A conference on Polynomial Maps and the Jacobian Conjecture

In honour of the mathematical work of Gary Meisters

May 9–10, 1997

Edited by Engelbert Hubbers
University of Nijmegen
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April 1998

Forword

I think it was in july 1996 when I was asked by the Department of Mathematics of the University of Nijmegen to organise a one day symposium for my thesis supervisor A.H.M. Levelt, at the occasion of his retirement. Almost immediately the idea crossed my mind to do the same for Gary, because I knew that he would become 65 on februari 17, 1997 and that he was close to his retirement. I e-mailed this idea to Bo Deng and Brian Coomes. Both reacted immediately very enthousiastic. Ofcourse everyone who knows Gary understands this enthousiasm!

At this place I like to thank Brian Coomes, Bo Deng, Jerry Johnson, David Pitts, and in particular the Department of Mathematics and Statistics at the University of Nebraska-Lincoln for all the support and organisation, which really made this conference a great success. And last, but certainly not least, a special word of thanks to Engelbert Hubbers who did all the editing work which made the appearence of this proceedings in this form possible!

I knew Gary personally since 1992 when my wife Sandra and our daughter Raïssa (then 5) had un unforgettable stay at Gary and Mary Ellen's house in Lincoln. We learned to know a man with great human warmth and exceptional humor. Furthermore he talked and talked and received from Raïssa the name 'King of the talking frogs' . It is this property, combined with great mathematical insight that made Gary into an apostal for the area called 'Polynomial automorphisms'. In particular he always came up with many questions and conjectures; for some of them he even rewarded money! This eventually led to the solution of the Markus-Yamabe Conjecture. His many pioneering papers (often with Czeslaw Olech) had a great influence on many research done in the area of polynomial automorphisms. This proceedings amply demonstrate this statement!

Finally, Gary and Mary Ellen I wish you many more happy and healthy years, together with Gary's NEXT computer, all his books, his many ideas and not to forget his great humor.

Arno van den Essen.

Barcelona, April 1998.

From left to right:

First row:

Anna Cima, Raïssa van den Essen, Kristi Lampe, Jie-Tai Yu,
L. Andrew Campbell, Marc Chamberland, Mary Ellen Meisters,
T. Parthasarathy, Sandra van den Essen, Daniel Daigle

Second row:

David Finston, David Wright, Brian Coomes, Pascal Adjamagbo,
Arno van den Essen, Gary Hosler Meisters, Czeslaw Olech, Jack K. Hale

Third row:

Engelbert Hubbers, Rodney Nillsen, Armengol Gasull, Sadahiro Saeki,
Allan Peterson, Jim Deveney, Robert McLeod, Carl Langenhop,
Gene Freudenburg, Carmen Chicone, Jerry Bebernes, Gianluca Gorni

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All pictures except the one above and the one on page 75 by Engelbert Hubbers. The photo above was taken by Gary's son in law Eric; the one on page 75 by Kristi Lampe.

PROGRAM

May 9

Morning session

C. Olech	:	How it al started ...
S. Saeki	:	On the Epicenter of TILFs
R. Nillsen	:	A Tilt at TILFs

Afternoon session

J. Bebernes	:	Nonlocal Problems Modelling the Formation of Shear Bands
L.A. Campbell	:	Picturing Pinchuk's Plane Polynomial Pair
B. Coomes	:	Shadowing and Polynomial Flows: Some 'Connections'
J.-T. Yu	:	Polynomial Retracts and the Jacobian Conjecture
A. Gasull	:	Sufficient Conditions for Global Asymptotic Stability

May 10

Morning session

K. Adjamagbo	:	TBA
T. Parthasarathy	:	Real Jacobian Conjecture and Cubic Linear Mappings
M. Chamberland	:	The Jacobian Conjecture and the Mountain Pass Theorem
D. Wright	:	On the 2-dimensional Jacobian Conjecture and Affine Varieties Containing $\mathbb{C}^2$.

Afternoon session

G. Gorni	:	On the Relations between Cubic-Homogeneous and Cubic-Linear Polynomial Mappings
G. Freudenburg	:	G_a-actions obtained by local slice constructions
E. Hubbers	:	Cubic Similarity in Dimension Five
A. van den Essen	:	The King of the Talking Frogs and Polynomial Automorphisms.

Table of Contents

A TILT AT TILFS **1**

Rod Nillsen

1 Introduction . . . 1
2 The basic problems . . . 2
3 The circle group case . . . 3
4 The case of the real line . . . 5
4.1 The first order difference space . . . 5
4.2 Higher order and fractional difference spaces . . . 6
5 Partial differential operators . . . 7
6 Multiplier operators . . . 9
7 Further applications . . . 9
7.1 The Riesz potential operators . . . 9
7.2 The Hilbert transform and related operators . . . 10
7.3 Wavelets . . . 11
8 Conclusion . . . 11

SHEAR BANDING—A STUDY IN NONLOCAL PROBLEMS **13**

Jerrold Bebernes

1 Introduction . . . 13
2 Nonlocal problems . . . 15
3 Existence-nonexistence for BVP(2.2) . . . 16
4 Finite time blowup . . . 16
5 Single point blowup . . . 18

PICTURING PINCHUK'S PLANE POLYNOMIAL PAIR **21**

L. Andrew Campbell

1 Introduction . . . 21
2 Asymptotics of Pinchuk's Map . . . 22
3 The Peretz Method . . . 25
4 The Asymptotic Variety of F . . . 30
5 The Asymptotic Flower of F . . . 31

RECOGNIZING AUTOMORPHISMS OF POLYNOMIAL ALGEBRAS **35**

Jie-Tai Yu

1 Introduction . . . 35
2 Gröbner basis approach . . . 36
3 The Jacobian conjecture . . . 36
4 Polynomial retracts and the $n = 2$ Jacobian conjecture . . . 37
5 Test polynomials . . . 39
6 Two open problems . . . 40

SOME REMARKS ON CONTINUOUS AND DISCRETE MARKUS-YAMABE PROBLEMS **43**

Anna Cima, Armengol Gasull and Francesc Mañosas

1 Introduction . . . 43
2 On quasi-homogeneous vector fields of degree one . . . 45
3 Continuous Case . . . 46
3.1 Examples of polynomial Markus-Yamabe vector fields having unbounded orbits . . . 46
3.2 Markus-Yamabe linear quasi-homogeneous vector fields and the existence of periodic orbits . . . 48
3.3 Genericity of the Markus-Yamabe Conjecture . . . 49
4 The discrete Case . . . 50
4.1 Examples and counterexamples in dimension two . . . 50
4.2 Examples of polynomial Markus-Yamabe maps having unbounded orbits . . . 50
4.3 Genericity and the relation with the Jacobian Conjecture . . 51
5 Global attractors and injectivity . . . 52

REAL JACOBIAN CONJECTURE AND CUBIC LINEAR MAPPINGS **57**

Thiruvenkatachari Parthasarathy

A MOUNTAIN PASS TO THE JACOBIAN CONJECTURE **59**

Marc Chamberland and Gary Hosler Meisters

ON THE 2-DIMENSIONAL JACOBIAN CONJECTURE AND AFFINE VARIETIES CONTAINING $\mathbb{C}^2$ **61**

David Wright

A NONLINEARIZABLE CUBIC-LINEAR MAPPING **63**

Gianluca Gorni

G_A-ACTIONS OBTAINED BY LOCAL SLICE CONSTRUCTIONS **69**

Gene Freudenburg

1 Introduction . . . 69
2 Local Slice Constructions . . . 70
3 Rank Three Examples . . . 72
3.1 Examples of Type (2,4n+1) . . . 72
3.2 Examples of Fibonacci Type . . . 73
4 The Graph of Kernels . . . 73

CUBIC SIMILARITY IN DIMENSION FIVE **75**

Engelbert Hubbers

1 Introduction . . . 75
2 Reduction to triangular matrices . . . 76
3 Meisters' representatives . . . 79

4 Classification of Drużkowski maps 80
4.1 Nilpotence index two 81
4.2 Nilpotence index three 83
4.3 Nilpotence index four 84
4.4 Nilpotence index five 84
5 Cubic similarity reduction 85
6 New representatives 87

THE KING OF THE TALKING FROGS AND POLYNOMIAL AUTOMORPHISMS **91**
Arno van den Essen
1 Preface 91
2 A short survey of Gary's work on polynomial automorphisms 92
2.1 Polynomial flows 93
2.2 Polynomial flows and the Jacobian Conjecture 93
2.3 The solution of the 2-dimensional Markus-Yamabe Conjecture for polynomial vector fields 94
2.4 Strong nilpotence 95
2.5 Power Similarity 95
3 The DMZ-Conjecture and the solution of the Markus-Yamabe Conjecture 96
4 Meisters' Cubic Linear Linearisation Conjecture and new counterexamples to the Markus-Yamabe Conjecture 101

A TILT AT TILFS

ROD NILLSEN
University of Wollongong

This paper is dedicated to Gary H. Meisters

Abstract. In this talk I will endeavour to give an overview of some aspects of the theory of Translation Invariant Linear Forms (TILFs) and associated Hilbert spaces of functions. In particular, I will discuss some of the early ideas and results of Gary Meisters in this area, and try to explain how these ideas have led to applications in various areas of harmonic analysis.

1 Introduction

Let $\mathbb{T}$ denote the circle group, $\mathbb{Z}$ the group of integers and $\mathbb{R}^n$ the n-dimensional Euclidean group. Let G denote any one of these groups (or any locally compact abelian group). Let $M(G)$ denote the measures on G of finite variation, and if $x \in G$ let δ_x denote the Dirac measure at x. Let X denote a vector space of functions or distributions on G such that if $\mu \in M(G)$ and $f \in X$, the convolution $\mu * f$ is defined and is such that $\mu * f \in X$. Let X' denote the algebraic dual of X and let $S \subseteq M(G)$. Then, a linear form $L \in X'$ is called *S-invariant*, or a SILF, if $L(\mu * f) = L(f)$ for all $\mu \in S$ and all $f \in X$.

If $x \in G$ and $f \in L^2(G)$, $\delta_x * f$ is the function $t \longmapsto f(t-x)$ and is called the *translation of f by x*. Then, when $S = \{\delta_x : x \in G\}$, an S-invariant form is called a *translation invariant linear form*, or a TILF, because it takes the same value on all translates of any given function. Thus, a TILF is a SILF!

In the case where $X = L^\infty(G)$, the positive and normalised TILFs are called invariant means (or Banach limits in the case when $G = \mathbb{Z}$), and are related to the theory of amenable groups.

Gary Meisters has obtained many interesting results about invariant linear forms in different contexts. One of these is a result of [8], which says that if $C_c^\infty(\mathbb{R})$ denotes all the complex-valued C^∞-functions on $\mathbb{R}$ which have compact support, then the only TILFs on $C_c^\infty(\mathbb{R})$ are the constant multiples of the TILF which is integration with respect to the usual Lebesgue measure. So, it follows that there are no discontinuous TILFs on $C_c^\infty(\mathbb{R})$. Gary also proved in [8] that if $\mathcal{E}(\mathbb{R})$ denotes the C^∞-functions of arbitrary support in $\mathbb{R}$, and if $\mathcal{S}(\mathbb{R})$ denotes the C^∞-functions which are rapidly decreasing and whose derivatives of all orders are rapidly decreasing, then neither $\mathcal{E}(\mathbb{R})$ nor $\mathcal{S}(\mathbb{R})$ have discontinuous TILFs.

The results in [8] are striking and elegant, and further information on them may be found in Jean Dieudonné's book [2, pp.208–209] and in [7, 11, 12]. However, in this paper I am mainly concerned to explain some developments which have arisen from work of Gary and Wolfgang Schmidt on the $L^2(G)$ case when G is a compact connected abelian group. It is not the purpose to present detailed proofs (which may or will be found elsewhere), but rather to look at some ideas and describe some of their applications.

2 The basic problems

Let X be a subspace of $L^2(G)$ and let S be a set of measures on G. Then the basic problem initially can be considered to be: *identify all S-invariant linear forms on X*. If we can't do this, at least let us try and say a few interesting things about them – for example, 0 is always an S-invariant form! Before proceeding, I would like to consider a refinement of this basic problem.

Let $\mathcal{D}(X,S)$ denote the vector subspace of X spanned by all vectors of the form $f-\mu*f$, for some $\mu \in S$ and $f \in X$. Thus, for $f \in X$, $f \in \mathcal{D}(X,S)$ if and only if there are $n \in \mathbb{N}$, $g_1, g_2, \ldots, g_n \in X$ and $\mu_1, \mu_2, \ldots, \mu_n \in S$ such that $f = \sum_{j=1}^n (g_j - \mu_j * g_j)$. When $S = \{\delta_x : x \in G\}$, $\mathcal{D}(X,S)$ is denoted by $\mathcal{D}(X)$. In the case where $G = \mathbb{R}$ and $S = \{\delta_x : x \in \mathbb{R}\}$, a function $f - \delta_x * f$ is a *first order difference*, as used to approximate derivatives of functions. For this reason, a general space $\mathcal{D}(X,S)$ is called a *difference space*.

The significance of the space $\mathcal{D}(X,S)$ lies in the immediate fact that, if $L \in X'$, L is S-invariant if and only if $L(\mathcal{D}(X,S)) = \{0\}$. This provides a characterization of the S-invariant forms, and provides further information about them. In fact, it was observed by Gary Meisters in 1973 [10] that there is a non-zero S-invariant form on X if and only if $\mathcal{D}(X,S) \neq X$, and that $\mathcal{D}(X,S)$ is dense in X if and only if the only continuous S-invariant form on X is 0. In view of these observations of Gary's, it is reasonable to regard the following problem as a refinement of the problem of identifying the S-invariant forms. The problem is: *characterize the space $\mathcal{D}(X,S)$ as a subspace of $L^2(G)$*.

3 The circle group case

The following result is due to Gary and Wolfgang Schmidt, and dates from 1972 [9]. As well as being a beautiful result in its own right, it can be regarded as the protoype for later results in a general L^2 context. Note that for a locally compact abelian group G, μ_G denotes a Haar measure on G, normalized in the case when G is compact.

Theorem 1 *If G is a compact and connected abelian group, and in particular if G is the circle group $\mathbb{T}$, then*

$$\mathcal{D}(L^2(G)) = \Big\{ f : f \in L^2(G) \text{ and } \int_G f \; d\mu_G = 0 \Big\}.$$

Thus, in this case, $\mathcal{D}(L^2(G))$ has codimension 1 in $L^2(G)$, and every TILF on $L^2(G)$ is continuous and is a multiple of the Haar measure on G. I would like to present an idea of the proof of this result, especially as the proof illustrates the general approach which later proved to be significant in the non-compact case.

Idea of proof. When G is compact, integration using the Haar measure is a TILF on $L^2(G)$, so that $\int_G f \; d\mu_G = 0$ for all $f \in \mathcal{D}(L^2(G))$. It is harder to prove that if $\int_G f \; d\mu_G = 0$, then $f \in \mathcal{D}(L^2(G))$. However, to this end, let $\widehat{\mu}$ denote the Fourier transform of the function or measure μ. Let $\widehat{G}$ be the dual group of G ($= \mathbb{Z}$, if $G = \mathbb{T}$). Then in [9], a Fourier transform argument is used to show that if $f \in L^2(G)$, $f \in \mathcal{D}(L^2(G))$ if and only if there are $x_1, x_2, \ldots, x_n \in G$ such that

$$\int_{\widehat{G}} \frac{|\widehat{f}(\gamma)|^2}{\sum_{j=1}^n |1 - \gamma(x_j)|^2} d\mu_{\widehat{G}}(\gamma) < \infty. \tag{1}$$

In fact, this characterization of $\mathcal{D}(L^2(G))$ is valid for non-compact G as well.

Now when (1) holds, there are $f_1, \ldots, f_n$ such that $f = \sum_{j=1}^n (f_j - \delta_{x_j} * f_j)$. However, the trouble is that for a given f, it is hard to tell whether there are $x_1, x_2, \ldots, x_n \in G$ such that (1) holds. We need a characterization which depends more directly upon f itself. To this end, let $\int_G f \; d\mu_G = 0$ and note that this means that $\widehat{f}(\widehat{e}) = 0$, where $\widehat{e}$ is the identity of $\widehat{G}$.

First, make some preliminary observations. Let $\gamma \in \widehat{G}$, $\gamma \neq \widehat{e}$. Then, the function $(x_1, \ldots, x_n) \longmapsto (\gamma(x_1), \ldots, \gamma(x_n))$ is continuous on the connected group G^n, so its range must be connected and in fact equals $\mathbb{T}^n$. So, if h is any continuous non-negative function on $\mathbb{T}^n$,

$$\begin{aligned} &\int_{G^n} h(\gamma(x_1), \ldots, \gamma(x_n)) d\mu_G(x_1) \ldots d\mu_G(x_n) \\ &= (2\pi)^{-n} \int_{[-\pi,\pi]^n} h(e^{i\theta_1}, \ldots, e^{i\theta_n}) \; d\theta_1 \ldots d\theta_n. \end{aligned}$$

Now, we use this and the assumed fact that $\widehat{f}(\widehat{e}) = 0$ to obtain

$$\begin{aligned}
& \int_{G^n} \left(\int_{\widehat{G}} \frac{|\widehat{f}(\gamma)|^2}{\sum_{j=1}^n |1-\gamma(x_j)|^2} d\mu_{\widehat{G}}(\gamma) \right) d\mu_G(x_1) \dots d\mu_G(x_n) \\
= \ & \int_{\widehat{G}} |\widehat{f}(\gamma)|^2 \left(\int_{G^n} \frac{d\mu_G(x_1) d\mu_G(x_2) \dots d\mu_G(x_n)}{\sum_{j=1}^n |1-\gamma(x_j)|^2} \right) d\mu_{\widehat{G}}(\gamma) \\
= \ & (2\pi)^{-n} \left(\int_{\widehat{G}} |\widehat{f}(\gamma)|^2 d\mu_{\widehat{G}}(\gamma) \right) \left(\int_{[-\pi,\pi]^n} \frac{d\theta_1 d\theta_2 \dots d\theta_n}{\sum_{j=1}^n |1-e^{i\theta_j}|^2} \right) \\
= \ & (2\pi)^{-n} \left(\int_{\widehat{G}} |\widehat{f}(\gamma)|^2 d\mu_{\widehat{G}}(\gamma) \right) \left(\int_{[-\pi,\pi]^n} \frac{d\theta_1 d\theta_2 \dots d\theta_n}{4\sum_{j=1}^n \sin^2 \theta_j/2} \right) \\
< \ & \infty,
\end{aligned}$$

as long as $n \geq 3$.

Thus, if we take $n = 3$, for almost all $(x_1, x_2, x_3) \in G^3$, (1) holds and it follows that any $f \in \mathcal{D}(L^2(G))$ is the sum of 3 first order differences.

This argument of Meisters and Schmidt showed there is a connection between estimating certain n-dimensional integrals and the characterization of spaces such as $\mathcal{D}(L^2(G))$. Note that in [9] it is shown that there is a function in $\mathcal{D}(L^2(\mathbb{T}))$ which is *never* the sum of 2 first order differences, so that the choice of 3 is best possible. Note also that the determination of sharp bounds on the required number of differences is related to problems of Diophantine approximation [9, 12].

