 0 & 1 \\ \frac{1}{3} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \text{ for } \begin{pmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \\ a_4 & b_4 & c_4 & d_4 \end{pmatrix}.$$

We can easily see that this matrix can be used without any loss of generality: let $x \in K^4$ be a vector such that $JH(x).x \neq 0$ and $\text{rank}(JH(x)) = 3$. So $\text{Ker}(JH(x)^3)$ has dimension three. Also $\text{Im}(JH(x))$ has dimension three. And since $\text{Im}(JH(x)) \subseteq \text{Ker}(JH(x)^3)$ we even have equality between those spaces. Naturally $x \in \text{Ker}(JH(x)^3)$, so also $x \in \text{Im}(JH(x))$. This means that there exists some $w \in K^4$ with $x = JH(x).w$. If we define $y = JH(x)^2.w$ and $z = JH(x)^3.w$ we get four independent vectors x, y, z

and w. There exists a linear invertible map A which maps e_1, e_2, e_3 and e_4 to x, y, z and w respectively. By replacing H by $A^{-1}HA$ we get the following system of equations

$$\begin{aligned} JH(e_1).e_4 &= e_1 \\ JH(e_1).e_1 &= JH(e_1)^2.e_4 = e_2 \\ JH(e_1).e_2 &= JH(e_1)^3.e_4 = e_3 \\ JH(e_1).e_3 &= JH(e_1)^4.e_4 = 0 \end{aligned}$$

and solving these equations gives the substitution given above.

After making this substitution in the system it reduced to a system that had no solutions. In fact this was not very surprising to us. In the similar case in three dimensions, $JH^3 = 0$ and $JH^2.X = 0$, there were no solutions either. Though we cannot prove it yet, we think that there is something like:

Conjecture 2.4 *Let $F : K^n \to K^n$ be a cubic homogeneous polynomial map that satisfies the Jacobian hypothesis. Then the case with* $\mathrm{rank}(JH) = n-1$ *and $JH^{n-1}.X = 0$ for all $X \in K^n$ cannot occur.*

It holds at least for $n \le 4$.

2.3.2 $JH^3.X \neq 0$

In this case we had no extra equations. The only thing that could help us to solve the system was the choice of the initial values. We substituted the matrix

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ \frac{1}{3} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

We found this substitution matrix in the following way. We can choose a vector $x \in K^4$ such that $JH(x)^3.x \neq 0$. Define $y = JH(x).x$, $z = JH(x)^2.x$ and $w = JH(x)^3.x$. Since these four vectors are independent, there exists a linear invertible map A with $A.e_1 = x$, $A.e_2 = y$, $A.e_3 = z$ and $A.e_4 = w$. By substituting H by $A^{-1}HA$ we get the following equations:

$$\begin{aligned} JH(e_1).e_1 &= e_2 \\ JH(e_1).e_2 &= JH(e_1)^2.e_1 = e_3 \\ JH(e_1).e_3 &= JH(e_1)^3.e_1 = e_4 \\ JH(e_1).e_4 &= JH(e_1)^4.e_1 = 0. \end{aligned}$$

Solving these equations leads to the substitution we mentioned above. This substitution lead to two solutions, depending on the choice of $g_4 = 0$ or $g_4 \neq 0$. These were the solutions we found respectively:

1. $$\begin{pmatrix} x_1 \\ x_2 - \frac{1}{3}x_1^3 \\ x_3 - x_1^2x_2 - e_3x_1x_2^2 - k_3x_2^3 \\ x_4 - x_1^2x_3 - e_4x_1x_2^2 - f_4x_1x_2x_3 - h_4x_1x_3^2 - k_4x_2^3 - l_4x_2^2x_3 \\ \quad - n_4x_2x_3^2 - q_4x_3^3 \end{pmatrix}$$

2. $\begin{pmatrix} x_1 \\ x_2 - \frac{1}{3}x_1^3 \\ x_3 - x_1^2x_2 - e_3x_1x_2^2 + g_4x_1x_2x_3 - k_3x_2^3 + m_4x_2^2x_3 + g_4^2x_2^2x_4 \\ x_4 - x_1^2x_3 - e_4x_1x_2^2 - \frac{2m_4}{g_4}x_1x_2x_3 - g_4x_1x_2x_4 - k_4x_2^3 \\ \quad - \frac{m_4^2}{g_4^2}x_2^2x_3 - m_4x_2^2x_4 \end{pmatrix}$

If we look at the first of these solutions we see that this polynomial map is triangular. Again, this is quite similar to what happened in dimension three. More precisely, if we omit F_4 and restrict ourselves to the indeterminates x_1, x_2 and x_3, we get indeed exactly the same polynomial map as in dimension three.

2.4 The classification theorem

In the previous sections we described the cases where $\text{rank}(JH) = 2$ or $\text{rank}(JH) = 3$. Before we shall combine these cases, we first give the already known results in case $\text{rank}(JH) = 1$.

Theorem 2.5 *Let $F = X - H : K^n \to K^n$ be a polynomial map with $\text{rank}(JH) = 1$. Then there exists $T \in GL_n(K)$ with*

$$T^{-1}FT = X - L(x_1, \ldots, x_r)$$

where $r < n$ and $L_i(x_1, \ldots, x_r) = 0$ for all $i \leq r$.

Proof: See [Bass et al. 82]. □

If we use this in our situation we get:

Corollary 2.6 *Let $F = X - H : K^4 \to K^4$ be a cubic homogeneous polynomial map with $\text{rank}(JH) = 1$. Then there exists $T \in GL_4(K)$ with*

$$T^{-1}FT = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 - a_4x_1^3 - b_4x_1^2x_2 - c_4x_1^2x_3 - e_4x_1x_2^2 - f_4x_1x_2x_3 \\ \quad - h_4x_1x_3^2 - k_4x_2^3 - l_4x_2^2x_3 - n_4x_2x_3^2 - q_4x_3^3 \end{pmatrix}.$$

Proof: We have to consider three cases: $r = 1, r = 2$ and $r = 3$.

- $r = 1$. With theorem 2.5 we know that there exists $A \in GL_4(K)$ with

  $$A^{-1}FA = \begin{pmatrix} x_1 \\ x_2 - L_2(x_1) \\ x_3 - L_3(x_1) \\ x_4 - L_4(x_1) \end{pmatrix}$$

  where $L_i(x_1)$ is cubic homogeneous. So in fact for all i we have $L_i(x_1) = \lambda_i x_1^3$. But now we know that there exists $B \in GL_4(K)$ with

  $$B^{-1}A^{-1}FAB = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 - \lambda_4 x_1^3 \end{pmatrix}$$

  and taking $T = AB$ completes the proof of this case.

- $r = 2$. In this case we know that there exists $A \in GL_4(K)$ with

$$A^{-1}FA = \begin{pmatrix} x_1 \\ x_2 \\ x_3 - L_3(x_1, x_2) \\ x_4 - L_4(x_1, x_2) \end{pmatrix}$$

where $L_3(x_1, x_2)$ and $L_4(x_1, x_2)$ are cubic homogeneous. But since $\text{rank}(JH) = 1$ we have $\det(J(L_3, L_4)) = 0$. And from this it follows that there exists some polynomial map of degree d, say $h(x, y)$ with $h(L_3, L_4) = 0$. But since both L_3 and L_4 are homogeneous, we have that $h_d(L_3, L_4) = 0$ where $h_d(x, y)$ is the homogeneous part of degree d of h. But we know how we can factor homogeneous maps of a certain degree in general. So here we can factor $h_d(x, y)$ into

$$h_3(x, y) = (\lambda_1 x + \mu_1 y)(\lambda_2 x + \mu_2 y) \cdots (\lambda_d x + \mu_d y).$$

And from $h_d(L_3, L_4) = 0$ we get that for at least one i

$$(\lambda_i L_3 + \mu_i L_4) = 0.$$

So also in this case there exists a $B \in GL_4(K)$ with

$$B^{-1}A^{-1}FAB = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 - L_4(x_1, x_2) \end{pmatrix}$$

and taking $T = AB$ completes the proof of the second case.

- $r = 3$. From theorem 2.5 it follows immediately that there exists $T \in GL_4(K)$ with

$$T^{-1}FT = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 - a_4x_1^3 - b_4x_1^2x_2 - c_4x_1^2x_3 - e_4x_1x_2^2 - f_4x_1x_2x_3 \\ \quad - h_4x_1x_3^2 - k_4x_2^3 - l_4x_2^2x_3 - n_4x_2x_3^2 - q_4x_3^3 \end{pmatrix}.$$

And this completes the proof of the corollary. □

Now we have enough information to present the theorem that gives a complete classification of all cubic homogeneous maps in dimension four that satisfy the Jacobian hypothesis.

Theorem 2.7 *Let $F = X - H$ be a cubic homogeneous polynomial map in dimension four, such that $\det(JF) = 1$. Then there exists some $T \in GL_4(K)$ with $T^{-1}FT$ is of one of the following forms:*

1. $$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 - a_4x_1^3 - b_4x_1^2x_2 - c_4x_1^2x_3 - e_4x_1x_2^2 - f_4x_1x_2x_3 \\ \quad - h_4x_1x_3^2 - k_4x_2^3 - l_4x_2^2x_3 - n_4x_2x_3^2 - q_4x_3^3 \end{pmatrix}$$

2. $$\begin{pmatrix} x_1 \\ x_2 - \frac{1}{3}x_1^3 - h_2x_1x_3^2 - q_2x_3^3 \\ x_3 \\ x_4 - x_1^2x_3 - h_4x_1x_3^2 - q_4x_3^3 \end{pmatrix}$$

$$3.\ \begin{pmatrix} x_1 \\ x_2 - \frac{1}{3}x_1^3 - c_1x_1^2x_4 + 3c_1x_1x_2x_3 - \frac{16q_4c_1^2-r_4^2}{48c_1^2}x_1x_3^2 - \frac{1}{2}r_4x_1x_3x_4 \\ \quad + \frac{3}{4}r_4x_2x_3^2 - \frac{r_4q_4}{12c_1}x_3^3 - \frac{r_4^2}{16c_1}x_3^2x_4 \\ x_3 \\ x_4 - x_1^2x_3 + \frac{r_4}{4c_1}x_1x_3^2 - 3c_1x_1x_3x_4 + 9c_1x_2x_3^2 - q_4x_3^3 - \frac{3}{4}r_4x_3^2x_4 \end{pmatrix}$$

$$4.\ \begin{pmatrix} x_1 \\ x_2 - \frac{1}{3}x_1^3 \\ x_3 - x_1^2x_2 - e_3x_1x_2^2 - k_3x_2^3 \\ x_4 - e_4x_1x_2^2 - k_4x_2^3 \end{pmatrix}$$

$$5.\ \begin{pmatrix} x_1 \\ x_2 - \frac{1}{3}x_1^3 + i_3x_1x_2x_4 - j_2x_1x_4^2 + s_3x_2x_4^2 + i_3^2x_3x_4^2 - t_2x_4^3 \\ x_3 - x_1^2x_2 - \frac{2s_3}{i_3}x_1x_2x_4 - i_3x_1x_3x_4 - j_3x_1x_4^2 - \frac{s_3^2}{i_3^2}x_2x_4^2 \\ \quad - s_3x_3x_4^2 - t_3x_4^3 \\ x_4 \end{pmatrix}$$

$$6.\ \begin{pmatrix} x_1 \\ x_2 - \frac{1}{3}x_1^3 - j_2x_1x_4^2 - t_2x_4^3 \\ x_3 - x_1^2x_2 - e_3x_1x_2^2 - g_3x_1x_2x_4 - j_3x_1x_4^2 - k_3x_2^3 - m_3x_2^2x_4 \\ \quad - p_3x_2x_4^2 - t_3x_4^3 \\ x_4 \end{pmatrix}$$

$$7.\ \begin{pmatrix} x_1 \\ x_2 - \frac{1}{3}x_1^3 \\ x_3 - x_1^2x_2 - e_3x_1x_2^2 - k_3x_2^3 \\ x_4 - x_1^2x_3 - e_4x_1x_2^2 - f_4x_1x_2x_3 - h_4x_1x_3^2 - k_4x_2^3 - l_4x_2^2x_3 \\ \quad - n_4x_2x_3^2 - q_4x_3^3 \end{pmatrix}$$

$$8.\ \begin{pmatrix} x_1 \\ x_2 - \frac{1}{3}x_1^3 \\ x_3 - x_1^2x_2 - e_3x_1x_2^2 + g_4x_1x_2x_3 - k_3x_2^3 + m_4x_2^2x_3 + g_4^2x_2^2x_4 \\ x_4 - x_1^2x_3 - e_4x_1x_2^2 - \frac{2m_4}{g_4}x_1x_2x_3 - g_4x_1x_2x_4 - k_4x_2^3 \\ \quad - \frac{m_4^2}{g_4^2}x_2^2x_3 - m_4x_2^2x_4 \end{pmatrix}$$

Proof: See the results in the previous sections and corollary 2.6. □

As a direct consequence of this theorem we have:

Corollary 2.8 *Let $F = X - H$ be a cubic homogeneous polynomial map from $K^4 \to K^4$ such that $\det(JF) = 1$ then (H_1, H_2, H_3, H_4) are linear dependent over K.*

Proof: We have seen that for such an F there always exists an invertible $T \in GL_4(K)$ such that

$$T^{-1}FT = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} - \begin{pmatrix} 0 \\ H_2' \\ H_3' \\ H_4' \end{pmatrix}$$

so in particular (H_1, H_2, H_3, H_4) are linear dependent over K. □

Furthermore we have:

Corollary 2.9 *Let $F = X - H$ be a cubic homogeneous polynomial map from $K^4 \to K^4$ such that $\det(JF) = 1$ then F is invertible.*

Proof: It was already known that such an F is invertible if the rank of JH equals zero or one. Using an implementation of theorem 1.24 one can easily check that all forms described in theorem 2.7 are invertible too. So now we know that each cubic homogeneous polynomial map $F : K^4 \to K^4$ with $\det(JF) = 1$ is invertible. $\square$

Chapter 3

Structurization in dimension two

3.1 Introduction

This chapter will give a description of a general form in two dimensions. If we take a look at the forms presented in theorem 2.7 we can see that all forms are essentially polynomial maps from A^2 to A^2, where A is a polynomial ring in two variables. For this reason we studied some properties of this kind of maps.

3.2 The Structure theorem

Not only could we transform each representative from theorem 2.7 to a two dimensional map, but we also found that they all could be written in a particular form. This was no coincidence since we were able to prove the following theorem:

Theorem 3.1 *Let A be a unique factorization domain and $H_1, H_2 \in A[x_1, x_2]$ such that $J(H_1, H_2)^2 = 0$. Then there exist $f(T) \in A[T]$ with $f(0) = 0$ and $\mu_1, \mu_2, c_1, c_2 \in A$ with*

$$\begin{pmatrix} H_1 \\ H_2 \end{pmatrix} = \begin{pmatrix} -\mu_2 f(\mu_1 x_1 + \mu_2 x_2) + c_1 \\ \mu_1 f(\mu_1 x_1 + \mu_2 x_2) + c_2 \end{pmatrix}.$$

Before we give the proof of this theorem we first present two lemma's.

Lemma 3.2 *If $A = K$ is a field then theorem 3.1 is true.*

Proof: See [Bass et al. 82]. □

Lemma 3.3 *Let A be a unique factorization domain and K its quotient field. Let $g(T) \in K[T] \setminus \{0\}$. Furthermore let $\mu_1, \mu_2 \in K$ such that*

$$g(\mu_1 x_1 + \mu_2 x_2) \in A[x_1, x_2].$$

Then there exist $\tilde{g}(T) \in A[T]$ and $\tilde{\mu_1}, \tilde{\mu_2} \in A$ such that

$$g(\mu_1 x_1 + \mu_2 x_2) = \tilde{g}(\tilde{\mu_1} x_1 + \tilde{\mu_2} x_2).$$

Proof: Choose $a, d \in A$ ($d \neq 0$) with $\gcd(a, d) = 1$ and $\tilde{\mu_1}, \tilde{\mu_2} \in A$ with $\gcd(\tilde{\mu_1}, \tilde{\mu_2}) = 1$ (and in particular $\gcd(d, \tilde{\mu_1} x_1 + \tilde{\mu_2} x_2) = 1$) such that

$$\mu_1 x_1 + \mu_2 x_2 = \frac{a(\tilde{\mu_1} x_1 + \tilde{\mu_2} x_2)}{d}.$$

This can be done since A is a unique factorization domain implies that also $A[x_1, x_2]$ is a unique factorization domain. Now write

$$g(T) = \sum_{i=0}^{j} a_i T^i$$

with $a_i \in K$ and some $j \in \mathbb{N}$. Given the assumptions it follows that for every homogeneous component it holds that:

$$a_i(\mu_1 x_1 + \mu_2 x_2)^i \in A[x_1, x_2].$$

If we write $a_i = \frac{p_i}{q_i}$ with $p_i \in A$ and $q_i \in A \setminus \{0\}$ and $\gcd(p_i, q_i) = 1$ we see that

$$\frac{p_i}{q_i} a^i \frac{(\tilde{\mu_1} x_1 + \tilde{\mu_2} x_2)^i}{d^i} \in A[x_1, x_2].$$

Now since $\gcd(a, d) = 1$ and $\gcd(d, \tilde{\mu_1} x_1 + \tilde{\mu_2} x_2) = 1$ we have that $d^i | p_i$ so $p_i = \tilde{p_i} d^i$ with $\tilde{p_i} \in A$. This results in the observation that

$$a_i(\mu_1 x_1 + \mu_2 x_2)^i = \frac{\tilde{p_i}}{q_i} a^i (\tilde{\mu_1} x_1 + \tilde{\mu_2} x_2)^i.$$

Now from $\gcd(p_i, q_i) = 1$ it follows that $q_i | a^i$, so $a^i = \alpha_i q_i$ with $\alpha_i \in A$. So now we have

$$a_i(\mu_1 x_1 + \mu_2 x_2)^i = \tilde{p_i} \alpha_i (\tilde{\mu_1} x_1 + \tilde{\mu_2} x_2)$$

and we can define

$$\tilde{g}(T) = \sum_{i=0}^{j} \tilde{p_i} \alpha_i T^i$$

which completes the proof of the lemma. □

Now we can continue with the proof of theorem 3.1.

Proof: Let K be the quotient field of A. Then we have $H_1, H_2 \in K[x_1, x_2]$ and according to lemma 3.2 there exist $f(T) \in K[T]$ with $f(0) = 0$ and $\mu_1, \mu_2, c_1, c_2 \in K$ with

$$\begin{pmatrix} H_1 \\ H_2 \end{pmatrix} = \begin{pmatrix} -\mu_2 f(\mu_1 x_1 + \mu_2 x_2) + c_1 \\ \mu_1 f(\mu_1 x_1 + \mu_2 x_2) + c_2 \end{pmatrix}.$$

Since $H_1, H_2 \in A[x_1, x_2]$ we have that $H_1(0,0) \in A$ and $H_2(0,0) \in A$. Using $f(0) = 0$ we see that

$$A^2 \ni \begin{pmatrix} H_1(0,0) \\ H_2(0,0) \end{pmatrix} = \begin{pmatrix} -\mu_2 f(\mu_1 0 + \mu_2 0) + c_1 \\ \mu_1 f(\mu_1 0 + \mu_2 0) + c_2 \end{pmatrix} = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}.$$

So in particular we have $c_1, c_2 \in A$. Combining this with the given fact that $H_1, H_2 \in A[x_1, x_2]$ we see that also $\mu_2 f(\mu_1 x_1 + \mu_2 x_2) \in A[x_1, x_2]$ and $\mu_1 f(\mu_1 x_1 + \mu_2 x_2) \in A[x_1, x_2]$. Multiplying by respectively x_2 and x_1 and adding the results gives us:

$$(\mu_1 x_1 + \mu_2 x_2) f(\mu_1 x_1 + \mu_2 x_2) \in A[x_1, x_2].$$

If we write

$$f(T) = \sum_{i=1}^{m} b_i T^i$$

with $b_i \in K$ and $m \in \mathbb{N}$ we can define

$$g(T) = Tf(T) = \sum_{i=1}^{m} b_i T^{i+1} \in K[T].$$

Since we have already seen that there exist some $\mu_1, \mu_2 \in K$ such that

$$g(\mu_1 x_1 + \mu_2 x_2) = (\mu_1 x_1 + \mu_2 x_2) f(\mu_1 x_1 + \mu_2 x_2) \in A[x_1, x_2]$$

we can now use lemma 3.3 and conclude that there exist

$$\tilde{g}(T) = \sum_{i=1}^{m} \tilde{b_i} T^{i+1} \in A[T]$$

and $\tilde{\mu_1}, \tilde{\mu_2} \in A$ such that

$$g(\mu_1 x_1 + \mu_2 x_2) = \tilde{g}(\tilde{\mu_1} x_1 + \tilde{\mu_2} x_2).$$

Now define $\tilde{f}(T) = \sum_{i=1}^{m} \tilde{b_i} T^i$ and we have $\tilde{g}(T) = T\tilde{f}(T)$. So now we have

$$\begin{aligned}
(\tilde{\mu_1} x_1 + \tilde{\mu_2} x_2)\tilde{f}(\tilde{\mu_1} x_1 + \tilde{\mu_2} x_2) &= \tilde{g}(\tilde{\mu_1} x_1 + \tilde{\mu_2} x_2) \\
&= g(\mu_1 x_1 + \mu_2 x_2) \\
&= (\mu_1 x_1 + \mu_2 x_2) f(\mu_1 x_1 + \mu_2 x_2).
\end{aligned}$$

From this it follows that

$$\tilde{\mu_1} x_1 \tilde{f}(\tilde{\mu_1} x_1 + \tilde{\mu_2} x_2) + \tilde{\mu_2} x_2 \tilde{f}(\tilde{\mu_1} x_1 + \tilde{\mu_2} x_2) = \mu_1 x_1 f(\mu_1 x_1 + \mu_2 x_2) + \mu_2 x_2 f(\mu_1 x_1 + \mu_2 x_2).$$

Note now that all monomials appearing in $f(\mu_1 x_1 + \mu_2 x_2)$ and in $\tilde{f}(\tilde{\mu_1} x_1 + \tilde{\mu_2} x_2)$ are

$$\left\{ x_1^i x_2^j : i + j = k, k = 1, \ldots, m \right\}.$$

But if we list the monomials appearing in $\mu_1 x_1 f(\mu_1 x_1 + \mu_2 x_2)$ we find the set

$$M_1 = \left\{ x_1^{i+1} x_2^j : i + j = k, k = 1, \ldots, m \right\}$$

and for $\mu_2 x_2 f(\mu_1 x_1 + \mu_2 x_2)$ we find

$$M_2 = \left\{ x_1^i x_2^{j+1} : i + j = k, k = 1, dots, m \right\}.$$

The same holds for the expressions with $\tilde{f}, \tilde{\mu_1}$ and $\tilde{\mu_2}$. Note that

$$D_1 = M_1 \setminus M_2 = \left\{ x_1^{i+1} : i = 1, \ldots, m \right\} \text{ and } D_2 = M_2 \setminus M_2 = \left\{ x_2^{i+1} : i = 1, \ldots, m \right\}.$$

Now if we take a look at the coefficients of a monomial $x_1^{i+1} \in D_1$, we see that these coefficients are completely determined by b_i and $\tilde{b_i}$. Since $x_1^{i+1} \in D_1$ we have that

$$\mu_1 x_1 b_i (\mu_1 x_1 + \mu_2 x_2)^i = \tilde{\mu_1} x_1 \tilde{b_i} (\tilde{\mu_1} x_1 + \tilde{\mu_2} x_2)^i.$$

But since this holds for every $i \in \{1, \ldots, m\}$ we can take the sum and find

$$\mu_1 x_1 f(\mu_1 x_1 + \mu_2 x_2) = \tilde{\mu_1} x_1 \tilde{f}(\tilde{\mu_1} x_1 + \tilde{\mu_2} x_2).$$

In an analogous way we can see that

$$\mu_2 x_2 f(\mu_1 x_1 + \mu_2 x_2) = \tilde{\mu_2} x_2 \tilde{f}(\tilde{\mu_1} x_1 + \tilde{\mu_2} x_2).$$

And from this we can easily deduce that

$$\begin{aligned}
\tilde{\mu_1} \tilde{f}(\tilde{\mu_1} x_1 + \tilde{\mu_2} x_2) &= \mu_1 f(\mu_1 x_1 + \mu_2 x_2) \\
\tilde{\mu_2} \tilde{f}(\tilde{\mu_1} x_1 + \tilde{\mu_2} x_2) &= \mu_2 f(\mu_1 x_1 + \mu_2 x_2).
\end{aligned}$$

So finally we have proven the existence of the desired $f(T) \in A[T]$ and the μ_1, μ_2, c_1 and $c_2 \in A$. □

3.3 Concrete transformations

The next example will show in detail that one of the eight representatives we found in the previous chapter for theorem 2.7 can be written in the way as described in theorem 3.1. For instance let us take the fifth representative.

Example 3.4 *Let F be the polynomial map*

$$\begin{pmatrix} x_1 \\ x_2 - \frac{1}{3}x_1^3 + i_3x_1x_2x_4 - j_2x_1x_4^2 + s_3x_2x_4^2 + i_3^2x_3x_4^2 - t_2x_4^3 \\ x_3 - x_1^2x_2 - \frac{2s_3}{i_3}x_1x_2x_4 - i_3x_1x_3x_4 - j_3x_1x_4^2 - \frac{s_3^2}{i_3^2}x_2x_4^2 \\ \quad - s_3x_3x_4^2 - t_3x_4^3 \\ x_4 \end{pmatrix}.$$

Since H_1 and H_4 equal zero we can see this four dimensional map as a two dimensional in the unknowns: x_2 and x_3. By substituting $x_2 = 0$ and $x_3 = 0$ we find the constant part of the homogeneous part of this map, i.e. the part that is not dependent of x_2 or x_3. This gives

$$\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} \frac{1}{3}x_1^3 + j_2x_1x_4^2 + t_2x_4^3 \\ j_3x_1x_4^2 + t_3x_4^3 \end{pmatrix}.$$

The remaining part of H is:

$$\begin{pmatrix} -i_3x_1x_2x_4 - s_3x_2x_4^2 - i_3^2x_3x_4^2 \\ x_1^2x_2 + \frac{2s_3}{i_3}x_1x_2x_4 + i_3x_1x_3x_4 + \frac{s_3^2}{i_3^2}x_2x_4^2 + s_3x_3x_4^2 \end{pmatrix}.$$

When we factor this we get the following polynomial maps:

$$-x_4\left(s_3x_2x_4 + i_3^2x_3x_4 + i_3x_1x_2\right)$$

and

$$\frac{i_3x_1 + s_3x_4}{i_3^2}\left(s_3x_2x_4 + i_3^2x_3x_4 + i_3x_1x_2\right).$$

If we move the factor i_3 in the first map to the outside of the parentheses and do in the second map just the opposite with $\frac{1}{i_3}$ and reorder the terms a bit we get:

$$-i_3x_4\left(\frac{i_3x_1 + s_3x_4}{i_3}x_2 + i_3x_4x_3\right)$$

respectively

$$\frac{i_3x_1 + s_3x_4}{i_3}\left(\frac{i_3x_1 + s_3x_4}{i_3}x_2 + i_3x_4x_3\right).$$

We see that with $\mu_1 = \frac{i_3x_1+s_3x_4}{i_3}$, $\mu_2 = i_3x_4$ and $f(T) = T$ we have written the homogeneous part in the desired way.