Meisters and Schmidt in fact showed that on any compact group with a finite number of components, any TILF on $L^2(G)$ is a multiple of Haar measure and so continuous. In 1973 Meisters [10] showed that the L^2-space of the Cantor group has discontinuous TILFs. Further results of Meisters and Bagget [11] and Johnson [3], produced a characterisation of the compact abelian groups G such that every TILF on $L^2(G)$ is continuous. Then Bourgain [1] showed that for $1 < p < \infty$, every TILF on $L^p(\mathbb{T})$ is continuous, and this was extended to other groups by Wai Lok Lo [5] who also gave an estimate of the required number of differences for higher order difference spaces of $L^p(\mathbb{T})$.

The result of Meisters and Schmidt suggests a reformulation of the basic problem for $L^2(G)$. For, in the circle group case let μ be the measure on $\widehat{\mathbb{T}} = \mathbb{Z}$ given by

$$\mu(\{x\}) = \begin{cases} 1, & \text{if } x \in \mathbb{Z} \text{ and } x \neq 0 \ ; \\ \infty, & \text{if } x = 0. \end{cases}$$

Then observe that for $f \in L^2(\mathbb{T})$, $\int_{\mathbb{Z}} |\widehat{f}|^2 d\mu < \infty$ if and only if $\widehat{f}(0) = 0$. So, for the circle group, their result can be stated as: *the Fourier transform maps $\mathcal{D}(L^2(\mathbb{T}))$ bijectively onto $L^2(\widehat{\mathbb{T}}, \mu)$*. Thus, on a general locally compact abelian group G, the basic problem for $L^2(G)$ may be considered to be: *describe a measure μ on $\widehat{G}$ such that the Fourier transform maps a difference space $\mathcal{D}(L^2(G), S)$ bijectively onto $L^2(\widehat{G}, \mu)$.*

4 The case of the real line

4.1 The first order difference space

In 1973, Gary Meisters [10] proved the following:

$$f \in \mathcal{D}(L^2(\mathbb{R})) \Longrightarrow \int_{-\infty}^{\infty} |\widehat{f}(x)|^2 |x|^{-2} dx < \infty. \tag{2}$$

It follows from this that $\mathcal{D}(L^2(\mathbb{R}))$ is a proper subspace of $L^2(\mathbb{R})$, and it is also dense in $L^2(\mathbb{R})$. So, as found in [10], *there are non-zero TILFs on $L^2(\mathbb{R})$ and every such TILF is discontinuous.* Further results for non-compact groups were obtained by Sadahiro Saeki [18], Gordon Woodward [21] and myself [14], all concerning the existence and even the profusion of discontinuous TILFs for non-compact cases such as $\mathbb{R}$ and, more generally, for non-compact amenable groups. On the other hand, G. Willis [20] showed that for a non-amenable group G, the only TILF on $L^p(G)$ (for $1 < p \leq \infty$) is 0.

Now (2) shows that Fourier transforms of functions in $\mathcal{D}(L^2(\mathbb{R}))$ have a certain precise behaviour *near the origin.* In fact, it was proved in [13] that the functions in $L^2(\mathbb{R})$ which are in $\mathcal{D}(L^2(\mathbb{R}))$ are *characterized* by the behaviour expressed in (2). So this result, together with Gary's earlier result in [10], gives the following.

Theorem 2 *Let $f \in L^2(\mathbb{R})$. Then*

$$f \in \mathcal{D}(L^2(\mathbb{R})) \Longleftrightarrow \int_{-\infty}^{\infty} |\widehat{f}(x)|^2 |x|^{-2} dx < \infty.$$

The space $\mathcal{D}(L^2(\mathbb{R}))$ is Hilbert, with the inner product given by

$$\langle f, g \rangle = \int_{-\infty}^{\infty} \widehat{f}(x)\overline{\widehat{g}(x)}(1 + |x|^{-2}) dx,$$

for all $f, g \in \mathcal{D}(L^2(\mathbb{R}))$. The Fourier transform maps $\mathcal{D}(L^2(\mathbb{R}))$ isometrically onto $L^2(\mathbb{R}, (1 + |x|^{-2}) dx)$.

Now the first order Sobolev space is denoted by $H^1(\mathbb{R})$ and consists of the functions in $L^2(\mathbb{R})$ whose derivatives are in $L^2(\mathbb{R})$. For $f \in L^2(\mathbb{R})$, $f \in H^1(\mathbb{R})$ if and only if $\int_{-\infty}^{\infty} |\widehat{f}(x)|^2 |x|^2 dx < \infty$, so functions in $H^1(\mathbb{R})$ are characterized by the behaviour of their Fourier transforms *at infinity.* The space $H^1(\mathbb{R})$ is Hilbert with inner product

$$\langle f, g \rangle = \int_{-\infty}^{\infty} \widehat{f}(x)\overline{\widehat{g}(x)}(1 + |x|^2) dx.$$

The spaces $\mathcal{D}(L^2(\mathbb{R}))$ and $H^1(\mathbb{R})$ are "complementary" in that the functions in the former are characterized by the behaviour of their Fourier transforms near the origin, while the functions in the latter are characterized by the behaviour of their Fourier transforms at infinity (see [15] for more aspects of this).

Now let D denote differentiation. Then, for $f \in H^1(\mathbb{R})$, $D(f)^\wedge(x) = ix\widehat{f}(x)$. It follows from this that *differentiation is a Hilbert space isometry from* $H^1(\mathbb{R})$ *onto* $\mathcal{D}(L^2(\mathbb{R}))$, so the latter space is the range of D in a natural sense. Since an element L of $L^2(\mathbb{R})'$ is a TILF if and only if $L(\mathcal{D}(L^2(\mathbb{R}))) = 0$, it follows by a Hamel basis argument that

$$D(H^1(\mathbb{R})) = \bigcap \Big\{ \text{ kernel } (L) : L \text{ is a TILF} \Big\},$$

which establishes a connection between TILFs and differentiation.

4.2 Higher order and fractional difference spaces

Now let $s > 0$, let $m \in \mathbb{N}$, and let α be a 2π-periodic function on $\mathbb{R}$ which has an absolutely convergent Fourier series. Assume that:

(i) for some $\delta > 0$,

$$\delta |x|^s \leq |\alpha(x)| \leq \delta^{-1} |x|^s, \text{ for all } x \text{ in } [-\delta, \delta];$$

(ii)

$$\int_{[-\pi,\pi]^m} \frac{dx_1 \dots dx_m}{\sum_{j=1}^m |\alpha(x_j)|^2} < \infty.$$

Let $S_\alpha = \{\delta_0 - \sum_{j=-\infty}^{\infty} \widehat{\alpha}(j)\delta_{-jy} : y \in \mathbb{R}\}$. Then it was proved in [13] that $\mathcal{D}(L^2(\mathbb{R}), S_\alpha)$ *consists precisely of all functions* $f \in L^2(\mathbb{R})$ *such that*

$$\int_{-\infty}^{\infty} |\widehat{f}(x)|^2 |x|^{-2s} dx < \infty.$$

Thus, as $\mathcal{D}(L^2(\mathbb{R}), S_\alpha)$ depends upon s but is independent of α, it may be denoted by $\mathcal{D}_s(L^2(\mathbb{R}))$. The following result is then analogous to Theorem 2 and extends it.

Theorem 3 *Let* $f \in L^2(\mathbb{R})$. *Then*

$$f \in \mathcal{D}_s(L^2(\mathbb{R})) \iff \int_{-\infty}^{\infty} |\widehat{f}(x)|^2 |x|^{-2s} dx < \infty.$$

The space $\mathcal{D}_s(L^2(\mathbb{R}))$ *is Hilbert, with the inner product given by*

$$\langle f, g \rangle = \int_{-\infty}^{\infty} \widehat{f}(x)\overline{\widehat{g}(x)}(1 + |x|^{-2s})dx, \ \textit{for } f, g \in \mathcal{D}_s(L^2(\mathbb{R})).$$

For $s \in \mathbb{N}$, let $H^s(\mathbb{R})$ denote the Sobolev space consisting of the functions in $L^2(\mathbb{R})$ all of whose derivatives of order at most n are in $L^2(\mathbb{R})$. Then $H^s(\mathbb{R})$ is Hilbert in the inner product

$$\langle f, g \rangle = \int_{-\infty}^{\infty} \widehat{f}(x)\overline{\widehat{g}(x)}(1 + |x|^{2s})dx, \text{ for } f, g \in \mathcal{D}_s(L^2(\mathbb{R})),$$

and it follows that D^s is an isometry from the Sobolev space $H^s(\mathbb{R})$ onto $\mathcal{D}_s(L^2(\mathbb{R}))$.

Now, let us make the definition that an S_α-invariant linear form, with S_α as above, is called an s-ILF (or sILF). Thus, for $s \in \mathbb{N}$,

$$D^s(H^s(\mathbb{R})) = \bigcap \Big\{ \text{ kernel } (L) : L \text{ is a } s\text{ILF} \Big\}.$$

Example 1 A function f in $L^2(\mathbb{R})$ is the second derivative of some function in $L^2(\mathbb{R})$ if and only if there are $x_1, \ldots, x_5 \in \mathbb{R}$, and $f_1, \ldots, f_5 \in L^2(\mathbb{R})$ such that $f = \sum_{j=1}^{5} \Big(f_j - 2^{-1}(\delta_{x_j} + \delta_{-x_j}) * f_j \Big)$. Also, a function f in $L^2(\mathbb{R})$ is the second derivative of some function in $L^2(\mathbb{R})$ if and only if $L(f) = 0$ for every $\{2^{-1}(\delta_x + \delta_{-x}) : x \in \mathbb{R}\}$-invariant form on $L^2(\mathbb{R})$.

5 Partial differential operators

The preceding discussion has established a connection between invariant forms, differentiation and difference subspaces, all in the context of one variable. The question arises as to what happens when several real variables are considered. In order to give an idea of what happens in this context, let V be a vector subspace of $\mathbb{R}^n$, let $|x|$ be the usual norm of $x \in \mathbb{R}^n$, and let $e_1, \ldots, e_r$ be an orthonormal basis for V. The *V-Laplacian* Δ_V is given by $\Delta_V = \sum_{j=1}^{r} D_{e_j}^2$, where D_{e_j} is differentiation in direction e_j. If P_V is the orthogonal projection onto V,

$$\Delta_V(f)^\wedge(x) = -|P_V(x)|^2 \widehat{f}(x).$$

Now let $V_1, V_2, \ldots, V_q$ be non-zero vector subspaces of $\mathbb{R}^n$, and let $s_1, s_2, \ldots, s_q$ be q strictly positive real numbers. Let

$$\begin{aligned}
\Upsilon &= \prod_{j=1}^{q} |P_{V_j}|^{s_j}, \\
\Psi &= \sum_{A \subseteq \{1,\ldots,q\}} \prod_{j \in A} |P_{V_j}|^{-s_j}, \\
\Theta &= \sum_{A \subseteq \{1,\ldots,q\}} \prod_{j \in A} |P_{V_j}|^{s_j}.
\end{aligned}$$

Let

$$W(L^2(\mathbb{R}^n), \Psi) = \Big\{ f \in L^2(\mathbb{R}^n) : \int_{\mathbb{R}^n} |\widehat{f}(x)|^2 \Psi^2(x) dx < \infty \Big\},$$

and similarly define $W(L^2(\mathbb{R}^n), \Theta)$. Then, using the fact that $\Upsilon\Psi = \Theta$, the operator $\prod_{j=1}^{q} |\Delta_{V_j}|^{s_j}$ may be defined and is an isometry from $W(L^2(\mathbb{R}^n), \Theta)$ onto $W(L^2(\mathbb{R}^n), \Psi)$. This definition is given by the requirement that

$$\Big(\prod_{j=1}^{q} |\Delta_{V_j}|^{s_j}(f) \Big)^\wedge = \Big(\prod_{j=1}^{q} |P_{V_j}|^{2s_j} \Big) \widehat{f},$$

for all $f \in W(L^2(\mathbb{R}^n), \Theta)$. The statement above that $\prod_{j=1}^q |\Delta_{V_j}|^{s_j}$ is an isometry results from routine manipulations of the definitions; but the point is that the space $W(L^2(\mathbb{R}^n), \Psi)$, the range of the operator, may be described alternatively as a "generalized" difference space which is analogous to the space $\mathcal{D}(L^2(\mathbb{R}))$ as it appears in Theorem 2. A corresponding description of the range is valid for the similarly defined operators $D_{u_1} D_{u_2} \ldots D_{u_r}$, for independent vectors $u_1, \ldots, u_r \in \mathbb{R}^n$. Before proceeding to technicalities, let us consider a special case.

Example 2 The Wave Operator is

$$\mathcal{W} = \frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2} = \left(\frac{\partial}{\partial x} - \frac{\partial}{\partial y}\right)\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y}\right) = D_{u_1} D_{u_2},$$

where $u_1 = (1,-1), u_2 = (1,1)$. The domain of $\mathcal{W}$ is the Sobolev-type space consisting of all $f \in L^2(\mathbb{R}^2)$ such that

$$\int_{\mathbb{R}^2} |\widehat{f}(x,y)|^2 \Big\{1 + |x-y| + |x+y| + |x-y| \cdot |x+y|\Big\}^2 dxdy < \infty.$$

The range of $\mathcal{W}$ consists of those functions in $L^2(\mathbb{R}^2)$ which are the sum of 9 functions, each of which is of the form $(x,y) \longmapsto g(x,y) - g(x+a, y+a) - g(x+b, y-b) + g(x+a+b, y+a-b)$, for some $a, b \in \mathbb{R}$ and some $g \in L^2(\mathbb{R})$. The range of $\mathcal{W}$ is the intersection of the kernels of all the linear forms which are $\{\delta_{(a,a)} + \delta_{(b,-b)} - \delta_{(a+b,a-b)} : a, b \in \mathbb{R}\}$-invariant.

Now for the more general case. Let $\alpha_1, \alpha_2, \ldots, \alpha_q$ be q continuous, complex valued 2π-periodic functions on $\mathbb{R}^n$ such that for some $\delta_1, \delta_2 > 0$,

$$\delta_1 |x|^{s_j} \leq |\alpha_j(x)| \leq \delta_2 |x|^{s_j},$$

for all $x \in [-\pi, \pi]$ and all $j = 1, 2, \ldots, q$. For each $j = 1, 2, \ldots, q$, let $m_j \in \mathbb{N}$ with $m_j > 2s_j$, and let $J_1, J_2, \ldots, J_q$ be q disjoint subintervals of $\mathbb{N}$ such that each J_j has m_j elements. Now consider the set of all functions f in $L^2(\mathbb{R}^n)$ such that f is equal to a sum of the form

$$\sum_{(k_1, \ldots, k_q) \in \prod_{j=1}^q J_j} \left(\prod_{j=1}^q \sum_{\ell=-\infty}^{\infty} \widehat{\alpha}_j(\ell) \delta_{-\ell y_{k_j}} \right) * \ h_{k_1 k_2 \ldots k_q},$$

where for each $k \in \{1, 2, \ldots, q\}$, $y_k \in V_k$; and for each $(k_1, \ldots, k_q) \in \prod_{j=1}^q J_j$, $h_{k_1 k_2 \ldots k_q} \in L^2(\mathbb{R}^n)$. This set of functions is a subset of $L^2(\mathbb{R}^n)$ which depends upon $V_1, \ldots, V_q$, $s_1, \ldots, s_q$ but is *independent* of $\alpha_1, \ldots, \alpha_q$ and $m_1, \ldots, m_q$. Accordingly, this set of functions is denoted by $\mathcal{D}_{s_1, s_2, \ldots, s_q}(L^2(\mathbb{R}^n), V_1, \ldots, V_q)$.

Theorem 4 *$\mathcal{D}_{s_1, s_2, \ldots, s_q}(L^2(\mathbb{R}^n), V_1, \ldots, V_q)$ is a vector subspace of $L^2(\mathbb{R}^n)$, and it is a Hilbert space in the inner product $[\cdot, \cdot]$ given by*

$$[f, g] = \int_{\mathbb{R}^n} \left(\sum_{A \subseteq \{1, 2, \ldots, q\}} \left[\prod_{j \in A} |P_{V_j}|^{-s_j} \right] \right) \widehat{f}(x) \overline{\widehat{g}(x)} \ dx.$$

In particular, in $\mathbb{R}^2$, if V_1 is the subspace spanned by $(-1,1)$ and V_2 is the one spanned by $(1,1)$, the range of $\mathcal{W}$, as in the above example, is the space $\mathcal{D}_{1,1}(L^2(\mathbb{R}^n), V_1, V_2)$.

More generally, if Θ, Ψ are the functions as before, Theorem 4 shows that $W(L^2(\mathbb{R}^n), \Psi) = \mathcal{D}_{s_1,s_2,\dots,s_q}(L^2(\mathbb{R}^n), V_1, \dots, V_q)$. Thus, we have

Theorem 5 *$\prod_{j=1}^q |\Delta_{V_j}|^{s_j}$ is an isometry from the space $W(L^2(\mathbb{R}^n), \Theta)$ onto the space $\mathcal{D}_{s_1,s_2,\dots,s_q}(L^2(\mathbb{R}^n), V_1, \dots, V_q)$.*

This result identifies the range of an operator $\prod_{j=1}^q |\Delta_{V_j}|^{s_j}$ as a "generalised difference space" when the domain of the operator is taken as the "Sobolev-type" space $W(L^2(\mathbb{R}^n), \Theta)$.

6 Multiplier operators

Partial differential operators are special cases of (unbounded) multiplier operators with mutipliers of the form $\prod_{j=1}^r |P_{V_j}|^{s_j}$ or $\prod_{j=1}^r \langle \cdot, e_j \rangle$, in the present context. A multiplier operator T on a space X is one for which there is a function φ such that $T(f)^\wedge = \varphi \widehat{f}$, for all $f \in X$. Recent work with Susumu Okada has shown that the ranges of a large class of multiplier operators on locally compact abelian groups may be described by means of "generalized difference spaces" on these groups. For example, the following result is proved in [17].

Theorem 6 *If T is a bounded multiplier operator on $L^2(G)$ with multiplier φ, there is a family of pseudomeasures $S = \{\mu_a : a \in \mathbb{R}\}$ on G such that:*

1. *$\mu_a * \mu_b = \mu_{a+b}$ for all $a, b \in \mathbb{R}$;*
2. *the range of T is the difference space $\mathcal{D}(L^2(G), S)$ and this space is a Hilbert space in the inner product $\langle \cdot, \cdot \rangle$ given by*

$$\langle f, g \rangle = \int_{\widehat{G}} \widehat{f}\, \overline{\widehat{g}}\, (1 + |\varphi|^{-2})\, d\mu_{\widehat{G}},$$

   for all f, g in the range of T; and
3. *for each f in the range of T, there are $a_1, a_2, a_3 \in \mathbb{R}$ and $f_1, f_2, f_3 \in L^2(G)$ such that $f = \sum_{j=1}^3 (f_j - \mu_{a_j} * f_j)$.*

Results with S. Okada have also been obtained which extend this sort of result to unbounded multiplier operators, the partial differential operators of Section 5 being special cases.

7 Further applications

7.1 The Riesz potential operators

Let $n, s \in \mathbb{N}$ with $0 < s < n/2$. The Riesz potential operator I_s of order s is given by

$$I_s(f)(x) = \int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-s}} dy,$$

for $x \in \mathbb{R}^n$. Then, as discussed in Chapter 7 of [19],

$$I_s(f)^\wedge(x) = \widehat{f}(x)/|x|^{n-s}.$$

The Sobolev space $W^s(L^2(\mathbb{R}^n))$ of order s on $\mathbb{R}^n$ consists of all functions $f \in L^2(\mathbb{R}^n)$ such that

$$||f|| = \left(\int_{\mathbb{R}^n} \left(1 + |x|^{2s}\right) |\widehat{f}(x)|^2 dx \right)^{1/2} < \infty,$$

and it is Hilbert in this norm $||\cdot||$. Note that $W^s(L^2(\mathbb{R})) = H^s(\mathbb{R})$, for $s \in \mathbb{N}$. The Laplace operator Δ is given by $\Delta = \sum_{j=1}^n \partial^2/\partial x_j^2$.