The following three tables give the values of c_1, c_2, μ_1, μ_2 and $f(T)$ of all eight representatives of theorem 2.7.[1]

	$c_1 \in K$
1	0
2	$\frac{1}{3}x_1^3 + h_2x_1x_3^2 + q_2x_3^3$
3	$\frac{1}{3}x_1^3 + \frac{16q_4c_1^2 - r_4^2}{48c_1^2}x_1x_3^2 + \frac{r_4q_4}{c_1}$
4	$x_1^2x_2 + \frac{1}{3}x_1^5 + e_3x_1x_2^2 + \frac{2}{3}e_3x_1^4x_2 + \frac{1}{9}e_3x_1^7 + k_3x_2^3 + k_3x_2^2x_1^3 + \frac{1}{3}k_3x_2x_1^6 + \frac{1}{27}k_3x_1^9$
5	$\frac{1}{3}x_1^3 + j_2x_1x_4^2 + t_2x_4^3$
6	$\frac{1}{3}x_1^3 + j_2x_1x_4^2 + t_2x_4^3$
7	$x_1^2x_2 + \frac{1}{3}x_1^5 + e_3x_1x_2^2 + \frac{2}{3}e_3x_1^4x_2 + \frac{1}{9}e_3x_1^7 + k_3x_2^3 + k_3x_2^2x_1^3 + \frac{1}{3}k_3x_2x_1^6 + \frac{1}{27}k_3x_1^9$
8	$x_1^2x_2 + \frac{1}{3}x_1^5 + e_3x_1x_2^2 + \frac{2}{3}e_3x_1^4x_2 + \frac{1}{9}e_3x_1^7 + k_3x_2^3 + k_3x_2^2x_1^3 + \frac{1}{3}k_3x_2x_1^6 + \frac{1}{27}k_3x_1^9$

[1] Since the maps with numbers four,seven and eight do not have two H_i-s equal to zero, we start in these cases by changing coordinates by substituting $x_2 = x_2 + \frac{1}{3}x_1^3$.

	$c_2 \in K$
1	$a_4x_1^3 + b_4x_1^2x_2 + e_4x_1x_2^2 + k_4x_2^3$
2	$x_1^2x_3 + h_4x_1x_3^2 + q_4x_3^3$
3	$x_1^2x_3 - \frac{r_4}{4c_1}x_1x_3^2 + q_4x_3^3$
4	$e_4x_1x_2^2 + \frac{2}{3}e_4x_1^4x_2 + \frac{1}{9}e_4x_1^7 + k_4x_2^3 + k_4x_2^2x_1^3 + \frac{1}{3}k_4x_2x_1^6 + \frac{1}{27}k_4x_1^9$
5	$j_3x_1x_4^2 + t_3x_4^3$
6	$j_3x_1x_4^2 + t_3x_4^3$
7	$e_4x_1x_2^2 + \frac{2}{3}e_4x_1^4x_2 + \frac{1}{9}e_4x_1^7 + k_4x_2^3 + k_4x_2^2x_1^3 + \frac{1}{3}k_4x_2x_1^6 + \frac{1}{27}k_4x_1^9$
8	$e_4x_1x_2^2 + \frac{2}{3}e_4x_1^4x_2 + \frac{1}{9}e_4x_1^7 + k_4x_2^3 + k_4x_2^2x_1^3 + \frac{1}{3}k_4x_2x_1^6 + \frac{1}{27}k_4x_1^9$

	$\mu_1 \in K$	$\mu_2 \in K$	$f(T) \in K[T]$
1	0	1	$(c_4x_1^2 + f_4x_1x_2 + l_4x_2^2)T$ $+ (h_4x_1 + n_4x_2)T^2$ $+ q_4T^3$
2	0	0	T
3	$-3c_1x_3$	$c_1x_1 + \frac{1}{4}r_4x_3$	$-\frac{1}{c_1}T$
4	0	0	T
5	$x_1 + \frac{s_3}{i_3}x_4$	i_3x_4	T
6	1	0	$(x_1^2 + g_3x_1x_4 + p_3x_4^2)T$ $+ (e_3x_1 + m_3x_4)T^2$ $+ k_3T^3$
7	1	0	$(x_1^2 + f_4x_1x_2 + \frac{1}{3}f_4x_1^4 + x_2^2l_4$ $+ \frac{2}{3}l_4x_2x_1^3 + \frac{1}{9}l_4x_1^6)T$ $+ (h_4x_1 + n_4x_2 + \frac{1}{3}n_4x_1^3)T^2$ $+ q_4T^3$
8	$\frac{m_4x_1^3+3m_4x_2+3g_4x_1}{3g_4}$	$\frac{g_4}{3}(3x_2 + x_1^3)$	T

Chapter 4

Drużkowski forms

4.1 Introduction

In this chapter we consider a special case of the cubic homogeneous polynomials in dimension four.

Definition 4.1 *A polynomial map* $F : K^n \to K^n$ *is on* Drużkowski form *if it has the form:*

$$F = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} - \begin{pmatrix} H_1 \\ H_2 \\ \vdots \\ H_n \end{pmatrix}$$

where each

$$H_i = (c_{1i}x_1 + c_{2i}x_2 + \cdots + c_{ni}x_n)^3$$

and $c_{ji} \in K$ *for* $i, j \in \{1, \dots, n\}$.

Naturally if F is on Drużkowski form it is also cubic homogeneous.

So of course the question arises: why look at a special case when you have already a complete description of the more general case. Well, the reason we examined the Drużkowski forms in dimension four was that we wanted to compare our results with the results of Meisters' power similarity research and check his conjecture and we wanted to look at Drużkowski forms in general.

4.2 A classification of four dimensional Drużkowski forms

We started our examination with the general Drużkowski form:

$$F = \begin{pmatrix} x_1 - (a_1x_1 + b_1x_2 + c_1x_3 + d_1x_4)^3 \\ x_2 - (a_2x_1 + b_2x_2 + c_2x_3 + d_2x_4)^3 \\ x_3 - (a_3x_1 + b_3x_2 + c_3x_3 + d_3x_4)^3 \\ x_4 - (a_4x_1 + b_4x_2 + c_4x_3 + d_4x_4)^3 \end{pmatrix}$$

In order to compute the equations described in [Wright 93] we needed the expanded form of F to determine the coefficient of each monomial. With these coefficients all put together in a 4×20 matrix, we computed the mentioned equations. Since there were only sixteen variables, we tried to solve this Drużkowski system without any preconditioning. However this didn't work out. The system was to complicated to solve. So we used the result of corollary 2.8. This corollary indicated that we could take one H_i equal to zero without loss of generality. For reasons which will become clear in section 4.3 we choose H_4 equal to zero, i.e. we substitute $a_4 = b_4 = c_4 = d_4 = 0$ in our original system. After this simplification we were able to solve the system completely. In order to reduce the number of solutions we used arguments of symmetry already at the moment of the computation of the solutions. Basically we used three assumptions:

1. $c_1 = 0$ and $c_2 = 0$. (This immediately implied $c_3 = 0$ also.)
2. $c_1 = 0$ and $c_2 \neq 0$.
3. $c_1 \neq 0$.

We first computed all solutions under the first assumption. This gave us three solutions. Secondly we computed the solutions under the second assumption, but because of the symmetry aspect, as soon as we found two of all a_i, b_i or d_i equal to zero, we left out this solution since it was basically one of the four solutions found before. This second assumption gave us two new solutions. Finally we computed the solutions under the third assumption, but because of the symmetry we only needed solutions which had none of the a_i, b_i, c_i or d_i ($i \in \{1, 2, 3\}$) equal to zero. This resulted in exactly one solution. This made up a total of six solutions. With these solutions we could formulate a classification theorem for the Drużkowski forms.

Theorem 4.2 *Let $F : K^4 \to k^4$ be a cubic homogeneous map on Drużkowski form. Then there exists a linear invertible T with $T^{-1}FT$ is one of the following six forms:*

1. $$\begin{pmatrix} x_1 - (d_1 x_4)^3 \\ x_2 - (a_2 x_1 + d_2 x_4)^3 \\ x_3 - (a_3 x_1 + b_3 x_2 + d_3 x_4)^3 \\ x_4 \end{pmatrix}$$

2. $$\begin{pmatrix} x_1 - (b_1 x_2 + d_1 x_4)^3 \\ x_2 - (d_2 x_4)^3 \\ x_3 - (a_3 x_1 + b_3 x_2 + d_3 x_4)^3 \\ x_4 \end{pmatrix}$$

3. $$\begin{pmatrix} x_1 - (a_1 x_1 - \frac{a_1^4}{a_2^3} x_2 + \frac{a_1 d_2}{a_2} x_4)^3 \\ x_2 - (a_2 x_1 - \frac{a_1^3}{a_2^2} x_2 + d_2 x_4)^3 \\ x_3 - (a_3 x_1 + b_3 x_2 + d_3 x_4)^3 \\ x_4 \end{pmatrix}$$

4. $$\begin{pmatrix} x_1 - (b_1 x_2 + \frac{d_3 b_1}{b_3} x_4)^3 \\ x_2 - (-\frac{c_2 b_3^3}{b_1^3} x_1 + c_2 x_3 + d_2 x_4)^3 \\ x_3 - (b_3 x_2 + d_3 x_4)^3 \\ x_4 \end{pmatrix}$$

5. $$\begin{pmatrix} x_1 - (d_1 x_4)^3 \\ x_2 - (\frac{a_3 c_2}{c_3} x_1 - \frac{c_3^3}{c_2^2} x_2 + c_2 x_3 + \frac{d_3 c_2}{c_3} x_4)^3 \\ x_3 - (a_3 x_1 - \frac{c_3^4}{c_2^3} x_2 + c_3 x_3 + d_3 x_4)^3 \\ x_4 \end{pmatrix}$$

6. $$\begin{pmatrix} x_1 - (\frac{b_1c_2^3+c_1c_3^3}{c_1^3}x_1 + b_1x_2 + c_1x_3 + \frac{c_1d_2}{c_2}x_4)^3 \\ x_2 - (\frac{c_2(b_1c_2^3+c_1c_3^3)}{c_1^4}x_1 + \frac{b_1c_2}{c_1}x_2 + c_2x_3 + d_2x_4)^3 \\ x_3 - (\frac{c_3(b_1c_2^3+c_1c_3^3)}{c_1^4}x_1 + \frac{b_1c_3}{c_1}x_2 + c_3x_3 + \frac{c_3d_2}{c_2}x_4)^3 \\ x_4 \end{pmatrix}$$

Proof: See the description above or section A.4. □

If we take the coefficients of these six solutions and put them in matrices A we see that the resulting matrices are of rank one, two or three. The first two have $\text{rank}(A) = 3$. Solution numbers three, four and five have $\text{rank}(A) = 2$. And solution number six has $\text{rank}(A) = 1$. This distinction between ranks is used in the next section.

4.3 Power similarity and a proof of a conjecture of Meisters

As stated before, the reason we examined these Drużkowski forms was to be able to compare these results with Meisters' result on power similarity. See [Meisters 93].

Until now we have always regarded maps $F : K^4 \to K^4$ where K was an arbitrary field with characteristic. However in this section we shall restrict ourselves to $K = \mathbb{C}$, since the power similarity property is defined in terms of $\mathbb{C}$, but any algebraically closed field would work.

Definition 4.3 *Let $F = X - (AX)^3$ and $G = X - (BX)^3$ be two polynomial maps on Drużkowski forms. Then the matrices $A, B \in \text{Mat}_n(\mathbb{C})$ are called* power similar *($A \sim_p B$) if there exists an invertible polynomial map T with $T^{-1}FT = G$.*

The idea behind power similarity is that one wants to use linear invertible maps T for transformations of coordinates of maps F on Drużkowski form. In general there is no need that the result $T^{-1}FT$ is again on Drużkowski form. So if $T^{-1}FT$ is indeed on Drużkowski form this is pretty special and this property deserves a special name: power similarity, introduced by Meisters in [Meisters 93].

Definition 4.3 is in terms of maps. It is also possible to look at power similarity on the level of matrices.

Proposition 4.4 *Let $F = X - (AX)^3$ and $G = X - (BX)^3$ be two polynomial maps on Drużkowski forms. Then the matrices A and B are power similar if and only if there exists $T \in GL_n(\mathbb{C})$ with $(ATX)^3 = T(BX)^3$.*

Proof: Let $F = X - (AX)^3$ and $G = X - (BX)^3$ be two polynomial maps on Drużkowski forms. Then the following statements are equivalent:

- A and B are power similar.
- There exists an invertible map T with $T^{-1}FT = G$.
- There exists an invertible map T with $T^{-1}(TX - (ATX)^3) = X - (BX)^3$.
- There exists an invertible map T with $X - T^{-1}(ATX)^3 = X - (BX)^3$.

- There exists an invertible map T with $T^{-1}(ATX)^3 = (BX)^3$.
- There exists an invertible matrix T with $T^{-1}(ATX)^3 = (BX)^3$.
- There exists an invertible matrix T with $(ATX)^3 = T(BX)^3$.

This proves the proposition. □

For $n = 4$ Meisters found six representatives and conjectured:

Conjecture 4.5 *The following six matrices form a complete system of representatives of* $\mathrm{Mat}_4(\mathbb{C})/\sim_p$*:*

$$J12 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$J22 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}, J23 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, N23 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$J34 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}, N34 = \begin{pmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Before we are going to prove this conjecture we shall explain the notation used here. The J comes from Jordan since all J matrices are on Jordan normal form. The N matrices represent the other nilpotent representatives of this power similarity equivalence relation. Furthermore the first number stands for the rank of the matrix. The second number gives the 'nilpotency index', i.e. the smallest power such that the representative raised to that power equals the zero matrix. E.g. $J23$ is a matrix on Jordan normal form of rank two with $J23^2 \neq 0$ and $J23^3 = 0$.

From these representatives we see also why we chose $H_4 = 0$ in the former section. Since all these representatives have a last row consisting of only zeroes, it is sometimes easy to see that a particular representative is power similar to one of our solutions, since it is simply a special case of it.

In order to check whether our solutions were equivalent to the given representatives of conjecture 4.5 we made a procedure `powsim` that computed the system of equations we got by taking a general matrix T with $\det(T) \neq 0$ and computing the matrix products from proposition 4.4 and comparing all coefficients. By looking at the rank we saw that there were at most three possibilities for each solution. By looking at the nilpotency index we could even restrict ourselves to two cases for each matrix. The resulting systems were easy to solve. So after we had found that our general solutions were all power similar to one of the six matrices of conjecture 4.5, we changed the procedure in such a way that it automatically solved the computed systems and -more important- that it gave the number of solutions to each system. Normally it is not very safe to solve a system of equations just by using the standard Maple procedure `solve`, since you can never be sure that it finds all solutions. However in this case it didn't matter if Maple found *all* solutions, as long as it found *a* solution. Only if Maple couldn't find a solution, we would have to look at these cases very secure to convince ourselves that there really were no solutions. Since that would mean that Meisters' six representatives were not complete. However Maple was able to find an invertible matrix T for each of our solutions that proved the power similarity of that particular solution to one of the six matrices of conjecture 4.5. But looking at these matrices T we saw that there were sometimes restrictions on the coefficients of our solutions. These restrictions had to prevent that divisions by zero occurred. This observation lead to the notion that we had to examine different cases inferred by these restrictions, e.g. if a solution for T meant a division by a_1, we also would have to find a solution for the system of equations we got by substituting $a_1 = 0$ to begin with. But before we give the results of this examination we first note the following.

If we look at the maps of theorem 4.2 we see that all maps can be regarded as a special case of the second map, i.e. for all maps F there exists an invertible T with $T^{-1}FT = G$, where F is any of the maps from theorem 4.2 and G is the map

$$\begin{pmatrix} x_1 - (b_1x_2 + d_1x_4)^3 \\ x_2 - (d_2x_4)^3 \\ x_3 - (a_3x_1 + b_3x_2 + d_3x_4)^3 \\ x_4 \end{pmatrix}$$

the second map in theorem 4.2. From definition 4.3 it follows immediately that these transformations carry through the power similarity properties.

The following table presents the actual mappings T for all maps from theorem 4.2 and the substitutions one has to make in the second map to get the special case.

	T	b_1	d_1	d_2	a_3	b_3	d_3
1	$\begin{pmatrix} x_2 \\ x_1 \\ x_3 \\ x_4 \end{pmatrix}$	a_2	d_2	d_1	b_3	a_3	d_3
2	$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}$	b_1	d_1	d_2	a_3	b_3	d_3
3	$\begin{pmatrix} x_1 \\ x_2 + (\frac{a_2}{a_1})^3x_1 \\ x_3 \\ x_4 \end{pmatrix}$	$-\frac{a_1^4}{a_2^3}$	$\frac{a_1d_2}{a_2}$	0	$a_3 + \frac{b_3a_2^3}{a_1^3}$	b_3	d_3
4	$\begin{pmatrix} x_3 \\ x_1 \\ x_2 + (\frac{b_3}{b_1})^3x_3 \\ x_4 \end{pmatrix}$	c_2	d_2	0	b_1	0	$\frac{b_1d_3}{b_3}$
5	$\begin{pmatrix} x_1 \\ x_3 \\ x_2 + (\frac{c_3}{c_2})^3x_3 \\ x_4 \end{pmatrix}$	0	d_1	0	$\frac{a_3c_2}{c_3}$	c_2	$\frac{d_3c_2}{c_3}$
6	$\begin{pmatrix} x_1 + (\frac{c_1}{c_2})^3x_3 \\ x_3 \\ x_2 + (\frac{c_3}{c_2})^3x_3 \\ x_4 \end{pmatrix}$	0	0	0	$-\frac{b_1c_2^4}{c_1^4} - \frac{c_2c_3^3}{c_1^3}$	c_2	d_2

In the matrix T it can happen that a certain variable has to be unequal to zero, in order to divide through it. For instance in the fourth row we divide through b_1. So we have to consider a case where $b_1 = 0$. In terms of the variables of the second map this means that we have to examine two cases: $a_3 \neq 0, b_3 = 0, d_2 = 0$ and $a_3 = 0, b_3 = 0, d_2 = 0, d_3 = 0$.

The following table gives the results of our research. A zero in a column means that we substituted a zero for the corresponding coefficient. A one means that we assumed that variable not equal to zero. Furthermore $\neq \frac{d_3 a_2}{a_3}$ means that this variable is not equal to that expression and -of course- if the $\neq$ doesn't show up in front of an expression the variable is equal to the given expression. If nothing appears in a column then that variable is completely free in case of the particular choices for the other variables in the same row.

If we now look at this table we see that the first case mentioned above is described in rows fourteen, fifteen and eighteen. Apparently the actual choice of b_3 is not important. The second case is described in rows three, four,six and ten.

Also the cases inferred by the other rows in the previous table are all described in this next table.

	a_3	b_1	d_2	b_3	d_1	d_3	
1	0	0	0	1	1		$J22$
2	0	0	0	1	0		$J12$
3	0	0	0	0	1		$J12$
4	0	0	0	0	0	1	$J12$
5	0	0	1	1			$J23$
6	0	0	1	0			$J12$
7	0	1	0	1	$\neq \frac{d_3 b_1}{b_3}$		$J22$
8	0	1	0	1	$\frac{d_3 b_1}{b_3}$		$J22$
9	0	1	0	0		1	$J22$
10	0	1	0	0		0	$J12$
11	0	1	1	1	$\neq \frac{d_3 b_1}{b_3}$		$N23$
12	0	1	1	1	$\frac{d_3 b_1}{b_3}$		$J23$
13	0	1	1	0			$J23$
14	1	0	0		1		$J23$
15	1	0	0		0		$J12$
16	1	0	1	$\neq -\frac{a_3 d_1^3}{d_2^3}$			$J23$
17	1	0	1	$-\frac{a_3 d_1^3}{d_2^3}$			$J22$
18	1	1	0				$J23$
19	1	1	1	1			$N34$
20	1	1	1	0			$J34$

From this table we obtain the important theorem:

Theorem 4.6 *Conjecture 4.5 is true.*

Proof: From the table above we learn that on one side all Meisters' representatives appear and on the other side no other matrices appear. So the set of six matrices presented in conjecture 4.5 is complete with respect to the relation $\sim_p$. □

4.4 General Drużkowski forms

We have the following interesting theorem concerning Drużkowski forms.

Theorem 4.7 *Let $r \in \mathbb{N}$. If the Jacobian Conjecture holds for every polynomial map $F : K^r \to K^r$ where F has the special form:*

$$F = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_r \end{pmatrix} - \begin{pmatrix} H_1(x_1, \ldots, x_r) \\ H_2(x_1, \ldots, x_r) \\ \vdots \\ H_r(x_1, \ldots, x_r) \end{pmatrix}$$

with $H_i = 0$ or $\deg(H_i) = 3$ (H_i homogeneous for all $i \in \{1, \ldots, r\}$) then for all $n \geq r$ and all $A \in \mathrm{Mat}_n(K)$ the Jacobian Conjecture holds for all Drużkowski forms

$$G = X - (AX)^3$$

with the rank of A equals r and $X = (x_1, \ldots, x_n)$.

Proof: Let $A \in \mathrm{Mat}_n(K)$ with $\mathrm{rank}(A) = r$. Then there exists some $T \in GL_n(K)$ such that AT is on column Echelonform. So the first r columns of AT are linearly independent and the other columns are zero. Then

$$ATX = \begin{pmatrix} \ell_1(x_1, \ldots, x_r) \\ \ell_2(x_1, \ldots, x_r) \\ \vdots \\ \ell_n(x_1, \ldots, x_r) \end{pmatrix}$$

where $\ell_i(x_1, \ldots, x_r)$ is some linear expression in $x_1, x_2, \ldots, x_r$. Now define $\tilde{G} = T^{-1}GT$. Then

$$\tilde{G} = X - T^{-1}(ATX)^3 = X - T^{-1} \begin{pmatrix} \ell_1^3(x_1, \ldots, x_r) \\ \ell_2^3(x_1, \ldots, x_r) \\ \vdots \\ \ell_n^3(x_1, \ldots, x_r) \end{pmatrix} = \begin{pmatrix} x_1 - h_1(x_1, \ldots, x_r) \\ x_2 - h_2(x_1, \ldots, x_r) \\ \vdots \\ x_n - h_n(x_1, \ldots, x_r) \end{pmatrix}$$

with h_i is a homogeneous polynomial of degree 3 over K. Note now that G is invertible if and only if $\tilde{G}$ is invertible and JG is invertible if and only if $J\tilde{G}$ is invertible. Finally, make a transformation of coordinates:

$$\begin{aligned} x'_{r+1} &= x_{r+1} - h_{r+1}(x_1, \ldots, x_r) \\ x'_{r+2} &= x_{r+2} - h_{r+2}(x_1, \ldots, x_r) \\ &\vdots \\ x'_n &= x_n - h_n(x_1, \ldots, x_r) \end{aligned}$$

On the new coordinates $x_1, \ldots, x_r, x'_{r+1}, \ldots, x'_n$ we get the following map:

$$\tilde{G} = \begin{pmatrix} x_1 - h_1(x_1, \ldots, x_r) \\ \vdots \\ x_r - h_r(x_1, \ldots, x_r) \\ x'_{r+1} \\ \vdots \\ x'_n \end{pmatrix}.$$

The invertibility of $\tilde{G}$ follows from the observation that $\tilde{G}$ is a trivial extension (in particular it doesn't change invertibility properties) of some $F : K^r \to K^r$ with the special form as described in the precondition of the theorem. And the assumption told us that such an F was invertible. □

If we restrict ourselves to the case where $n = 4$ we can deduce the following theorem:

Theorem 4.8 *The Jacobian Conjecture holds for all Drużkowski forms*

$$G = X - (AX)^3$$

where $X = (x_1, \ldots, x_n)$ *and* $\operatorname{rank}(A) \leq 4$.

Proof: From corollary 2.9 it follows that the Jacobian Conjecture holds for all polynomial maps $F : K^4 \to K^4$ of the special form

$$F = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} - \begin{pmatrix} H_1(x_1, x_2, x_3, x_4) \\ H_2(x_1, x_2, x_3, x_4) \\ H_3(x_1, x_2, x_3, x_4) \\ H_4(x_1, x_2, x_3, x_4) \end{pmatrix}.$$

Combining this with theorem 4.7 completes the proof. □

As was written in chapter 1 so far this theorem was only true if the rank or corank of A was smaller than three. If we combine these theorems we get the following result:

Theorem 4.9 *Let* $F : K^n \to K^n$ *be a polynomial map on Drużkowski form that satisfies the Jacobian hypothesis. Then if* $n \leq 7$ *the map* F *is invertible.*

Proof: Let A be the matrix of the coefficients of F. Then if $\operatorname{rank}(A) \leq 4$ we use theorem 4.8 and we know that F is invertible. If $\operatorname{rank}(A) > 4$ we have that $\operatorname{corank}(A) < 3$ and we can use theorem 1.18 to prove the invertibility of F. □

Chapter 5

Applications

5.1 Introduction

In this chapter we will use our classifications to test some conjectures or to describe some corollaries.

5.2 Exponents

One of the things we would like to know is the answer to the question whether the eight maps from theorem 2.7 can be written as the exponent of a locally nilpotent derivation.