Theorem 7 *[13, 14]. The operator $|\Delta|^{s/2}$ is an isometry from $W^s(L^2(\mathbb{R}^n))$ onto the difference space $\mathcal{D}_s(L^2(\mathbb{R}^n))$, and its inverse is the Riesz potential operator of order s. Also, $\mathcal{D}_s(L^2(\mathbb{R}^n))$ consists of the functions f in $L^2(\mathbb{R}^n)$ such that $I_s(f) \in L^2(\mathbb{R}^n)$.*

7.2 The Hilbert transform and related operators

The Hilbert transform on $L^2(\mathbb{R})$ arises from convolution by the kernel $x \longmapsto 1/\pi x$. Now let s be an even non-negative integer, and consider the function

$$K_{s,y} : x \longmapsto \frac{1}{\pi x \prod_{k=1}^{s/2} (x^2 - k^2 y^2)}.$$

Owing to the identity

$$\sum_{k=0}^{s} \binom{s}{k} \frac{(-1)^k}{x - ky} = \frac{(-1)^s s! y^s}{\prod_{k=0}^{s} (x - ky)},$$

convolution by $K_{s,y}$ defines a bounded operator $H_{s,y}$ on $L^2(\mathbb{R})$ in the same way as the Hilbert transform. In fact the Hilbert transform is the case $s = 0$.

Theorem 8 *[14]. Let $y \in \mathbb{R}, y \neq 0$. The operator $H_{2,y}$ on $L^2(\mathbb{R})$ is given by convolution by the kernel $x \longmapsto 1/\pi x(x^2 - y^2)$. This operator has multiplier $x \longmapsto -2iy^{-2} sign(x) \sin^2(xy/2)$. The range of this operator consists of all functions in $L^2(\mathbb{R})$ which can be expressed in the form $g - 2^{-1}(\delta_y + \delta_{-y}) * g$ for some $g \in L^2(\mathbb{R})$. That is, the range is the intersection of the kernels of all the $\{\delta_y + \delta_{-y})/2\}$-invariant linear forms on $L^2(\mathbb{R})$.*

Whereas this result describes the range of $H_{2,y}$ in terms of certain second order differences, convolution by the kernel $x \longmapsto 1/\pi(x^2 - y^2)$ has a range which can be expressed in terms of *first* order differences.

7.3 Wavelets

Let $\mathbb{R}^*$ denote the non-zero real numbers. If $h \in L^2(\mathbb{R})$, we define a function U_h from $L^2(\mathbb{R})$ into the functions on $\mathbb{R} \times \mathbb{R}^*$ by

$$U_h(f)(a,b) = |a|^{-1/2} \int_{-\infty}^{\infty} f(x) \; \overline{h\left(\frac{x-b}{a}\right)} \; dx,$$

for all $f \in L^2(\mathbb{R})$ and all $a \in \mathbb{R}^*, b \in \mathbb{R}$. The function U_h is linear and is called the *wavelet transform* with *wavelet* h. A standard identity in the theory of the wavelet transform, which is analogous to the Plancherel Theorem, is

$$\int_{-\infty}^{\infty} \int_{\infty}^{\infty} \frac{|U_h(f)(a,b)|^2}{|a|^2} \, dadb = \frac{1}{2\pi} \left(\int_{-\infty}^{\infty} \frac{|\widehat{h}(x)|^2}{|x|} dx \right) \left(\int_{-\infty}^{\infty} |f(x)|^2 dx \right).$$

This singles out the wavelets h which have the property that

$$\int_{-\infty}^{\infty} |\widehat{h}(x)|^2 |x|^{-1} dx < \infty,$$

and such wavelets are called *admissible*, in which case, the wavelet transform maps $L^2(\mathbb{R})$ into $L^2(\mathbb{R}^* \times \mathbb{R}, |a|^{-2} dadb)$. It is clear from what has been said earlier that h *is admissible* if and only if $h \in \mathcal{D}_{1/2}(L^2(\mathbb{R}))$. Equivalently, h is admissible if and only if there is $g \in W^{1/2}(L^2(\mathbb{R}))$ such that $|D|^{1/2}(g) = h$. Alternatively, if we think of a TILF as being a "1-invariant linear form", h is an admissible wavelet if and only if for any 1/2-ILF, L say, $L(h) = 0$. Further aspects of wavelets and difference spaces may be found in [14, 15, 16].

8 Conclusion

The ideas in Gary Meisters' early papers on TILFs have proved to be of central and essential importance for subsequent work. In particular, the connection between TILFs and the estimation of certain specific integrals in $\mathbb{R}^n$, developed by Gary with Wolfgang Schmidt [9], has proved to be the cornerstone in developing the theory and applications of SILFs and TILFs in the non-compact case. Details of many of the ideas presented in this paper for the non-compact case, together with proofs, can be found in detail in [13, 14]. Since the publication of [14], some other work in this area may be found in [4, 5, 6, 15, 16, 17]. The expository paper [15] discusses the case of the real line, including proofs of the fundamental results for this case. It is fair to say that without Gary's input into this area of harmonic analysis, I would certainly not have carried out this work, so I owe him a very great debt; and not only because of this, but also owing to his warm and generous resonse to this work.

It may seem a long way from the concept of a TILF to the characterization of the ranges of differential and multiplier operators, to which the concept has led. However, as I tried to explain in [15], if one looks at the fundamental result of Gary and Wolfgang Schmidt in [9] from the appropriate viewpoint, it may be regarded as characterizing the range of d/dx on the first order Sobolev space of $L^2([-\pi, \pi])$

as the space consisting precisely of those functions expressible in a certain way as a sum of first order differences. So, their result can be regarded as establishing a precise relationship between the usual or "continuous" calculus on $[-\pi, \pi]$ and the "discrete" calculus which is based upon finite differences. That is, from this viewpoint, their result can be looked upon as unifying or reconciling "opposite" concepts, the continuous and the discrete. Still another way of thinking of their result is by regarding it as characterizing the "admissible" wavelets on the circle group, well in advance of the theory of wavelets!

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SHEAR BANDING—A STUDY IN NONLOCAL PROBLEMS

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Dedicated to Gary Meisters in honor of his sixty-fifth anniversary.

1 Introduction

The formation of shear bands in materials has important implications to a variety of physical processes. These bands are observed in very thin zones and are generally regarded as a precursor to material failure. Shear band formation is caused by the heat generated in regions with highest strain rate. With insufficient time for diffusion of this heat, a localized thermal softening of the material occurs which enhances plastic flow in a thin zone. This adiabatic strain localization can be modelled as nonlinear thermally-activated reaction-diffusion equations. This leads to a class of nonlocal parabolic problems and their associated time-independent steady-state counterparts.

This lecture describes and explains some of the results found in Bebernes-Talaga [1], Bebernes-Lacey [2], and Bebernes-Li-Talaga [3]. The modelling discussed in the following is based on the work of Burns ([4],[5]).

Consider loading a thin-walled tube of metal of length d in torsion with ends held at constant temperature T_0 and the tube having initial temperature T_0. One end is fixed and the other end is twisted at a constant rate $v = v_0$. If z denotes the axial coordinate, t time, $w(z,t)$ the linear displacement, $v = w_t$ the velocity, $\gamma(z,t) = w_z(z,t)$ the shear strain, and $\tau(z,t)$ the shear stress, then the thermovisco-plastic shear model is given by the following system of conservation laws:

$$\begin{aligned}
&\varphi v_t = \tau_z && \text{(Momentum)}\\
&b\tau_t = v_z - \gamma_t && \text{(Elasticity)}\\
&T_z = \lambda T_{zz} + \mu^{p-1}\cdot\tau\cdot\gamma_t && \text{(Energy)}\\
&\gamma_t = \Phi(\tau,\gamma,T) && \text{(Constitutive)}
\end{aligned} \tag{1.1}$$

where $\varphi,\ b,\ \lambda,\ \mu$, and p are constants.

If $\varphi << 1$ and $b << 1$, then the model simplifies to the quasi-static model:

$$\begin{aligned}
&\tau_z = 0\\
&v_z = \gamma_t\\
&T_t = \lambda T_{zz} + \mu^{p-1}\cdot T\cdot\gamma_t\\
&\gamma_t = \Phi(\tau,\gamma,T)
\end{aligned} \tag{1.2}$$

from which we observe that the stress is only time-dependent, $\tau = \tau(t)$. When the stress-strain law is in the plastic regime (Marchand-Duffy, [6]), $\tau = \tau(t) = \tau_0$ is approximately constant. If the (plastic) strain rate is given by the Arrhenius law:

$$\gamma_t = v_z = \mu\exp\left(\frac{-\Delta H(\tau)}{KT}\right) \tag{1.3}$$

where ΔH is the activation enthalpy and K is Boltzman's constant, then, from (1.2),(1.3) the mathematical model for the shearing process reduces to a reaction-diffusion equation which describes the energy balance coupled with a compatibility equation

$$\begin{gathered}
T_t - \lambda T_{zz} = \tau\mu^p\exp\left(\tfrac{-\Delta H}{KT}\right)\\
v_z = \mu\exp\left(\tfrac{-\Delta H}{KT}\right)\\
\begin{aligned}
&T(0,t) = T(d,t) = T_0 && v(0,t) = 0\\
&T(z,0) = T_0 && v(d,t) = v_0
\end{aligned}
\end{gathered} \tag{1.4}$$

By integrating the compatibility equation, (1.4) reduces to the nonlocal problem

$$\begin{gathered}
T_z = \lambda T_{zz} + \tau\mu^p\frac{\exp\left(\frac{-\Delta H}{KT}\right)}{\left(\int_0^d \exp\left(\frac{-\Delta H}{KT}\right)dz\right)^p}\\
T(0,t) = T(d,t) = T_0,\\
T(z,0) = T_0.
\end{gathered} \tag{1.5}$$

This reduces to the nondimensional model

$$\begin{gathered}
\theta_t - \theta_{xx} = \frac{\delta}{\left(\int_{-1}^1 \exp\left(\frac{-\beta}{1+\theta}\right)dx\right)^p}\cdot\exp\left(\frac{-\beta}{1+\theta}\right)\\
\theta(-1,t) = 0 = \theta(1,t)\\
\theta(x,0) = \theta_0(x) \geq 0
\end{gathered} \tag{1.6}$$

where $\beta = \dfrac{\Delta H(\tau_0)}{KT_0}$ and $\delta = \tau_0 \mu^p \geq 0$.

It has been experimentally verified ([6]) that a typical value of β is 40 so $\varepsilon = \beta^{-1} << 1$ and the reciprocal of the strain rate sensitivity can play a role in shear banding similar to the activation energy in combustion theory. Setting $\theta(x,t) = \varepsilon u(x,t) + O(\varepsilon^2)$, then to first-order, (1.6) becomes

$$
\begin{aligned}
u_t - u_{xx} &= \frac{\delta}{\left(\int_{-1}^{1} e^u dx\right)^p} \cdot e^u \\
u(-1,t) &= 0 = u(1,t) \\
u(x,0) &= u_0(x) \geq 0
\end{aligned}
\tag{1.7}
$$

where $\delta > 0$ and $p \geq 0$. Thus, the problem of shear band localization can be modelled by a nonlocal parabolic problem.

In the experimental study of shear band formation by Marchand and Duffy [6], a thin-walled tubular specimen of steel was loaded at a strain rate large enough to produce shear banding. During the shear band formation, temperature measurements were made and photographs taken of the specimens to provide strain measurements at several locations along the tube at different times. A narrow shear band is seen to form. This narrow band of high strain which often precedes failure in materials was seen to form near the axial midpoint on the surface of the tube as the temperature there increased dramatically.

The question of this lecture is does the nonlocal model (1.7) predict these experimental observations? Mathematically this would be answered affirmatively if solution to (1.7) blows up in finite time and if information can be given about the blow-up set.

2 Nonlocal problems

We are led to consider nonlocal problems of the form

$$
\begin{aligned}
u_t - \Delta u &= \frac{\delta f(u)}{\left(\int_\Omega f(u) dx\right)^p} \quad , \qquad x \in \Omega,\ t > 0 \\
u(x,0) &= u_0(x) \geq 0 \quad , \qquad x \in \Omega \\
u(x,t) &= 0 \quad , \qquad x \in \partial\Omega,\ t > 0
\end{aligned}
\tag{2.1}
$$

and the associated steady-state problem

$$
\begin{aligned}
-\Delta u &= \frac{\delta f(u)}{\left(\int_\Omega f(u) dx\right)^p} \quad , \quad x \in \Omega \\
u(x) &= 0 \quad , \qquad x \in \partial\Omega
\end{aligned}
\tag{2.2}
$$

where $p \geq 0,\ \delta > 0,\ \Omega \subset \mathbb{R}^n$ is a bounded domain with smooth boundary $\partial\Omega$, f is positive and Lipschitz continuous and $u_0(x) \geq 0$ is in $L^2(\Omega)$ with $u_0(x) = 0$ on $\partial\Omega$. The following standard results for classical nonlinear partial differential equations carry over to the nonlocal problems (2.1) and (2.2):

1) Any solution of IBVP(2.1) or BVP(2.2) is positive for $x \in \Omega$ with outer normal derivative $\frac{\partial u}{\partial N} \leq 0,\ x \in \partial\Omega$.
2) For $u_0 \in L^2(\Omega)$, $\sup u_0(x) < \infty$, IBVP(2.1) has a unique, nonextendable classical solution $u(x,t)$ on $\Omega \times [0,T)$ where either $T = +\infty$ or $T < +\infty$ and $\lim_{t\to T} \sup_{x\in\Omega} u(x,t) = +\infty$.
3) For $\Omega = B_1(0)$, (a) any solution of BVP(2.2) is radially symmetric and radially decreasing; (b) if $u_0(x)$ is radially symmetric and radially decreasing, then the solution of $u(x,t)$ of IBVP(2.1) is also for each $t \in [0,T)$.

3 Existence-nonexistence for BVP(2.2).

Let $f(u) = e^u$. The following theorems are proven in [2]. Nonexistence for BVP(2.2) should give information about nonexistence of global solutions for IBVP(2.1).

Theorem 1 *For* $\Omega = B_1(0) \subset \mathbb{R}^1$*:*

a) if $p \geq 1$, *BVP (2.2) has a unique solution for all* $\delta > 0$*;*
b) if $0 \leq p < 1$, *then there exists* $\delta^* > 0$ *such that BVP (2.2) has: i) two solutions for* $\delta < \delta^*$*; ii) one solution for* $\delta = \delta^*$*; and iii) no solution for* $\delta > \delta^*$*.*

Theorem 2 *For* $\Omega = B_1(0) \subset \mathbb{R}^2$*,*

a) if $p > 1$, *BVP(2.2) has a unique solution for all* $\delta > 0$*.*
b) if $p = 1$, *BVP(2.2) has a unique solution for* $\delta < \delta^* = 8\pi$ *and no solution for* $\delta \geq \delta^*$*.*
c) if $0 \leq p < 1$, *then there exists* $\delta^* > 0$ *such that BVP(2.2) has: i) two solutions for* $\delta < \delta^*$*, ii) a unique solution for* $\delta = \delta^*$*, and iii) no solution for* $\delta > \delta^*$*.*

By using Pohozaev's identity, nonexistence results can be extended to strictly star-shaped domains. A domain $\Omega \subset \mathbb{R}^n,\ n \geq 2$, is *strictly star-shaped* (containing 0) if there exists $a > 0$ such that

$$x \cdot N \geq a \int_{\partial\Omega} ds, \quad \text{for } x \in \partial\Omega,\ N \text{ unit outer normal.}$$

Theorem 3 *For* $p \leq 1$, *spatial dimension* $n \geq 2$, *BVP(2.2) has no solution for* $\delta > \dfrac{2n}{a}|\Omega|^{p-1}$*.*

4 Finite time blowup

Consider IBVP(2.1) and the associated steady-state problem BVP(2.2) when $f(u) = e^u,\ p < 1,\ n = 1$ or 2, and $\Omega \subset \mathbb{R}^n$ such that Theorem 1, 2, or 3 is valid. Then there exists a critical $\delta^* > 0$ such that for $\delta > \delta^*$, no solution of BVP(2.2) exists.

For spatial dimensions $n = 1$ or 2, IBVP(2.1) defines a local semiflow in $H_0^1(\Omega)$. For $0 < p < 1$, this local semiflow has a Lyapanov functional given by

$$V[u](t) = \frac{1}{2}\int_\Omega |\nabla u|^2 dx + \frac{\delta}{p-1}\left(\int_\Omega e^u dx\right)^{1-p} \tag{4.1}$$

and the semiflow is gradient-like in the sense that for any $t \in [0, T)$

$$\int_0^t ||u_t||_2^2 + V[u](t) = V[u_0]. \tag{4.2}$$

Theorem 4 *For $\delta > \delta^*$, the solution $u(t, u_0)$ of IBVP(2.1) blows up in finite time $T < \infty$.*

The proof is given in [2] and is based on ideas of Marek [7]. An outline of the proof can be given by stating the following five lemmas.

Lemma 1 *If u is a global solution of IBVP(2.1), then there exists $\kappa = \kappa(u_0)$ such that*

$$||u(t, u_0)||_2 = \left(\int_\Omega |u|^2 dx\right)^{\frac{1}{2}} \le \kappa \quad \textit{for all } t \ge 0.$$

Lemma 2 *If* $||u(t, u_0)|| \equiv \left(\int_\Omega |\nabla u|^2 dx\right)^{\frac{1}{2}} \to \infty$ *as* $t \to t_m$, *then* $t_m < \infty$.

Lemma 3 *If $u(t, u_0)$ is a global solution with the ω-limit set $\omega(u_0) \ne \varphi$, and if $w \in \omega(u_0)$ is any equilibrium solution, then $||w|| \le K(u_0)$.*

Lemma 4 *If $u(t, u_0)$ is a global solution with*

$$\liminf_{t\to\infty} ||u(t, u_0)|| < \infty, \quad \limsup_{t\to\infty} ||u(t, u_0)|| = \infty,$$

then for any B sufficiently large, there exists an equilibrium solution w with $||w|| = B$.

Lemma 5 *If $u(t, u_0)$ is global, then $\sup_{t\ge 0} ||u(t, u_0)|| < \infty$ and $\sup_{t \ge \tau} |u(t, u_0)| < \infty$ for any $\tau > 0$.*

Lemma 5 follows from the previous lemmas. For if $u(t, u_0)$ is global, then, by Lemma 1, $||u(t, u_0)|| \not\to \infty$. By Lemmas 3 and 4, $\limsup_{t\to\infty} ||u|| \ne +\infty$ so $\sup_{t \ge 0} ||u(t, u_0)|| < \infty$. But then Lemma 5 implies solution of BVP(2.2) exists. But, by assumption $\delta > \delta^*$; therefore no such solution exists. Thus $u(t, u_0)$ must blow up in finite time.