In order to find this answer we start with the assumption that a map F can be written as

$$F = \exp(D)$$

where D is some locally nilpotent derivation. Actually this notation is not completely correct. This has to do with the different types of the two terms. The type of the lefthandside is $F : K^4 \to K^4$ whereas the type of the righthandside is $\exp(D) : K[x_1, x_2, x_3, x_4] \to K[x_1, x_2, x_3, x_4]$. So we should have written[1]

$$F^* = \exp(D)$$

where F^* is defined as

$$\begin{aligned} F^* : K[x_1, x_2, x_3, x_4] &\to K[x_1, x_2, x_3, x_4] \\ g &\mapsto g(F_1, F_2, F_3, F_4). \end{aligned}$$

But since we have $F^*(x_i) = F_i$ for $i = 1, \ldots, 4$, the intention of what we mean if we write $F = \exp(D)$ is completely clear. So in the rest of this paper we shall sometimes abuse this notation.

[1] See also [Essen 92b].

In order to find this D we have to compute the logarithm of $\exp(D)$, i.e. the logarithm of F^*. We do this in the following (purely formal) way:

$$\begin{aligned} D &= \log(\exp(D)) \\ &= \log(F^*) \\ &= \log(I + (F^* - I)) \\ &= (F^* - I) - \frac{(F^* - I)^2}{2} + \frac{(F^* - I)^3}{3} - \frac{(F^* - I)^4}{4} + \cdots \end{aligned}$$

If we write $\Delta_H = F^* - I$ we get

$$D = \sum_{i=1} \frac{(-1)^{i+1}}{i} \Delta_H^i.$$

Lemma 5.1 *Let D be a locally nilpotent derivation. Then there exists $p \in \mathbb{N}$ such that $\Delta_H^p(x_i) = 0$ for $i = 1, \ldots, 4$.*

Proof: We have

$$\begin{aligned} \Delta_H &= F^* - I \\ &= \exp(D) - I \\ &= D + \frac{D^2}{2} + \frac{D^3}{3!} + \cdots \end{aligned}$$

and since D is locally nilpotent we know that for each i there exists a p_i with $D^{p_i}(x_i) = 0$. So if we define $p = \max p_i$ then $D^p(X) = 0$. From this it follows that $\Delta_H(X)$ is a finite sum of powers of D applied on X. After p compositions of $\Delta_H(X)$ we see that the smallest power in the summation is D^p, so $\Delta_H^p(X) = 0$. □

From lemma 5.1 we know that $\Delta_H^p(X)$ must be equal to zero for some $p \in \mathbb{N}$. So the next thing we do is determine whether this $\Delta_H(X)$ vanishes after a finite number of iterations. We used a procedure **deltahnumber** to find the smallest number p with $\Delta_H^p(X) = 0$.

The first seven forms gave positive results, i.e. our procedure terminated because it had found such a p.

```
> for i from 1 to 7
> do
>   print(i,deltahnumber(G[i],20));
> od;
```

```
1, 3

2, 3

3, 3

4, 6

5, 4

6, 6

7, 15
```

The second argument gives a maximum bound to force termination of the procedure.

The eighth map gave a negative result:

```
> deltahnumber(G[8],100);
Error, (in deltahnumber) Not vanished after 100 iterations
```

So it is most unlikely that this Δ_H^n will vanish after any finite number of iterations. So this F is most probably not of the form $\exp(D)$. We shall get back to this point later on.

Now that we have seen that D is a finite sum of powers of Δ_H we can compute D itself. We used the procedure `computeD` for this purpose.

But since we used only a formal construction of this D, assuming that it existed, we have to check that indeed the equation

$$F = \exp(D)$$

holds.

In order to compute $\exp(D)$ we use the powerseries

$$\exp(D) = \sum_{i=0} \frac{D^i}{i!}.$$

If D is indeed locally nilpotent then this sum is only a finite sum. If it is not, this sum does not necessarily exist. However we used our procedure `computeexpD` to compute $\exp(D)$ with the hope that it would terminate. And indeed it did terminate in all seven cases. After this we had to compare the results of this procedure with the original maps. In fact we made one procedure `testexp` that computes, given a list of polynomial maps, respectively the derivation D, the map $\exp(D)$ and tests whether the original map equals $\exp(D)$. The first seven forms of theorem 2.7 all terminated and returned the boolean 'true'.

So to answer the question at the beginning of this section:

Theorem 5.2 *The first seven forms of theorem 2.7 can be written as* $\exp(D)$ *where* D *is some locally nilpotent derivation.*

Proof: It is easy to check that the following seven derivations are locally nilpotent and have one of the first seven maps from theorem 2.7 as their exponent. □

The notation we will use here is as follows. Each derivation is given by a list of four polynomials. For instance the result of `computeD(G[2])` gives the list:

```
                  3             2         3           2               2         3
[0, - 1/3 x1  - h2 x1 x3  - q2 x3 , 0, - x1  x3 - h4 x1 x3  - q4 x3 ]
```

This stands for the derivation:

$$0\frac{\partial}{\partial x_1} + (-\tfrac{1}{3}x_1^3 - h_2x_1x_3^2 - q_2x_3^3)\frac{\partial}{\partial x_2} + 0\frac{\partial}{\partial x_3} + (-x_1^2x_3 - h_4x_1x_3^2 - q_4x_3^3)\frac{\partial}{\partial x_4}.$$

```
> computeD(G[1]);
[0, 0, 0,

           3          2             2                  2                             2         3
- a4 x1  - b4 x1  x2 - c4 x1  x3 - e4 x1 x2  - f4 x1 x2 x3 - h4 x1 x3  - k4 x2

                2               2          3
     - l4 x2  x3 - n4 x2 x3  - q4 x3

]
> computeD(G[2]);
                    3                  2          3            2                     2          3
     [0, - 1/3 x1  - h2 x1 x3  - q2 x3 , 0, - x1  x3 - h4 x1 x3  - q4 x3 ]
> computeD(G[3]);
                                                                                                   2    2
                 3                    2                                              2            x1 x3  r4
[0, - 1/3 x1  - x4 c1 x1  + 3 c1 x1 x2 x3 - 1/3 x1 x3  q4 + 1/48 ----------
                                                                                                          2
                                                                                                        c1

                                                                                                  3                         2       2
                                                                       2               r4 q4 x3               x4 r4  x3
         - 1/2 r4 x1 x3 x4 + 3/4 r4 x2 x3  - 1/12 --------- - 1/16 ----------,
                                                                                          c1                         c1

0,
                                    2
        2                    r4 x1 x3                                                           2           3                  2
- x1  x3 + 1/4 --------- - 3 c1 x1 x3 x4 + 9 c1 x2 x3  - q4 x3  - 3/4 r4 x3  x4
                             c1
]
> computeD(G[4]);
                            3
   [0, - 1/3 x1 ,
                2                        2            3                5                       4                            7
       - x1  x2 - e3 x1 x2  - k3 x2  - 1/6 x1  - 1/3 e3 x1  x2 - 1/54 e3 x1

                                     2     3                                  6
            - 1/2 k3 x2  x1  - 1/18 k3 x2 x1 ,

                                   2            3                          4                           7                      2     3
       - e4 x1 x2  - k4 x2  - 1/3 e4 x1  x2 - 1/54 e4 x1  - 1/2 k4 x2  x1

                                                   6
            - 1/18 k4 x2 x1                                                                                    ]
> computeD(G[5]);
                  3                                           2                     2        2           2             3
[0, - 1/3 x1  + i3 x1 x2 x4 - j2 x1 x4  + s3 x2 x4  + i3  x3 x4  - t2 x4

                              4                                 2    3                          5                          2     5
         + 1/6 i3 x1  x4 + 1/6 s3 x4  x1  + 1/2 s3 x4  t2 + 1/2 i3  x4  t3

                              2     3                                   4                                    4
         + 1/2 i3 x1  x4  j2 + 1/2 i3 x1 x4  t2 + 1/2 s3 x4  j2 x1

                         2     4
         + 1/2 i3  x4  j3 x1,

                                                                                                                  2          2
       2                    s3 x1 x2 x4                                                           2     s3  x2 x4                             2
    - x1  x2 - 2 ----------- - i3 x1 x3 x4 - j3 x1 x4  - ---------- - s3 x3 x4
                                  i3                                                                           2
                                                                                                             i3

                                          2     2     3
                     3                s3  x4  x1                        5               2            3                           5
         - t3 x4  - 1/6 ----------- - 1/6 x1  - 1/2 x1  t2 x4  - 1/2 s3 x4  t3
                                              2
                                            i3

                                                                              4                         2     3                            4
                         3               2                     s3 x1  x4      s3 x1  x4  j2      s3 x1 x4  t2
         - 1/2 x1  j2 x4  - 1/3 --------- - ------------- - ------------
                                                                         i3                      i3                       i3
```

$$- 1/2\ i3\ x1^{2}\ x4^{3}\ j3 - 1/2\ i3\ x1\ x4^{4}\ t3 - 1/2\ \frac{s3^{2}\ x4^{4}\ j2\ x1}{i3^{2}}$$

$$- 1/2\ \frac{s3^{2}\ x4^{5}\ t2}{i3^{2}} - 1/2\ s3\ x4^{4}\ j3\ x1,$$

0]

```
> computeD(G[6]);
```

$$[0, - 1/3\ x1^{3} - j2\ x1\ x4^{2} - t2\ x4^{3},$$

$$- m3\ x2^{2}\ x4 - 1/2\ k3\ x2^{2}\ x1^{3} - p3\ x2\ x4^{2} - j3\ x1\ x4^{2} - e3\ x1^{2}\ x2\ j2\ x4^{2}$$

$$- e3\ x1\ x2^{2} - 1/2\ p3\ x4^{4}\ j2\ x1 - 1/18\ k3\ x2\ x1^{6} - 1/2\ k3\ x2\ j2^{2}\ x1^{2}\ x4^{4}$$

$$- 3/2\ k3\ x2^{2}\ t2\ x4^{3} - x1^{2}\ x2 - k3\ x2^{3} - t3\ x4^{3} - g3\ x1\ x2\ x4$$

$$- 1/3\ e3\ x1^{4}\ x2 - 1/6\ x1^{5} - 1/54\ e3\ x1^{7} - 1/2\ x1^{2}\ t2\ x4^{3} - 1/2\ x1^{3}\ j2\ x4^{2}$$

$$- 1/54\ m3\ x4\ x1^{6} - 1/6\ m3\ x4^{7}\ t2^{2} - 1/6\ p3\ x4^{2}\ x1^{3} - 1/2\ p3\ x4^{5}\ t2$$

$$- e3\ x1\ x2\ t2\ x4^{3} - 1/9\ e3\ x1^{5}\ j2\ x4^{2} - 1/9\ e3\ x1^{4}\ t2\ x4^{3}$$

$$- 1/6\ e3\ x1^{3}\ j2^{2}\ x4^{4} - 1/3\ e3\ x1^{2}\ j2\ x4^{5}\ t2 - 1/6\ e3\ x1\ t2^{2}\ x4^{6}$$

$$- 1/6\ g3\ x1^{4}\ x4 - 1/2\ g3\ x1^{2}\ x4^{3}\ j2 - 1/2\ g3\ x1\ x4^{4}\ t2$$

$$- 1/3\ k3\ x2\ x1^{4}\ j2\ x4^{2} - 1/2\ k3\ x2\ t2^{2}\ x4^{6} - 3/2\ k3\ x2^{2}\ j2\ x1\ x4^{2}$$

$$- 1/3\ k3\ x2\ x1^{3}\ t2\ x4^{3} - k3\ x2\ j2\ x1\ x4^{5}\ t2 - 1/3\ m3\ x4\ x2\ x1^{3}$$

$$- m3\ x4^{3}\ x2\ j2\ x1 - m3\ x4^{4}\ x2\ t2 - 1/9\ m3\ x4^{3}\ x1^{4}\ j2 - 1/9\ m3\ x4^{4}\ x1^{3}\ t2$$

$$- 1/6\ m3\ x4^{5}\ j2^{2}\ x1^{2} - 1/3\ m3\ x4^{6}\ j2\ x1\ t2,$$

0]

```
> computeD(G[7]);
```

$$[0, - 1/3\ x1^{3},$$

$$- x1^{2}\ x2 - e3\ x1\ x2^{2} - k3\ x2^{3} - 1/6\ x1^{5} - 1/3\ e3\ x1^{4}\ x2 - 1/54\ e3\ x1^{7}$$

$$- 1/2\ k3\ x2^{2}\ x1^{3} - 1/18\ k3\ x2\ x1^{6},$$

$$- l4\ x2^{2}\ x3 - n4\ x2\ x3^{2} + 1/168399\ q4\ x1^{23}\ k3^{2} - h4\ x1\ x3^{2} - 1/6\ f4\ x1^{6}\ x2$$

$$+ 2/8505\ h4\ x1^{15}\ k3 - 1/6\ f4\ x1^{4}\ x3 - 1/2\ f4\ x1^{3}\ x2^{2} - 1/2\ x1^{2}\ k3\ x2^{3}$$

$$- 1/2\ x1^{3}\ e3\ x2^{2} + 1/5670\ n4\ x1^{15}\ e3 - 1/6\ n4\ e3^{2}\ x1^{2}\ x2^{5}$$

$$- 1/3\ n4\ e3\ x1\ x2^{6}\ k3 - 1/3\ n4\ x1^{5}\ x3\ x2 - 5/18\ n4\ x1^{4}\ e3\ x2^{5}\ k3$$

$$- 1/3\ n4\ x1^{2}\ x2^{5}\ k3 - 5/9\ n4\ x1^{4}\ x3\ e3\ x2^{2} - 7/9\ n4\ x1^{3}\ x3\ k3\ x2^{3}$$

$$- 1/6\ n4\ x1^{6}\ x2^{3}\ e3 - 2/9\ n4\ x1^{5}\ x2^{4}\ k3 - 1/9\ n4\ x1^{5}\ e3^{2}\ x2^{4}$$

$$- 1/6\ n4\ x1^{3}\ k3^{2}\ x2^{6} - n4\ x3\ x1^{2}\ x2^{2} - n4\ x3\ e3\ x1\ x2^{3} - n4\ x3\ k3\ x2^{4}$$

$$- 1/6\ n4\ x1^{4}\ x2^{3} - 1/6\ n4\ k3^{2}\ x2^{7} - 1/6\ n4\ x1^{3}\ x3^{2} - 1/18\ n4\ x1^{7}\ x2^{2}$$

$$- 1/3\ n4\ x1^{3}\ x2^{4}\ e3 - e4\ x1\ x2^{2} - 1/3\ l4\ x2^{3}\ x1^{4}\ e3 - 4/9\ l4\ x2^{4}\ x1^{3}\ k3$$

$$- 1/54\ l4\ x1^{6}\ x3 - 1/54\ l4\ x1^{8}\ x2 - 1/2\ l4\ x2^{3}\ x1^{2} - 1/2\ l4\ x2^{5}\ k3$$

$$- 2/9\ l4\ x2^{2}\ x1^{5} - 1/2\ l4\ x2^{4}\ e3\ x1 - 1/3\ l4\ x2\ x1^{3}\ x3$$

$$- 1/18\ l4\ x1^{7}\ e3\ x2^{2} - 1/9\ l4\ x1^{6}\ k3\ x2^{3} - 1/6\ h4\ x1\ k3^{2}\ x2^{6}$$

$$- h4\ x1^{2}\ x3\ e3\ x2^{2} - h4\ x1\ x3\ k3\ x2^{3} - 1/3\ h4\ x1^{4}\ x2^{3}\ e3$$

$$- 1/3\ h4\ x1^{3}\ x2^{4}\ k3 - 1/6\ h4\ x1^{3}\ e3^{2}\ x2^{4} - 1/3\ h4\ x1^{2}\ e3\ x2^{5}\ k3$$

$$- 5/18\ f4\ x1^{5}\ e3\ x2^{2} - 7/18\ f4\ x1^{4}\ k3\ x2^{3} - 1/2\ f4\ x1\ k3\ x2^{4}$$

$$- h4\ x1^{3}\ x3\ x2 - 1/2\ f4\ x1^{2}\ e3\ x2^{3} - 1/2\ x1^{4}\ x2 + 1/2835\ q4\ x1^{15}$$

$$- 3/2\ q4\ x3^{2}\ e3\ x1\ x2^{2} - 3/2\ q4\ x3^{2}\ x1^{2}\ x2 - 3/2\ q4\ x3^{2}\ k3\ x2^{3}$$

$$- q4\ x3\ x1^{2}\ x2^{4}\ k3 - 1/2\ q4\ x3\ e3^{2}\ x1^{2}\ x2^{4} - q4\ x3\ e3\ x1\ x2^{5}\ k3$$

$$- q4\ x3\ x1^{3}\ x2^{3}\ e3 - 1/2\ q4\ x3\ k3^{2}\ x2^{6} - 1/2\ q4\ x3\ x1^{4}\ x2^{2}$$

$$+ 7/60\ q4\ k3\ e3^{2}\ x1^{5}\ x2^{6} + 1/20\ q4\ k3^{3}\ x2^{8}\ x1^{3} - 1/270\ q4\ k3\ x3^{2}\ x1^{9}$$

$$+ 1/18\ q4\ k3\ x1^{10}\ x2^{3} + \frac{31}{3780}\ q4\ k3\ x1^{13}\ x2^{2} - q4\ k3\ x3^{2}\ x2^{2}\ x1^{3}$$

$$- 2/15\ q4\ k3^{2}\ x3\ x2^{4}\ x1^{6} + 1/270\ q4\ k3^{2}\ x3\ x2^{3}\ x1^{9} - 1/6\ q4\ k3\ x3^{2}\ x2\ x1^{6}$$

$$- 1/2\ q4\ k3^{2}\ x3\ x2^{5}\ x1^{3} - 1/10\ q4\ k3\ x3\ x1^{8}\ x2^{2} + 1/270\ q4\ k3\ x3\ x1^{11}\ x2$$

$$+ 2/15\ q4\ k3^{2}\ e3\ x1^{4}\ x2^{7} + 7/45\ q4\ k3^{2}\ e3\ x1^{7}\ x2^{6} + \frac{11}{378}\ q4\ k3^{2}\ x1^{11}\ x2^{4}$$

$$+ 7/60\ q4\ k3\ e3^{2}\ x1^{8}\ x2^{5} + \frac{23}{756}\ q4\ k3\ e3^{2}\ x1^{11}\ x2^{4} + 1/15\ q4\ k3^{3}\ x2^{7}\ x1^{6}$$

$$+ \frac{13}{540}\ q4\ k3^{3}\ x2^{6}\ x1^{9} + 1/270\ q4\ k3\ x3\ e3\ x1^{10}\ x2^{2}$$

$$- 8/45\ q4\ k3\ x3\ e3\ x1^{7}\ x2^{3} + \frac{13}{270}\ q4\ k3^{2}\ e3\ x1^{10}\ x2^{5} + 7/60\ q4\ k3^{2}\ x1^{8}\ x2^{5}$$

$$- 5/6\ q4\ k3\ x3\ e3\ x1^{4}\ x2^{4} - 1/6\ q4\ x1^{7}\ x3\ x2 - 1/2\ q4\ x1^{6}\ x3\ e3\ x2^{2}$$

$$+ 1/15\ q4\ x1^{8}\ x2^{3}\ e3 + 1/12\ q4\ x1^{7}\ x2^{4}\ k3 + 1/12\ q4\ x1^{7}\ e3^{2}\ x2^{4}$$

$$- 2/3\ q4\ x1^{5}\ x3\ k3\ x2^{3} + 7/60\ q4\ x1^{5}\ k3^{2}\ x2^{6} - 1/3\ q4\ x1^{5}\ x3^{2}$$

$$+ 1/60\ q4\ x1^{9}\ x2^{2} - 1/1701\ q4\ x1^{18}\ e3\ k3\ x2 + 1/5\ q4\ x1^{6}\ e3\ k3\ x2^{5}$$

$$+ 1/6\ q4\ x1^{9}\ e3\ k3\ x2^{4} + \frac{32}{945}\ q4\ x1^{12}\ e3\ k3\ x2^{3} - 1/630\ q4\ x1^{15}\ e3\ k3\ x2^{2}$$

$$- 1/756\ q4\ e3^{2}\ x1^{14}\ k3\ x2^{3} - 1/90\ n4\ x1^{8}\ e3^{2}\ x2^{3} + 1/270\ n4\ x1^{11}\ e3^{2}\ x2^{2}$$

$$+ 1/2430\ n4\ x1^{14}\ e3^{2}\ x2 + 1/36\ q4\ e3^{3}\ x1^{9}\ x2^{4} + \frac{17}{2835}\ q4\ e3^{3}\ x1^{12}\ x2^{3}$$

$$- 1/11340\ q4\ e3^{3}\ x1^{15}\ x2^{2} - 1/18\ q4\ e3^{2}\ x1^{8}\ x3\ x2^{2} + 1/5670\ q4\ e3^{2}\ x1^{14}\ x3$$

$$- 1/3\ q4\ e3^{2}\ x1^{5}\ x3\ x2^{3} - 1/11340\ q4\ e3^{2}\ x1^{16}\ x2 + 1/30\ q4\ e3^{3}\ x1^{6}\ x2^{5}$$

$$- x1^{2}\ x3 - q4\ x3^{3} - k4\ x2^{3} + 2/2835\ q4\ x1^{14}\ k3\ x3 - 1/2835\ q4\ x1^{16}\ k3\ x2$$

$$- 5/2268\ q4\ x1^{14}\ k3^{2}\ x2^{3} - f4\ x1\ x2\ x3 - 1/102060\ n4\ x1^{17}\ e3^{2}$$

$$- 1/135\ n4\ x1^{9}\ e3\ x2^{2} + 1/270\ n4\ x1^{12}\ e3\ x2 - 1/6\ n4\ x1^{6}\ k3\ x3\ x2^{2}$$

$$- 1/162\ n4\ x1^{9}\ k3\ x3\ x2 + 1/2430\ n4\ x1^{12}\ k3\ x3 - 1/54\ n4\ x1^{8}\ k3\ x2^{3}$$

$$+ 7/810\ n4\ x1^{11}\ k3\ x2^{2} + 1/1215\ n4\ x1^{14}\ k3\ x2 - \frac{11}{270}\ n4\ x1^{7}\ k3\ e3\ x2^{4}$$

$$+ \frac{11}{810}\ n4\ x1^{10}\ k3\ e3\ x2^{3} + 1/405\ n4\ x1^{13}\ k3\ e3\ x2^{2}$$

$$+ 1/17010\ q4\ k3^{3}\ x2^{2}\ x1^{21} - \frac{191}{591080490}\ q4\ k3^{3}\ x1^{27} - 2/9\ x1^{6}\ e3\ x2$$

$$+ 1/810\ h4\ x1^{13}\ k3^{2}\ x2^{2} - 1/324\ q4\ e3\ x1^{13}\ k3^{2}\ x2^{4}$$

$$- \frac{22}{8505}\ q4\ e3\ x1^{16}\ k3^{2}\ x2^{3} - 8/8505\ q4\ e3^{2}\ x1^{17}\ k3\ x2^{2}$$

$$- 1/7290\ n4\ e3\ x1^{16}\ k3\ x2 - 1/76545\ n4\ e3\ x1^{19}\ k3 - 2/9\ h4\ x1^{6}\ x3$$

$$- 1/18\ h4\ x1^{8}\ x2 - 1/25515\ q4\ x1^{19}\ e3^{2} - 1/30\ n4\ k3^{2}\ x1^{6}\ x2^{5}$$

$$+ \frac{17}{1620}\ n4\ k3^{2}\ x2^{4}\ x1^{9} + 7/2430\ n4\ k3^{2}\ x2^{3}\ x1^{12} - 1/4860\ n4\ k3^{2}\ x2^{2}\ x1^{15}$$

$$- 1/21870\ n4\ k3^{2}\ x2\ x1^{18} + 1/918540\ n4\ k3^{2}\ x1^{21} - 1/25515\ n4\ x1^{17}\ k3$$

$$- 1/54\ n4\ x1^{8}\ x3 - 2/27\ n4\ e3\ x1^{7}\ x3\ x2 - 1/810\ n4\ e3\ x1^{10}\ x3$$

$$+ 1/405\ q4\ e3\ x1^{13}\ k3\ x3\ x2 - 1/3\ x1^{5}\ k3\ x2^{2} - 1/18\ x1^{8}\ k3\ x2$$

$$- 1/270\ h4\ x1^{11} - 1/945\ q4\ x1^{17}\ k3^{2}\ x2^{2} + 1/102060\ q4\ x1^{20}\ k3^{2}\ x2$$

$$+ 1/18\ q4\ e3^{2}\ x1^{10}\ x2^{3} + \frac{17}{1890}\ q4\ e3^{2}\ x1^{13}\ x2^{2} - 1/51030\ q4\ k3^{2}\ x3\ x1^{18}$$

$$- 1/540\ q4\ k3^{3}\ x2^{5}\ x1^{12} - \frac{11}{5670}\ q4\ k3^{3}\ x2^{4}\ x1^{15} + 1/102060\ q4\ k3^{3}\ x2^{3}\ x1^{18}$$

$$+ 1/270\ q4\ k3^{2}\ x3\ x2^{2}\ x1^{12} - 1/9\ x1^{7} + 1/270\ q4\ x1^{14}\ e3\ x2 - 1/810\ x1^{11}\ k3$$

$$- 1/108\ f4\ x1^{9} - 1/54\ x1^{9}\ e3 - 1/6\ h4\ x1^{5}\ x2^{2} - 1/17010\ q4\ x1^{19}\ k3$$

$$- 1/27\ f4\ x1^{8}\ e3\ x2 - 1/3\ e4\ x1^{4}\ x2 - 1/2\ k4\ x2^{2}\ x1^{3} - 1/18\ k4\ x2\ x1^{6}$$

$$- 2/9\ h4\ x1^{6}\ k3\ x2^{3} - 1/6\ h4\ x1^{7}\ e3\ x2^{2} - 1/1620\ f4\ x1^{11}\ e3$$

$$+ 1/102060\ q4\ e3\ x1^{19}\ k3^{2}\ x2^{2} + 1/25515\ q4\ e3\ x1^{22}\ k3^{2}\ x2$$

$$- 1/90\ q4\ x1^{10}\ x3 + 1/180\ q4\ x1^{12}\ x2 + 1/4860\ l4\ x1^{13}\ e3$$

$$+ 1/102060\ l4\ x1^{15}\ k3 + 1/30\ q4\ e3\ x1^{11}\ x2^{2} - 1/18\ q4\ e3\ x1^{7}\ x3^{2}$$

```
                 7                 9                      13                 11
   - 1/54 e4 x1  - 1/18 q4 e3 x1  x3 x2 + 1/1620 n4 x1   + 1/1620 l4 x1

                     19    2                9     2               12
   - 1/153090 h4 x1   k3  - 1/324 l4 x1  k3 x2  + 1/972 l4 x1   k3 x2

                  3   18                  4   2                   15   2
   - 2/25515 q4 e3  x1   x2 - 2/3 q4 e3 x1  x3  x2 + 1/17010 h4 x1   e3

                  13                  6   2   3             9   2   2
   + 1/4860 f4 x1   k3 - 1/9 h4 x1  e3  x2  - 1/54 h4 x1  e3  x2

                10                   4   2   5              7   2   4
   - 1/54 h4 x1   e3 x2 - 1/6 h4 x1  k3  x2  - 2/45 h4 x1  k3  x2

                 10   2   3           7                        9       2
   + 1/810 h4 x1   k3  x2  - 1/9 h4 x1  k3 x3 x2 - 1/30 h4 x1  k3 x2

                 12                  5          4             8         3
   + 1/810 h4 x1   k3 x2 - 5/18 h4 x1  k3 e3 x2  - 8/135 h4 x1  k3 e3 x2

                 11          2           4        2              10
   + 1/810 h4 x1   k3 e3 x2  - 2/3 h4 x1  k3 x3 x2  - 1/405 h4 x1   k3 x3

        23         2   23                   7      2              10
   + ------- q4 e3  x1   k3 - 1/12 f4 x1  k3 x2  - 1/324 f4 x1   k3 x2
     5051970

                5                       8                        14
   - 4/9 h4 x1  e3 x2 x3 - 1/27 h4 x1  e3 x3 + 1/1215 h4 x1   e3 k3 x2

]
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We promised that we would come back to the last form of the classification theorem. We are going to prove that for almost all values of the coefficients this F cannot be written as the exponent of some locally nilpotent derivation D. So we can restrict ourselves to $\mathbb{C}$ in stead of K. Furthermore we recall a theorem by Baire:

Theorem 5.3 *Let $f_1, f_2, \ldots$ be a sequence of non-zero polynomials in $\mathbb{C}[x_1, \ldots, x_n]$. Then*

$$\mathbb{C}^n \not\subset \bigcup_{m=1}^{\infty} V(f_m)$$

where $V(f_m)$ denotes the set of zeroes of f_m.

We also present the following lemma.

Lemma 5.4 *Let F be the map*

$$\left(\begin{array}{l} x_1 \\ x_2 - \frac{1}{3}x_1^3 \\ x_3 - x_1^2 x_2 - e_3 x_1 x_2^2 + g_4 x_1 x_2 x_3 - k_3 x_2^3 + m_4 x_2^2 x_3 + g_4^2 x_2^2 x_4 \\ x_4 \ - x_1^2 x_3 - e_4 x_1 x_2^2 - \frac{2m_4}{g_4} x_1 x_2 x_3 - g_4 x_1 x_2 x_4 - k_4 x_2^3 \\ \qquad - \frac{m_4^2}{g_4^2} x_2^2 x_3 - m_4 x_2^2 x_4 \end{array} \right).$$

Then for $n \geq 0$ the term

$$\left(\left(\tfrac{1}{3}\right)^n x_1^{4n} x_2^2 x_4 - n \left(\tfrac{1}{3}\right)^{n+1} x_1^{4n+3} x_2 x_4 \right) g_4^{n+2}$$

appears in the third row of $\Delta_H^{n+1}(X)$.

Proof: We give a sketch of the proof with induction to n. If $n = 0$ we have that

$$\Delta_H^{n+1}(X)|_3 = \Delta_H(X)|_3 = -x_1^2 x_2 - e_3 x_1 x_2^2 + g_4 x_1 x_2 x_3 - k_3 x_2^3 + m_4 x_2^2 x_3 + g_4^2 x_2^2 x_4.$$

It is easy to see that this satisfies the claim of the lemma in case $n = 0$. Now let the induction hypothesis be that the lemma is true for certain n. So we have that

$$\left(\left(\tfrac{1}{3}\right)^n x_1^{4n} x_2^2 x_4 - n \left(\tfrac{1}{3}\right)^{n+1} x_1^{4n+3} x_2 x_4 \right) g_4^{n+2}$$

appears in the third row of $\Delta_H^{n+1}(X)$. If we now compute[2] $\Delta_H^{n+2}(X)|_3$, we see that there are three steps that contribute to the desired term:

- $\left(\frac{1}{3}\right)^n x_1^{4n} x_2^2 x_4 g_4^{n+2}$ with the substitutions:
  - All x_1-s are substituted by x_1.
  - One x_2 is substituted by x_2.
  - One x_2 is substituted by $-\frac{1}{3}x_1^3$.
  - The x_4 is substituted by $-g_4 x_1 x_2 x_4$.

  This gives the term:

  $$\left(\tfrac{1}{3}\right)^n x_1^{4n} x_2 \left(-\tfrac{1}{3}x_1^3\right)\left(-g_4 x_1 x_2 x_4\right) g_4^{n+2}$$

  which can be reordered to

  $$\left(\tfrac{1}{3}\right)^{(n+1)} x_1^{4(n+1)} x_2^2 x_4 g_4^{(n+1)+2}.$$

- $\left(\frac{1}{3}\right)^n x_1^{4n} x_2^2 x_4 g_4^{n+2}$ with the substitutions:
  - All x_1-s are substituted by x_1.
  - All x_2-s are substituted by $-\frac{1}{3}x_1^3$.
  - The x_4 is substituted by $-g_4 x_1 x_2 x_4$.

  This gives the term:

  $$\left(\tfrac{1}{3}\right)^n x_1^{4n} \left(-\tfrac{1}{3}x_1^3\right)^2 \left(-g_4 x_1 x_2 x_4\right) g_4^{n+2}$$

  and this can be reordered to

  $$-\left(\tfrac{1}{3}\right)^{(n+1)+1} x_1^{4(n+1)+3} x_2 x_4 g_4^{(n+1)+2}.$$

- $-n\left(\frac{1}{3}\right)^{n+1} x_1^{4n+3} x_2 x_4 g_4^{n+2}$ with the substitutions:
  - All x_1-s are substituted by x_1.
  - The x_2 is substituted by $-\frac{1}{3}x_1^3$.
  - The x_4 is substituted by $-g_4 x_1 x_2 x_4$.

  This gives the term:

  $$-n\left(\tfrac{1}{3}\right)^n x_1^{4n+3} \left(-\tfrac{1}{3}x_1^3\right)\left(-g_4 x_1 x_2 x_4\right) g_4^{n+2}$$

  and this can be reordered to

  $$-n\left(\tfrac{1}{3}\right)^{(n+1)+1} x_1^{4(n+1)+3} x_2 x_4 g_4^{(n+1)+2}.$$

If we add these three parts we get:

$$\left(\left(\tfrac{1}{3}\right)^{(n+1)} x_1^{4(n+1)} x_2^2 x_4 - (n+1)\left(\tfrac{1}{3}\right)^{(n+1)+1} x_1^{4(n+1)+3} x_2 x_4\right) g_4^{(n+1)+2}.$$

Together with the step for $n = 0$ this proves the lemma. □

Arguing in a similar way one can prove that the term

$$\left(\left(\tfrac{1}{3}\right)^n x_1^{4n} x_2^2 x_4 - n\left(\tfrac{1}{3}\right)^{n+1} x_1^{4n+3} x_2 x_4\right) g_4^{n+2}$$

does not cancel with any other monomial. As a consequence we obtain:

[2] $\Delta_H^{n+2}(X)|_3$ stands for the third row of the vector $\Delta_H^{n+2}(X)$.

Corollary 5.5 *For all $p \in \mathbb{N}$ we have that:*

$$0 \neq \Delta_H^p(X)|_3 \in \mathbb{C}[e_3, e_4, k_3, k_4, g_4, m_4][g_4^{-1}][x_1, x_2, x_3, x_4].$$

This leads to the following theorem:

Theorem 5.6 *Let $\tilde{F}$ be the map*

$$\begin{pmatrix} x_1 \\ x_2 - \frac{1}{3}x_1^3 \\ x_3 - x_1^2 x_2 - \tilde{e}_3 x_1 x_2^2 + \tilde{g}_4 x_1 x_2 x_3 - \tilde{k}_3 x_2^3 + \tilde{m}_4 x_2^2 x_3 + \tilde{g}_4^2 x_2^2 x_4 \\ x_4 - x_1^2 x_3 - \tilde{e}_4 x_1 x_2^2 - \frac{2\tilde{m}_4}{\tilde{g}_4} x_1 x_2 x_3 - \tilde{g}_4 x_1 x_2 x_4 - \tilde{k}_4 x_2^3 \\ \quad - \frac{\tilde{m}_4^2}{\tilde{g}_4^2} x_2^2 x_3 - \tilde{m}_4 x_2^2 x_4 \end{pmatrix}.$$

Then there is no locally nilpotent derivation D such that $\tilde{F} = \exp(D)$.

Proof: From Baire's theorem (theorem 5.3) it follows that there exists a point

$$(\tilde{e}_3, \tilde{e}_4, \tilde{k}_3, \tilde{k}_4, \tilde{g}_4, \tilde{m}_4) \in \mathbb{C}^6$$

with $\tilde{g}_4 \neq 0$ and such that $\Delta_{\tilde{H}}^p(X)|_3$ (where $\tilde{H}$ is H evaluated at this particular point) is a non-zero polynomial in $\mathbb{C}[x_1, x_2, x_3, x_4]$ for all $p \in \mathbb{N}$. Lemma 5.1 now completes the proof. $\square$

But we do have some positive news concerning this particular map.

Theorem 5.7 *Let F be the map*

$$\begin{pmatrix} x_1 \\ x_2 - \frac{1}{3}x_1^3 \\ x_3 - x_1^2 x_2 - e_3 x_1 x_2^2 + g_4 x_1 x_2 x_3 - k_3 x_2^3 + m_4 x_2^2 x_3 + g_4^2 x_2^2 x_4 \\ x_4 - x_1^2 x_3 - e_4 x_1 x_2^2 - \frac{2m_4}{g_4} x_1 x_2 x_3 - g_4 x_1 x_2 x_4 - k_4 x_2^3 \\ \quad - \frac{m_4^2}{g_4^2} x_2^2 x_3 - m_4 x_2^2 x_4 \end{pmatrix}.$$

Then

$$F = \exp(D_1) \circ \exp(D_2)$$

where D_1 and D_2 are the locally nilpotent derivations

$$\begin{aligned} D_1 &= -\tfrac{1}{3}x_1^3 \frac{\partial}{\partial x_2} \\ D_2 &= \Big(\tfrac{1}{2} g_4 x_1^3 x_2^2 + \tfrac{1}{2}(g_4 e_3 + m_4) x_1^2 x_2^3 - x_1^2 x_2 + \tfrac{1}{2}\left(m_4 e_3 + g_4 k_3 + g_4^2 e_4\right) x_1 x_2^4 \\ &\quad - e_3 x_1 x_2^2 + g_4 x_1 x_2 x_3 + \tfrac{1}{2}\left(g_4^2 k_4 + m_4 k_3\right) x_2^5 - k_3 x_2^3 + m_4 x_2^2 x_3 + g_4^2 x_2^2 x_4\Big) \frac{\partial}{\partial x_3} \\ &\quad + \Big(-\tfrac{1}{2} x_1^4 x_2 + \left(-\tfrac{1}{2} e_3 - \tfrac{m_4}{g_4}\right) x_1^3 x_2^2 + \left(-\tfrac{m_4 e_3}{g_4} - \tfrac{m_4^2}{2g_4^2} - \tfrac{1}{2}k_3 - \tfrac{1}{2} g_4 e_4\right) x_1^2 x_2^3 \\ &\quad - x_1^2 x_3 + \left(-\tfrac{m_4 k_3}{g_4} - \tfrac{1}{2} m_4 e_4 - \tfrac{m_4^2 e_3}{g_4^2} - \tfrac{1}{2} g_4 k_4\right) x_1 x_2^4 - e_4 x_1 x_2^2 - \tfrac{2m_4}{g_4} x_1 x_2 x_3 \\ &\quad - g_4 x_1 x_2 x_4 + \tfrac{1}{2}\left(m_4 k_4 - \tfrac{m_4^2 k_3}{g_4^2}\right) x_2^5 - k_4 x_2^3 - \tfrac{m_4^2}{g_4^2} x_2^2 x_3 - m_4 x_2^2 x_4\Big) \frac{\partial}{\partial x_4}. \end{aligned}$$

Proof: It is easy to verify this by computation. $\square$

Note that D_1 and D_2 do not commute. If they would have, also this F could have been written as an exponent of a local nilpotent derivation D, namely $D = D_1 + D_2$.

Furthermore we can note that the results of this section have the following consequence:

Theorem 5.8 *Let F be a cubic homogeneous map that satisfies the Jacobian hypothesis. Then*

$$F = \exp(D_1) \circ \cdots \circ \exp(D_n)$$

for some $n \leq 2$.

Proof: Combining the results from theorem 5.2 and theorem 5.7 we see that this theorem holds with $n = 1$ or $n = 2$. □

This theorem suggests that the following weaker form of the tame generators conjecture is true (see for instance [Joseph 76], [Essen 92a] and [Essen 92b]):

Conjecture 5.9 *Every element of $Aut_{\mathbb{C}}\mathbb{C}[X]$ is a finite product of linear automorphisms and automorphisms of the form $\exp(D)$, where D is locally nilpotent on $\mathbb{C}[X]$.*

5.3 Strong nilpotency

If we look at the quadratic homogeneous polynomial maps, $F = X - Q$, we know that nilpotency implies strong nilpotency for these maps in dimensions two, three and four. See [Meisters 91, Lemma 1,page 6]. For cubic homogeneous polynomial maps we don't know an analogon of this theorem. But now that we have a complete description of all cubic homogeneous polynomial maps which satisfy the Jacobian hypothesis, i.e. have nilpotent JH, we can easily check this property.

In dimension three we only have one representative, described in [Wright 93]. It is triangular, hence strong nilpotent.

Theorem 5.10 *For all cubic homogeneous polynomial maps $F = X - H : K^3 \to K^3$ we have that JH is nilpotent implies JH is strong nilpotent.*

Proof: By [Wright 93] we know that for each cubic homogeneous polynomial map $F = X - H$ there exists $T \in GL_3(K)$ with

$$T^{-1}HT = \begin{pmatrix} 0 \\ \frac{1}{3}x_1^3 \\ x_1^2x_2 + d_3x_1x_2^2 + g_3x_2^3 \end{pmatrix}.$$

And this last map is on triangular form, so the theorem follows. □

In dimension four we had eight representatives and most of them were a little bit too complex to check by hand without making computation errors. So we did this class by computer. The result of this examination is presented in the following theorem.

Theorem 5.11 *There exist cubic homogeneous polynomial maps $F = X - H : K^4 \to K^4$ with JH is nilpotent but with JH is not strong nilpotent.*

Proof: Though it suffices to give only one example that is not strong nilpotent we shall give here the complete output of the computercheck.[3] Since it is very obvious that the first representative of theorem 2.7 is on triangular form, and thus has a strong nilpotent Jacobian matrix JH, we shall not regard this representative, but only examine the other seven.

[3]This output is the result of the procedure `strongnilclass4` which is based on the procedure `strongnilpotent`.

From solution two, four and seven in the (beautiful) Maple output given below it is clear that the nilpotency of JH does not imply the strong nilpotency of JH. □

```
               ########## JH has rank two and JH^2.X=0 ##########
                                solution 1
                           Strong nilpotency of
 [         0          0                         0                        0 ]
 [                                                                         ]
 [      2         2                                     2                  ]
 [    x1  + h2 x3     0            2 h2 x1 x3 + 3 q2 x3                  0 ]
 [                                                                         ]
 [         0          0                         0                        0 ]
 [                                                                         ]
 [                         2                   2                           ]
 [ x3 (2 x1 + h4 x3)  0  x1  + h4 x1 x3 + q4 x3  + x3 (x1 h4 + 2 q4 x3)  0 ]
                                   true


                                solution 2
                           Strong nilpotency of
                               [0, 0, 0, 0]
                                     2
         %2          %1 (8 c1 x1 + 12 c1  x4 - x3 r4)
 [1/12 ---- + 1/48 -------------------------------, - 3/4 %1 x3,
        c1                           2
                                   c1

                                     2
     r4 %2         %1 (- x1 r4 - 36 c1  x2 + 3 r4 c1 x4 + 8 c1 q4 x3)
 1/48 ----- + 1/48 --------------------------------------------------,
         2                                      2
       c1                                     c1

                     2
      %1 (12 x1 c1  + 3 x3 r4 c1)
 1/48 ---------------------------]
                     2
                   c1
                               [0, 0, 0, 0]
                            2
      x3 (8 c1 x1 + 12 c1  x4 - x3 r4)               2
 [1/4 --------------------------------, - 9 c1 x3 ,
                     c1
                                      2
      %2         x3 (- x1 r4 - 36 c1  x2 + 3 r4 c1 x4 + 8 c1 q4 x3)
 1/4 ---- + 1/4 --------------------------------------------------,
      c1                              c1
                     2
      x3 (12 x1 c1  + 3 x3 r4 c1)
 1/4 ---------------------------]
                   c1
%1 :=                            4 c1 x1 + x3 r4
                 2           2                          2
%2 := 4 c1 x1  + 12 c1  x1 x4 - x3 x1 r4 - 36 x3 c1  x2 + 3 x3 r4 c1 x4
                   2
     + 4 c1 q4 x3
                                   false
```

########## JH has rank two and JH^2.X<>0 ##########

solution 3

Strong nilpotency of

$$\begin{bmatrix} 0 & 0 & 0 & 0 \\ x1^{2} & 0 & 0 & 0 \\ x2\,(e3\,x2 + 2\,x1) & k3\,x2^{2} + e3\,x1\,x2 + x1^{2} + x2\,(2\,k3\,x2 + e3\,x1) & 0 & 0 \\ e4\,x2^{2} & 2\,x2\,(k4\,x2 + e4\,x1) + k4\,x2^{2} & 0 & 0 \end{bmatrix}$$

true

solution 4

Strong nilpotency of

$[0, 0, 0, 0]$

$$[x1^{2} - i3\,x2\,x4 + j2\,x4^{2},\ - i3\,x1\,x4 - s3\,x4^{2},\ - x4^{2}\,i3^{2},\ - i3\,x1\,x2 + 2\,j2\,x1\,x4 - 2\,s3\,x2\,x4 - 2\,i3^{2}\,x3\,x4 + 3\,t2\,x4^{2}]$$

$$\left[\frac{2\,x1\,x2\,i3^{2} + 2\,s3\,x2\,x4\,i3 + i3^{3}\,x3\,x4 + j3\,x4^{2}\,i3^{2}}{i3^{2}},\ \frac{x1^{2}\,i3^{2} + 2\,s3\,x1\,x4\,i3 + s3^{2}\,x4^{2}}{i3^{2}},\ \frac{i3^{3}\,x1\,x4 + s3\,x4^{2}\,i3^{2}}{i3^{2}},\ (2\,s3\,x1\,x2\,i3 + i3^{3}\,x1\,x3 + 2\,j3\,x1\,x4\,i3^{2} + 2\,s3^{2}\,x2\,x4 + 2\,s3\,x3\,x4\,i3^{2} + 3\,t3\,x4^{2}\,i3^{2}) \big/ i3^{2}\right]$$

$[0, 0, 0, 0]$

false

solution 5

Strong nilpotency of

$[0, 0, 0, 0]$

$$[x1^{2} + j2\,x4^{2},\ 0,\ 0,\ 2\,j2\,x1\,x4 + 3\,t2\,x4^{2}]$$

$$[2\,x2\,x1 + e3\,x2^{2} + g3\,x2\,x4 + j3\,x4^{2},\ x1^{2} + 2\,e3\,x1\,x2 + g3\,x1\,x4 + 3\,k3\,x2^{2} + 2\,m3\,x2\,x4 + p3\,x4^{2},\ 0,\ g3\,x1\,x2 + 2\,j3\,x1\,x4 + m3\,x2^{2} + 2\,p3\,x2\,x4 + 3\,t3\,x4^{2}]$$

$[0, 0, 0, 0]$

true

```
########## JH has rank three and JH^3.X<>0 ##########

                    solution 6

               Strong nilpotency of

                   [0, 0, 0, 0]

                      2
                   [x1 , 0, 0, 0]

                     2              2
[x2 (e3 x2 + 2 x1), k3 x2  + e3 x1 x2 + x1  + x2 (2 k3 x2 + e3 x1), 0, 0]

                   2                  2
  [2 x3 x1 + e4 x2  + f4 x2 x3 + h4 x3 ,

                                       2                    2
     2 e4 x1 x2 + f4 x1 x3 + 3 k4 x2  + 2 l4 x2 x3 + n4 x3 ,

       2                                  2                       2
     x1  + f4 x1 x2 + 2 h4 x1 x3 + l4 x2  + 2 n4 x2 x3 + 3 q4 x3 , 0]

                       true



                    solution 7

               Strong nilpotency of

                   [0, 0, 0, 0]

                      2
                   [x1 , 0, 0, 0]

                              2                                  2
[x2 (2 x1 + e3 x2 - g4 x3), x1  + e3 x1 x2 - g4 x1 x3 + k3 x2  - x2 m4 x3

           2                                          2
    - x2 g4  x4 + x2 (e3 x1 + 2 k3 x2 - m4 x3 - g4  x4), x2 (- g4 x1 - x2 m4)

          2   2
    , - x2  g4 ]

    2                  2   2                       3
 2 g4  x1 x3 + e4 x2  g4  + 2 m4 x2 x3 g4 + g4  x2 x4                       2
[-----------------------------------------------------, (2 e4 x1 x2 g4
                            2
                          g4

                       3                    2   2         2                     2
    + 2 m4 x1 x3 g4 + g4  x1 x4 + 3 k4 x2  g4  + 2 m4  x2 x3 + 2 m4 x2 x4 g4 )

      /    2     2   2                       2   2     3               2   2
     /  g4 , g4  x1  + 2 m4 x1 x2 g4 + m4  x2    g4  x1 x2 + m4 x2  g4
    /         --------------------------------, ----------------------]
                              2                           2
                            g4                          g4

                       false
```

The reason why we included all results and not only one solution that was not strong nilpotent, is that we now can see that in fact in all three classes with solutions[4] there are both examples of strong nilpotent maps and of not-strong nilpotent maps.

[4]The three classes meant here are the three cases with solutions of the four cases we regarded separately in chapter 2. They are separated in the output given above by '##########'-s.

5.4 Iteration of H

In [Meisters 91] the following question is raised by Meisters:

Question 5.12 *Let $F = X - H : K^5 \to K^5$ be a cubic homogeneous map, with nilpotent JH but not strongly nilpotent. Let m denote the nilpotency index of JH. Do we have that $H^{m-2} = 0$?*

With H^m we mean m times the composition of H:

$$H^m = \underbrace{H \circ H \circ \cdots \circ H}_{m}.$$

We first checked the situation in dimension four. We wrote a procedure `iterationtest` that gives a list with three entries as result: the first entry tells us whether the analogon of the question 5.12 is true, the second gives the nilpotency index and the third gives the iteration index, i.e. the number of iterations of H you need to get the constant map $[0, \ldots, 0]$. Of course, the first entry is very redundant from the last two: just subtract the third from the second entry and compare it with two.

This was the result: (`G` holds all eight representatives of theorem 2.7)