5 Single point blowup

Theorem 4 for $\Omega = (-1,1) \subset \mathbb{R}^1$, $\delta > \delta^*$, $p < 1$ where $u(x,t)$ represents the temperature perturbation for the shear banding model (1.7) tells us that thermal runaway or blowup occurs in finite time T. The question of where blowup occurs is answered by the following theorem.

Theorem 5 *For $\Omega = B_1(0)$, initial data $u_0(x)$ radially symmetric and decreasing, $f(u) = e^u$, $n = 1$ or 2, $p < 1$, and $\delta > \delta^*$, then the blowup set for IBVP(2.1) consists of a single point $x = 0$.*

The details of the proof of this theorem appear in [3]. Here we will sketch an outline of the main ingredients of the proof.

The equation (2.1) for radially symmetric and radially decreasing initial data $u_0(x)$ becomes, with $r = |x|$,

$$u_t = \frac{1}{r^{n-1}}(r^{n-1}u_r)_r + \delta k(t)e^u \tag{5.1}$$

where $k(t) = \left(\int_{B_1(0)} e^u dx\right)^{-p}$ with $u_r < 0$ on $\{0 < r < 1\} \times [0,T)$.

Assume there exists $\bar{x} \neq 0$ in $B_1(0)$ such that

$$\lim_{t \to T^-} u(\bar{x},t) = +\infty,$$

then $k(t) \to 0$ as $t \to T^-$ and $k'(t) \leq 0$ for t sufficiently near T.

On $[0,1] \times [t^*,T)$ for t^* sufficiently near T,

$$J(r,t) \equiv r^{n-1}u_r(r,t) + \eta c(r)F(u,t) \tag{5.2}$$

for $\eta \in (0,1)$ satisfies

$$J_t + \frac{n-1}{r}J_r - J_{rr} - AJ = D. \tag{5.3}$$

For $F(u,t) = e^{\beta u} \cdot k(t)$, any $\beta < 1$, and $c(r) = r^n$, $D \leq 0$ so Maximum Principle arguments can be used to show $J(r,t) \leq 0$ and hence

$$u_r \leq -\eta r e^{\beta u} k(t). \tag{5.4}$$

Let φ be the normalized positive eigenfunction for the first eigenvalue of

$$\begin{aligned} -\Delta\varphi &= \lambda\varphi\ , \quad x \in B_r(0)\ , \quad 0 < r \leq 1, \\ \varphi &= 0\ , \qquad x \in \partial B_r(0)\ , \end{aligned}$$

with $\int_{B_r} \varphi dx = 1$.

Using Jensen's inequality, one can prove

Lemma 6 *For each $r \in (0,1]$, there exists $C(r) > 0$ such that the solution $u(x,t)$ of IBVP (2.1) satisfies:*

$$\int_{B_r} u\varphi dx \leq \frac{1}{\alpha} \ln \frac{1}{T-t} + C(r) \tag{5.5}$$

where $\alpha = 1 - p$.

Let $0 < r_1 < |\bar{x}| < r_2 < 1$, $A = \{r_1 < |x| < r_2\}$, and $B = B_b(\bar{x})$ where $b = 1/2 \min[r_2 - |\bar{x}|, |\bar{x}| - 1]$, then $B \subset A \subset B_1(0)$. Let ψ be the normalized first eigenfunction of

$$\begin{aligned} -\Delta\psi = \lambda\psi \ , \quad & x \in B \\ \psi = 0 \ , \quad & x \in \partial B \end{aligned}$$

with $\displaystyle\int_B \psi dx = 1$. Then the following dichotomy holds: Either

a) there exists $\varepsilon_0 \in (0,1)$ such that

$$\frac{\displaystyle\int_B e^u \psi dx}{\left(\displaystyle\int_\Omega e^u dx\right)^p} \leq C(\bar{x}) \left(\frac{1}{T-t}\right)^{1-\varepsilon_0} \tag{5.6}$$

for all t sufficiently near T, or

b) for any $\varepsilon \in (0,1)$, there exists a sequence $\{t_n\}$, $t_n \to T^-$, such that

$$\frac{\displaystyle\int_B e^{u(x,t_n)} \psi(x) dx}{\left(\displaystyle\int_\Omega e^{u(x,t_n)} dx\right)^p} \geq \left(\frac{1}{T-t_n}\right)^{1-\varepsilon} . \tag{5.7}$$

If a) holds, multiply (2.1) by ψ and integrate over B. Using Green's Second Identity, (6.6), and Lemma 6, we arrive at

$$\frac{d}{d_t}\left(\int_B u\psi dx\right) \leq C_1 \ln\left(\frac{1}{T-t}\right) + C_2 \left(\frac{1}{T-t}\right)^{1-\varepsilon_0} \tag{5.8}$$

which implies $\displaystyle\int_B uxdx \leq C$ for all $t \in [0,T]$. But $\bar{x} \in B$ is assumed to be a blowup point and $u(r,t)$ is radially decreasing. This is a contradiction. We conclude b) must hold. From (5.7), we have

$$\left(\int_\Omega e^u dx\right)^{2p} \leq C \left(\frac{1}{T-t_n}\right)^{\frac{2p}{\alpha}\left(1+\frac{\alpha\varepsilon}{p}\right)} \tag{5.9}$$

and, for any $\beta \in (0,1)$,

$$\left(\int_B e^u \psi dx\right)^{2\beta} \geq \frac{1}{C}\left(\frac{1}{T-t_n}\right)^{\left(\frac{1-\varepsilon}{2}\right)2\beta} . \tag{5.10}$$

Using the Lyapunov functional (4.1) and (4.2) we have

$$V[u_0] \geq \frac{1}{2}\int_\Omega |\nabla u|^2 dx - \frac{\delta}{\alpha}\left(\int_\Omega e^u dx\right)^\alpha. \tag{5.11}$$

Also,

$$J(r,t) = r^{n-1}u_r + \eta r^n e^{\beta u} k(t) \leq 0 \tag{5.12}$$

for $\eta > 0$ sufficiently small and $\beta \in (0,1)$.

From (5.11) and (5.12), we have

$$V[u_0] \geq \frac{1}{2}\eta^2 k^2(t)\int_\Omega r^2 e^{2\beta u} dx - \frac{\delta}{\alpha}\left(\int_\Omega e^u dx\right)^\alpha. \tag{5.13}$$

Therefore, for any $\varepsilon > 0$ there exists $\{t_n\}, t_n \to T^-$ such that

$$V[u_0] \geq C\left(\frac{1}{T-t_n}\right)^{\frac{1-\varepsilon}{\alpha}2\beta - \frac{2p}{\alpha} - 2\varepsilon} - D\left(\frac{1}{T-t_n}\right)^{1+\frac{\varepsilon\alpha}{p}} \tag{5.14}$$

with $\beta \in (0,1)$. for $\varepsilon > 0$ sufficiently small and $\beta < 1$ sufficiently near 1, since $\frac{2}{\alpha} - \frac{2p}{\alpha} < 1$, we see that the right hand side of (5.13) tends to $+\infty$ as $t_n \to T$. This is a contradiction and we must conclude that the only blowup point is the origin.

For the shear banding model (1.7), Theorem 5 tells us that the temperature perturbation $u(x,t)$ becomes unbounded only at the origin.

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PICTURING PINCHUK'S PLANE POLYNOMIAL PAIR

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Abstract. Sergey Pinchuk discovered a class of pairs of real polynomials in two variables that have a nowhere vanishing Jacobian determinant and define maps of the real plane to itself that are not one-to-one. This paper describes the asymptotic behavior of one specific map in that class. The level of detail presented permits a good geometric visualization of the map. Errors in an earlier description of the image of the map are corrected (the complement of the image consists of two, not four, points). Techniques due to Ronen Peretz are used to verify the description of the asymptotic variety of the map.

1 Introduction

The strong real Jacobian conjecture stated that every polynomial map from $\mathbb{R}^n$ to $\mathbb{R}^n$ with nowhere vanishing Jacobian determinant is univalent (one-to-one). This

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conjecture was refuted (for $n = 2$ and hence all larger n) in 1994 by Sergey Pinchuk, who provided a class of counterexamples [5]. One of the counterexamples is a map $F = (P, Q) : \mathbb{R}^2 \to \mathbb{R}^2$ with $P(x, y)$ and $Q(x, y)$ polynomials of total degree 10 and 25, respectively [6]. That particular map is the primary focus of this paper. It can be described as follows.

Let $t = xy - 1, h = t(xt + 1), f = ((h + 1)/x)(xt + 1)^2, P = f + h, Q = -t^2 - 6th(h + 1) - 170fh - 91h^2 - 195fh^2 - 69h^3 - 75fh^3 - (75/4)h^4$. Then $F(x, y) = (P(x, y), Q(x, y))$ is a real polynomial map from $\mathbb{R}^2$ to itself; its Jacobian determinant is everywhere positive; and it is not univalent.

This map has been considered elsewhere, in particular in [1], where the assertion was made that $F(\mathbb{R}^2)$ consists of all of $\mathbb{R}^2$ except for exactly four points. There were errors and oversights in the calculations, and two of the four points cited are actually in the image. In this paper, the complement of the image is identified as consisting of the points $(0, 0)$ and $(-1, -163/4)$ only.

The asymptotic behavior of the map is studied. In particular, the asymptotic variety of F (as defined by Ronen Peretz in [4]) is computed using Peretz's technique. Denote it by AV$[F]$. For the particular F studied here, AV$[F]$ admits a parameterization by two polynomials of degree two and five in a single variable.

The *asymptotic flower* of F (new terminology, see [2]) is the inverse image under F of AV$[F]$. Denote it by AF$[F]$. By construction, the restriction of F to a mapping from $\mathbb{R}^2 \setminus \text{AF}[F]$ to $\mathbb{R}^2 \setminus \text{AV}[F]$ is a proper map. For this particular F, it reduces to homeomorphisms of four simply connected domains in $\mathbb{R}^2$, each mapping onto one of the two simply connected components of $\mathbb{R}^2 \setminus \text{AV}[F]$. This description provides a good geometric visualization of the map (and is supplemented by graphics).

2 Asymptotics of Pinchuk's Map

Pinchuk's map $F(x, y) = (P(x, y), Q(x, y))$ is most easily studied by considering the fibers $P = c$ of the map P, because P only has degree 10, whereas Q has degree 25. The following information and table are excerpted from [1]. The fiber $P = 0$ has five components and $P = -1$ has four components. In both cases ($c = 0$ and $c = -1$) the fibers can be computed and their components parameterized explicitly without great difficulty, because the polynomial $P - c$ factorizes simply. The other fibers are parameterized by the rational curve

$$x(h) = \frac{(c - h)(h + 1)}{(c - 2h - h^2)^2}$$

$$y(h) = \frac{(c - 2h - h^2)^2(c - h - h^2)}{(c - h)^2}$$

For a fixed value c, the components of the fiber $P = c$ are the images the map $h \mapsto (x(h), y(h))$ for values of h between successive poles (which occur when $h = c$ or $c - 2h - h^2 = 0$; no cancellation occurs as long as c is neither 0 nor -1). The table below summarizes the data on number of components and the range of Q for all fibers $P = c$. Q is always monotone (hence one-to-one) on any component of a fiber $P = c$, because the Jacobian determinant of P and Q is everywhere nonzero.

$P = c$	Ranges of Q on the components
$c > 0$	$(+\infty, q-),\ (q-, q+),\ (q+, -\infty),\ (-\infty, +\infty)$
$c = 0$	$(0, 208),\ (-\infty, 0),\ (0, +\infty),\ (-\infty, 0),\ (208, +\infty)$
$-1 < c < 0$	$(+\infty, q-),\ (q-, -\infty),\ (-\infty, q+),\ (q+, +\infty)$
$c = -1$	$(-\infty, -163/4),\ (-\infty, -163/4),\ (-163/4, +\infty),\ (-163/4, +\infty)$
$c < -1$	$(+\infty, -\infty),\ (-\infty, +\infty)$
Legend: (a, b) denotes the open interval from $\min(a, b)$ to $\max(a, b)$; $q+$ $(q-)$ = the value of Q at $h = -1 + \sqrt{1+c}$ (resp., $-1 - \sqrt{1+c}$);	

Table 1. Ranges of Q on fibers $P = c$ for Pinchuk's map

Remark 1 In [1] there was a typographical error in the formula for $x(h)$ (the term in the denominator was not squared). The computations leading to the results in Table 1 used the correct parameterization (the one shown above). Also, in [1] one of the points listed as not in the image of F was the point $(0, 208)$. However, a glance at the table shows that this cannot be correct; the value 208 lies in $(0, +\infty)$, which is the range of Q on one the components of the fiber $P = 0$.

Remark 2 In [1], the parameterization by $x(h), y(h)$ was introduced without any indication of how it arises. It comes from a straightforward process of solving the equations that define P, first for x and then for y. For example, if $P = c$ then the first step is $c = f + h$, then $c - h = ((h+1)/x)(xt+1)^2 = ((h+1)/x)(h/t)^2$, hence $xt^2 = (h+1)h^2/(c-h)$. From the defining equations again, $t = h - xt^2$, which allows solving for t in terms of h, then for $x = xt^2/t^2$, and finally for $y = (t+1)/x$.

The finite endpoints of ranges of Q occur precisely due to components of a fiber along which the x or y component blows up, but $Q(x, y)$ does not. Denote $x(h), y(h)$ by $x(c, h), y(c, h)$ to capture the dependence on c. Then one has the following rational identities

$$P(x(c,h), y(c,h)) = c$$

$$\begin{aligned} Q(x(c,h), y(c,h)) = \frac{1}{4(c-h)^2}\{&197h^6 + (416 - 726c)h^5 \\ &+ (252 - 1684c + 825c^2)h^4 + (-1224c + 2040c^2 - 300c^3)h^3 \\ &+ (1648c^2 - 780c^3)h^2 + (-680c^3)h\} \end{aligned}$$

The identities can be verified simply by substitution. For $c \neq 0, -1$ it can be checked that Q blows up when h tends to c (one of the poles of the parameterization), but not when h tends to either of the values $-1 + \sqrt{1+c}$, $-1 - \sqrt{1+c}$ (which are also poles of the parametrization, as they are the zeroes of $c - 2h - h^2$). The respective values of q are denoted by $q+$,$q-$. Of course, they depend on c. By definition, the asymptotic variety of a map [4] consists of points in the image plane that are limits of the images of points along a curve that tends to infinity in the original plane. By that definition, for each $c \neq 0, -1$ the points $(c, q+)$ and $(c, q-)$ are in the asymptotic variety, AV$[F]$, of the map F. These points can be obtained by simply substituting $c = 2h + h^2$ into the above rational identities for P and Q. To make life simple, u and v will be used as coordinates in the image plane, with x and y reserved for points in the original plane. Carrying out the indicated substitution yields the following parameterized curve in the image plane

$$u = P = c = 2h + h^2$$

$$v = Q = -(1/4)(1736h^3 + 1044h^2 + 1155h^4 + 300h^5)$$

The values of h that lead to $c = 0, -1$ are $h = -1$ $(c = 0)$, $h = 0$ $(c = 0)$, and $h = -2$ $(c = 0)$. The corresponding points (u, v) arising from the above parameterization are, respectively, $(-1, -163/4)$,$(0, 0)$, and $(0, 208)$. From Table 1, these points all belong to the AV$[F]$. So the entire curve lies in AV$[F]$. Using Peretz's technique of standard asymptotic identities, it will be shown below that this is, in fact, the entire asymptotic variety AV$[F]$. Figure 1 is a depiction of the variety.

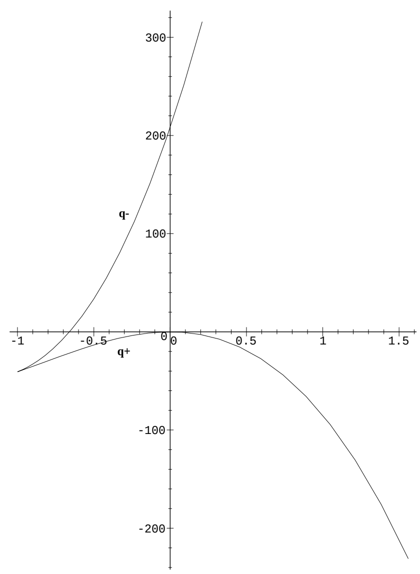

Figure 1. The asymptotic variety of Pinchuk's map.

The figure illustrates the fact that $q-$ is the larger of the two values of Q for a given c, whereas $q+$ is the smaller (except at $c = -1$, where they coincide). This can be seen easily by using a parametrization in terms of $w = h + 1$, for which the upper portion corresponds to $w < 0$ and the lower portion to $w > 0$.

Remark 3 In [1] it was claimed that there was a point of the form $u = c, v = d$ for some $-1 < c < 0$ which was not in the image of F, because the values of $q+$ and $q-$ supposedly coincided at that point. (That is, $d = q+ = q-$, hence d would not lie in any of the ranges of Q on the fiber $P = c$.) From the above figure, it is clear that there is no point where $q+$ or $q-$ coincide for $c > -1$.

Table 1 can be rewritten using the fact that $q+ < q-$ for $c \neq 0, -1$ to put all the intervals (a, b) that are ranges of Q on fibers of P in canonical form - that is, with $a < b$. The result is

$P = c$	Ranges of Q on the components
$c > 0$	$(-\infty, q+),\ (q+, q-),\ (q-, +\infty),\ (-\infty, +\infty)$
$c = 0$	$(0, 208),\ (-\infty, 0),\ (0, +\infty),\ (-\infty, 0),\ (208, +\infty)$
$-1 < c < 0$	$(-\infty,\ q+),\ (q+,\ +\infty),\ (-\infty,\ q-),\ (q-,\ +\infty)$
$c = -1$	$(-\infty, -163/4),\ (-\infty, -163/4),\ (-163/4, +\infty),\ (-163/4, +\infty)$
$c < -1$	$(-\infty, +\infty),\ (-\infty, +\infty)$

Table 2. Ranges of Q on fibers $P = c$ for Pinchuk's map - rewritten

This clearly shows that the only points omitted from the image of F are the points $(-1, -163/4)$ and $(0, 0)$.

3 The Peretz Method

This section uses the techniques described in Ronen Peretz's paper [4] to derive conditions that must be satisfied by any asymptotic values of the polynomial P. In the next section, those conditions will be used to show that AV$[F]$ is exactly the curve shown in Figure 1.

Observe first that the highest (total) degree term in P is x^6y^4, so P satisfies the Peretz normalization criterion $\deg(P) = \deg_x(P) + \deg_y(P)$. This implies that P has only x or y-finite asymptotic curves. In fact, P can only have asymptotic curves with $x \to \pm\infty$ and $y \to 0$, or vice versa. To search for y-finite asymptotic curves and the corresponding asymptotic values, follow the steps outlined by Peretz. That is, first write

$$P(x, y) = P_6x^6 + P_5x^5 + P_4x^4 + P_3x^3 + P_2x^2 + P_1x + P_0$$

where $P_0, \ldots, P_6$ are polynomials in y. This yields

$$\begin{aligned}
P_6 &= y^4 \\
P_5 &= -4y^3 \\
P_4 &= 3y^3 + 6y^2 \\
P_3 &= -7y^2 - 4y \\
P_2 &= 3y^2 + 5y + 1 \\
P_1 &= -3y - 1 \\
P_0 &= y
\end{aligned}$$

Then, assuming that P tends to the (finite) value C along an asymptotic curve (one that tends to infinity in the domain space), write down the Peretz assertions

$$\begin{aligned}
&P_6x^6 + P_5x^5 + P_4x^4 + P_3x^3 + P_2x^2 + P_1x + P_0(0) \to C \\
&P_6x^5 + P_5x^4 + P_4x^3 + P_3x^2 + P_2x + P_1(0) \to 0 \\
&P_6x^4 + P_5x^3 + P_4x^2 + P_3x + P_2(0) \to 0 \\
&P_6x^3 + P_5x^2 + P_4x + P_3(0) \to 0 \\
&P_6x^2 + P_5x + P_4(0) \to 0 \\
&P_6x + P_5(0) \to 0 \\
&P_6(0) \to 0
\end{aligned}$$

These follow from the fact that if a product of two factors tends to a finite limit and one factor is x (which tends to $\pm\infty$), then the other factor tends to 0.