```
> for i from 1 to 8 do iterationtest(G[i]) od;
                                [false, 2, 2]
                                [false, 2, 2]
                                [false, 3, 2]
                                [false, 3, 3]
                                [false, 3, 2]
                                [false, 3, 3]
                                [false, 4, 4]
                                [false, 4, 3]
```

From this result we see immediately that question 5.12 has certainly not a positive answer in dimension four. But we do have the following result:

Theorem 5.13 *Let $F = X - H : K^4 \to K^4$ be a cubic homogeneous map that satisfies the Jacobian hypothesis. Let m denote its nilpotency index. Then we have that either $H^{m-1} = 0$ or $H^m = 0$.*

If we now look back at the results we found in section 5.3 we see that the three cubic homogeneous maps F that have an H such that H^{m-1} is identically zero, are exactly the three maps where JH is not strong nilpotent. So if we restrict ourselves to the not strongly nilpotent maps, we actually do get a very similar result as question 5.12:

Theorem 5.14 *Let $F = X - H : K^4 \to K^4$ be a cubic homogeneous map, with nilpotent JH but not strongly nilpotent. Let m denote the nilpotency index of JH. Then $H^{m-1} = 0$.*

Proof: There are only three maps that satisfy the conditions of this theorem. It is easy to verify that they also satisfy the claim of this theorem. □

Though the existence of theorem 5.13 may seem to justify that the answer to question 5.12 is positive it actually does the opposite. It provides us with examples of polynomial maps $F : K^4 \to K^4$ with a Jacobian matrix JH that is not strongly nilpotent. And with these examples we can build new maps $G : K^5 \to K^5$ which satisfy the conditions of question 5.12 but do not satisfy the claim of it.

Example 5.15 *Let F be the map*

$$\begin{pmatrix} x_1 \\ x_2 - \frac{1}{3}x_1^3 \\ x_3 - x_1^2 x_2 - e_3 x_1 x_2^2 + g_4 x_1 x_2 x_3 - k_3 x_2^3 + m_4 x_2^2 x_3 + g_4^2 x_2^2 x_4 \\ x_4 - x_1^2 x_3 - e_4 x_1 x_2^2 - \frac{2m_4}{g_4} x_1 x_2 x_3 - g_4 x_1 x_2 x_4 - k_4 x_2^3 \\ \quad - \frac{m_4^2}{g_4^2} x_2^2 x_3 - m_4 x_2^2 x_4 \end{pmatrix}$$

then we have that $\det(JF) = 1$ *and JH is nilpotent but not strongly nilpotent. The nilpotency index of JH is four. Furthermore $H \circ H \circ H = [0, 0, 0, 0]$, so the iteration index is three. In a trivial way we can extend this map F to $G : K^5 \to K^5$ by adding a fifth component to F, namely $x_5 - H_5$, with H_5 equal to zero. If we denote the homogeneous part of G by $\tilde{H}$ we get that $J\tilde{H}$ is still nilpotent and still not strongly nilpotent. Furthermore we have that the nilpotency index of $J\tilde{H}$ is still four and its iteration index still three.*

This observation leads to the following theorem:

Theorem 5.16 *The answer to question 5.12 is: no.*

Proof: Follows immediately from example 5.15. □

5.5 Differential equations

If we look at the system of differential equations given by

$$\begin{cases} \dot{y_1}(t) = P_1(y_1(t), \dots, y_n(t)) \\ \quad \vdots \\ \dot{y_n}(t) = P_n(y_1(t), \dots, y_n(t)) \end{cases}$$

(where $P_i \in \mathbb{R}[x_1, \dots, x_n]$) or abbreviated given by

$$\dot{y} = P(y(t))$$

and with initial values $y(0) = x$ for some $x \in \mathbb{R}^n$ and $P_i(0) = 0$ for all $i \in \{1, \dots, n\}$, then we see immediately that the constant function $y(t) = 0$ is a solution of this system. But now the question arises whether this constant solution is globally asymptotic stable or not.

In some cases we know the answer to this question:

Theorem 5.17 *Let $F = X - H : \mathbb{R}^n \to \mathbb{R}^n$ be a polynomial automorphism and let $P(y(t)) = -(JF)^{-1}(y(t))F(y(t))$. Then the constant solution $y(t) = 0$ is a globally asymptotic stable solution of $\dot{y} = P(y(t))$.*

Proof: See [Meisters et al. 92, page 4]. □

From this theorem it follows that our maps from theorem 2.7 also represent a class of systems of differential equations with globally asymptotic stable solutions.

Chapter 6

Dimension five

6.1 Introduction

Naturally, the good results we obtained in dimension four encouraged us to examine dimension five also. However in dimension four the most general cubic homogeneous map had $20 \times 4 = 80$ variables. But in dimension five we have $35 \times 5 = 175$ variables. So there is an enormous increment of complexity of the corresponding system of equations. Even if we used -like we did in dimension four- a matrix with some initial values we obtained by a linear invertible map, we still had 150 variables left. So this didn't give us much hope or expectation for this case being solved. But it turned out to be even worse. Indeed we were not able to solve this system, but not since it was simply too large to solve. It was even too large to draw up! At least it was for the method described in [Wright 93]. After a week of computation we still hadn't found all w_r^β-s. We had problems concerning both the memory size and the processor time. This was probably caused by the enormous number of different β-s on one side and the number of determinants of 4×4 and 5×5 matrices we had to compute on the other side.

So we made a further restriction. Instead of cubic homogeneous maps we examined linear cubic homogeneous maps.

Definition 6.1 *An n-dimensional polynomial map F is called* linear cubic homogeneous *if it has the form*

$$F_i = x_i - H_i$$

where H_i is homogeneous of degree three and linear in each x_j, for $i, j = 1, \ldots, n$.

Since this is clearly a subclass of the cubic homogeneous class, we could use the same method to compute all w_r^β-s. But in this case we had only $10 \times 5 = 50$ variables. And since there were lots of zero columns in the matrices of which the determinants had to be computed, these computations were simpler both in storage and in time aspects. In particular, we were able to draw up this system. Unfortunately, we couldn't solve this system completely. But we did find several solutions and we shall describe them in the next section.

6.2 Linear cubic homogeneous maps

The most general linear cubic homogeneous map is given by

$$F = X - H$$

where for $u = 1, \ldots, 5$

$$\begin{aligned} H_u &= a_u x_1 x_2 x_3 + b_u x_1 x_2 x_4 + c_u x_1 x_2 x_5 + d_u x_1 x_3 x_4 + e_u x_1 x_3 x_5 + \\ &\quad f_u x_1 x_4 x_5 + g_u x_2 x_3 x_4 + h_u x_2 x_3 x_5 + i_u x_2 x_4 x_5 + j_u x_3 x_4 x_5. \end{aligned}$$

With this general map we computed the corresponding system of equations. We tried to solve it in the usual way, i.e. look for easy equations with our procedures `es` solve them and substitute their (mostly unique) solutions in the original system. However we were not able to find a complete solution of this system. At least not within an acceptable period of time. After a search of approximately two weeks, we had found several solutions, but there was no accurate estimation of how much more we had to solve. It seemed as if we had only solved a minor part of the system at the moment we stopped our examination. But in the solutions we did find we could see some general patterns, though.

In fact we found 222 solutions. If we look at these solutions we see that they all have one, two or three H_i-s equal to zero.

If we look at the maps with three zeroes, we see that we have maps of the form described in chapter 3: a map of dimension two. So we can use theorem 3.1 to describe these maps.

Example 6.2 *Let F be the map*

$$\begin{pmatrix} x_1 - \frac{b_1 h_1}{i_1} x_1 x_2 x_3 - b_1 x_1 x_2 x_4 - \frac{b_1 j_1}{i_1} x_1 x_3 x_4 - g_1 x_2 x_3 x_4 \\ \quad - h_1 x_2 x_3 x_5 - i_1 x_2 x_4 x_5 - j_1 x_3 x_4 x_5 \\ x_2 \\ x_3 \\ x_4 \\ x_5 + \frac{b_1^2 h_1}{i_1^2} x_1 x_2 x_3 + \frac{b_1^2}{i_1} x_1 x_2 x_4 + \frac{b_1^2 j_1}{i_1^2} x_1 x_3 x_4 - g_5 x_2 x_3 x_4 \\ \quad + \frac{b_1 h_1}{i_1} x_2 x_3 x_5 + b_1 x_2 x_4 x_5 + \frac{b_1 j_1}{i_1} x_3 x_4 x_5 \end{pmatrix}$$

and regard this as a polynomial map $(H_1, H_2) \in K[x_1, x_5] \times K[x_1, x_5]$. With the notation of theorem 3.1 we can write this two dimensional map as

$$\begin{pmatrix} H_1 \\ H_2 \end{pmatrix} = \begin{pmatrix} -\mu_2 f(\mu_1 x_1 + \mu_2 x_5) + c_1 \\ \mu_1 f(\mu_1 x_1 + \mu_2 x_5) + c_2 \end{pmatrix}.$$

where

$$\begin{aligned} \mu_1 &= b_1 \\ \mu_2 &= i_1 \\ c_1 &= g_1 x_2 x_3 x_4 \\ c_2 &= g_5 x_2 x_3 x_4 \\ f(T) &= -\frac{h_1 x_2 x_3 + x_2 x_4 i_1 + j_1 x_3 x_4}{i_1^2} T. \end{aligned}$$

This was one arbitrary example of the 32 solutions with three H_i-s equal to zero.

If we now take a look at the solutions with two homogeneous parts equal to zero, we see that this is the major part of the 222 solutions, namely 141. Although they do have three parts not equal to zero, when we computed the rank of the corresponding JH it turned out that this rank was always equal to two. So there had to be some C with $C \circ F \circ C^{-1} = F'$ where F' has only two real homogeneous parts, and hence we were back in the two dimension case.

Example 6.3 *Let F be the map*

$$\begin{pmatrix} x_1 \\ x_2 - a_2x_1x_2x_3 - d_2x_1x_3x_4 \\ x_3 \\ x_4 - a_4x_1x_2x_3 - d_4x_1x_3x_4 - e_4x_1x_3x_5 \\ x_5 + \frac{2d_2a_4a_2+a_2^3+d_2a_4d_4}{d_2e_4}x_1x_2x_3 + \frac{a_2d_4+d_2a_4+a_2^2+d_4^2}{e_4}x_1x_3x_4 \\ \quad + (a_2+d_4)x_1x_3x_5 \end{pmatrix}$$

then $\operatorname{rank}(JH) = 2$. *Furthermore if we let C be the map*

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 + \frac{d_2a_4+a_2^2}{d_2e_4}x_2 + \frac{a_2+d_4}{e_4}x_4 \end{pmatrix}$$

we get

$$C \circ F \circ C^{-1} = \begin{pmatrix} x_1 \\ x_2 - a_2x_1x_2x_3 - d_2x_1x_3x_4 \\ x_3 \\ x_4 + \frac{a_2^2}{d_2}x_1x_2x_3 + a_2x_1x_3x_4 - e_4x_1x_3x_5 \end{pmatrix}$$

which has only two non-zero homogeneous components.

Note that in example 6.3 the map $C \circ F \circ C^{-1}$ is again a linear cubic homogeneous map. Unfortunately this does not hold in general as we can learn from the next example:

Example 6.4 *Let F be the map*

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 + b_4x_1x_2x_3 - b_3x_1x_2x_4 \\ x_4 + \frac{b_4^2}{b_3}x_1x_2x_3 - b_4x_1x_2x_4 \\ x_5 - a_5x_1x_2x_3 - b_5x_1x_2x_4 - d_5x_1x_3x_4 - g_5x_2x_3x_4 \end{pmatrix}$$

and let C be the map

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 - \frac{b_3}{b_4}x_4 \\ x_4 \\ x_5 \end{pmatrix}.$$

Then we have

$$C \circ F \circ C^{-1} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 + \frac{b_4^2}{b_3}x_1x_2x_3 \\ x_5 - a_5x_1x_2x_3 + (-\frac{a_5b_3}{b_4} - b_5)x_1x_2x_4 - d_5x_1x_3x_4 \\ \quad - \frac{d_5b_3}{b_4}x_1x_4^2 - g_5x_2x_3x_4 - \frac{g_5b_3}{b_4}x_2x_4^2 \end{pmatrix}$$

and we see that this map is not linear cubic homogeneous.

Now if we take a look at the third class of maps we found -the ones with only one homogeneous part equal to zero- we see that in this set of 49 maps there are 48 maps with $\text{rank}(JH) = 2$ and one with $\text{rank}(JH) = 1$. In particular we see that there is no map with $\text{rank}(JH) \geq 3$. Let us first take a look at the one example of rank one.

Example 6.5 *Let F be the map*

$$\begin{pmatrix} x_1 - \frac{e_4 g_1}{h_4} x_1 x_3 x_4 + \frac{h_4 e_2}{e_4} x_1 x_3 x_5 - g_1 x_2 x_3 x_4 + \frac{h_4^2 e_2}{e_4^2} x_2 x_3 x_5 \\ x_2 + \frac{e_4^2 g_1}{h_4^2} x_1 x_3 x_4 - e_2 x_1 x_3 x_5 + \frac{e_4 g_1}{h_4} x_2 x_3 x_4 - \frac{h_4 e_2}{e_4} x_2 x_3 x_5 \\ x_3 \\ x_4 + \frac{e_4^3 g_1}{e_2 h_4^2} x_1 x_3 x_4 - e_4 x_1 x_3 x_5 + \frac{e_4^2 g_1}{e_2 h_4} x_2 x_3 x_4 - h_4 x_2 x_3 x_5 \\ x_5 + \frac{e_4^5 g_1^2}{e_2^2 h_4^4} x_1 x_3 x_4 - \frac{g_1 e_4^3}{e_2 h_4^2} x_1 x_3 x_5 + \frac{e_4^4 g_1^2}{e_2^2 h_4^3} x_2 x_3 x_4 - \frac{e_4^2 g_1}{e_2 h_4} x_2 x_3 x_5 \end{pmatrix}$$

and let C be the map

$$\begin{pmatrix} x_1 \\ x_2 + \frac{e_4}{h_4} x_1 \\ x_3 \\ x_4 + \frac{e_4^2}{e_2 h_4} x_1 \\ x_5 + \frac{g_1 e_4^4}{e_2^2 h_4^3} x_1 \end{pmatrix}$$

then we get

$$C \circ F \circ C^{-1} = \begin{pmatrix} x_1 - g_1 x_2 x_3 x_5 + \frac{e_2 h_4^2}{e_4^2} x_2 x_3 x_5 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix}.$$

Since this was an example of rank one it is natural that each $C_i = x_i + c_i x_j$ for all i and some j. But in the examples with rank two we also found something like that. In general each

$$C_i = x_i + c_{ij} x_j + c_{ik} x_k,$$

but in our examples we found that either $c_{ij} = 0$ or $c_{ik} = 0$ for each i.

If we now take a look at one of the examples with $\text{rank}(JH) = 2$ we can illustrate what we mean by this.

Example 6.6 *Let F be the map*

$$\begin{pmatrix} x_1 \\ x_2 - \frac{b_2 e_2}{f_2} x_1 x_2 x_3 - b_2 x_1 x_2 x_4 - e_2 x_1 x_3 x_5 - f_2 x_1 x_4 x_5 \\ x_3 - \frac{b_3 e_2}{f_2} x_1 x_2 x_3 - b_3 x_1 x_2 x_4 \\ x_4 + \frac{b_3 e_2^2}{f_2^2} x_1 x_2 x_3 + \frac{b_3 e_2}{f_2} x_1 x_2 x_4 \\ x_5 + \frac{b_2^2 e_2}{f_2^2} x_1 x_2 x_3 + \frac{b_2^2}{f_2} x_1 x_2 x_4 + \frac{b_2 e_2}{f_2} x_1 x_3 x_5 + b_2 x_1 x_4 x_5 \end{pmatrix}$$

and let C be the map

$$\begin{pmatrix} x_1 \\ x_2 + \frac{f_2}{b_2} x_5 \\ x_3 + \frac{f_2}{e_2} x_4 \\ x_4 \\ x_5 \end{pmatrix}$$

then we get

$$C \circ F \circ C^{-1} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 + \frac{b_3 e_2^2}{f_2^2} x_1 x_2 x_3 - \frac{b_3 e_2^2}{b_2 f_2} x_1 x_3 x_5 \\ x_5 + \frac{b_2^2 e_2}{f_2^2} x_1 x_2 x_3 \end{pmatrix}.$$

If we compare the C from this example with the C from example 6.3 we can see what we meant by the remark made above concerning the special form of the transformation map C in this class where there was only one zero homogeneous part. This special form of C where for each i

$$C_i = x_i + c_{ij} x_j + c_{ik} x_k$$

either $c_{ij} = 0$ or $c_{ik} = 0$ is a consequence of the fact that the four non-zero homogeneous parts always appear in two pairs, where the two components of a pair can be transformed to each other by multiplication with a constant. In particular it does not happen that one of the H_i-s is a linear sum of two other H_i-s with two non-zero constants.

All other maps in this class followed the same special structure of C.

6.3 Future work

There is still a lot of research left to be done in this area. Since we didn't solve the linear cubic homogeneous system in dimension five completely, this is the first aspect that comes into mind. As we explained before the reason that we didn't solve this system completely was not because it is inherent too difficult, but more because of the practical reason that we didn't have enough time to solve it. So we think that with enough time and patience it can be solved completely. However we do think that this class will not give any really new maps, such as a map with three independent H_i-s.

The next step is more interesting. This is the case where we omit the linearity demand and examine the complete class of cubic homogeneous maps in dimension five. But as we have seen in the introduction of this section it is even difficult to draw up this system, since it takes a very long time to compute all determinants. So it might be very interesting for this dimension -and higher dimensions of course- to look for an alternative way to compute this system. It should be a way without all those expensive determinants of 5×5 matrices.

Now if one has been able to compute this complete system there are several ways to continue. The first is to try to solve the system completely, but since we have seen it was difficult -in time- to solve the linear system, this is probably too opportunistic, even though we can substitute a 5×5 matrix with initial values in this case.

A second continuation with probably more success, is the approach we took in chapter 4. There we solved the Drużkowski system and with this solution we were able to find for instance a complete set of representatives for the power similarity relation. But if we take a look at [Meisters 91] we see that this is probably a large set and thus probably a lot of work.

A third approach is probably the most interesting. In this case we don't intend to find a complete classification. We are only interested in just one counterexample to the Jacobian Conjecture. It may seem a bit strange that after we found the affirmative results in dimension four, we still think the Jacobian Conjecture in general is not true.

But we can explain this. If we look at the research in dimension three, it was all very simple since there weren't many cases. If we look at the research in dimension four, we see that we have to distinguish several cases. This is basically the consequence of the fact that in dimension three, there is only one interesting Jordan normal form:

$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

whereas there are three forms with rank at least two in dimension four:

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \text{ and } \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Of course in dimension five we have even more of these forms. But if we are looking for a counterexample we don't have to examine all of these cases. From the results we found it is obvious that if the rank of JH is too small or too large there is not much chance that this will give a counterexample. The reason everything went well according to the Jacobian Conjecture in dimension three and four is that there is not enough room for strange things. If $\text{rank}(JH) = 2$ it is so low that everything is fixed and if $\text{rank}(JH) = 3$ it is so high that again everything is fixed. Now in dimension five we think that if $\text{rank}(JH) = 2$ or $\text{rank}(JH) = 4$ everything is in order with the Jacobian Conjecture, but if $\text{rank}(JH) = 3$ there might just be enough freedom for something strange to happen. In fact we even have an interesting matrix with initial values in mind for this case. But unfortunately we have not been able so far to compute the complete system of so-called w_r^β-s. So we have not been able to test this interesting matrix.

Appendix A

Procedures

A.1 Introduction

This appendix shows the code of the Maple procedures we used and it also describes the way we used them. We used Maple V Release 2.

A.2 The concrete implementation

The following procedures are all in a file called `part5`. This must be read in the beginning of a Maple session.

Some initializations:

```
X:=[x1,x2,x3,x4];
alfabet:=['a','b','c','d','e','f','g','h','i',
          'j','k','l','m','n','o','p','q','r',
          's','t','u','v','w','x','y','z']:

readlib(write):
```

The 'linalg' package defines a new 'trace' so we have to make a copy of the original one. The if-construction is necessarily since otherwise if we read this file more than once, we would still loose the original trace-procedure and get two copies of the trace from the 'linalg' package.

```
if op(rtrace)=rtrace
then
  rtrace:=op(trace);
fi;

with(linalg):
read(jacobi):
```

A.2.1 Procedures for manipulation of partitions

The procedure **partitions**:

Input: Two integers number and n.

Output: A list with all partitions of number over n places.

```
partitions := proc(number,n)
 local i,j,k,parts,tail;
 if n = 1
 then
   RETURN([[number]])
 elif number = 0
 then
   RETURN([[0$n]])
 fi;
 parts := [];
 for i from number by -1 to 0
 do
   j := number - i;
   tail := partitions(j,n-1);
   for k from 1 to nops(tail)
   do
     parts := [op(parts),[i,op(tail[k])]]
   od
 od;
 eval(parts);
end:
```

The procedure **sumpart**:

Input: Two partitions p1 and p2.

Output: One partition p1+p2.

```
sumpart := proc(p1,p2)
 local i;
 if nops(p1) <> nops(p2)
 then
   ERROR(`Partitions have different lengths`)
 fi;
 ['p1[i]+p2[i]'$'i'=1..nops(p1)];
end:
```

The procedure **subpart**:

Input: Two partitions p1 and p2.

Output: One partition p1-p2.

```
subpart := proc(p1,p2)
 local i;
 if nops(p1) <> nops(p2)
 then
   ERROR(`Partitions have different lengths`)
 fi;
 ['p1[i]-p2[i]'$'i'=1..nops(p1)];
end:
```

Note that although the procedures sumpart and subpart were originally written for partitions, they can be used to add or subtract all kinds of lists with the same length.

The procedure **pospart**:

Input: A partition p.

Output: A boolean which is true iff all elements of p are non-negative.

```
pospart := proc(p)
 local i;
 for i from 1 to nops(p)
 do
   if p[i] < 0
   then
     RETURN(false)
   fi
 od;
 true;
end:
```

The procedure **pairs**:

Input: A list of partitions parts, a single partition sum and a number n.

Output: A list of lists of n partitions which form the partition sum when added. All partitions must come from the original list parts.

```
pairs := proc(parts,sum,n)
 local total,poss,rest,diff,tail,i;
 if n = 1
 then
   if member(sum,parts)
   then
     RETURN([[sum]])
   else
     RETURN([])
   fi
 fi;
 total := [];
 rest := parts;
 while rest <> []
 do
   poss := rest[1];
   rest := [op(2..nops(rest),rest)];
   diff := subpart(sum,poss);
   if pospart(diff)
   then
     tail := pairs(rest,diff,n-1);
     if tail <> []
     then
       for i from 1 to nops(tail)
       do
         total := [op(total),[poss,op(tail[i])]]
       od
     fi
   fi
 od;
 eval(total);
end:
```

A.2.2 Procedures to compute each w_r^β

The procedure **firstsummand**:

Input: An integer r and a list of integers numbers of the form [1,..,n]

Output: A list of all possible choices of r different integers from the list numbers. The first summand in the definition of w_r^β runs over this list.

```
firstsummand := proc(r,numbers)
 local n,rest,total,head,tail,i;
 n := nops(numbers);
 if r = 1
 then
   RETURN(['[numbers[i]]'$'i'=1..n])
 elif r = n
 then
   RETURN([numbers])
 fi;
 rest := numbers;
 total := [];
 while rest <> []
 do
   head := rest[1];
   rest := [op(2..n,rest)];
   n := nops(rest);
   tail := firstsummand(r-1,rest);
   for i from 1 to nops(tail)
   do
     total := [op(total),[head,op(tail[i])]]
   od
 od;
 eval(total);
end:
```

The procedure **secondsummand**:

Input: An integer r, one of the elements coming from the result of the procedure firstsummand (ti), a partition beta, a list of partitions parts and a number n which indicates the length of each partition.

Output: A list of list of r partitions which have as a sum beta plus the index partition $\varepsilon_{t_1}+\cdots+\varepsilon_{t_r}$. This is a list over which the second summation runs in the definition of w_r^β.

```
secondsummand := proc(r,ti,beta,parts,n)
 local eps,j;
 eps := [];
 for j from 1 to n
 do
   if member(j,ti)
   then
     eps := [op(eps),1]
   else
```

```
      eps := [op(eps),0]
    fi
 od;
 eps := sumpart(eps,beta);
 eval(pairs(parts,eps,r));
end:
```

The procedure **smallpart**:

Input: The matrix M, an element of the first summand i, an element of the second summand j, the dimension of the matrices that appear behind the summation signs in the definition of w_r^β and a list of partitions.

Output: The product of the two determinants in the definition of w_r^β.

```
smallpart := proc(M,i,j,r,parts)
 local A,B,k,l,pos;
 option remember;
 A := array(1..r,1..r,[[0$r]$r]);
 B := array(1..r,1..r,[[0$r]$r]);
 for k from 1 to r
 do
   for l from 1 to r
   do
     A[k,l] := j[l][i[k]];
     if member(j[l],parts,'pos')
     then
       B[k,l] := M[i[k],pos]
     else
       ERROR(`Current partition not valid`)
     fi
   od
 od;
 eval(det(A)*det(B));
end:
```

The procedure **wrbeta**:

Input: Integers r and n, a partition beta, a matrix M with the coefficients of the map, and a list parts of partitions.

Output: The value of w_r^β.

```
wrbeta := proc(r,n,beta,M,parts)
 local fs,sum,i,ss,j;
 print(w_r_beta,r,beta);
#  this print statement shows at what point the
#  process is running at the moment
 fs := firstsummand(r,['i'$'i'=1..n]);
 sum := 0;
 for i in fs
 do
   ss := secondsummand(r,i,beta,parts,n);
   for j in ss
   do
     sum := sum + smallpart(M,i,j,r,parts)
   od
 od;
```

```
 sum := ((-1)^r)*sum;
 eval(sum);
end:
```

A.2.3 The main procedure

The procedure `simplifyM`:

Input: A matrix M with the coefficients of the original map and the dimension n.

Output: A list containing a matrix and a system of equations. The matrix is the same as M, unless the dimension is 3. In that case, the one solution, is substituted in the original M.

```
simplifyM := proc(M,n)
 local parts1,N,i,system,parts2,j,solution;
 parts1 := partitions(3,n);
 N := evalm(M);
 system := {};
 solution := {};
 for i from 1 to n
 do
   parts2 := partitions(2*i,n);
   for j in parts2
   do
     system := {op(system),wrbeta(i,n,j,N,parts1)}
   od;
 od;
 if n=3
 then
   solution := solvesystem(system);
   N:= subs(solution,evalm(N));
 fi;
 [evalm(N),system];
end:
```

A.2.4 Procedures used to solve system of equations

The procedure `easysystem1`:

Input: A system of equations.

Output: A set of equations of the form single variables times some constant or powers of variables times some constant.

```
easysystem1 := proc(sys)
 local n,easy,i,stype;
 n := nops(sys);
 easy := {};
 for i from 1 to n
 do
```

```
   stype := whattype(sys[i]);
   if member(stype,{`string`})
   then
     easy := {op(easy),sys[i]}
   elif (member(stype,{`*`})) and
        (nops(sys[i])=2) and
        (member(whattype(op(1,sys[i])),{`integer`,`fraction`}) or
         member(whattype(op(2,sys[i])),{`integer`,`fraction`}))
   then
     easy := {op(easy),sys[i]}
   elif member(stype,{`^`})  and nops(indets(sys[i]))=1
   then
     easy := {op(easy),sys[i]}
   fi
 od;
 eval(factor(easy));
end:
```

The procedure **easysystem2**:

Input: A system of equations.

Output: A set of equations where the equations must have at least one linear term and at most 10 terms. Furthermore all equations with at most 2 terms are picked.

```
easysystem2 := proc(sys)
 local n,easy,i,stype,j;
 n := nops(sys);
 easy := {};
 for i from 1 to n
 do
   stype := whattype(sys[i]);
   if member(stype,{`*`,`string`})
   then
     easy := {op(easy),sys[i]}
   elif
     member(stype,{`+`})
   then
     if nops(sys[i]) < 3
     then
       easy := {op(easy),sys[i]}
     elif nops(sys[i]) < 10
     then
       for j from 1 to nops(sys[i])
       do
         if member(whattype(op(j,sys[i])),{`string`})
         then
           easy := {op(easy),sys[i]};
           break
         elif
           member(whattype(op(j,sys[i])),{`*`})
         then
           if (nops(op(j,sys[i]))=2) and
              ((member(whattype(op(1,op(j,sys[i]))),{`integer`}) and
               member(whattype(op(2,op(j,sys[i]))),{`string`})) or
               (member(whattype(op(2,op(j,sys[i]))),{`integer`})) and
               member(whattype(op(2,op(j,sys[i]))),{`string`}))
           then
             easy := {op(easy),sys[i]};
             break
```

```
          fi
        fi
      od
    fi
  fi
 od;
 print(nops(easy));
 eval(factor(easy));
end:
```

The procedure **easysystem3**:

Input: A system of equations sys, two sets of indeterminates ind and new.

Output: All equations that have at more indeterminates than the ones from ind, and that have no other indeterminates than ind union new. In practice ind is normally the empty set.