Look, from the bottom up, for the first assertion in which the term before $\to$ is not zero. This is the assertion containing $P_2(0)$. Write the assertion out in full. After judicious factorization it is

$$(yx)^4 - 4(yx)^3 + (3y + 6)(yx)^2 + (-7y - 4)(yx) + 1 \to 0$$

and since $y \to 0$, this implies that $xy \to r$, where r is a root of

$$r^4 - 4r^3 + 6r^2 - 4r + 1 = (r - 1)^4$$

Remark 4 Justification. First, the power product xy must remain bounded, otherwise the expression would not approach a finite limit. Next, even if the polynomial has multiple distinct roots, the image of the curve must ultimately remain near a single root, otherwise the value of the expression would not be ultimately small (it would not tend to zero because of the trips from one root to another, necessarily involving points away from the roots). Similar reasoning applies later, when other power products than xy are considered.

Thus $xy \to 1$, which means that $(y - 1/x)x \to 0$. In other words

$$y = 1/x + o(1/x)$$

Next denote the error term by z, so that $y = 1/x + z$. Substitute $y = 1/x + z$ into the Peretz assertions to obtain the following ones

$$
\begin{aligned}
3z^2x^2 + x^6z^4 + 2z^2x^3 + 3z^3x^4 + 3zx &\to C \\
z^4x^5 + 3z^3x^3 + 2z^2x^2 + 3z^2x + 6z + 3/x &\to 0 \\
z^4x^4 + 3z^3x^2 + 2z^2x - 5z - 4/x &\to 0 \\
x^3z^4 + (-3z^2 + z^2(3+3z))x - 2z + 2z(3+3z) + 3z^2 + (3+9z)/x + 3/x^2 &\to 0 \\
z^4x^2 - 6z^2 - 8z/x - 3/x^2 &\to 0 \\
z^4x + 4z^3 + 6z^2/x + 4z/x + 1/x^3 &\to 0 \\
0 &\to 0
\end{aligned}
$$

Using the facts that $1/x \to 0$, $z \to 0$, and $xz \to 0$, these assertions can be immediately simplified to

$$
\begin{aligned}
x^6z^4 + 2x^3z^2 + 3x^4z^3 &\to C \\
x^5z^4 &\to 0
\end{aligned}
$$

plus five additional trivial assertions of the form $0 \to 0$. The fact that $x^5z^4 \to 0$ imposes the requirement that $z^4 = o(x^{-5})$. It follows that $z = o(|x|^{-5/4})$. No specific data on the form of the error term is implied. Finally, the assertion $(x^3z^2)^2 + (2 + 3xz)(x^3z^2) \to C$, together with $xz \to 0$, means that x^3z^2 tends to a root r of $r^2 + 2r - C = 0$. If $x^3z^2 \to r$ then $|z| = |r|^{1/2}|x|^{-3/2} + o(|x|^{-3/2})$. Since $5/4 < 3/2$, any such z automatically satisfies the $z = o(|x|^{-5/4})$ requirement.

To sum up, the following *necessary* requirements on the asymptotic behavior along a y-finite asymptotic curve for P with asymptotic limit C have been derived.

$$y = x^{-1} + s|x|^{-3/2} + o(|x|^{-3/2})$$

where $|s| = |r|^{1/2}$ and r is a root of $r^2 + 2r - C = 0$. If $r \neq 0$, then only one of the two possible choices of s occurs for a given asymptotic curve. However, either choice will lead to a curve with the right properties, since in either case $x^3z^2 \to r$.

To verify that these conditions suffice, denote again by z the (new) error term.

$$y = x^{-1} + s|x|^{-3/2} + z$$

Compute $P(x, x^{-1} + sx^{-3/2})$. The result is

$$s^4 + 2s^2 + (3s^3 + 3s)x^{-1/2} + (3s^2 + 1)x^{-1} + sx^{-3/2}$$

This is a correct formula for what must happen if $x \to +\infty$. If $x \to -\infty$ instead, $x^{-3/2}$ must be replaced by $|x|^{-3/2} = (-x)^{-3/2}$. Compute $P(x, x^{-1} + s(-x)^{-3/2})$. The result is

$$s^4 - 2s^2 + (-3s^2 + 3s)(-x)^{-1/2} + (3s^2 - 1)(-x)^{-1} + s(-x)^{-3/2}$$

To obtain the corresponding asymptotic identities in Peretz's standard form, substitute $1/x^2$ for x and y for s in the first, and $-1/x^2$ for x and y for s in the second, obtaining

$$P(1/x^2, yx^3 + x^2) = y^4 + 2y^2 + (3y^3 + 3y)x + (3y^2 + 1)x^2 + yx^3$$
$$P(-1/x^2, yx^3 - x^2) = y^4 - 2y^2 + (3y^3 - 3y)x + (3y^2 - 1)x^2 + yx^3$$

Remark 5 In [4] Peretz claims that to find all the asymptotic values of a polynomial P corresponding to y-finite asymptotic curves, it suffices to consider asymptotic identities of the form $P(1/x^k, yx^N + a_{N-1}x^{N-1} + \ldots + a_0) = a(x,y) \in \mathbb{R}[x,y]$. This appears to be an oversight. As this case shows, one must consider asymptotic identities involving $\pm 1/x^k$ when k is even, otherwise asymptotic values obtained as $x \to -\infty$ will be missed. It turns out that both P and Q satisfy asymptotic identities for each of the two asymptotic curves above. The (u,v) coordinates of points in AV$[F]$ are obtained by substituting $x = 0$ in the right hand sides of the asymptotic identities, and allowing y to vary. The right hand side of the first identity for P reduces to $y^4 + y^2$ for $x = 0$, so one can obtain only points with $u \geq 0$. In fact, one obtains the points in Figure 1 on the $q+$ portion of the curve, starting at $(0,0)$ and going to the right. The remaining points in AV$[F]$ all derive from the identities for the second asymptotic curve $(-1/x^2, yx^3 - x^2)$.

Next consider the case in which z, the error term, is not identically zero. As an illustration, compute $P(x, x^{-1} + sx^{-3/2} + z)$. The result is the same result obtained when $z = 0$ plus the following additional terms

$$z^4x^6 + 4sz^3x^{9/2} + 3z^3x^4 + (6s^2z^2 + 2z^2)x^3 + 9sz^2x^{5/2} + 3z^2x^2 + (4s^3z + 4sz)x^{3/2} + (3z + 9s^2z)x + 6szx^{1/2} + z$$

Each of these terms tends to zero as a consequence of $z = o(|x|^{-3/2})$. So these are indeed all asymptotic curves, and the limiting asymptotic value obtained is independent of the form of z as long as $z = o(|x|^{-3/2})$. No new asymptotic limits are found. However, formally different asymptotic identities can be derived. For instance, from $y = x^{-1} + ax^{-3/2} + bx^{-2}$ one obtains the following asymptotic identity when $1/x^2$ is substituted for x and y is substituted for b

$$P(1/x^2, yx^4 + ax^3 + x^2) = x^4y^4 + (4x^3a + 3x^4)y^3 + (9x^3a + 6a^2x^2 + 3x^4 + 2x^2)y^2 + (6x^3a + x^4 + 4ax + 4a^3x + 3x^2 + 9a^2x^2)y + a^4 + 3ax + 3a^2x^2 + x^2 + x^3a + 3a^3x + 2a^2$$

Setting $x = 0$ in the right hand side to see what asymptotic limits are obtained yields $a^4 + 2a^2$, the same set of limit values as for the previous asymptotic identity. Note that the free parameter a yields the asymptotic values here, whereas all the y terms disappear if x is set equal to zero.

To look for x-finite asymptotic values, similar steps are taken, but there are fewer such steps since the powers of y extend only up to y^4. The first assertion, from the bottom up, that has a nonzero constant term on the left is (suitably rearranged)

$$(x^2y)^3 + (x^2y)^2(-4x+3) + (x^2y)(6x^2 - 7x + 3) + 1 \to 0$$

and as $x \to 0$ this implies that x^2y tends to a root r of the equation

$$r^3 + 3r^2 + 3r + 1 = (r+1)^3 = 0$$

Thus the first approximation is

$$x = -y^{-1/2} + o(y^{-1/2})$$

with $y \to +\infty$ the only possibility. Substitute $x = -y^{-1/2} + z$ into the Peretz assertions to obtain

$$\begin{aligned} z^6y^4 + 18z^4y^3 - 4y^3z^5 + 6z^4y^2 - 47z^3y^2 + 36y^2z^2 + 8y \\ - 4z^3y - 44zy + 41z^2y + 11 - 12z + 20z^4y^{(5/2)} + 12z^2y^{(1/2)} \\ - 24zy^{(3/2)} - 34zy^{(1/2)} + 61z^2y^{(3/2)} - 24z^3y^{(3/2)} - 32z^3y^{(5/2)} \\ - 6z^5y^{(7/2)} + 14y^{(1/2)} + 4y^{(-1/2)} \\ \to 0 \end{aligned}$$

$$\begin{aligned} z^6y^3 + 18z^4y^2 - 4z^5y^2 + 6z^4y - 47z^3y + 36z^2y + 8 \\ + 36z^2 - 41z + 6y^{(-1)} + 20z^4y^{(3/2)} - 24z^3y^{(1/2)} - 24zy^{(1/2)} \\ + 61z^2y^{(1/2)} - 24zy^{(-1/2)} - 6z^5y^{(5/2)} - 32z^3y^{(3/2)} + 11y^{(-1)} \\ \to 0 \end{aligned}$$

$$\begin{aligned} - 18zy^{(-1/2)} + 40z^2y^{(-1/2)} - 20zy^{(-1)} + z^6y^2 - 6z^5y^{(3/2)} + 18z^4y \\ - 32z^3y^{(1/2)} - 4yz^5 + 20z^4y^{(1/2)} + 4y^{(-3/2)} - 40z^3 + 33z^2 + 4y^{(-1)} \\ \to 0 \end{aligned}$$

$$\begin{aligned} z^6y - 6z^5y^{(1/2)} + 15z^4 - 20z^3y^{(-1/2)} + 15z^2y^{(-1)} - 6zy^{(-3/2)} + y^{(-2)} \\ \to 0 \end{aligned}$$

and the trivial assertion $0 \to 0$. Every term containing a monomial of the form z^my^n tends to zero if $m \geq 2n$. The last two assertions collapse to the trivial $0 \to 0$. However, the next one from the bottom up reduces to $8 \to 0$. Since that cannot happen, it follows that there are no x-finite asymptotic limits.

4 The Asymptotic Variety of F

In the previous section it was shown that the asymptotic curves $(1/x^2, yx^3+x^2)$ and $(-1/x^2, yx^3-x^2)$, both defined for $x \neq 0$, are a basis for the asymptotic values of P, in the sense that every asymptotic value of P arises as a limit along one of these curves. These curves are also asymptotic curves that yield finite asymptotic values for Q. Specifically, one has the asymptotic identities

$$\begin{aligned} Q(1/x^2, yx^3+x^2) = {} & -(1/4)(1736y^6+1044y^4+1155y^8+300y^10) \\ & -(x/4)(5700y^7+6692y^5+2792y^3+1800y^9) \\ & -(x^2/4)(9636y^4+11250y^6+4500y^8+2432y^2) \\ & -(x^3/4)(6000y^7+680y+11100y^5+6140y^3) \\ & -(x^4/4)(4500y^6+1460y^2+5475y^4) \\ & -(x^5/4)(1800y^5+1080y^3)-75x^6y^4 \end{aligned}$$

$$\begin{aligned} Q(-1/x^2, yx^3-x^2) = {} & +(1/4)(1736y^6-1044y^4-1155y^8+300y^10) \\ & +(x/4)(-5700y^7+6692y^5-2792y^3+1800y^9) \\ & +(x^2/4)(9636y^4-11250y^6+4500y^8-2432y^2) \\ & +(x^3/4)(6000y^7-680y-11100y^5+6140y^3) \\ & +(x^4/4)(4500y^6+1460y^2-5475y^4) \\ & +(x^5/4)(1800y^5-1080y^3)+75x^6y^4 \end{aligned}$$

Substituting $x=0$ to obtain the asymptotic values, both here and in the asymptotic identities for P, yields the following two parameterized curves that together make up the whole asymptotic variety

$$u = y^4+2y^2,\ v = -(1/4)(1736y^6+1044y^4+1155y^8+300y^{10})$$

$$u = y^4-2y^2,\ v = (1/4)(1736y^6-1044y^4-1155y^8+300y^{10})$$

These two parameterizations can be combined into one by putting $y^2 = h$ (for $h \geq 0$) in the first, and $y^2 = -h$ (for $h \leq 0$) in the second, which yields exactly the parameterization considered before

$$u = h^2+2h,\ v = -(1/4)(1736h^3+1044h^2+1155h^4+300h^5)$$

Remark 6 The functions t,h, and f introduced in the definition of P and Q all satisfy asymptotic identities with respect to each of the above two asymptotic curves. As Peretz remarked in [4, 3], examples such as Pinchuk's arise from finding pairs of polynomials with a nowhere vanishing Jacobian determinant in a real subalgebra of $\mathbb{R}[x,y]$ consisting of polynomials all of which have one or more shared asymptotic curves for which they satisfy asymptotic identities.

5 The Asymptotic Flower of F

In [2] the authors consider (primarily polynomial) maps of the real plane to itself that are proper. The *flower* of a map is the inverse image of the set of critical values (and a value is critical precisely if it is the image of a point at which the Jacobian determinant vanishes). Away from the flower the map is locally a covering map (proper and a local homeomorphism). In fact, it is a a covering map (over its image) on any connected component of the complement of the flower. In the case of Pinchuk's map the flower as defined above is empty, but the covering property fails to hold because the map is not proper. This suggests calling the above flower the *critical flower* and introducing as well the *asymptotic flower*, defined as the inverse image of the set of asymptotic values of the map. On the complement of the asymptotic flower, the restricted map to the complement of the set of asymptotic values is proper. This is because, by definition, as the asymptotic flower is approached, the image of a point will tend to infinity in the image plane or to an asymptotic value. But since the codomain of the restricted map is the complement of the set of asymptotic values, this means that the image is tending to infinity relative to that codomain. (Note. Asymptotic values can be defined in terms of limits of sequences as well on manifolds, using local pathwise connectedness to produce the appropriate curves. More general definitions are possible as well.) On each component of the complement of the *total flower* (the union of the critical and asymptotic flowers) the restricted map to the complement of the critical and asymptotic values will be a covering map over its image.

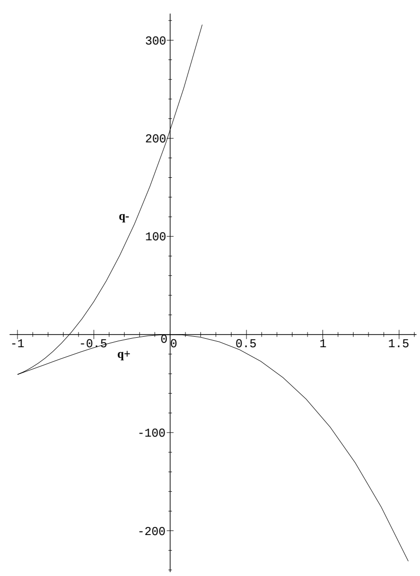

Figure 2. Three curves in the asymptotic variety.

Since the asymptotic variety AV[F] has been identified for the F at hand, it remains only to compute its inverse image to obtain AF$[F]$, the asymptotic flower of F. By consulting Table 2, it becomes clear that

- exactly two points, $(0,0)$ and $(-1,-163/4)$, have no inverse images
- every other point of AV$[F]$ has exactly one inverse image
- every point not in AV$[F]$ has exactly two inverse images

This follows from a case by case check, cases corresponding to rows of the table. If one removes the two points that have no inverse images from AV$[F]$, it breaks up into three connected curves. Call them $C1, C2, C3$, as follows.

$C1$ is the $q-$ curve, starting at $(-1,-163/4)$ and continuing up and to the right. $C2$ is the portion of the $q+$ curve starting at $(0,0)$ and continuing down and to the left, ending at $(-1,-163/4)$. Finally, $C3$ is the portion of the $q+$ curve ending at $(0,0)$ and arriving from down and to the right. Starting and ending points mentioned are not actually points of these curves, since they represent precisely points that were removed. The descriptions also imply orientations for the the three curves. Figure 2 shows the curves and their orientations.

Each point of each of the three curves has exactly one inverse image. Furthermore, as a starting or ending point, finite or infinite, is approached, the inverse image point tends to infinity. Thus the inverse image of each of $C1, C2, C3$ is a smooth curve in the plane (no singularities and no self-intersections) that tends to infinity at either end. Call these curves $D1, D2, D3$. By definition, AF$[F] = D1 \cup D2 \cup D3$, where the curves are considered as point sets.

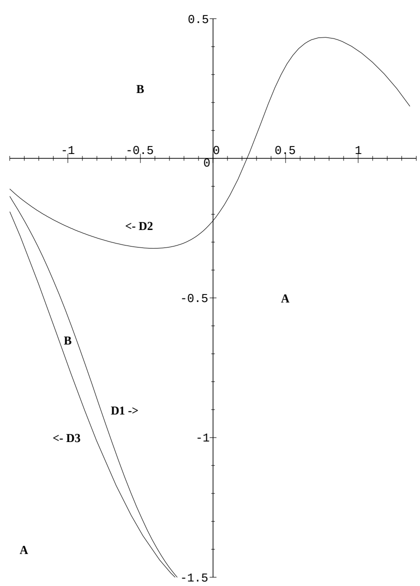

Figure 3. The asymptotic flower of Pinchuk's map.

Each of these curves $D1, D2, D3$ divides the plane into two simply connected parts (the Jordan curve Theorem), which may be described as the regions left and right of the curve, using the induced orientations to define left and right. Removing the curves $D1, D2, D3$ thus leaves exactly four simply connected open components. The restriction $F : \mathbb{R}^2 \setminus \mathrm{AF}[F] \to \mathbb{R}^2 \setminus \mathrm{AV}[F]$, maps each component into either the region L to the left of $\mathrm{AV}[F]$ or into the region R to its right. Each region mapped into L is, in fact, mapped homeomorphically onto L, because we are dealing with a covering of a simply connected region. Similarly for R. Label a connected component of $\mathbb{R}^2 \setminus \mathrm{AF}[F] = \mathbb{R}^2 \setminus (D1 \cup D2 \cup D3)$ with A if it maps to L, and with B if it maps to R. Two of the regions are labeled A, and two are labeled B. The global data of the map (the relations between the domains and curves) are best explained by a figure. Figure 3 depicts the component curves of $\mathrm{AF}[F]$ and their orientations, and also labels the regions defined by the curves. Figure 3 uses nonlinear scaling to produce a more comprehensible picture; the values plotted are actually the arctangents of the coordinates x and y. The figure was generated by solving for the inverse images of a large number of points on $\mathrm{AV}[F]$. The labeling of the regions can be checked by computing the images of a few points not in the flower (and can also be deduced to a large extent from the fact that F is orientation preserving, since its Jacobian determinant is everywhere positive).

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RECOGNIZING AUTOMORPHISMS OF POLYNOMIAL ALGEBRAS

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Abstract. We discuss how to recognize whether an endomorphism of a polynomial algebra is an automorphism through three different approaches: Gröbner basis, the Jacobian conjecture and test polynomials.

1 Introduction

Let K be a field. To avoid complications, we assume throughout this paper that the characteristic of K is zero. Let $K[X] := K[x_1, \ldots, x_n]$ be the polynomial algebra in n variables over K. Let $F := (f_1, \ldots, f_n) \in (K[X])^n$ be an n-tuple. Obviously, $\varphi : p(X) \to p(F)$ is an endomorphism of $K[X]$. On the otherhand, every endomorphism of $K[X]$ may be defined in that way. To slightly abuse the language, sometimes we say that $F := (f_1, \ldots, f_n)$ is an endomorphism of $K[X]$.

The main problem considered in this paper is: given $\varphi : X \to F$ an endomorphism of $K[X]$, how to recognize whether φ is an automorphism?

Key words and phrases. Endomorphisms, automorphisms, Gröbner basis, Jacobian conjecture, test polynomials.

We shall discuss this problem in this paper via three different approaches: Gröbner basis, Jacobian conjecture and test polynomials.

The paper is organized as follows: Section 2 introduces the Gröbner basis approach given by van den Essen. In Section 3, we introduce the Jacobian conjecture and present our recent result on the 'positive' and 'negative' case of the conjecture. In Section 4, we give a new appoach on the $n = 2$ Jocobian conjecture via polynomial retracts. Section 5 deals with the test polynomial approach. Finally, in the concluding Section 6, we propose two open problems related to the Jacobian conjecture.

In the sequal we sometimes denote $(x_1, \ldots, x_n)$ by X, $(y_1, \ldots, y_n)$ by Y, $(f_1, \ldots, f_n)$ by F, and $(g_1, \ldots, g_n)$ by G.

2 Gröbner basis approach

In 1990, Arno van den Essen [8] proved the following theorem.

Theorem 1 *Let $\varphi : x_i \to f_i$ be an endomorphism of $K[x_1, \ldots, x_n]$. Then φ is an automorphism if and only if the reduced Gröbner basis of the ideal generated by*

$$\{y_1 - f_1, \ldots, y_n - f_n\}$$

in the polynomial ring $K[x_1, \ldots, x_n, y_1, \ldots, y_n]$ under the lexicographic ordering

$$x_1 > \cdots > x_n > y_1 > \cdots > y_n$$

is

$$\{x_1 - g_1, \ldots, x_n - g_n\}$$

where $g_i \in K[y_1, \ldots, y_n]$. Moreover, if $X \to F$ is an automorphism, and if we define $G := (g_1, \ldots, g_n)$. Then $Y \to G$ is the inverse automorphism of F.

Note that Theorem 1 gives an algorithm to decide whether an endomorphism of $K[X]$ is an automorphism.

3 The Jacobian conjecture

If $\varphi : X \to F$ is an automorphism of $K[X]$ and $\varphi^{-1} : X \to G$ is the inverse. Then

$$G \circ F = X.$$

Hence

$$J(G \circ F) = J(X)$$

where J denotes the usual Jacobian (matrix) operator. By the chain rule,

$$J(G)(F)J(F) = I$$

where I is the identity matrix of order n. Hence

$$J(F) \in GL_n(K[X]).$$

The Jacobian conjecture is that the converse of the above statement is true.

Conjecture 1 (The Jacobian conjecture) *Let $\varphi : X \to F$ be an automorphism of $K[X]$. If $J(F) \in GL_n(K[X])$. Then φ is an automorphism.*

Formulated by O. Keller [12] in 1939, the conjecture is still open for $n \geq 2$ (the $n = 1$ conjecture is obviously true), to the best of our knowledge.

For arbitrary n, O. Keller [12] himself proved the birational case (i.e. with the addtional condition that $K(X) = K(F)$) in 1939. In 1973, L.A. Campbell [3] proved the Galois case of the conjecture (i.e. with the additional condition that $K(X)/K(F)$ is a Galois extension). In 1980, S.S.-S. Wang [21] proved the quadratic case of the Jacobian conjecture. In 1982, H. Bass, E. Connell and D. Wright [2] reduce the Jacobian conjecture to the cubic homogeneous case. Namely, to solve the conjecture, one only needs to consider the case $F = X + H$ where every monomial in H is cubic (that implies that $J(H)$ is a nilpotent matrix). For $n = 2$, T.T. Moh [14] proved the conjecture for the case $\max\{\deg(f_1), \deg(f_2)\} \leq 100$. For a history and background of the Jacobian conjecture, see [2].

We have recently reduced the Jacobian conjecture to the so-called 'positive case' and solved the 'negative case'. In order to present these results, first note that it is well-known that in order to solve the Jacobian conjecture, we only need to consider the case $K = \mathbb{C}$, the field of complex numbers and we only need to prove that $F : \mathbb{C}^n \to \mathbb{C}^n$ is injective; see, for instance, [2]. In fact, by the following well-known fact, we only need to consider the case $K = \mathbb{R}$, the field of real numbers.

Let $F := (f_1, \ldots, f_n) : \mathbb{C}^n \to \mathbb{C}^n$ be a differentiable map. Then naturally F may be viewed as a map $\overline{F}$ from $\mathbb{R}^{2n} \to \mathbb{R}^{2n}$. Moreover, F is injective if and only if $\overline{F}$ is injective and $\det(J(\overline{F})) = (\det(J(F))^2$.

Then in 1995 we proved the following two theorems in [22]:

Theorem 2 *To solve the Jacobian conjecture, one only needs to consider the case $f_i = x_i + H_i^{(2)} + H_i^{(3)} + H_i^{(4)} \in \mathbb{R}[x_1, \ldots, x_n]$ where $H_i^{(j)}$ are homogeneous of degree j and all coefficients in f_i are nonnegative.*

Theorem 3 *Let $f_i = x_i - H_i \in \mathbb{R}[x_1, \ldots, x_n]$ where $J(f_1, \ldots, f_n) \in GL_n(\mathbb{R}[x_1, \ldots, x_n])$, $ord(H_i) \geq 2$ and all coefficients of H_i are nonnegative. Then $\varphi : x_i \to f_i$ is an automorphism.*

4 Polynomial retracts and the $n = 2$ Jacobian conjecture

In this section we focus on the $n = 2$ Jacobian conjecture with a new approach via polynomial retracts. Let $K[x, y]$ be the polynomial algebra in two variables over a field K of characteristic 0. A subalgebra R of $K[x, y]$ is called a *retract* if it satisfies any of the following equivalent conditions:

- **R1** There is an idempotent endomorphism (a *retraction*, or *projection*) φ of $K[x, y]$ that $\varphi(K[x, y]) = R$.
- **R2** There is a homomorphism $\varphi : K[x, y] \to R$ that fixes every element of R.
- **R3** $K[x, y] = R \oplus I$ for some ideal I of the algebra $K[x, y]$.

R4 $K[x,y]$ is a projective extension of R in the category of K-algebras. In other words, there is a splitting exact sequence $1 \to I \to K[x,y] \to R \to 1$, where I is the same ideal as in (R3) above.

Example 1 K; $K[x,y]$; any subalgebra of the form $K[p]$, where $p \in K[x,y]$ is a *coordinate* polynomial (i.e. $K[p,q] = K[x,y]$ for some polynomial $q \in K[x,y]$). There are other, less obvious, examples of retracts: if $p = x + x^2y$, then $K[p]$ is a retract of $K[x,y]$, but p is not coordinate since it has a fiber $\{p = 0\}$ which is reducible, and therefore is not isomorphic to a line.

The very presence of several equivalent definitions of retracts shows how natural these objects are.

In [5], Costa has proved that every proper retract of $K[x,y]$ (i.e. one different from K and $K[x,y]$) has the form $K[p]$ for some polynomial $p \in K[x,y]$, i.e. is isomorphic to a polynomial K-algebra in one variable. A natural problem now is to characterize somehow those polynomials $p \in K[x,y]$ that generate a retract of $K[x,y]$. Since the image of a retract under any automorphism of $K[x,y]$ is again a retract, it would be reasonable to characterize retracts up to an automorphism of $K[x,y]$, i.e. up to a "change of coordinates". We give an answer to this problem in [19] as follows

Theorem 4 *Let $K[p]$ be a retract of $K[x,y]$. There is an automorphism ψ of $K[x,y]$ that takes the polynomial p to $x+y\cdot q$ for some polynomial $q = q(x,y)$. A retraction for $K[\psi(p)]$ is given then by $x \to x + y \cdot q$; $y \to 0$.*

Our proof of this result is based on the famous Abhyankar-Moh Theorem of embeddings of the line in the plane [1].

Theorem 4 yields another characterization of retracts of $K[x,y]$ (see [19]):

Proposition 1 *A polynomial $p \in K[x,y]$ generates a retract of $K[x,y]$ if and only if there is an endomorphism of $K[x,y]$ that takes p to x.*

Although the form to which any retract can be reduced by Theorem 4 might seem rather general, it is in fact quite restrictive, and has an interesting application to the $n = 2$ Jacobian conjecture.

Now we formulate the following conjecture in [19].

Conjecture 2 *If $p, q \in K[x,y]$ with $J(p,q) \in GL_2(K[x,y])$, then $K[p]$ is a retract of $K[x,y]$.*

Proposition 2 *(see [19]) Conjecture 2 is equivalent to the $n = 2$ Jacobian conjecture.*

Another application of retracts to the $n = 2$ Jacobian conjecture (somewhat indirect though) is based on the "φ^∞-trick" familiar in combinatorial group theory (see [13]). For an endomorphism φ of $K[x,y]$ denote by $\varphi^\infty(K[x,y]) = \bigcap_{k=1}^{\infty} \varphi^k(K[x,y])$ the *stable image* of φ. Then we have:

Theorem 5 *(see [19]) Let φ be an endomorphism of $K[x,y]$. If the Jacobian matrix of φ is invertible, then either φ is an automorphism, or $\varphi^{\infty}(K[x,y]) = K$.*

Our proof of Theorem 5 is based on a recent result of Formanek [10].

Obviously, if φ fixes a polynomial $p \in K[x,y]$, then $p \in \varphi^{\infty}(K[x,y])$. Therefore, we have ([18]):

Proposition 3 *Suppose φ is an endomorphism of $K[x,y]$ with invertible Jacobian matrix. If $\varphi(p) = p$ for some non-constant polynomial $p \in K[x,y]$, then φ is an automorphism.*

This yields the following reformulation of the Jacobian conjecture: if φ is an endomorphism of $K[x,y]$ with invertible Jacobian matrix, then for some automorphism α, the mapping $\alpha \cdot \varphi$ fixes a non-constant polynomial.

5 Test polynomials

In this section we introduce another approach to recognize automorphisms of polynomial algebras via test polynomials. Let A be an algebraic object. An element $a \in A$ is called a *test element* for automorphisms of A if for any endomorphism φ of A such that $\varphi(a) = a$, then φ is an automorphism. This definition was explicitly given by V. Shpilrain [18] in 1994, but the history of test elements goes back to Nielsen in 1918 and Dicks in 1982. A classical result of Nielsen [17] states that an endomorphism $x \to f;\ y \to g$ of the free group F_2 with two generators x, y is an automorphism if and only if $[f,g]$ is conjugate to $[x,y]$. Hence the commutator $[x,y] = xyx^{-1}y^{-1}$ is a test element of F_2. Dicks [6] proved a similar result for the free associative algebra $K\langle x,y\rangle$ of rank two: an endomorphism $(x,y) \to (f,g)$ of $K\langle x,y\rangle$ is an automorphism if and only if $[f,g] = \alpha[x,y]$ where $\alpha \in K^*$. Hence $[x,y] = xy - yx$ is a test polynomial of $K\langle x,y\rangle$. Obviously any element in a proper retract of an algebraic object is not a test element. For a free algebraic object A generated (freely) by n elements, define the *rank* of an element $a \in A$ as the minimum munber $m \le n$ such that a belongs to a free subobject generated (freely) by m free generators of A. It is easy to see that a test element of a free algebraic object A generated freely by n elements must have maximum rank n. Naturally one may ask the question to determine all test elements of an algebraic object. This problem has been solved for both finitely generated free groups and Lie algebras. Turner [20] proved that test elements of a finitely generated free group are precisely those elements not contained in a proper retract of the group. Very recently, we have obtained a similar result for free Lie algebras in [16].

Theorem 6 *Test elements of a finitely generated free Lie algebra are precisely those elements not contained in a proper retract of the Lie algebra*

The proofs of the above results on test elements of free groups and Lie algebras rely heavily on the fact that every subgroup (subalgebra, repectively) of a free group (free Lie algebra, respectively) is again a free group (free Lie algebra, respectively).

In polynomial and free associative algebra cases, the problem is much harder, since obviously there are subalgebras of $K[X]$ ($K\langle X\rangle$, respectively) that are not polynomial algebras (free associative algebras, respectively).

The polynomial $x_1^2 + \cdots + x_n^2$ is the first example of test polynomial for $\mathbb{R}[x_1, \ldots, x_n]$; it was given by van den Essen and V. Shpilrain [9]. However, in [7] we showed that it is not a test polynomial for $\mathbb{C}[x_1, \ldots, x_n]$. Therefore, whether a polynomial in $K[X]$ is a test polynomial depends on the properties of the ground field K.

In [7] we obtained some test polynomials for both $K[X]$ and $K\langle X\rangle$. Unfortunately, all the test polynomials of $K[X]$ we know can only recognize linear automorphisms. Hence we may ask whether there exists a 'non-trivial test polynomial' for the polynomial algebra. On the otherhand, for free associative algebras, we obtained in [7] some 'non-trivial' test polynomials.

Theorem 7 *$[x_1, x_2] \ldots [x_{2n-1}, x_{2n}]$ is a test polynomial for the free associative algebra $K[x_1, \ldots, x_{2n}]$ recognizing the automorphisms that fix all but one or two of the variables.*

In [7], a *test vector space* W of $K\langle X\rangle$ is defined as follows. An endomorphism $X \to F$ of $K\langle X\rangle$ is an automorphism if and only if $w(F) \in W$ for every $w(X) \in W$ and $w(F)$ is not the zero polynomial.

Then we determined in [7] all test vector space of a free associative algebra.

Theorem 8

i) W is a test vector space of $K\langle x, y\rangle$ if and only if W is spanned on a finite set of powers $[x, y]^k$ of the commutator $[x, y]$.
ii) For $n > 2$ there is no test vector space in $K\langle x_1, \ldots, x_n\rangle$.

The proofs of both Theorem 7 and 8 are based on the result of Dicks in [6] and non-commutative algebra technics developed in P.M. Cohn [4].

The problems to determine all test polynomials of $K[x_1, \ldots, x_n]$ or $K\langle x_1, \ldots, x_n\rangle$ remains open, even for the case $n = 2$.

6 Two open problems

In this section, we propose two open problems which are closely related to the Jacobian conjecture.

$p \in K[x_1, \ldots, x_n]$ is called a *coordinate polynomial* if there exist $p_2, \ldots, p_n \in K[x_1, \ldots, x_n]$ such that $K[p, p_2, \ldots, p_n] = K[x_1, \ldots, x_n]$. Or, equivalently, there exists an automorphism φ of $K[x_1, \ldots, x_n]$ such that $\varphi(p) = x_1$.

Problem 1 Is it true that any endomorphism φ of $K[x_1, \ldots, x_n]$ taking coordinate polynomials to coordinate polynomials, is actually an automorphism?

In [9], van den essen and Shpilrain give a positive answer to the problem 1 for the case $n = 2$. Moreover, they observed that in general, $J(\varphi) \in GL_n(K[x_1, \ldots, x_n])$.

Problem 2 Let $F = (F_1, \dots, F_n) \in (K[X_1, \dots, X_n])^n$ with each F_i irreducible and let $p_{ij} \in K[t_1, \dots, t_{n-1}]$ with zero constant terms for $i, j = 1, \dots, n$ such that the $n \times n$ matrix

$$(F_j(p_{i1}, \dots, p_{in}))_{\substack{i=1,\dots,n \\ j=1,\dots,n}} = \begin{pmatrix} 0 & t_1 & t_2 & \dots & t_{n-2} & t_{n-1} \\ t_1 & 0 & t_2 & \dots & t_{n-2} & t_{n-1} \\ t_1 & t_2 & 0 & \dots & t_{n-2} & t_{n-1} \\ & \dots & & \dots & & \dots \\ t_1 & t_2 & t_3 & \dots & 0 & t_{n-1} \\ t_1 & t_2 & t_3 & \dots & t_{n-1} & 0 \end{pmatrix}.$$

Is $X \to F$ an automorphism of $K[X_1, \dots, X_n]$?

In [15], McKay, Moh and Wang give a positive answer to Problem 2 for the case $n = 2$. Then in [11], Jelonek proved that $J(F) \in GL_n(K[x_1, \dots, x_n])$.

Note in some sense, Problem 1 and 2 are 'dual problems'.

Remark 1 The author was notified by Arno van den Essen that Problem 1 and 2 have recently been solved with positive solutions by Z. Jenolek.

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SOME REMARKS ON CONTINUOUS AND DISCRETE MARKUS-YAMABE PROBLEMS

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Dedicated to Gary H. Meisters

Abstract. The aim of this paper paper is twofold. By one side it summarizes some of the results of the three authors (some of them made also in collaboration with A. van den Essen, E. Hubbers and J. Llibre, see the references [Cima & others]) related with injectivity and Markus-Yamabe problems. On the other hand we point out some open problems related with the above papers, discussing finally which is the relation between injectivity and global asymptotic stability.

1 Introduction

Before starting with a brief survey about the actual situation of Markus-Yamabe (MY) problems we would like to stress how the work of Gary Meisters and Czeslaw Olech has strongly influenced our interest into them. Our first contact with the

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problems was motivated by a visit of J. Sotomayor to Barcelona around 1984. In that visit he explained us the MY Conjecture and gave us a copy of [20]. From that moment we started to be interested in the MY problems. After several years appeared the famous paper [17]. From that paper and again motivated by a visit of J. Sotomayor to Barcelona, appeared the paper [13]. R. Fessler and C. Gutierrez explained us that they were also motivated by [17] and [13] to continue the study of MY Conjecture. In fact they separately proved it for the plane, see [12, 14]. At this point M. Sabatini organized a meeting in Trento. Later A. van den Essen organized another meeting in Curaçao. It was in both places were we enter in contact with most people interested in MY problems and also was the starting point of the paper [4]. In this last paper the authors presented a polynomial counterexample to the MY Conjecture for dimension greater than two.

Here we describe which are the MY problems.

Let $F : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ be a $\mathcal{C}^\infty$ map. We say that F satisfies:

(i) The *Markus-Yamabe assumption*, (MYA), if for any $x \in \mathbb{R}^n$, the jacobian of F at x has all its eigenvalues with negative real part.

(ii) The *discrete Markus-Yamabe assumption*, (DMYA), if for any $x \in \mathbb{R}^n$, $DF(x)$ has all its eigenvalues with modulus less than one.

Markus-Yamabe problems.