```
easysystem3 := proc(sys,ind,new)
 local i;
 easy:={};
 for i from 1 to nops(sys)
 do
   if ((nops(indets(sys[i]) minus ind) > 0) and
      (nops(indets(sys[i]) minus (ind union new)) = 0))
   then
     easy:={op(easy),sys[i]};
   fi;
 od;
 print(nops(easy));
 eval(factor(easy));
end:
```

The procedure **easysystem4**:

Input: A substitution a and a set of equations b.

Output: The set of equations you get after substitution of a in b and simplifying -in case of fractions- and finally expanding in order to achieve that the procedures es1,es2 and es3 work properly.

```
es4:=proc(a,b)
 local c;
 c:=expand(simplify(subs(a,b)));
 print(nops(c),nops(indets(c)));
 c;
end:
```

The procedure **easysystem5**. This is basically the same as easysystem3, but we do not factor here. Sometimes the factorization takes too long.

```
easysystem5 := proc(sys,ind,new)
 local i;
 easy:={};
 for i from 1 to nops(sys)
 do
   if ((nops(indets(sys[i]) minus ind) > 0) and
```

```
      (nops(indets(sys[i]) minus (ind union new)) = 0))
   then
     easy:={op(easy),sys[i]};
   fi;
 od;
 eval(easy);
end:
```

The procedure **solvesystem**:

Input: A system of equations sys.

Output: A solution for as far as there exists only one solution. This procedure can not be used very much.

```
solvesystem := proc(sys)
 local system,easy1,easy2,solution,solution2,solution3;
 system := sys;
 solution3 := {};
 easy1 := easysystem2(system);
 while easy1 <> {}
 do
   easy2 := easysystem1(system);
   while easy2 <> {}
   do
     system := system minus easy2;
     solution := solve(easy2);
     solution2 := {solution};
     if nops(solution2) > 1
     then
       solution := `intersect`('solution2[i]'$'i'=1..nops(solution2));
       system := system union easy2
     elif nops(solution2) = 0
     then
       ERROR(`No solution for this system`)
     fi;
     solution3 := subs(solution,solution3) union solution;
     system := subs(solution,eval(system));
     system := system minus {0};
     easy2 := easysystem1(system)
   od;
   easy1 := subs(solution3,eval(easy1));
   system := system minus easy1;
   solution := solve(easy1);
   solution2 := {solution};
   if nops(solution2) > 1
   then
     solution := `intersect`('solution2[i]'$'i'=1..nops(solution2));
   elif nops(solution2) = 0
   then
     ERROR(`No solution for this system`)
   fi;
   solution3 := subs(solution,solution3) union solution;
   system := subs(solution,eval(system));
   system := system minus {0};
   easy1 := easysystem2(system)
 od;
 system := subs(solution3,eval(sys));
 solution := solve(system);
 solution2 := {solution};
```

```
 if nops(solution2) > 1
 then
   solution := `intersect`('solution2[i]'$'i'=1..nops(solution2));
 elif nops(solution2) = 0
 then
   ERROR(`No solution for this system`)
 fi;
 solution3 := subs(solution,solution3) union solution;
 eval(solution3);
end:
```

Abbreviations: it turned out that **easysystem** is a difficult word to type, so we introduced some abbreviations with **es**:

```
es1 := op(easysystem1):
es2 := op(easysystem2):
es3 := op(easysystem3):
es5 := op(easysystem5):
```

The procedure **es6**:

Input: A system of equations c.

Output: A list of two elements. The first element is a list of indices in the system of equations with descending number of variables. The second element is this smallest number of variables.

```
es6:=proc(c)
 local i,min,index;
 min:=nops(indets(c));
 index:=[];
 for i from 1 to nops(c)
 do
   if nops(indets(c[i])) > 0 and
      nops(indets(c[i])) <= min
   then
      min:=nops(indets(c[i]));
      index:=[op(index),i];
   fi;
 od;
 [index,min];
end:
```

A.2.5 General tools

The procedure **maakhomogeen**:

Input: A list L of maps X-H.

Output: A list of maps H. It gives the homogeneous part of the maps.

```
maakhomogeen:=proc(L)
 local i,HL,j;
 HL:=[];
 for i from 1 to nops(L)
 do
   HL:=[op(HL),factor(['x.j-L[i][j]'$'j'=1..nops(L[1])])];
 od;
 HL:
end:
```

The procedure **maakjacobiaan**:

Input: A list L of homogeneous maps H.

Output: A list of jacobianmatrices JH.

```
maakjacobiaan:=proc(L)
 local i,NL;
 NL:=[];
 for i from 1 to nops(L)
 do
   NL:=[op(NL),jacobian(L[i],['x.j'$'j'=1..nops(L[1])])]:
 od;
 NL:
end:
```

The procedure **genereeralg**:

Input: A natural number n.

Output: The most general polynomial map in dimension n which is cubic homogeneous. As a side effect the coefficients of the map are placed in a global matrix M.

```
genereeralg:=proc(n)
 local i,j,par,mon,X;
 X:=['x.i'$'i'=1..n];
 par := partitions(3,n);
 M:=array(1..n,1..nops(par));
 for i from 1 to n
 do
   H.i:=0;
 od;
 for i from 1 to nops(par)
 do
   mon:=maakmonoom(X,par[i],n);
   for j from 1 to n
   do
     H.j:=H.j + Y.j.i * mon;
     M[j,i] := Y.j.i;
   od;
 od;
 ['x.i-H.i'$'i'=1..n];
end:
```

The procedure **maakmonoom**:

Input: A list of variables var, a partition par and a number n.

Output: The monomial $a_1^{p_1}\cdots a_n^{p_n}$, where var=$[a_1,\ldots,a_n]$ and par=[p1,..,pn].

```
maakmonoom:=proc(var,par,n)
 local i,mon;
 mon:=1;
 for i from 1 to n
 do
   mon:=mon*var[i]^par[i];
 od;
end:
```

The procedure `maakopl`:

Input: A list of substitutions list and a single substitution sub.

Output: A list of substitutions derived by substituting each element of list in the original sub and union this with the element of list, so that we enlarge the substitution sub by the one from list and also apply the substitution from list to sub.

```
maakopl:=proc(list,sub)
 local i,list2;
 list2:=[];
 for i from 1 to nops(list)
 do
   list2:=[op(list2),subs(list[i],sub) union list[i]]
 od;
 list2:
end:
```

The procedure `bepaalopl`:

Input: A list,set or single variable var and a list of substitutions oo.

Output: A list with the indices of the list oo which do not yield zero after substitution on var.

```
bepaalopl:=proc(var,oo)
 local i;
 lijst:=[]:
 for i from 1 to nops(oo)
 do
   if not(member(0,convert(subs(oo[i],var),set)))
   then
     lijst:={op(lijst),i}:
   fi;
 od;
 lijst
end:
```

The procedure `maakdruz`:

Input: A natural number n.

Output: The most general polynomial map F in dimension n on Drużkowski form. As a side effect the coefficients of this map are placed in the global matrix M.

```
maakdruz:=proc(n)
 local i,F,H,term,H2,par;
 par:=partitions(3,n):
 M:=array(1..n,1..nops(par));
 H:=[];
 F:=[];
 for i from 1 to n
 do
   term:=0;
   for j from 1 to n
   do
     term:=term + cat(alfabet[j],i)*x.j;
   od;
   H:=[op(H),term^3];
   F:=[op(F),x.i - H[i]];
 od;
 H2:=['sort(expand(H[i]),['x.j'$'j'=1..n])'$'i'=1..n];
 for i from 1 to n
 do
   for j from 1 to nops(par)
   do
     M[i,j]:= coeffs(op(j,H2[i]),['x.j'$'j'=1..n]);
   od;
 od;
 F;
end:
```

The procedure **powersim**:

Input: A matrix A and its nilpotency index nil.

Output: This procedure computes a system of equations that should hold if A is power similar to one of the six matrices from Meisters. They are separated by their rank and nilpotency index. If possible these systems (at most 3) are solved automatically by Maple. If this gives some problems one should comment out the appropriate rules. The systems are stored in the global variables sys1,sys2 and sys3. So one can always try to solve these systems by hand. The automatic solutions are stored in the global variables opsys1,opsys2 and opsys3.

```
powersim:= proc(A,nil)
 local i,j,r,T,X,powermatrices,ATX3,TBX3,B,C;
 T:=array([[p1,q1,r1,s1],[p2,q2,r2,s2],[p3,q3,r3,s3],[p4,q4,r4,s4]]);
 X:=[x1,x2,x3,x4];
 var:={p1, p2, q1, s1, r1, q2, s2, r2, p4, p3, q3, s3, r3, q4, s4, r4,z};
 sys1:={z*det(T)-1};
 sys2:={z*det(T)-1};
 sys3:={z*det(T)-1};
 powermatrices:=[[array([[0,1,0,0],[0,0,0,0],[0,0,0,0],[0,0,0,0]])],
                [array([[0,1,0,0],[0,0,0,0],[0,0,0,1],[0,0,0,0]]),
                 array([[0,1,0,0],[0,0,1,0],[0,0,0,0],[0,0,0,0]]),
                 array([[0,0,1,0],[0,0,1,1],[0,0,0,1],[0,0,0,0]])],
                [array([[0,1,0,0],[0,0,1,0],[0,0,0,1],[0,0,0,0]]),
                 array([[0,1,1,0],[0,0,1,0],[0,0,0,1],[0,0,0,0]])]];
 r:=rank(A);
 ATX3:=evalm(A*(T*X));
 ATX3:=['ATX3[j]^3'$'j'=1..4];
 for i from 1 to nops(powermatrices[r])
 do
   B:=powermatrices[r][i];
   TBX3:=evalm(B*X);
```

```
    TBX3:=['TBX3[j]^3'$'j'=1..4];
    TBX3:=evalm(T*TBX3);
    C:=['collect(ATX3[j]-TBX3[j],[x1,x2,x3,x4],distributed)'$'j'=1..4];
    for j from 1 to 4
    do
      sys.i:={op(sys.i),coeffs(C[j],[x1,x2,x3,x4])};
    od;
  od;
  for i from nops(powermatrices[r])+1 to 3
  do
    sys.i:={1};
  od;
  if (r=2) and (nil=2)
  then
    sys2:={1};
    sys3:={1};
  elif (r=2) and (nil=3)
  then
    sys1:={1};
  fi:
  opsys1:=[solve(sys1,var)];
  opsys2:=[solve(sys2,var)];
  opsys3:=[solve(sys3,var)];
  print(r,nops(opsys1),nops(opsys2),nops(opsys3));
end:
```

The procedure `trans`:

Input: Two matrices A and B.

Output: A general matrix T. This matrix T has been used to draw up a system of equations that should hold if T^{-1} A T = B. This system is stored in the global variable sys1.

```
trans:=proc(A,B)
 local i,T,X;
 T:=array([[p1,q1,r1,s1],[p2,q2,r2,s2],[p3,q3,r3,s3],[p4,q4,r4,s4]]);
 X:=[x1,x2,x3,x4];
 var:={p1, p2, q1, s1, r1, q2, s2, r2, p4, p3, q3, s3, r3, q4, s4, r4,z};
 sys1:={z*det(T)-1};
 TA:=evalm(T*A);
 BT:=evalm(B*T);
 C:=[''collect(TA[i,j]-BT[i,j],[x1,x2,x3,x4],distributed)'$'j'=1..4'$'i'=1..4];
 for j from 1 to nops(C)
 do
   sys1:={op(sys1),coeffs(C[j],[x1,x2,x3,x4])};
 od;
 evalm(T);
end:
```

The procedure `maakdruzmat`:

Input: A list dru with the solutions of the Drużkowski system as they were found after the substitution c4 = 0, d4 = 0, b4 = 0, a4 = 0.

Output: A list of matrices of the coefficients of the Drużkowski forms.

```
maakdruzmat:=proc(dru)
 local i,A,M,een;
 A:=array([[a1,b1,c1,d1],[a2,b2,c2,d2],[a3,b3,c3,d3],[a4,b4,c4,d4]]):
 M:=[]:
 een:={c4 = 0, d4 = 0, b4 = 0, a4 = 0};
 for i from 1 to nops(dru)
 do
   M:=[op(M),subs(een,dru[i],evalm(A))];
 od;
end:
```

The procedure **strongnilpotent**:

Input: A map F=X-H.

Output: A boolean that indicates whether JH is strong nilpotent or not.

```
strongnilpotent := proc(F)
 local Alfa,i,Z,n,H,j,N,prod;
 Alfa:=['A','B','C','D','E','F','G','H','I',
        'J','K','L','M','N','O','P','Q','R',
        'S','T','U','V','W','X','Y','Z']:
 n:=nops(F);
 Z:=array([[0$n]$n]);
 H:=maakhomogeen([F]);
 N:=op(maakjacobiaan(H));
 prod:=subs({'x.j=cat(Alfa[1],j)'$'j'=1..n},evalm(N));
 for i from 2 to n
 do
   prod:=evalm(prod*subs({'x.j=cat(Alfa[i],j)'$'j'=1..n},evalm(N)));
 od;
 print(`Strong nilpotency of`);
 print(evalm(N));
 print(linalg[equal](prod,Z));
end:
```

The procedure **strongnillist**:

Input: A list G of maps.

Output: The result of the strong-nilpotency test is written to a file 'strnil' for all elements of G.

```
strongnillist:=proc(G)
 local i;
 open(strnil);
 for i from 1 to nops(G)
 do
   print(`solution `.i);
   strongnilpotent(G[i]);
 od;
 close(strnil);
end:
```

The procedure **strongnilclass4**:

Input: The seven solutions from the classification in a list G.

Output: The result of the nilpotency test is written to the file strnilclass4.

```
strongnilclass4:=proc(G)
 local i;
 open(strnilclass4);
 print(`########## JH has rank two and JH^2.X=0 ##########`);
 for i from 1 to 2
 do
   print(`solution `.i);
   strongnilpotent(G[i]);
 od;
 print(`########## JH has rank two and JH^2.X<>0 ##########`);
 for i from 3 to 5
 do
   print(`solution `.i);
   strongnilpotent(G[i]);
 od;
 print(`########## JH has rank three and JH^3.X<>0 ##########`);
 for i from 6 to 7
 do
   print(`solution `.i);
   strongnilpotent(G[i]);
 od;
 close(strnilclass4);
end:
```

The procedure `nilpotencyindex`:

Input: A matrix M with dimension n x n.

Output: The nilpotency index of M. (If M is nilpotent, otherwise an error message.)

```
nilpotencyindex:=proc(M,n)
 local i,Z;
 Z:=array([[0$n]$n]);
 for i from 1 to n
 do
   if linalg[equal](Z,evalm(M^i))
   then RETURN(i)
   fi
 od;
 ERROR(`Matrix not nilpotent`);
end:
```

The procedure `iterationtest`:

Input: A map F=X-H.

Output: A list with a boolean that indicates the validity of some conjecture and with the nilpotency index of JH and with the iteration number of H.

```
iterationtest := proc(F)
 local i,n,H,N,K,j;
 n:=nops(F);
 H:=op(maakhomogeen([F]));
 N:=op(maakjacobiaan([H]));
 ni:=nilpotencyindex(N,n);
 it:=iterationnumber(H);
 [evalb(ni-it=2),ni,it]
end:
```

The procedure **iterationnumber**:

Input: A homogeneous map H.

Output: The smallest number n (n < max) with H^n=0. If such an n doesn't exist an error message is returned.

```
iterationnumber := proc(H)
 local i,max,K;
 max := 20;
 n:=nops(H);
 K:=H:
 for i from 2 to max
 do
   K:=compose(K,H,['x.j'$'j'=1..n]);
   if linalg[equal](K,[0$n])
   then RETURN(i)
   fi;
 od;
 ERROR(`Not vanished after `.max.` iterations`);
end:
```

The procedure **deltah**:

Input: A list gee and a map ef.

Output: The list (ef-I)(gee) where I is the identical map.

```
deltah:=proc(gee,ef)
 local cee,i,X;
 X:=['x.i'$'i'=1..nops(ef)];
 cee := compose(gee,ef,X);
 ['cee[i]-gee[i]' $ ('i' = 1 .. nops(gee))]
end:
```

The procedure **deltahnumber**:

Input: A map F and a number max.

Output: The smallest number n (n<max) with deltah^n=0. If this doesn't exist an error message is returned

```
deltahnumber:= proc(F,max)
 local i,K;
 n:=nops(F);
 K:=['x.i'$'i'=1..n];
 for i from 2 to max
 do
   K:=deltah(K,F);
   if linalg[equal](K,[0$n])
   then RETURN(i)
   fi;
 od;
 ERROR(`Not vanished after `.max.` iterations`);
end:
```

The procedure **computeD**:

Input: A polynomial map F.

Output: A locally nilpotent derivation D with F=exp(D). (If such a derivation exists.)

```
computeD:=proc(F)
 local i,j,D,K,n;
 n:=nops(F);
 D:=[0$n];
 K:=['x.i'$'i'=1..n];
 K:=deltah(K,F);
 j:=1;
 while not(linalg[equal](K,[0$n]))
 do
   D:=['D[i]+(-1)^(j-1)/j*(K[i])'$'i'=1..n];
   K:=deltah(K,F);
   j:=j+1;
 od;
 D;
end:
```

The procedure **computeexpD**:

Input: A locally nilpotent derivation D.

Output: The map exp(D).

```
computeexpD:=proc(D)
 local X,i,j,expD,K;
 n:=nops(D);
 K:=['x.i'$'i'=1..n];
 X:=K;
 expD:=K;
 j:=1;
 while not(linalg[equal](K,[0$n]))
 do
   K:=['applyder(K[i],D,X)'$'i'=1..n];
   expD:=['expD[i]+1/j!*(K[i])'$'i'=1..n];
   j:=j+1;
 od;
 expD;
end:
```

The procedure **transform**:

Input: A map F.

Output: The map F on the new coordinates where x2=x2+1/3*x1^3.

```
transform:=proc(F)
 subs(x2=x2+1/3*x1^3,F);
end:
```

The procedure **struct**:

Input: A map F, two indices a and b.

Output: The homogeneous part of F, taken as a map in K[xa,xb]. In global variables H,R,D and JD this map is structured according to the structure theorem in chapter 3.

```
struct:=proc(F,a,b)
 H:=op(maakhomogeen([F]));
 R:=subs({x.a=0,x.b=0},H);
 D:=[H[a]-R[a],H[b]-R[b]];
 JD:=jacobian(D,[x.a,x.b]);
 D:=factor(D);
end:
```

The procedure **ordern**:

Input: A map F, the variables X and the dimension n.

Output: A system of equations that must hold if we assume that F^n=I.

```
ordern:=proc(F,X,n)
 local i,FF,HH;
 FF:=compose(['x.i'$'i'=1..nops(F)],F,X);
 for i from 2 to n
 do
   FF:=compose(FF,F,X);
 od;
 HH:=['x.i-FF[i]'$'i'=1..4];
 HH:=['collect(HH[i],X,distributed)'$'i'=1..nops(F)];
 ef:={};
 for i from 1 to nops(F)
 do
   ef:={op(ef),coeffs(HH[i],X)}
 od;
end:
```

The procedure **ordertest**:

Input: A map F, the variables X and a special variable t.

Output: The result of a partial solution of the ordern set of equations if it is substituted in the special variable t. For instance we wanted to see that for all n g4=0 was necessary. So we took t=g4, and saw that it was always zero after the substitution.

```
ordertest:=proc(F,X,t)
 local i,max;
 max:=30:
 for i from 1 to max
 do
   ordern(F,X,i):
   solve(es1(ef)):
   print(i,subs(",t));
 od:
end:
```

The procedure **gelijk**:

Input: Two lists of equal length.

Output: A boolean that indicates whether the two lists were equal.

```
gelijk:=proc(F,G)
 linalg[equal](simplify(['F[i]-G[i]'$'i'=1..nops(F)]),[0$nops(F)]);
end:
```

The procedure **testexp**:

Input: A list G of polynomial maps.

Output: The result of the comparison between the elements of G and the exp(D)'s as they were computed for each element of G.

```
testexp:=proc(G)
 local i,D,eD;
 for i from 1 to nops(G)
 do
   D:=computeD(G[i]);
   eD:=computeexpD(D);
   print(i,gelijk(G[i],eD));
 od;
end:
```

A.3 Solving the general case

This section shows the way we were able to solve the general case of the cubic homogeneous polynomial maps in dimension four.

A.3.1 The system of equations for rank two

We start with some initialization.

```
read part5;
with(linalg):
X:=[x1,x2,x3,x4]:

# generic polynomial map for n=4
H1 := a1*x1^3 + b1*x1^2*x2 + c1*x1^2*x3 + d1*x1^2*x4 + e1*x1*x2^2 +
      f1*x1*x2*x3 + g1*x1*x2*x4 + h1*x1*x3^2 + i1*x1*x3*x4 + j1*x1*x4^2 +
      k1*x2^3 + l1*x2^2*x3 + m1*x2^2*x4 + n1*x2*x3^2 + o1*x2*x3*x4 +
      p1*x2*x4^2 + q1*x3^3 + r1*x3^2*x4 + s1*x3*x4^2 + t1*x4^3:
H2 := a2*x1^3 + b2*x1^2*x2 + c2*x1^2*x3 + d2*x1^2*x4 + e2*x1*x2^2 +
      f2*x1*x2*x3 + g2*x1*x2*x4 + h2*x1*x3^2 + i2*x1*x3*x4 + j2*x1*x4^2 +
      k2*x2^3 + l2*x2^2*x3 + m2*x2^2*x4 + n2*x2*x3^2 + o2*x2*x3*x4 +
      p2*x2*x4^2 + q2*x3^3 + r2*x3^2*x4 + s2*x3*x4^2 + t2*x4^3:
H3 := a3*x1^3 + b3*x1^2*x2 + c3*x1^2*x3 + d3*x1^2*x4 + e3*x1*x2^2 +
      f3*x1*x2*x3 + g3*x1*x2*x4 + h3*x1*x3^2 + i3*x1*x3*x4 + j3*x1*x4^2 +
      k3*x2^3 + l3*x2^2*x3 + m3*x2^2*x4 + n3*x2*x3^2 + o3*x2*x3*x4 +
      p3*x2*x4^2 + q3*x3^3 + r3*x3^2*x4 + s3*x3*x4^2 + t3*x4^3:
H4 := a4*x1^3 + b4*x1^2*x2 + c4*x1^2*x3 + d4*x1^2*x4 + e4*x1*x2^2 +
```

```
      f4*x1*x2*x3 + g4*x1*x2*x4 + h4*x1*x3^2 + i4*x1*x3*x4 + j4*x1*x4^2 +
      k4*x2^3 + l4*x2^2*x3 + m4*x2^2*x4 + n4*x2*x3^2 + o4*x2*x3*x4 +
      p4*x2*x4^2 + q4*x3^3 + r4*x3^2*x4 + s4*x3*x4^2 + t4*x4^3:

F1 := x1 - H1:
F2 := x2 - H2:
F3 := x3 - H3:
F4 := x4 - H4:

H:=[H1,H2,H3,H4]:
F:=[F1,F2,F3,F4]:

M := array(1..4,1..20,[[a1,b1,c1,d1,e1,f1,g1,h1,i1,j1,
                         k1,l1,m1,n1,o1,p1,q1,r1,s1,t1],
                        [a2,b2,c2,d2,e2,f2,g2,h2,i2,j2,
                         k2,l2,m2,n2,o2,p2,q2,r2,s2,t2],
                        [a3,b3,c3,d3,e3,f3,g3,h3,i3,j3,
                         k3,l3,m3,n3,o3,p3,q3,r3,s3,t3],
                        [a4,b4,c4,d4,e4,f4,g4,h4,i4,j4,
                         k4,l4,m4,n4,o4,p4,q4,r4,s4,t4]]);
```

Now we continue by computing JH,JH^2,JH^3 and $JH^2.X$. Furthermore we collect all entries of JH^3 and $JH^2.X$ and put these twenty polynomials in a list. From this list we extract the coefficients of all monomials and put them in two systems: `af` contains all equations from $JH^3 = 0$ and `bf` contains all equations from $JH^2.X = 0$. The set `cf` is the union of `af` and `bf`.

```
N := jacobian(H,[x1,x2,x3,x4]):

N2:=evalm(N^2):
N3:=evalm(N2*N):

N2X:=evalm(N2*X):

N3N2X:=[]:
for i from 1 to 4
do
  for j from 1 to 4
  do
    N3N2X:=[op(N3N2X),collect(N3[i,j],X,distributed)]
  od
od:

for i from 1 to 4
do
  N3N2X:=[op(N3N2X),collect(N2X[i],X,distributed)]
od:

af:={}:
for i from 1 to 16
do
  af:={op(af),coeffs(N3N2X[i],X)}
od:

bf:={}:
for i from 17 to 20
do
  bf:={op(bf),coeffs(N3N2X[i],X)}
od:
```

```
cf := af union bf:
```

A.3.2 $JH^3 = 0$ and $JH^2.X = 0$

In this case we had two matrices to substitute in the system.