(i) **MYC(n)** *(Markus-Yamabe Conjecture, [19]). Let* F *be a* $\mathcal{C}^\infty$ *vector field defined on* $\mathbb{R}^n$ *satisfying the MYA. If* $F(p) = 0$, *then the critical point* p *is a global attractor of* $\dot{x} = F(x)$.

(ii) **DMYQ(n)** *(Discrete Markus-Yamabe Question, [7, 22]). Let* F *be a* $\mathcal{C}^\infty$ *map from* $\mathbb{R}^n$ *into itself such that* $F(0) = 0$ *and satisfying the DMYA. Is it true that the fixed point* 0 *is a global attractor for the discrete dynamical system generated by* F*?*

As it has already been told, the MYC is true for $n = 2$ and false for $n \geq 3$ (even for polynomial F), see for instance [4] and the references therein, or even better the Web Address [16].

The present situation of the DMYQ is the following: The answer is yes for $n = 1$ and no for $n \geq 2$. In fact for $n = 2$ the answer is yes for polynomial F, while the example which has not 0 as a global attractor is given by a rational map. For $n \geq 3$ it is possible to give a polynomial F without any global attractor. See [4, 7] and the references therein.

In the first part of this paper (Sections 2, 3 and 4) we present a survey of several results about the MYC and the DMYQ, specially the results which are related with the notion of "quasi-homogeneity" of polynomials. At the same time we present some related open problems which we think that are interesting by themselves. Although they will be stated more precisely in the paper we list them here:

(1) Is the MYC(n) generically true for polynomial vector fields ? (Problem 2).

(2) Are there quasi-homogeneous linear maps $N : \mathbb{C}^n \longrightarrow \mathbb{C}^n$ with $DN(x)$ nilpotent and with all its n components linearly independent over $\mathbb{C}$? (Problem 1).

(3) Which is the answer to the DMYQ for entire maps $F : \mathbb{C}^2 \longrightarrow \mathbb{C}^2$? (Problem 3).

As we will see in Section 3.2 the answer to Question 2 is relevant for knowing if there are polynomial MY vector fields with periodic orbits or invariant tori. It is well known that there are smooth non polynomial counterexamples with a periodic orbit, see [2, 3].

This part of the paper is organized as follows. Section 2 deals with quasi-homogeneous objects and their properties. The proof of these properties can be found in [8]. Section 3 is devoted to the continuous case (MYC) while in Section 4 we treat with the discrete case (DMYQ).

The last part of the paper is included in Section 5. There we study the relation between injectivity and the existence of critical points which are global attractors. As far as we know the results of this section are new.

2 On quasi-homogeneous vector fields of degree one

This section is a summary of some results of [8]. See also [1, Chap. 1]. It is said that $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ is a *quasi-homogeneous* function with weights $\alpha_1, \alpha_2, \ldots, \alpha_n$ and *quasi-degree* d if

$$f(\lambda^{\alpha_1} x_1, \lambda^{\alpha_2} x_2, \ldots, \lambda^{\alpha_n} x_n) = \lambda^d f(x_1, x_2, \ldots, x_n) \quad , \quad \text{for all } \lambda > 0.$$

The weights can be taken as non zero real numbers.

It is said that $F = (F_1, F_2, \ldots, F_n) : \mathbb{R}^n \to \mathbb{R}^n$ is a *quasi- homogeneous* vector field (resp. map) with weights $\alpha_1, \alpha_2, \ldots, \alpha_n$ and *quasi-degree* d if each F_i is a quasi-homogeneous function with weights $\alpha_1, \alpha_2, \ldots, \alpha_n$ and *quasi-degree* $\alpha_i + d - 1$. Quasi-homogeneous vector fields (resp. maps) of degree one are called *linear quasi-homogeneous vector fields (resp. maps)*. From now on we deal with them.

Given the weights $\alpha_1, \alpha_2, \ldots, \alpha_n$ we define the "semi straight line" which passes through a point $x \in \mathbb{R}^n$, as

$$L_x = \{(\lambda^{\alpha_1} x_1, \lambda^{\alpha_2} x_2, \ldots, \lambda^{\alpha_n} x_n) : \lambda \in \mathbb{R}^+\}.$$

We note that if $\alpha_i \alpha_j > 0$ for all $i, j = 1, 2, \ldots, n$, then 0 belongs to the limit set of L_x. If the weights have different sign then, in general, it is not so. The above fact makes a big difference between the two different situations.

Finally, consider $\dot{x} = F(x)$ and $x \in \mathbb{R}^n$. The solution which passes through x is called of *exponential type* if

$$x(t) = (x_1 e^{m_1 t}, x_2 e^{m_2 t}, \ldots, x_n e^{m_n t})$$

for some $m_1, m_2, \ldots, m_n \in \mathbb{R}$.

Proposition 1 *([8]) Let F be a linear quasi-homogeneous vector field with weights $\alpha_1, \alpha_2, \ldots, \alpha_n$ and consider the differential system $\dot{x} = F(x)$. The following hold:*

(i) *F is invariant by the change of variables $\bar{x}_i = \lambda^{\alpha_i} x_i$.*

(ii) *If $\alpha_i \alpha_j > 0$ for all $i, j = 1, 2, \ldots, n$, then the knowledge of the orbits near the origin determines the global phase portrait of F. Furthermore, if 0 is locally asymptotically stable, then 0 is a global attractor.*

(iii) *Let L_x be the semi straight line which passes through x. Then L_x is invariant by the flow of $\dot{x} = F(x)$ if and only if the solution which passes through x is of the form $x_k(t) = x_k e^{m_k t}$ where $m_k = c\alpha_k$, $k = 1, 2, \ldots, n$, $c \in \mathbb{R}$. Furthermore these solutions of exponential type can be found by solving solve the nonlinear system of equations*

$$F_k(x) = c\alpha_k x_k, \qquad k = 1, 2, \ldots, n$$

where c is a real number.

Observe that a Corollary of (ii) of the above proposition is that if we assume that F is linear quasi-homogeneous with $\alpha_i\alpha_j > 0$ for all $i, j = 1, 2, \ldots, n$, and it satisfies the MYA then 0 is a global attractor.

For the case of discrete dynamical systems similar results can be developed. Let F be a linear quasi-homogeneous map and consider the discrete dynamical system

$$x^{(m)} = F(x^{(m-1)}) \quad , \quad x^{(0)} \in \mathbb{R}^n \quad , \quad m \in \mathbb{N}.$$

An orbit of this system is the set $\{x^{(m)} : m \in \mathbb{N}\}$. We say that the orbit which begins at $x^{(0)}$ is of *exponential type* if there exist some constants $a_1, a_2, \ldots, a_n$ such that

$$x^{(m)} = (x_1^{(0)} a_1^m, x_2^{(0)} a_2^m, \ldots, x_n^{(0)} a_n^m).$$

If for $i = 1, 2, \ldots, n$, $|a_i| < 1$ then $\lim_{m\to\infty} x^{(m)} = 0$, while if $|a_i| > 1$ and $x_i^0 \neq 0$ for some $i = 1, 2, \ldots, n$ then $\lim_{m\to\infty} ||x^{(m)}|| = \infty$. A "Semi straight line" L_x is now invariant if $F(L_x) \subset L_x$.

For this type of dynamical systems we have a result similar to Proposition 1. (iii):

Proposition 2 *([8]). Let F be a linear quasi-homogeneous vector field with weights $\alpha_1, \alpha_2, \ldots, \alpha_n$ and let L_x be the semi straight line which passes through x. Then L_x is invariant by the discrete dynamical system generated by F if and only if the orbit which begins at x is of exponential type. Furthermore to find the invariant straight lines it suffices to solve the system of equations*

$$F_k(x) = \lambda^{\alpha_k} x_k, \qquad k = 1, 2, \ldots, n,$$

where λ is a real positive number.

3 Continuous Case

3.1 Examples of polynomial Markus-Yamabe vector fields having unbounded orbits

The first polynomial counterexample to the MYC which appeared in the literature is the following (see [4]):

$$\begin{cases} \dot{x} = & -x + z(x + yz)^2 \\ \dot{y} = & -y - (x + yz)^2 \\ \dot{z} = & -z \end{cases}$$

This is a linear quasi-homogeneous MY vector field with weights 1,2,-1. To find invariant straight lines we solve the system:

$$\begin{cases} -x + z(x+yz)^2 & = cx \\ -y - (x+yz)^2 & = 2cy \\ -z & = -cz \end{cases}$$

which gives $c = 1$ and the set of solutions $\{(x, \frac{-x^2}{27}, \frac{18}{x}) : x \in \mathbb{R} \setminus \{0\}\}$. We note that the above set can be described as $L_{\mathbf{x}} \cup L_{\mathbf{x}'}$ where $\mathbf{x} = (18, -12, 1)$ and $\mathbf{x}' = (-18, -12, -1)$. Hence, we obtain the solutions of exponential type $x(t) = \pm 18e^t$, $y(t) = -12e^{2t}$, $z(t) = \pm e^{-t}$. Clearly $||(x(t), y(t), z(t))|| \to \infty$ as $t \to \infty$ and 0 is not a global attractor.

Observe that the main idea of the above example consists into taking linear quasi-homogeneous vector fields of the form $\lambda x + H(x)$ where $H(x)$ is a nilpotent map and $\lambda < 0$. In [8] the authors used this idea to obtain a family of vector fields which contains the above one. In [10] the author is able to find counterexamples of the same form in $\mathbb{R}^n, n \geq 5$ but with $H(x)$ homogeneous of degrees 2 and 3 and linear quasi-homogeneous. These counterexamples are also relevant from the point view of the relation between the MY Conjecture and the Jacobian Conjecture, see [18, 23]. Both families of counterexamples are described in the sequel.

Proposition 3 *([8]). The family of vector fields*

$$\dot{x} = (\lambda x_1 - bx_3^m(ax_1x_3^l + bx_2x_3^m)^k, \lambda x_2 + ax_3^l(ax_1x_3^l + bx_2x_3^m)^k, \lambda x_3, \ldots, \lambda x_n)$$

satisfy the following properties:

(1) *They are linear quasi-homogeneous with weights*

$$(\alpha_1, \alpha_2, \ldots, \alpha_n) = (m + kl, l + km, 1 - k, \ldots, 1 - k),$$

for all $a, b, \lambda \in \mathbb{R}, k, l, m \in \mathbb{N}$.

(2) *They satisfy the MYA, for all $\lambda \in \mathbb{R}$ with $\lambda < 0$.*

(3) *They have unbounded orbits for all $\lambda \in \mathbb{R}$ with $\lambda < 0$, $k \geq 2$ an even number, $l, k, l - m \in \mathbb{N}$ different from zero and for all $a, b \in \mathbb{R}$.*

Inside this family, the vector field which has less degree is obtained by considering $k = 2, m = 0$ and $l = 1$, what gives degree five.

Proposition 4 *([10]). The families of vector fields defined for $x \in \mathbb{R}^n, n \geq 5$,*

$$F_1(x) = -x + Q(x) \qquad \textit{and} \qquad F_2(x) = -x + x_5 Q(x),$$

where

$$Q(x) = (x_2x_5, x_1^2 - x_4x_5, x_2^2, 2x_1x_2 - x_3x_5, 0, \ldots, 0),$$

define vector fields which satisfy the MYA and which have orbits that tend to infinity when the time goes to infinity.

3.2 Markus-Yamabe linear quasi-homogeneous vector fields and the existence of periodic orbits

In this section we try to find a polynomial MY vector field with a periodic orbit. To this end we discuss how can be usefull a generalization to the complex of the method used in [8] to obtain MY vector fields with unbounded orbits.

Take $x = (x_1, x_2 \dots x_n) \in \mathbb{C}^n = \mathbb{R}^{2n}$ and a linear quasi-homogeneous vector field with real weights $\alpha_1, \alpha_2 \dots \alpha_n$ and of the form

$$\lambda x + N(x),$$

with $N(x) = (N_1(x), N_2(x), \dots, N_n(x))$ and $DN(x)$ a nilpotent matrix. When $\lambda = -1 + \beta i$, $0 \neq \beta \in \mathbb{R}$ the vector field is a MY vector field. If the system of equations

$$(\alpha_1 x_1, \alpha_2 x_2 \dots, \alpha_n x_n) i = \lambda x + N(x),$$

had some solution $0 \neq x \in \mathbb{C}^n$ then the differential equation

$$\dot{x} = \lambda x + N(x),$$

would be a MY vector fields with particular solutions of the form

$$x(t) = (x_1 \mathrm{e}^{i\alpha_1 t}, x_2 \mathrm{e}^{i\alpha_2 t}, \dots, x_n \mathrm{e}^{i\alpha_n t}).$$

Observe that the above solutions would give either periodic orbits or invariant tori in $\mathbb{R}^{2n}$ for the differential equation considered.

Also note that if $N_1(x), N_2(x), \dots, N_n(x)$ are linearly dependent over $\mathbb{C}$ then, through a linear change of variables, the differential equation can be written as follows:

$$\dot{z} = \lambda z + (M_1(z), M_2(z), \dots, M_{n-1}(z), 0),$$

where $z = (z_1, z_2 \dots, z_n) \in \mathbb{C}^n$ and $M = (M_1, M_2, \dots, M_{n-1}, 0)$ with DM also nilpotent. Then a periodic solution or an invariant torus of the differential equation must be included in the hyperplane $z_n = 0$. Hence we can restrict our attention to a new differential equation in $\mathbb{C}^{n-1}$,

$$\dot{w} = \lambda w + (M_1(w,0), M_2(w,0), \dots, M_{n-1}(w,0)),$$

where $w = (z_1, z_2, \dots, z_{n-1}) \in \mathbb{C}^{n-1}$ and the map $M^*(w) := (M_1(w,0), M_2(w,0), \dots, M_{n-1}(w,0))$ has again its Jacobian matrix $DM^*(x)$ nilpotent.

The conclusion is that an example in $\mathbb{C}^n$ of the form $\dot{x} = \lambda x + N(x)$ with $N = (N_1, N_2, \dots, N_n)$ such that $DN(x)$ is a nilpotent map and $N_1, N_2, \dots, N_n$ linearly dependent over $\mathbb{C}$ gives an example in dimension $n-1$.

So to obtain examples in "minimal" dimension we need maps with nilpotent Jacobian matrix which components are linearly independent. It is easy to see that for $n = 2$ there are no such kind of examples. As far as we know, the only examples in the literature, for $n \geq 3$, are constructed by van den Essen in [9]. Unfortunately these examples are not linear quasi-homogeneous. We formulate the following question:

Problem 1 Are there linear quasi-homogeneous maps $N = (N_1, N_2, \dots, N_n) : \mathbb{C}^n \to \mathbb{C}^n$ with nilpotent Jacobian matrix $DN(x)$ and such that $N_1, N_2, \dots, N_n$ are linearly independent over $\mathbb{C}$ for some $n \geq 3$?

3.3 Genericity of the Markus-Yamabe Conjecture

In the set of all polynomial maps of fixed degree we define the following topology. We denote by $\mathcal{X}_m$ the set of all polynomial maps $F = (P^1, P^2, \ldots, P^n) : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ such that $\deg(P^i) \leq m$. By identifying $\mathcal{X}_m$ with $\mathbb{R}^M$, where M is the number of all coefficients of $P^1, P^2, \ldots, P^n$, we endow $\mathcal{X}_m$ with the so called *coefficient topology.*

Inside $\mathcal{X}_m$ we consider the set of all the maps which satisfy the MYA and we denote by $\mathrm{Int}\{MYA\}$ the interior of this set. The following is an open problem.

Problem 2 If $F \in \mathrm{Int}\{MYA\}$ and $F(0) = 0$, is it true that 0 is globally asymptotically stable?

We think that the answer to this question is very relevant in the setting of the problem. In fact, if we consider the interior the set $\{F \in \mathcal{X}_m : F \text{ satisfies the RJA }\}$ where RJA means the Real Jacobian assumption (i. e., $\det DF(x) \neq 0$, for all $x \in \mathbb{R}^n$), then we know that all the maps in this set are injective (see [5]). Hence, in spite of the fact that the Real Jacobian Conjecture is false (see [21]) it is true for a big subset of maps which satisfy the hypothesis. What about the MY Conjecture?

The first step is to give a characterization of $\mathrm{Int}\{MYA\}$. If $F = (P^1, P^2, \ldots, P^n) \in \mathcal{X}_m$, we denote by $F_m = (P^1_m, P^2_m, \ldots, P^n_m)$, where P^i_m is the homogeneous part of P^i of degree m. The answer is the following.

Proposition 5 *([5]) The following statements hold.*

(i) *If F satisfies the MYA then $DF_m(x)$ has all the eigenvalues with non-positive real part at each $x \in \mathbb{R}^n$.*

(ii) *Let $F \in \mathcal{X}_m$. Then $F \in Int\{MYA\}$ if and only if $DF_m(x)$ has all the eigenvalues with negative real part at each $x \neq 0$.*

(iii) *The set $Int\{MYA\}$ is non empty if and only if m is odd.*

Assuming that F satisfies the MYA, the existence of a critical point globally asymptotically stable implies that $\dot{x} = F(x)$ has a unique critical point and that there are not orbits of $\dot{x} = F(x)$ which scape at infinity.

From Proposition 5 we can see that if $F \in \mathrm{Int}\{MYA\}$ then $F + c \in \mathrm{Int}\{MYA\}$ for all $c \in \mathbb{R}$ and hence the uniqueness of the critical point is equivalent to the injectivity of F. In fact, each $F \in \mathrm{Int}\{MYA\}$ is injective (see Theorem 1).

Concerning the stability of the orbits of $\dot{x} = F(x)$ with F satisfying the MYA, let a be an infinite critical point of $p(F)$ (the Poincaré compactification of F) and let λ_a be the eigenvalue associated to the eigenvector not contained in the tangent space at infinity. It is easy to prove that $\lambda_a < 0$ implies the existence of some unbounded orbit which tends to infinity in the direction determined by a (see [5]).

We summarize the results in the following theorem.

Theorem 1 *([5]) Let $F \in Int\{MYA\}$ and consider $\dot{x} = F(x)$. Then:*

(i) *$F : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ is bijective.*

(ii) *For each a infinite critical point of F, $\lambda_a > 0$ where λ_a is the eigenvalue associated to the eigenvector not contained in the tangent space at infinity.*

4 The discrete Case

4.1 Examples and counterexamples in dimension two

Let $F:\mathbb{R}^2 \longrightarrow \mathbb{R}^2$ be a map satisfying the DMYA. If F is a polynomial map, then we get the following.

Theorem 2 *([7]) Let $F:\mathbb{R}^2 \longrightarrow \mathbb{R}^2$ be a polynomial map satisfying the DMYA. Then it exists a unique fixed point of F which is a global attractor for the discrete dynamical system generated by F.*

The proof of Theorem 2 is based in the fact that each polynomial map satisfying the hypothesis of the theorem, through an affine transformation, is a triangular map. For this kind of maps, the result is true in any dimension (see Theorem A in [7]).

The next question is to know if the result is true for a more general class of maps. The following proposition gives a negative answer for the class of the diffeomorphisms. From the dynamical point of view the class of diffeomorphisms is interesting because we can consider the positive time (the future: $x_1 = F(x_0)$) and the negative time (the past: $x_{-1} = F^{-1}(x_0)$).