```
sub1:={a1=0,a2=1/3,a3=0,a4=0,b1=0,b2=0,b3=0,b4=0,c2=0,c3=0,c4=1,
       d1=0,d2=0,d3=0,d4=0};


sub2:={a1=0,a2=1/3,a3=0,a4=0,b1=0,b2=0,b3=0,b4=0,c1=1,c2=0,c3=0,c4=0,
       d1=0,d2=0,d3=0,d4=0};

df:=es4(sub1,cf):

es1(df);
ee:=solve(");
df:=es4(",df):

es1(df);
solve(");
ee:=subs(",ee) union ":
df:=es4(",df):

es1(df);
solve(");
ee:=subs(",ee) union ":
df:=es4(",df):

es1(df);
solve(");
ee:=subs(",ee) union ":
df:=es4(",df):

es1(df);
solve(");
ee:=subs(",ee) union ":
df:=es4(",df):

es6(df);
df3:=es3(df,{},{c1,f2});
oo1:=[solve(df3,{f2})];"[1]:
ee:=subs(",ee) union ":
df:=es4(",df):

es6(df);
df3:=es3(df,{},{i1,c1,h3,r3});
oo2:=[solve(df3)];"[1]:
ee:=subs(",ee) union ":
df:=es4(",df):

es6(df);
df3:=es3(df,{},{c1,h3,n1});
oo3:=[solve(df3)];"[1]:
ee:=subs(",ee) union ":
df:=es4(",df):
```

```
es6(df);
df3:=es3(df,{},{c1,o2});
oo4:=[solve(df3)];"[1]:
ee:=subs(",ee) union ":
df:=es4(",df):

df3:=es3(df,{},{c1,h3,j2});
oo5:=[solve(df3)];"[1]:
ee:=subs(",ee) union ":
df:=es4(",df):

df3:=es3(df,{},{c1,n4,h3});
oo6:=[solve(df3)];
zf:=df:
```

At this point we have to choose one of the following cases:

1. $c_1 = 0$ and $n_4 = 0$
2. $c_1 = 0$ and $n_4 \neq 0$
3. $c_1 \neq 0$

The first case:

```
eea:=subs(oo6[1],ee) union oo6[1]:
df:=es4(eea,zf):
es1(df);
solve(");
eea:=subs(",eea) union ":
df:=es4(",df):

es1(df);
solve(");
eea:=subs(",eea) union ":
df:=es4(",df):

df3:=es3(df,{},{i4,h3});
oo7:=[solve(df3)];
"[1];
eea:=subs(",eea) union ":
df:=es4(",df):

es1(df);
solve(");
eea:=subs(",eea) union ":
df:=es4(",df):

es1(df);
solve(");
eea:=subs(",eea) union ":
df:=es4(",df):

es1(df);
solve(");
eea:=subs(",eea) union ":
df:=es4(",df):

es1(df);
```

```
solve(");
eea:=subs(",eea) union ":
df:=es4(",df):

sol1:=subs(eea,sub1) union eea:
```

The set **sol1** represents a solution.

The second case:

```
eeb:=subs(oo6[2],ee) union oo6[2]:
df:=es4(eeb,zf):
es1(df);
solve(");
eeb:=subs(",eeb) union ":
df:=es4(",df):

es1(df);
solve(");
```

No solutions in this case since we have a contradiction with $n_4 \neq 0$.

The third case:

```
eec:=subs(oo6[3],ee) union oo6[3]:
df:=es4(eec,zf):

es1(df);
solve(");
eec:=subs(",eec) union ":
df:=es4(",df):

df3:=es3(df,{},{c1,i4});
oo8:=[solve(df3,{i4})];"[1]:
eec:=subs(",eec) union ":
df:=es4(",df):

df3:=es3(df,{},{t2,s1});
oo9:=[solve(")];"[1]:
eec:=subs(",eec) union ":
df:=es4(",df):

df3:=es3(df,{},{c1,t2});
oo10:=[solve(")];"[1]:
eec:=subs(",eec) union ":
df:=es4(",df):

df3:=es3(df,{},{c1,s4});
oo11:=[solve(df3)];"[1]:
eec:=subs(",eec) union ":
df:=es4(",df):

df3:=es3(df,{},{i2,h1,n2}):
oo12:=[solve(")];"[1]:
eec:=subs(",eec) union ":
df:=es4(",df):

df3:=es3(df,{},{c1,h1,r1}):
oo13:=[solve(df3)];
# c1<>0 so only the second solution is valid
"[2]:
```

```
eec:=subs(",eec) union ":
df:=es4(",df):

df3:=es3(df,{},{h4,c1,r1});
oo14:=[solve(df3)];"[1]:
eec:=subs(",eec) union ":
df:=es4(",df):

df3:=es3(df,{},{i2,c1,h4}):
oo15:=[solve(df3)];"[1]:
eec:=subs(",eec) union ":
df:=es4(",df):

df3:=es3(df,{},{c1,h4,r4}):
oo16:=[solve(df3)];"[1]:
eec:=subs(",eec) union ":
df:=es4(",df):

df3:=es3(df,{},{h4,c1,s2});
oo17:=[solve(df3)];"[1]:
eec:=subs(",eec) union ":
df:=es4(",df):

df3:=es3(df,{},{q1,c1,q4});
oo18:=[solve(df3)];"[1]:
eec:=subs(",eec) union ":
df:=es4(",df):

df3:=es3(df,{},{c1,h4,r2,h2}):
oo19:=[solve(df3)];"[1]:
eec:=subs(",eec) union ":
df:=es4(",df):

df3:=es3(df,{},{h2,c1,h4,q4});
oo20:=[solve(df3)];"[1]:
eec:=subs(",eec) union ":
df:=es4(",df):

factor(df);
oo21:=[solve(df,{q2})];"[1]:
eec:=subs(",eec) union ":
df:=es4(",df):

sol2:=subs(eec,sub1) union eec:
```

The set `sol2` represents the second and last -as will follow from the next case- solution of this case.

The second substitution matrix yields the following result:

```
df:=es4(sub2,cf):

es1(df);
solve(");
df:=es4(",df):

es1(df);
solve(");
df:=es4(",df):

es1(df);
```

```
solve(");
df:=es4(",df):

es1(df);
solve(");
df:=es4(",df):

es6(df);
df3:=es3(df,{},{h3});
oo1:=[solve(df3)];
df:=es4(oo1[1],df):

es1(df);
solve(");
df:=es4(",df):

df3:=es3(df,{},{n1});
oo2:=[solve(df3)];
```

It is easy to check that this system of equations `df3` has no solutions.

So for this case we have a list of two solutions:

```
sol:=[sol1,sol2]:
```

A.3.3 $JH^3 = 0$ and $JH^2.X \neq 0$

In this case there was only one substitution matrix. Note further that we only use `af` in this case.

```
sub3:={a1=0,a2=1/3,b2=0,b3=1,b1=0,c2=0,d1=0,c3=0,c4=0,
      a3=0,d2=0,d3=0,d4=0,a4=0,b4=0,c1=0};

df:=es4(sub3,af):
es1(df);
ee:=solve(");
df:=es4(",df):

es1(df);
solve(");
ee:=subs(",ee) union ":
df:=es4(",df):

df3:=es3(df,{},{h3,f2});
oo1:=[solve(df3)];"[1]:
ee:=subs(",ee) union ":
df:=es4(",df):

df3:=es3(df,{},{e1,h3});
oo2:=[solve(df3)];"[1]:
ee:=subs(",ee) union ":
df:=es4(",df):

df3:=es3(df,{},{g2,i3,e2,f3}):
oo3:=[solve(df3)];"[1]:
ee:=subs(",ee) union ":
df:=es4(",df):
```

```
df3:=es3(df,{},{h3,n1});
oo4:=[solve(df3)];"[1]:
ee:=subs(",ee) union ":
df:=es4(",df):

es1(df);
solve(");
ee:=subs(",ee) union ":
df:=es4(",df):

es1(df);
solve(");
ee:=subs(",ee) union ":
df:=es4(",df):

df3:=es3(df,{},{m1,i3,g4});
oo5:=[solve(df3)];"[1]:
ee:=subs(",ee) union ":
df:=es4(",df):

df3:=es3(df,{},{t1,j2,j4});
oo6:=[solve(df3)];"[1]:
ee:=subs(",ee) union ":
df:=es4(",df):

df3:=es3(df,{},{e4,e2,l4,g4});
oo7:=[solve(df3)];"[1]:
ee:=subs(",ee) union ":
df:=es4(",df):

es1(df);
solve(");"[1];
ee:=subs(",ee) union ":
df:=es4(",df):

es1(df);
solve(");
ee:=subs(",ee) union ":
df:=es4(",df):

es1(df);
solve(");"[1];
ee:=subs(",ee) union ":
df:=es4(",df):

es1(df);
solve(");
ee:=subs(",ee) union ":
df:=es4(",df):

df3:=es3(df,{},{k1,l2});
oo8:=[solve(df3)];
zf:=df:
```

Here we have to make a choice:

1. $k_1 = 0$
2. $k_1 \neq 0$

The first choice gives the following path:

```
eea:=subs(oo8[1],ee) union oo8[1]:
df:=es4(eea,zf):

df3:=es3(df,{},{o2,r3});
oo9:=[solve(df3)];"[1]:
eea:=subs(",eea) union ":
df:=es4(",df):

df3:=es3(df,{},{s2,i3});
oo10:=[solve(df3)];"[1]:
eea:=subs(",eea) union ":
df:=es4(",df):

df3:=es3(df,{},{i3,p4});
oo11:=[solve(df3)];
zf1:=df:
```

At this point we have to make a choice again. Together with the assumption made before we now have:

1. $k_1 = 0$ and $p_4 = 0$
2. $k_1 = 0$ and $p_4 \neq 0$

```
eeaa:=subs(oo11[2],eea) union oo11[2]:
df:=es4(eeaa,zf1):
es1(df);
solve(");"[1];
eeaa:=subs(",eeaa) union ":
df:=es4(",df):

df3:=es3(df,{},{p2,s3});
oo12:=[solve(df3)];"[1]:
eeaa:=subs(",eeaa) union ":
df:=es4(",df):

df3:=es3(df,{},{e4,i3,o4});
oo13:=[solve(df3)];"[1]:
zf2:=df:
ff:=eeaa:
```

Again another choice:

1. $k_1 = 0, p_4 = 0$ and $e_4 = 0$
2. $k_1 = 0, p_4 = 0$ and $e_4 \neq 0$

```
ffa:=subs(oo13[2],ff) union oo13[2]:
df:=es4(ffa,zf2):

df3:=es3(df,{},{l2,n3}):
oo14:=[solve(df3)];"[1]:
ffa:=subs(",ffa) union ":
df:=es4(",df):

es1(df);
solve(");
ffa:=subs(",ffa) union ":
```

```
df:=es4(",df):

es1(df);
solve(");
ffa:=subs(",ffa) union ":
df:=es4(",df):

df3:=es3(df,{},{i3,k4});
oo15:=[solve(df3)];
zf3:=df:
```

Again another choice:

1. $k_1 = 0, p_4 = 0, e_4 = 0$ and $i_3 = 0$
2. $k_1 = 0, p_4 = 0, e_4 = 0$ and $i_3 \neq 0$

```
ffaa:=subs(oo15[1],ffa) union oo15[1]:

es1(df);
solve(");
ffaa:=subs(",ffaa) union ":
df:=es4(",df):

es1(df);
solve(");
ffaa:=subs(",ffaa) union ":
df:=es4(",df):

es1(df);
solve(");
ffaa:=subs(",ffaa) union ":
df:=es4(",df):

df3:=es3(df,{},{t2,k4});
oo16:=[solve(df3)];
zf4:=df:
gg:=ffaa:
```

And the final choice in this branch:

1. $k_1 = 0, p_4 = 0, e_4 = 0, i_3 = 0$ and $k_4 = 0$
2. $k_1 = 0, p_4 = 0, e_4 = 0, i_3 = 0$ and $k_4 \neq 0$

```
gga:=subs(oo16[2],gg) union oo16[2]:
df:=es4(gga,zf4):
subs(sub3,gga,F);
sol3:=subs(gga,sub3) union gga:
```

At this point we have found the first solution of this case: `sol3`.

The choice $k_1 = 0, p_4 = 0, e_4 = 0, i_3 = 0$ and $k_4 \neq 0$ gives:

```
ggb:=subs(oo16[1],gg) union oo16[1]:
df3:=es3(df,{},{j2,k4});
{j2=0}:
ggb:=subs(",ggb) union ":
```

```
df:=es4(",df);

df3:=es3(df,{},{k4,j3,p3,t3,g3,m3});
{j3=0,p3=0,t3=0,g3=0,m3=0};
ggb:=subs(",ggb) union ":

subs(sub3,ggb,F);
sol4:=subs(ggb,sub3) union ggb:
```

Here we have a second solution, **sol4**.

Now we are on our way back to the top of the tree of choices. We go on with the assumption $k_1 = 0, p_4 = 0, e_4 = 0$ and $i_3 \neq 0$.

```
ffab:=subs(oo15[2],ffa) union oo15[2]:
df:=es4(ffab,zf3):
es1(df);
solve(");
ffab:=subs(",ffab) union ":
df:=es4(",df):

df3:=es3(df,{},{i3,e3});
{e3=0};
ffab:=subs(",ffab) union ":
df:=es4(",df):

df3:=es3(df,{},{m2,i3,o3});
oo17:=[solve(df3)];
#only first solution valid (i3<>0)
oo17[1];
ffab:=subs(",ffab) union ":
df:=es4(",df):

df3:=es3(df,{},{i3,m3});
{m3=0};
ffab:=subs(",ffab) union ":
df:=es4(",df):

df3:=es3(df,{},{i3,k3});
{k3=0};
ffab:=subs(",ffab) union ":
df:=es4(",df):

df3:=es3(df,{},{i3,g3,s3});
oo18:=[solve(df3,{g3})];
"[1]:
ffab:=subs(",ffab) union ":
df:=es4(",df):

df3:=es3(df,{},{s3,i3,p3});
oo19:=[solve(df3,{p3})];"[1]:
ffab:=subs(",ffab) union ":
df:=es4(",df):

subs(sub3,ffab,F);
sol5:=subs(ffab,sub3) union ffab:
```

This was the third solution of this system. Now further with $k_1 = 0, p_4 = 0$ and $e_4 \neq 0$.

```
ffb:=subs(oo13[1],ff) union oo13[1]:
df:=es4(ffb,zf2):
```

```
es1(df);
solve(");
ffb:=subs(",ffb) union ":
df:=es4(",df):

df3:=es3(df,{},{l2,n3});
oo20:=[solve(df3)];"[1]:
ffb:=subs(",ffb) union ":
df:=es4(",df):

es1(df);
solve(");
ffb:=subs(",ffb) union ":
df:=es4(",df):

es1(df);
solve(");
ffb:=subs(",ffb) union ":
df:=es4(",df):

df3:=es3(df,{},{m2,e4,j2});
{m2=0,j2=0};
ffb:=subs(",ffb) union ":
df:=es4(",df):

es1(df);
solve(");
ffb:=subs(",ffb) union ":
df:=es4(",df):

df3:=es3(df,{},{g3,e4});
{g3=0};
ffb:=subs(",ffb) union ":
df:=es4(",df):

df3:=es3(df,{},{j3,t2,m3,e4});
{m3=0,j3=0,t2=0};
ffb:=subs(",ffb) union ":
df:=es4(",df):

oo21:=[solve(df,{p3,t3})];"[1]:
ffb:=subs(",ffb) union ":
df:=es4(",df):

subs(sub3,ffb,F);
sol6:=subs(ffb,sub3) union ffb:
```

Now `sol6` contains the fourth solution. Further with $k_1 = 0$ and $p_4 \neq 0$.

```
eeab:=subs(oo11[1],eea) union oo11[1]:
df:=es4(eeab,zf1):

es1(df);
solve(");
eeab:=subs(",eeab) union ":
df:=es4(",df):

es1(df);
solve(");
eeab:=subs(",eeab) union ":
```

```
df:=es4(",df):

df3:=es3(df,{},{l2,n3});
oo22:=[solve(df3)];"[1]:
eeab:=subs(",eeab) union ":
df:=es4(",df):

es1(df);
solve(");
eeab:=subs(",eeab) union ":
df:=es4(",df):

es1(df);
solve(");
eeab:=subs(",eeab) union ":
df:=es4(",df):

df3:=es3(df,{},{j3,p4,t4,t3});
{j3=0,t3=0,t4=0};
eeab:=subs(",eeab) union ":
df:=es4(",df):

es1(df);
solve(");
eeab:=subs(",eeab) union ":
df:=es4(",df):

es1(df);
solve(");
```

This path leads to a contradiction with the assumption $p_4 \neq 0$. Further with $k_1 \neq 0$.

```
eeb:=subs(oo8[2],ee) union oo8[2]:
df:=es4(eeb,zf):

df3:=es3(df,{},{o2,r3});
oo23:=[solve(df3)];"[1]:
eeb:=subs(",eeb) union ":
df:=es4(",df):

df3:=es3(df,{},{k1,k2,m2,r3});
{m2=0,k2=0,r3=0};
eeb:=subs(",eeb) union ":
df:=es4(",df):

es1(df);
solve(");
eeb:=subs(",eeb) union ":
df:=es4(",df):

df3:=es3(df,{},{k1,i3,e2});
{e2=0,i3=0};
eeb:=subs(",eeb) union ":
df:=es4(",df):

es1(df);
solve(");
```

Also this path leads to a contradiction and since we are now back at the top of the tree we have found all solutions of this system.

```
sol:=[op(sol),sol3,sol4,sol5,sol6]:
```

A.3.4 The system of equations for rank three

In the rank three cases we didn't use JH^4 itself; we only used an equivalent set of equations described in [Wright 93]. So here we have to calculate some new systems `af`,`bf` and `cf`.

```
sub4:={a1=0,a2=1/3,a3=0,a4=0,b1=0,b2=0,b3=1,b4=0,
       c1=0,c2=0,c3=0,c4=0,d1=1,d2=0,d3=0,d4=0};
sub5:={a1=0,a2=1/3,a3=0,a4=0,b1=0,b2=0,b3=1,b4=0,
       c1=0,c2=0,c3=0,c4=1,d1=0,d2=0,d3=0,d4=0};

af:=simplifyM(M,4)[2]:

N3X:=evalm(N3*X):
N3XX:=[];

for i from 1 to 4 do
N3XX:=[op(N3XX),collect(N3X[i],X,distributed)]:od:

bf:={};
for i from 1 to 4 do
bf:={op(bf),coeffs(N3XX[i],X)}: od:

cf:=af union bf:
```

A.3.5 $JH^4 = 0$ and $JH^3.X = 0$

The process goes similar as in the previous sections. So we omit the comments between the different parts of the Maple session. There are only some comments to indicate in what branch of the tree we are working.

```
df := es4(sub4,cf):

ee:=sub4;

es1(df);
solve(");
ee:=subs(",ee) union ":
df:=es4(",df):

es1(df);
solve(");
ee:=subs(",ee) union ":
df:=es4(",df):

df3:=es3(df,{},{g2,i3});
oo1:=[solve(df3)];"[1]:
ee:=subs(",ee) union ":
df:=es4(",df):

df3:=es3(df,{},{j4});
```

```
oo2:=[solve(df3)];"[1]:
ee:=subs(",ee) union ":
df:=es4(",df):

df3:=es3(df,{},{e4,f1,g4,i1,g1,i2}):
oo3:=[solve(df3)];"[1]:
ee:=subs(",ee) union ":
df:=es4(",df):

df3:=es3(df,{},{h3,f2,e2,f3,e4,h2}):
oo4:=[solve(df3)];"[1]:
ee:=subs(",ee) union ":
df:=es4(",df):

es1(df);
solve(");
ee:=subs(",ee) union ":
df:=es4(",df):

es1(df);
solve(");
ee:=subs(",ee) union ":
df:=es4(",df):

df3:=es3(df,{},{e1,h3,n2,q3});
oo5:=[solve(df3)];"[1]:
ee:=subs(",ee) union ":
df:=es4(",df):

df3:=es3(df,{},{o2,r3});
oo6:=[solve(df3)];"[1]:
ee:=subs(",ee) union ":
df:=es4(",df):

df3:=es3(df,{},{n4,q3,h3});
oo7:=[solve(df3)];
zf:=df:
##############################n4=0
eea:=subs(oo7[1],ee) union oo7[1]:
df:=es4(eea,zf):
df3:=es3(df,{},{q1,h3,e4});
oo8:=[solve(df3)];
zf1:=df:
##############################n4=0,h3=0
eeaa:=subs(oo8[1],eea) union oo8[1]:
df:=es4(eeaa,zf1):
df3:=es3(df,{},{n1,l4});
oo9:=[solve(df3)];"[1]:
eeaa:=subs(",eeaa) union ":
df:=es4(",df):

df3:=es3(df,{},{n1,e2,e4});
oo10:=[solve(df3)];"[1]:
eeaa:=subs(",eeaa) union ":
df:=es4(",df):

es1(df);
solve(");
eeaa:=subs(",eeaa) union ":
df:=es4(",df):

es1(df);
```

```
solve(");"[1];
eeaa:=subs(",eeaa) union ":
df:=es4(",df):

df3:=es3(df,{},{i3,o4,e4,r1});
oo11:=[solve(df3)];"[1]:
eeaa:=subs(",eeaa) union ":
df:=es4(",df):

es1(df);
solve(");
eeaa:=subs(",eeaa) union ":
df:=es4(",df):

es1(df);
solve(");
eeaa:=subs(",eeaa) union ":
df:=es4(",df):

df3:=es3(df,{},{l2,n3,r3});
oo12:=[solve(df3)];"[1]:
eeaa:=subs(",eeaa) union ":
df:=es4(",df):

df3:=es3(df,{},{k2,l3,k1,e2});
oo13:=[solve(df3)];"[1]:
eeaa:=subs(",eeaa) union ":
df:=es4(",df):

es1(df);
solve(");
eeaa:=subs(",eeaa) union ":
df:=es4(",df):

es1(df);
solve(");
eeaa:=subs(",eeaa) union ":
df:=es4(",df):

es1(df);
solve(");
eeaa:=subs(",eeaa) union ":
df:=es4(",df):

es1(df);
solve(");
eeaa:=subs(",eeaa) union ":
df:=es4(",df):

df3:=es3(df,{},{p2,s3,j1,i2,m2,o3});
oo14:=[solve(df3)];"[1]:
eeaa:=subs(",eeaa) union ":
df:=es4(",df):

df3:=es3(df,{},{m1,i2,e3});
oo15:=[solve(df3)];
{m1=0};
eeaa:=subs(",eeaa) union ":
df:=es4(",df):

df3:=es3(df,{},{p4,i2,m2,k3});
oo16:=[solve(df3)];"[1]:
```

```
eeaa:=subs(",eeaa) union ":
df:=es4(",df):

es1(df);
solve(");
eeaa:=subs(",eeaa) union ":
df:=es4(",df):

df3:=es3(df,{},{p1,i3}):
oo17:=[solve(df3)];"[1]:
#no solutions
##############################n4=0,h3<>0
eeab:=subs(oo8[2],eea) union oo8[2]:
df:=es4(eeab,zf1):

es1(df);
solve(");
eeab:=subs(",eeab) union ":
df:=es4(",df):

df3:=es3(df,{},{h3,k4,n1});
oo18:=[solve(df3)];
#no solutions h3<>0
##############################n4<>0
eeb:=subs(oo7[2],ee) union oo7[2]:
df:=es4(eeb,zf):

es1(df);
#no solutions n4<>0
```

So this case had no solutions.

A.3.6 $JH^4 = 0$ and $JH^3.X \neq 0$

```
df:=es4(sub5,af):

ee:=sub5;
es1(df);
solve(");
ee:=subs(",ee) union ":
df:=es4(",df):

es1(df);
solve(");
ee:=subs(",ee) union ":
df:=es4(",df):

df3:=es3(df,{},{h1,i2,f1,g2,j3}):
oo1:=[solve(df3)];"[1]:
ee:=subs(",ee) union ":
df:=es4(",df):

df3:=es3(df,{},{g2,i3,j4,m2,o3,p4});
oo2:=[solve(df3)];"[1]:
ee:=subs(",ee) union ":
df:=es4(",df):
```

```
df3:=es3(df,{},{o2,r3,s4,e1,f2,g3,i3,j4,h2}):
oo3:=[solve(df3)];"[1]:
ee:=subs(",ee) union ":
df:=es4(",df):

df3:=es3(df,{},{e2,f3,g4,p2,s3,t4}):
oo4:=[solve(df3)];"[1]:
ee:=subs(",ee) union ":
df:=es4(",df):

df3:=es3(df,{},{e1,g3,h3,i4,e1,k2,l3,m4,j3,n2,q3,r4}):
oo5:=[solve(df3)];"[1]:
ee:=subs(",ee) union ":
df:=es4(",df):

df3:=es3(df,{},{s1,i3,j3,h2});
oo6:=[solve(df3)];"[1]:
ee:=subs(",ee) union ":
df:=es4(",df):

df3:=es3(df,{},{h2, t2, i3, j3});
oo7:=[solve(df3)];"[1]:
ee:=subs(",ee) union ":
df:=es4(",df):

df3:=es3(df,{},{p1, e1, h2, i3, j3}):
oo8:=[solve(df3[2])];
"[1]:
ee:=subs(",ee) union ":
df:=es4(",df):

df3:=es3(df,{},{m1,i3,g4,h2,g3,e1,i3,h2,l2,n3,o4}):
oo9:=[solve(df3)];"[1]:
ee:=subs(",ee) union ":
df:=es4(",df):

df3:=es3(df,{},{e1, h2, i3, j3});
oo10:=[solve(df3)];
zf:=df:
############################j3=0
eea:=subs(oo10[1],ee) union oo10[1]:
df:=es4(eea,zf):
df3:=es3(df,{},{h2,i3,r1});
oo11:=[solve(df3)];"[1]:
eea:=subs(",eea) union ":
df:=es4(",df):

df3:=es3(df,{},{h2,i3,s2});
oo12:=[solve(df3)];"[1]:
eea:=subs(",eea) union ":
df:=es4(",df):

df3:=es3(df,{},{i3,h2,t3});
oo13:=[solve(df3)]:
zf1:=df:
############################j3=0,h2=0
eeaa:=subs(oo13[1],eea) union oo13[1]:
df:=es4(eeaa,zf1):

es1(df);
solve(");
eeaa:=subs(",eeaa) union ":
```

```
df:=es4(",df):

df3:=es3(df,{},{s3,t4});
oo14:=[solve(df3)];"[1]:
eeaa:=subs(",eeaa) union ":
df:=es4(",df):

es1(df);
solve(");
eeaa:=subs(",eeaa) union ":
df:=es4(",df):

es1(df);
solve(");
eeaa:=subs(",eeaa) union ":
df:=es4(",df):

df3:=es3(df,{},{p3,s4,r3});
oo15:=[solve(df3)];
zf2:=df:
ff:=eeaa:

############################j3=0,h2=0,p3=0
ffa:=subs(oo15[1],ff) union oo15[1]:
df3:=es3(df,{},{r3,s4});
oo16:=[solve(df3)];"[1]:
ffa:=subs(",ffa) union ":
df:=es4(",df):

df3:=es3(df,{},{m2,p4});
oo17:=[solve(df3)];"[1]:
ffa:=subs(",ffa) union ":
df:=es4(",df):

df3:=es3(df,{},{q2,m3});
oo18:=[solve(df3)];
zf3:=df:
############################j3=0,h2=0,p3=0,m3=0
ffaa:=subs(oo18[2],ffa) union oo18[2]:
df:=es4(ffaa,zf3):
df3:=es3(df,{},{g3,h3,e1});
oo19:=[solve(df3)];

zf4:=df:
gg:=ffaa:
############################j3=0,h2=0,p3=0,m3=0,g3=0
gga:=subs(oo19[1],gg) union oo19[1]:
df3:=es3(df,{},{e1, n1, h3});
oo20:=[solve(df3[1])];
"[1];
gga:=subs(",gga) union ":
df:=es4(",df):

df3:=es3(df,{},{h3,e1,h3,e1,q2,n2});
oo21:=[solve(df3)];"[1]:
gga:=subs(",gga) union ":
df:=es4(",df):

es1(df);
solve(");
gga:=subs(",gga) union ":
df:=es4(",df):
```

```
es1(df);
solve(");
"[1];
gga:=subs(",gga) union ":
df:=es4(",df):

df3:=es3(df,{},{n3,o4});
oo22:=[solve(df3)];"[1]:
gga:=subs(",gga) union ":
df:=es4(",df):

df3:=es3(df,{},{k1, e2, g4});
oo23:=[solve(df3)];"[1]:
gga:=subs(",gga) union ":
df:=es4(",df):

oo24:=[solve(df)];"[1]:
gga:=subs(",gga) union ":
subs(gga,F);

sol7:=gga:

############################j3=0,h2=0,p3=0,m3=0,g3<>0
ggb:=subs(oo19[2],gg) union oo19[2]:
df:=es4(",zf4):

df3:=es3(df,{},{n1,q2,q3,g3});
oo25:=[solve(df3)];
#no solutions g3<>0
############################j3=0,h2=0,p3=0,m3<>0
ffab:=subs(oo18[1],ffa) union oo18[1]:
df:=es4(",zf3):
df3:=es3(df,{},{n2,q3});
oo26:=[solve(df3)];"[1]:
ffab:=subs(",ffab) union ":
df:=es4(",df):

df3:=es3(df,{},{e1, n1, g3, h3});
oo27:=[solve(df3)];"[1]:
ffab:=subs(",ffab) union ":
df:=es4(",df):

es1(df);
solve(");
ffab:=subs(",ffab) union ":
df:=es4(",df):

df3:=es3(df,{},{n3,m3});
{n3=0}:
ffab:=subs(",ffab) union ":
df:=es4(",df):

df3:=es3(df,{},{m3,o4});
{o4=0};
ffab:=subs(",ffab) union ":
df:=es4(",df):

df3:=es3(df,{},{m3,n4,h4});
{h4=0,n4=0};
ffab:=subs(",ffab) union ":
df:=es4(",df):
```

```
df3:=es3(df,{},{m3,q4});
{q4=0};
ffab:=subs(",ffab) union ":
df:=es4(",df):

df3:=es3(df,{},{k1, e2, m3, g4});
oo28:=[solve(df3)];"[1]:
ffab:=subs(",ffab) union ":
df:=es4(",df):
#note from now on g4<>0 (since m3=g4^2)

df3:=es3(df,{},{k2, l3, f4, g4});
oo29:=[solve(df3[1])];
"[2];
ffab:=subs(",ffab) union ":
df:=es4(",df):

oo30:=[solve(df)];
"[1];
ffab:=subs(",ffab) union ":
df:=es4(",df):

subs(ffab,F);

sol8:=ffab:

############################j3=0,h2=0,p3<>0
ffb:=subs(oo15[2],ff) union oo15[2]:
df:=es4(",zf2):
df3:=es3(df,{},{q2,p3});
{q2=0};
ffb:=subs(",ffb) union ":
df:=es4(",df):

df3:=es3(df,{},{n2,q3});
oo31:=[solve(df3)];"[1]:
ffb:=subs(",ffb) union ":
df:=es4(",df):

df3:=es3(df,{},{p3,q4});
{q4=0};
ffb:=subs(",ffb) union ":
df:=es4(",df):

df3:=es3(df,{},{p3,e1,g3});
oo32:=[solve(df3)];
zf5:=df:
############################j3=0,h2=0,p3<>0,e1=0
ffba:=subs(oo32[1],ffb) union oo32[1]:
df:=es4(ffba,zf5):
df3:=es3(df,{},{n1, g3, h3});
oo33:=[solve(df3)];"[1]:
ffba:=subs(",ffba) union ":
df:=es4(",df):

es1(df);
#no solutions p3<>0
############################j3=0,i3=0,p3<>0,e1<>0
ffbb:=subs(oo32[2],ffb) union oo32[2]:
df:=es4(ffbb,zf5):
#note from now on g3<>0 since p3=-1/2 g3^2
```

```
df3:=es3(df,{},{g3,e1});
{g3=0};
# no solutions g3<>0
############################j3=0,i3<>0
eeab:=subs(oo13[2],eea) union oo13[2]:
df:=es4(",zf1):

df3:=es3(df,{},{i3,t4});
{t4=0};
eeab:=subs(",eeab) union ":
df:=es4(",df):

df3:=es3(df,{},{i3,g3,s3});
oo33:=[solve(df3)];
"[2]:
eeab:=subs(",eeab) union ":
df:=es4(",df):

df3:=es3(df,{},{p3,i3,s3});
oo34:=[solve(df3,{p3})];
"[1]:
eeab:=subs(",eeab) union ":
df:=es4(",df):

df3:=es3(df,{},{i3,p4,s3,s4});
oo35:=[solve(df3,{p4})];
"[1]:
eeab:=subs(",eeab) union ":
df:=es4(",df):

df3:=es3(df,{},{i3,o1,s3});
oo36:=[solve(df3)];
"[2];
eeab:=subs(",eeab) union ":
df:=es4(",df):

df3:=es3(df,{},{i3,s3,e1,h3});
oo37:=[solve(df3)];
"[3];
eeab:=subs(",eeab) union ":
df:=es4(",df):

es1(df);
solve(");
eeab:=subs(",eeab) union ":
df:=es4(",df):

es1(df);
solve(");
eeab:=subs(",eeab) union ":
df:=es4(",df):

df3:=es3(df,{},{i3,m2,m3,g4});
oo38:=[solve(df3)];
"[2]:
eeab:=subs(",eeab) union ":
df:=es4(",df):

df3:=es3(df,{},{i3,s4,e3,k1,k3});
oo39:=[solve(df3)];
"[2]:
eeab:=subs(",eeab) union ":
```

```
df:=es4(",df):

df3:=es3(df,{},{i3,e4});
oo40:=[solve(df3)];
#no solutions i3<>0
#############################j3<>0
eeb:=subs(oo10[2],ee) union oo10[2]:
df:=es4(eeb,zf):

df3:=es3(df,{},{r1, s2, h2, g3, j3, i3});
oo41:=[solve(df3)];"[1]:
eeb:=subs(",eeb) union ":
df:=es4(",df):

df3:=es3(df,{},{s2, h2, g3, h3, j3, i3,t3});
oo42:=[solve(df3)];"[1]:
eeb:=subs(",eeb) union ":
df:=es4(",df):

df3:=es3(df,{},{o1, h2, g3, h3, j3, i3, g4});
oo43:=[solve(df3)];"[1]:
eeb:=subs(",eeb) union ":
df:=es4(",df):

df3:=es3(df,{},{h2, g3, h3, j3, i3, t4});
oo44:=[solve(df3,{t4})];"[1]:
eeb:=subs(",eeb) union ":
df:=es4(",df):

df3:=es3(df,{},{o1, h2, g3, h3, j3, i3}):
oo45:=[solve(df3)];
zf6:=df:
####################################j3<>0,h2=-3/4 i3
eeba:=subs(oo45[1],eeb) union oo45[1]:
df:=es4(eeba,zf6):

df3:=es3(df,{},{s3, o1, e2, g3, h3, j3, i3}):
oo46:=[solve(df3)];"[1]:
eeba:=subs(",eeba) union ":
df:=es4(",df):

df2:=es2(df):
oo47:=[solve(df2)];"[1]:
eeba:=subs(",eeba) union ":
df:=es4(",df):

df3:=es3(df,{},{s3, g3, h3, j3, i3}):
oo48:=[solve(df3)]:
zf7:=df:
hh:=eeba:
# four choices
################################j3<>0,s3=0
hha:=subs(oo48[1],hh) union oo48[1]:
df:=es4(hha,zf7):
df3:=es3(df,{},{j3,p3,s4,r3});
oo49:=[solve(df3)];
"[3];
hha:=subs(",hha) union ":
df:=es4(",df):

df3:=es3(df,{},{p4,j3});
{p4=0};
```

```
hha:=subs(",hha) union ":
df:=es4(",df):

df3:=es3(df,{},{r3, j3, i3, f4});
oo50:=[solve(df3)];"[1]:
hha:=subs(",hha) union ":
df:=es4(",df):

df3:=es3(df,{},{m2, j3, i3, e4});
oo51:=[solve(df3)];"[1]:
hha:=subs(",hha) union ":
df:=es4(",df):

df3:=es3(df,{},{q2, r2, r3, k1, n1, m2, j3, i3}):
oo52:=[solve(df3)];"[1]:
hha:=subs(",hha) union ":
df:=es4(",df):

df3:=es3(df,{},{j3,n2});
oo53:=[solve(df3,{n2})];"[1]:
hha:=subs(",hha) union ":
df:=es4(",df):

df3:=es3(df,{},{j3,q3});
oo54:=[solve(df3)];
#no solutions since j3<>0
################################j3<>0,s3<>0(= i3<>0, 8j3g3<>i3^2)
hhb:=subs(oo48[2],hh) union oo48[2]:
df:=es4(hhb,zf7):

df3:=es3(df,{},{p3, s4, g3, j3, i3}):
oo55:=[solve(df3)]:
"[1]:
hhb:=subs(",hhb) union ":
df:=es4(",df):

df3:=es3(df,{},{q2, r2, r3, n1, g3, j3, i3, f4}):
oo56:=[solve(df3)];
subs(hhb,oo56[1],s3);
#no solutions since s3<>0
################################j3<>0,s3<>0(= i3<>0, 8j3g3<>i3^2)
hhc:=subs(oo48[3],hh) union oo48[3]:
df:=es4(hhc,zf7):

df3:=es3(df,{},{p3, s4, g3, j3, i3}):
oo57:=[solve(df3)];
"[1]:
hhc:=subs(",hhc) union ":
df:=es4(",df):

df3:=es3(df,{},{q2, r2, r3, n1, g3, j3, i3, f4}):
oo58:=[solve(df3)];
subs(hhc,oo58[1],s3);
#no solutions since s3<>0
################################j3<>0,s3<>0(= i3<>0, 24j3g3<>16h3j3-i3^2)
hhd:=subs(oo48[4],hh) union oo48[4]:
df:=es4(hhd,zf7):

df3:=es3(df,{},{p4, p3, s4, g3, h3, j3, i3}):
oo59:=[solve(df3)]:
zf8:=df:
################################
```

```
hhda:=subs(oo59[1],hhd) union oo59[1]:
df:=es4(hhda,zf8):

df3:=es3(df,{},{r3, m2, g3, h3, j3, i3, e4, f4}):
oo60:=[solve(df3)];"[1]:
hhda:=subs(",hhda) union ":
df:=es4(",df):

df3:=es3(df,{},{k1, m2, g3, h3, j3, i3});
oo61:=[solve(df3)];"[1]:
hhda:=subs(",hhda) union ":
df:=es4(",df):

df3:=es3(df,{},{r2, r3, n1, m2, g3, h3, j3, i3}):
oo62:=[solve(df3)]:
nops(");
for i from 1 to 9 do
subs(hhda,oo62[i],s3) od;
#so only solution 6 and 8 are valid
zf9:=df:
ii:=hhda:
###############################
iia:=subs(oo62[6],ii) union oo62[6]:
df:=es4(iia,zf9):

df3:=es3(df,{},{q2, r2, q3, n2, h3, j3, i3, o4}):
oo63:=[solve(df3)]:
nops(");
#in order to check that there are really no solutions in these 15 equations:
df4:=es3(df3,{},{q2, r2, n2, h3, j3, i3}):
oo64:=[solve(df4)];
df3:=es4(oo64[1],df3):
df4:=es3(df3,{},{r2, q3, h3, j3, i3});
oo65:=[solve(df4)];
#so no solutions in df4 so we can restrict ourselves again
df3:=df4:
oo66:=[solve(df3[4])];
df3:=es4(oo66[1],df3);
df3:=factor(df3);
oo67:=[solve(df3[1])];
df3:=factor(es4(oo67[1],df3));
oo68:=[solve(df3[2])];
df3:=factor(es4(oo68[1],df3));
#this leads to a contradiction since j3<>0
###############################
iib:=subs(oo62[8],ii) union oo62[8]:
df:=es4(iib,zf9):

df3:=es3(df,{},{q2, r2, n2, h3, j3, i3, o4}):
oo69:=[solve(df3)]:
subs(iib,oo69[1],s3);
subs(iib,oo69[2],s3);
oo69[1]:
iib:=subs(",iib) union ":
df:=es4(",df):

df3:=es3(df,{},{r2, q3, h3, j3, i3, m3}):
oo70:=[solve(df3)]:
#no solutions in 14 equations
df4:=df3:
oo71:=[solve(df4[1])]:
df4:=factor(es4(oo71[2],df4)):
```

```
oo72:=[solve(df4[10])];
subs(iib,oo71[2],oo72[1],s3);
#this yields zero so certainly no valid solutions
################################
hhdb:=subs(oo59[2],hhd) union oo59[2]:
df:=es4(hhdb,zf8):

df3:=es3(df,{},{r2, p3, r3, n1, m2, g3, j3, i3, e4, f4}):
oo73:=[solve(df3)]:
subs(hhdb,oo73[1],s3);
subs(hhdb,oo73[2],s3);
oo73[2]:
hhdb:=subs(",hhdb) union ":
df:=es4(",df):

df3:=es3(df,{},{q2, r2, k1, n2, g3, j3, i3}):
oo74:=[solve(df3)]:
subs(hhdb,oo74[1],s3);
oo74[1]:
hhdb:=subs(",hhdb) union ":
df:=es4(",df):

df3:=es3(df,{},{r2, q3, g3, j3, i3}):
oo75:=[solve(df3)]:
#no solutions in 5 equations
df4:=df3:
oo76:=[solve(df4[1])]:
subs(hhdb,oo76[1],s3);
subs(hhdb,oo76[2],s3);
df4:=factor(es4(oo76[1],df4)):
oo77:=[solve(df4[2])]:
subs(hhdb,oo76[1],oo77[1],s3);
df4:=factor(es4(oo77[1],df4)):
oo78:=[solve(df4[3])]:
#no solutions j3<>0
####################################j3<>0,h2<>-3/4 i3
eebb:=subs(oo45[2],eeb) union oo45[2]:
df:=es4(eebb,zf6):

df3:=es3(df,{},{q1, r2, s3, e2, h2, h3, j3, i3, h4}):
oo79:=[solve(df3)]:
"[1]:
eebb:=subs(",eebb) union ":
df:=es4(",df):

df3:=es3(df,{},{p3, s4, h2, h3, j3, i3}):
oo80:=[solve(df3)];"[1]:
eebb:=subs(",eebb) union ":
df:=es4(",df):

df3:=es3(df,{},{p4, h2, h3, j3, i3}):
oo81:=[solve(df3)]:
"[2]:
eebb:=subs(",eebb) union ":
df:=es4(",df):

df3:=es3(df,{},{r2, h2, h3, j3, i3, f4}):
oo82:=[solve(df3)]:
"[1]:
eebb:=subs(",eebb) union ":
df:=es4(",df):
```

```
df3:=es3(df,{},{r2, r3, l1, h2, m2, e3, h3, j3, i3}):
oo83:=[solve(df3)]:
"[1]:
eebb:=subs(",eebb) union ":
df:=es4(",df):

df3:=es3(df,{},{s4, h2, h3, j3, i3}):
oo84:=[solve(df3[3])]:
"[1]:
eebb:=subs(",eebb) union ":
df:=es4(",df):

df3:=es3(df,{},{r2, n1, h2, h3, j3, i3, e4,r3}):
df5:=es3(df,{},{h2, h3, j3, i3}):
df3:=df3 minus df5:
oo85:=[solve(df3)]:
for i from 1 to 6 do subs(oo85[i],[h2,i3]) od;
#four choices
jj:=eebb:
zf10:=df:
##############################r2<>0
jja:=subs(oo85[1],jj) union oo85[1]:
df:=es4(jja,zf10):

df3:=es3(df,{},{k1,j3,r2,h3});
oo86:=[solve(df3)]:
"[2]:
jja:=subs(",jja) union ":
df:=es4(",df):

es1(df);
solve(");
jja:=subs(",jja) union ":
df:=es4(",df):
df3:=es3(df,{},{j3,r2});
# no solutions since both j3 and r2 <> 0
##############################r2=0
jjb:=subs(oo85[2],jj) union oo85[2]:
df:=es4(jjb,zf10):

es1(df);
solve(");
jjb:=subs(",jjb) union ":
df:=es4(",df):

df3:=es3(df,{},{j3,q3,m2,i3,k1});
oo87:=[solve(df3,{q3,m2,i3,k1})];"[1]:
jjb:=subs(",jjb) union ":
df:=es4(",df):

df3:=es3(df,{},{j3,n2});
oo88:=[solve(df3)];
# no solutions since j3<>0
##############################r2<>0,h3<>0
jjc:=subs(oo85[4],jj) union oo85[4]:
df:=es4(jjc,zf10):

df3:=es3(df,{},{h2, h3, i3}):
oo89:=[solve(df3)]:
#since h2<>-3/4 i3 only first solution possible, but
subs(oo89[1],jjc):
#Error, division by zero
```

```
#so no solutions
###############################r2<>0,h3=0
jjd:=subs(oo85[6],jj) union oo85[6]:
jjd;
%1;
[allvalues(")];
jjda:=subs(%1="[1],jjd):
jjdb:=subs(%1=""[2],jjd):
###############################
jjda:=simplify(jjda):
df:=es4(jjda,zf10):

df3:=es3(df,{},{r2});
oo90:=[solve(df3)];
#no solution since r2<>0
###############################
jjdb:=simplify(jjdb):
df:=es4(jjdb,zf10):

df3:=es3(df,{},{r2});
oo91:=[solve(df3)];
#no solution since r2<>0
###############################
```

To give an indication of the complexity of this solve process: at the end of this session Maple showed us the following characteristics:

```
bytes used=3229462752, alloc=113946236, time=46606.15
```

And this was not even the complete process. Because of a reboot of our machine we couldn't do it in one session.

A.4 Solving the Drużkowski system

The druzkowski system was computed with **simplifyM**. Here is the session in which we solved the system:

```
read part5;
F:=maakdruz(4);
H:=['x.i-F[i]'$'i'=1..4];
H:=expand(H);
H:=sort(H);
M:=array(1..4,1..20);
for i from 1 to 4
do
  for j from 1 to 20
  do
    M[i,j]:=coeffs(op(j,H[i]),[x1,x2,x3,x4]):
  od
od:
df:=simplifyM(M,4)[2]:
```

```
read druzkowskistelsel:
cf:=df:
een:={a4=0,b4=0,c4=0,d4=0};
cf:=es4(een,df):
df:=cf:
df3:=es3(df,{},{a1, b2, c3, c1, c2});
oo1:=[solve(df3)];
ffa:=oo1[1]:
ffb:=oo1[2]:
ffc:=oo1[3]:
###########################ffc: c1=0,c2=0
df:=es4(ffc,df):
df3:=es3(df,{},{a1,a2,b2});
oo2:=[solve(df3)];
ffca:=subs(oo2[1],ffc) union oo2[1]:
ffcb:=subs(oo2[2],ffc) union oo2[2]:
###############################ffca: a2<>0
df:=es4(ffca,cf):
df3:=es3(df,{},{a1,b1,a2});
oo4:=[solve(df3)];
ffcaa:=subs(oo4[1],ffca) union oo4[1]:
ffcab:=subs(oo4[2],ffca) union oo4[2]:
###################################ffcaa: a1=0
df:=es4(ffcaa,df):
subs(een,ffcaa,F);
###################################ffcab: a1<>0
df:=es4(ffcab,cf):
oo5:=[solve(df)];
#only second solution is valid (a1<>0)
ffcab:=subs(oo5[2],ffcab) union oo5[2]:
subs(een,ffcab,F);
###############################ffcb: a2=0
df:=es4(ffcb,df):
oo3:=[solve(df)];
ffcb := subs(oo3[1],ffcb) union oo3[1]:
subs(een,ffcb,F);
###########################ffb: c1=0,c2<>0
df:=es4(ffb,cf):
df3:=es3(df,{},{c3, b3, c2});
oo6:=[solve(df3)];
ffba:=subs(oo6[1],ffb) union oo6[1]:
ffbb:=subs(oo6[2],ffb) union oo6[2]:
###############################ffba: c3=0
df:=es4(ffba,df):
es1(df);
ffba:=subs({a1=0},ffba) union {a1=0}:
df:=es4(ffba,df):
df3:=es3(df,{},{c2,a3,b3,b1});
oo7:=[solve(df3)];
#only first and fourth solution valid (c2<>0)
ffbaa:=subs(oo7[1],ffba) union oo7[1]:
ffbab:=subs(oo7[4],ffba) union oo7[4]:
###################################ffbaa: a3=0
df:=es4(ffbaa,df):
df3:=es3(df,{},{c2,b1,a2,b3});
oo8:=[solve(df3,{a2,b1,b3})];
ffbaaa:=subs(oo8[1],ffbaa) union oo8[1]:
```

```
ffbaab:=subs(oo8[2],ffbaa) union oo8[2]:
ffbaac:=subs(oo8[3],ffbaa) union oo8[3]:
#######################################ffbaaa: a2=0
# no new solution: symmetry a1=a2=a3=a4=0 to c1=c2=c3=c4=0
#######################################ffbaab: a2<>0,b1<>0
df:=es4(ffbaab,cf):
oo9:=[solve(df3,{b3,d1,d3})];
ffbaaba:=subs(oo9[1],ffbaab) union oo9[1]:
ffbaabb:=subs(oo9[2],ffbaab) union oo9[2]:
###########################################ffbaaba: b3=0
# no new solution: symmetry a1=a2=a3=a4=0 to c1=c2=c3=c4=0
###########################################ffbaabb: b3<>0
df:=es4(ffbaabb,cf):
subs(een,ffbaabb,F);
#######################################ffbaac: a2<>0,b1=0
# no new solution: symmetry b1=b2=b3=b4=0 to c1=c2=c3=c4=0
###################################ffbab: a3<>0
# no new solution: symmetry b1=b2=b3=b4=0 to c1=c2=c3=c4=0
###############################ffbb: c3<>0
df:=es4(ffbb,cf):
df3:=es3(df,{},{c3,c2,a2,a3});
oo10:=[solve(df3,{a2})];
ffbb:=subs(oo10[1],ffbb) union oo10[1]:
df:=es4(ffbb,df):
es1(df);
ffbb:=subs({a1=0},ffbb) union {a1=0}:
df:=es4(ffbb,df):
df3:=es3(df,{},{b1,a3,c3});
oo11:=[solve(df3,{b1,a3})];
ffbba:=subs(oo11[1],ffbb) union oo11[1]:
ffbbb:=subs(oo11[2],ffbb) union oo11[2]:
###################################ffbba: a3=0
# no new solution: symmetry a1=a2=a3=a4=0 to c1=c2=c3=c4=0
###################################ffbbb: a3<>0
df:=es4(ffbbb,cf):
df3:=es3(df,{},{c3,c2,d2,d3});
oo12:=[solve(df3,{d2})];
ffbbb:=subs(oo12[1],ffbbb) union oo12[1]:
df:=es4(ffbbb,df);
subs(een,ffbbb,F);
###########################ffa: c1<>0
df:=es4(ffa,cf):
df3:=es3(df,{},{b2, c3, c1, a2, c2, a3}):
oo13:=[solve(df3[1],{a2})];
ffa:=subs(oo13[1],ffa) union oo13[1]:
df:=es4(ffa,df):
df3:=es3(df,{},{b2, c3, c1, c2, a3}):
oo14:=[solve(df3,{b2,a3})];
ffaa:=subs(oo14[1],ffa) union oo14[1]:
ffab:=subs(oo14[2],ffa) union oo14[2]:
###############################ffaa:
# no new solution a1=0
###############################ffab:
df:=es4(ffab,cf):
df3:=es3(df,{},{c1, b2, c2, b1, c3, b3}):
df3[8];
oo15:=[solve(df3[8],{b2,b3})];
```

```
ffaba:=subs(oo15[1],ffab) union oo15[1]:
ffabb:=subs(oo15[2],ffab) union oo15[2]:
###################################ffaba:
# no new solution a1=0
###################################ffabb:
df:=es4(ffabb,df):
df3:=es3(df,{},{c1, b2, c2, b1, c3}):
df3[11];
oo16:=[solve(df3,{b2})];
#only second and third valid
ffabba:=subs(oo16[2],ffabb) union oo16[2]:
ffabbb:=subs(oo16[3],ffabb) union oo16[3]:
#######################################ffabba:
# no new solution a1=0
#######################################ffabbb:
df:=es4(ffabbb,df):
oo17:=[solve(df[5],{d3})];
ffabbb:=subs(oo17[1],ffabbb) union oo17[1]:
df:=es4(ffabbb,df):
oo18:=[solve(df,{d1,b1})];
#only second and third valid
ffabbba:=subs(oo18[2],ffabbb) union oo18[2]:
ffabbbb:=subs(oo18[3],ffabbb) union oo18[3]:
###########################################ffabbba:
df:=es4(ffabbba,df);
subs(een,ffabbba,F);
###########################################ffabbbb:
# no new solution a1=0

ffabbba:=simplify(ffabbba):
druzo:=[ffcaa,ffcab,ffcb,ffbaabb,ffbbb,ffabbba]:

G:=[];
for i from 1 to 6 do
G:=[op(G),subs(een,druzo[i],F)]:
od:

D:=maakdruzmat(druzo):
powersim(D[6],2):
opsys1[1];
```

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