Proposition 6 *([7]) Let $G_a:\mathbb{R}^2 \longrightarrow \mathbb{R}^2$ be defined by*

$$G_a(x,y) = \left(-ax - \frac{ky^3}{1+x^2+y^2}\,,\, -ay - \frac{kx^3}{1+x^2+y^2}\right)$$

where $k \in \left(1, \frac{2}{\sqrt{3}}\right)$. Then for a small enough, the map G_a is a global diffeomorphism from $\mathbb{R}^2$ to $\mathbb{R}^2$ which satisfies the following properties:

(i) *For all $x \in \mathbb{R}^n$ and for all λ eigenvalue of $(DG_a)(x)$, $|\lambda| < 1$.*

(ii) *$G_a(0) = 0$ and there exists $q \in \mathbb{R}^n$, $q \neq 0$ which satisfies $G_a^4(q) = q$.*

Another question related with this problem, which has the attention of some mathematicians is the following.

Problem 3 Let $F:\mathbb{C}^2 \longrightarrow \mathbb{C}^2$ be an entire map with all the eigenvalues of $DF(x)$ with modulus less than one at each $x \in \mathbb{C}^2$. Is it true that there exists an unique fixed point of F which is a global attractor for the discrete dynamical system generated by F?

4.2 Examples of polynomial Markus-Yamabe maps having unbounded orbits

The first polynomial map satisfying the DMYA and having unbounded orbits was given by A. van den Essen and E. Hubbers (see [11]) and it is the following:

$$F(x,y,z,w) = \left(\frac{1}{2}x + w(xz+yw)^2, \frac{1}{2}y - (xz+yw)^2, \frac{1}{2}z + w^2, \frac{1}{2}w\right).$$

Inspired in this example we developed the theory of section 2 and it was possible to construct the counterexamples in dimension three.

This is a linear quasi-homogeneous map with weights -5, -4, 2, 1. To find invariant straight lines we solve the system

$$\begin{cases} \frac{1}{2}x + w(xz+yw)^2 & = \lambda^{-5}x \\ \frac{1}{2}y - (xz+yw)^2 & = \lambda^{-4}y \\ \frac{1}{2}z + w^2 & = \lambda^2 z \\ \frac{1}{2}w & = \lambda w \end{cases}$$

which gives $\lambda = \frac{1}{2}$, $x = \frac{(31)^2 63}{(32)^3} w^{-5}$, $y = \frac{31(63)^2}{8(32)^2} w^{-4}$ and $z = -4w^2$. The set of solutions is $L_{\mathbf{x}} \cup L_{\mathbf{x}'}$ where $\mathbf{x} = \left(\frac{(31)^2 63}{(32)^3}, \frac{31(63)^2}{8(32)^2}, -4, 1\right) = (x_0, y_0, z_0, w_0)$ and $\mathbf{x}' = (-x_0, y_0, z_0, -w_0)$. And one unbounded orbit is $\left((32)^m x_0, (16)^m y_0, (\frac{1}{4})^m z_0, (\frac{1}{2})^m w_0\right)$.

In fact, we have the analogous of Proposition 3 for the discrete case.

Proposition 7 *([8]) The family of maps*

$$F(x) = (\lambda x_1 - b x_3^m (a x_1 x_3^l + b x_2 x_3^m)^k, \lambda x_2 + a x_3^l (a x_1 x_3^l + b x_2 x_3^m)^k, \lambda x_3, \ldots, \lambda x_n)$$

satisfy the following properties:

(1) *For all $a, b, \lambda \in \mathbb{R}, k, l, m \in \mathbb{N}$ they are linear quasi-homogeneous with weights*

$$(\alpha_1, \alpha_2, \ldots, \alpha_n) = (m + kl, l + km, 1 - k, \ldots, 1 - k).$$

(2) *For all $\lambda \in \mathbb{R}$ with $\lambda < 0$ they satisfy the DMYA.*

(3) *For all $\lambda \in \mathbb{R}$ with $|\lambda| < 1$, $k \geq 2$ an even number, $l, k, l - m \in \mathbb{N}$ different from zero and for all $a, b \in \mathbb{R}$ the discrete dynamical system generated by F has unbounded orbits.*

Of course a similar result to Proposition 4 holds for iteration of maps.

4.3 Genericity and the relation with the Jacobian Conjecture

The first result about polynomial maps satisfying the DMYA is the following.

Lemma 1 *([7]) Let F be a polynomial map from $\mathbb{R}^n$ to itself such that it satisfies the DMYA. Then the characteristic polynomial of $(DF)_x$ is independent of x.*

Let $F \in \mathcal{X}_m$ be a polynomial map of degree m and assume that F satisfies the DMYA. The above lemma, in particular implies that $\det(DF_m)_x \equiv 0$ for all $x \in \mathbb{R}^n$. It is clear that if we perturb slightly F inside $\mathcal{X}_m$ we obtain some maps which dont satisfy the above property. So, we can assert that $\mathrm{Int}\{DMYA\} = \emptyset$. Hence, in this setting, it has no sense to speak about genericity.

On the other hand, assuming that F satisfies the DMYA, the existence of a fixed point of F being a global attractor for the dynamical system generated by F clearly implies that this fixed point is unique. Hence, we can formulate a problem weaker than the DMYQ as follows:

Conjecture 1 (Fixed Point Conjecture, FPC) *Let $F : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ be a polynomial map satisfying the DMYA. Then F has a unique fixed point.*

Next theorem shows that this problem is equivalent to the celebrated Jacobian Conjecture, which can be established as follows.

Conjecture 2 (Jacobian Conjecture, JC) *Let $F : \mathbb{C}^n \longrightarrow \mathbb{C}^n$ be a polynomial map with $detDF(x) \in \mathbb{C}^* = \mathbb{C} \setminus \{0\}$ at each $x \in \mathbb{C}^n$. Then F is invertible.*

Theorem 3 *([7])*

$$JC \quad \text{is equivalent to} \quad FPC$$

The conclusion is that the DMYA has no global implications from the dynamical point of view but perhaps has implications on the injectivity of certain maps.

To end this section we want to state a result which in some sense is a reciprocal to JC for $\mathcal{C}^\infty$ maps.

Theorem 4 *([6, Thm. D]) Let $F : \mathbb{R}^n \to \mathbb{R}^n$ be a $\mathcal{C}^\infty$ injective map. Then $detDF(x)$ does not change sign.*

5 Global attractors and injectivity

Let $F : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ be a $\mathcal{C}^\infty$ map and consider the differential system $\dot{x} = F(x)$. Assume that $F(0) = 0$ and that 0 is a local attractor. In this section we study the problem of which additional conditions imply that 0 is a global attractor. Clearly, if F satisfies the MYA then we know that it is so.

We begin with two simple examples. Consider the following integrable systems:

$$\begin{cases} \dot{x} = -x + x^2 y, \\ \dot{y} = -y, \end{cases} \tag{5.1}$$

and

$$\begin{cases} \dot{x} = -x + xy, \\ \dot{y} = -y. \end{cases} \tag{5.2}$$

System (5.1) has some particular solutions of the type $x(t) = \frac{2}{k} e^t$, $y(t) = ke^{-t}$ which let us to say that 0 is not a global attractor. The general solution of (5.2) is $x(t) = x_0 e^{-t - y_0(e^{-t}-1)}$, $y(t) = y_0 e^{-t}$ and hence 0 is a global attractor for system (5.2). One difference between systems (5.1) and (5.2) is the injectivity. While $F_1(x, y) = (-x + x^2 y, -y)$ is not injective, $F_2(x, y) = (-x + xy, -y)$ is injective.

In the sequel we discuss the effect of adding the injectivity condition to the condition of being a local attractor. Note that if F is an injective map then $\det DF(x)$ does not change sign, see Theorem 4. So, if 0 is a local hyperbolic attractor we get that $\det DF(x) \geq 0$ for all $x \in \mathbb{R}^2$. Observe that this last condition (with the strict inequality) together with the condition $\mathrm{trace}(DF(x)) < 0$ are the MY conditions in the plane.

As we shall see, the injectivity condition is not enough to assure the global asymptotic stability. But it is so for some special systems.

Proposition 8

(i) *Consider the family of systems*

$$\begin{cases} \dot{x} = -x + f(x,y), \\ \dot{y} = -Ay, \end{cases} \tag{5.3}$$

where A is some positive real number and $f(x,y)$ smooth and beginning at least with second order terms. If the map $F(x,y) := (-x + f(x,y), -Ay)$ is injective then 0 is a global attractor of system (5.3).

(ii) *There are systems of the form*

$$\begin{cases} \dot{x} = -x + f(x,y), \\ \dot{y} = -Ay + g(x,y), \end{cases} \tag{5.4}$$

with A a some positive real number and $f(x,y)$ and $g(x,y)$ polynomials and beginning at least with second order terms, such that the map $(-x + f(x,y), -Ay + g(x,y))$ is injective and 0 is a local (but not global) attractor for system (5.4).

Proof. F injective implies that $\frac{\partial}{\partial x}(-x + f(x,y)) \leq 0$, i. e. $-1 + \frac{\partial f}{\partial x}(x,y) \leq 0$. The linear part of $F(x,y)$ is:

$$\begin{pmatrix} -1 + \frac{\partial f}{\partial x}(x,y) & * \\ 0 & -A \end{pmatrix}.$$

So, $\det(DF)(x,y) = A(1 - \frac{\partial f}{\partial x}(x,y)) \geq 0$ and $\operatorname{tr}(DF)(x,y) = -A - 1 + \frac{\partial f}{\partial x}(x,y) < 0$. Therefore the vector field is "almost" MY and injective. By using results of Olech's paper (see [20, Thm. 5]) we also can assert that 0 is a global attractor. This proves statement (i).

In order to prove (ii) consider the Liénard equation:

$$\begin{cases} \dot{x} = -y, \\ \dot{y} = x + \lambda y + y^2. \end{cases} \tag{5.5}$$

The origin is a critical point and its eigenvalues are

$$\mu_{1,2} = \frac{\lambda \pm \sqrt{\lambda^2 - 4}}{2}.$$

If we choose $\lambda < -2$ we get two real negative eigenvalues and so the origin is a local attractor for (5.5).

The phase portrait of system (5.5) is drawn in [15]. From that study it is clear that the origin is not a global attractor. On the other hand, trough the linear change which diagonalizes the matrix, system (5.5) becomes:

$$\begin{cases} \frac{du}{dt} = \mu_1 u + P(u,v), \\ \frac{dv}{dt} = \mu_2 v + Q(u,v), \end{cases}$$

for some polynomials P and Q. Trough the change of time $s = -\mu_1 t$ we get the system:

$$\begin{cases} \frac{du}{ds} = -u + f(u,v), \\ \frac{dv}{ds} = -\frac{\mu_2}{\mu_1} v + g(u,v). \end{cases} \tag{5.6}$$

Since systems (5.5) and (5.6) are topologically equivalents, they have the same phase-portrait. Hence system (5.6) provides an example which proves (ii). □

A way of interpreting the above proposition is saying that, for planar vector fields, the injectivity of a vector field is related to the existence of a global attractor just for "Markus-Yamabe" vector fields.

We end by observing that the counterexample given in [4] (see also Section 3.1) shows that there is no relation between injectivity and global asymptotic stability in $\mathbb{R}^n, n \geq 3$.

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REAL JACOBIAN CONJECTURE AND CUBIC LINEAR MAPPINGS

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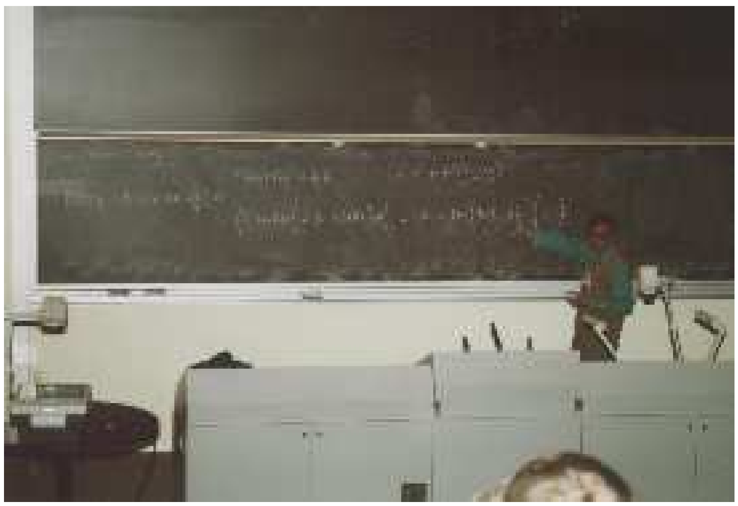

Abstract. Let A be a real square matrix of order n. With each A we associate cubic linear maps, called Drużkowski maps, from $\mathbb{R}$ to $\mathbb{R}$ defined by $F(x) = x + (Ax)$ where the i-th coordinate $F_i(x)$ of $F(x)$ is given by

$$F_i(x) = x_i + (a_{i1}x_1 + \cdots + a_{in}x_n)^3.$$

It is well known that if Drużkowski maps are injective for every A in every dimension n then the well known Jacobian Conjecture holds good. Unfortunately the nonvanishing of the Jacobian alone of the Drużkowski map will not suffice for the injectivity of such maps as shown by an example of Pinchuk.

Here we want to study the injectivity of Drużkowski maps in some special cases by placing some conditions on A. In particular Drużkowski has shown that if the rank of A is n then F is injective. In fact we show that if A is nonsingular (or rank of A is n) then the Jacobian must be a weak P-matrix (or a singular P_0-matrix, all proper principal minors of the Jacobian is nonnegative) and consequently such a Drużkowski map must be injective. Our proof is different from that of Drużkowski.

In the Drużkowski map if the Jacobian of F does not vanish of every $x \in \mathbb{R}^n$, then it puts some restrictions on the matrix A. For example A cannot be an N-matrix

or almost P or $A < 0$ etc. In view of Pinchuk's example it is not clear how useful this information will be. In fact it will be nice to know the rank of A in Pinchuk's example. (Recently professor Engelbert Hubbers has made some progress in this connection and he has given a bound for the dimension n in Pinchuk's example.)

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A MOUNTAIN PASS TO THE JACOBIAN CONJECTURE

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Abstract. This paper presents a new injectivity theorem and a new open question. The main result of the paper is proved by means of the Mountain Pass Lemma and states that if all the eigenvalues of $F'(\mathbf{x})F'(\mathbf{x})^T$ are bounded away from zero for all $\mathbf{x} \in \mathbb{R}^n$, where $F : \mathbb{R}^n \to \mathbb{R}^n$ is a class C^1 map, then F is injective. This was discovered in a joint attempt by the authors to prove a stronger result conjectured by the first author: Namely, that a sufficient condition for injectivity of class C^1 maps F of $\mathbb{R}^n$ into itself is that all the eigenvalues of $F'(\mathbf{x})$ are bounded away from zero on $\mathbb{R}^n$. If true, it would imply (via *Reduction-of-degree*) *injectivity of polynomial maps* $F : \mathbb{R}^n \to \mathbb{R}^n$ *satisfying the hypothesis*, $\det F'(\mathbf{x}) \equiv 1$, of the celebrated Jacobian Conjecture of Ott-Heinrich Keller. The paper ends with several examples to illustrate a variety of cases and known counterexamples to some natural questions.

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ON THE 2-DIMENSIONAL JACOBIAN CONJECTURE AND AFFINE VARIETIES CONTAINING $\mathbb{C}^2$

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Abstract. The two-dimensional Jacobian Conjecture is equivalent to the non-existence of a map of complex varieties $f : V \to \mathbb{C}^2$ where V is an affine variety properly containing $\mathbb{C}^2$ as an open subvariety, f restricted to $\mathbb{C}^2$ has constant non-zero Jacobian determinant, $V - \mathbb{C}^2$ is a (possibly singular) rational curve whose normalization is $\mathcal{C}^1$, and V admits a map to CP^1 making it a $\mathcal{C}^1$-bundle over CP^1. We show the non-existence of such a map f for a large class of such affine varieties V.

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A NONLINEARIZABLE CUBIC-LINEAR MAPPING

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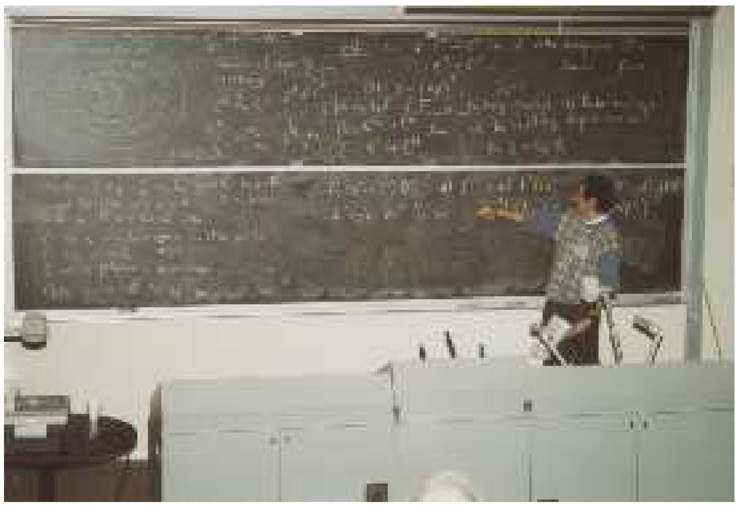

Arno van den Essen in his paper [3] produces, among other things, the following cubic-homogeneous polynomial automorphism of $\mathbb{C}^5$:

$$f\colon \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} \mapsto \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} + \begin{pmatrix} x_2 x_5^2 \\ x_1^2 x_5 - x_4 x_5^2 \\ x_2^2 x_5 \\ 2x_1 x_2 x_5 - x_3 x_5^2 \\ 0 \end{pmatrix} \tag{0.1}$$

that has the following property: for all $\lambda \in \mathbb{C} \setminus \{0\}$, $|\lambda| \neq 1$ there exists no analytic automorphism $k_\lambda\colon \mathbb{C}^5 \to \mathbb{C}^5$ that linearizes λf, in the sense that $\lambda f(k_\lambda(x)) = k_\lambda(\lambda x)$ for all $x \in \mathbb{C}^5$. It is therefore a counterexample to a conjecture that originated in [1].

The same paper in theorem 2.3 states that there exists a cubic-linear polynomial automorphism F of $\mathbb{C}^{17}$ with the same property: for all $\lambda \in \mathbb{C} \setminus \{0\}$, $|\lambda| \neq 1$ there exists no analytic automorphism $K_\lambda\colon \mathbb{C}^{17} \to \mathbb{C}^{17}$ such that $\lambda F(K_\lambda(X)) = K_\lambda(\lambda X)$ for all $X \in \mathbb{C}^{17}$. This F is then a counterexample to a more special conjecture that

*The author is very glad to have been able to attend the conference in honour of Prof. Gary Meisters and is grateful to Gary himself, Bo Deng, Roger Wiegand and their families for their kindness and hospitality.

was advanced in [5]. However the F is not actually exhibited, but only a hint at its construction is given, following the general method described in [4].

Now we have carried out all the calculations and here are the results. The map F is defined as $F(X) := X - (AX)^{*3}$, where the exponent means the componentwise cubic power, and the matrix A is given as

$$A = \frac{1}{12}\begin{pmatrix}
0 & 0 & -4 & 2 & 0 & 0 & 0 & 0 & 2 & 4 & -2 & 0 & 0 & 0 & 0 & 2 & 0 \\
0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 4 & 0 & 0 & -2 & 0 & 0 & 0 & 0 \\
0 & -4 & 0 & 0 & -1 & 0 & 1 & 2 & 0 & 0 & 0 & 1 & 0 & -1 & 2 & 0 & 0 \\
4 & 0 & 0 & 0 & 0 & -2 & 0 & 0 & 0 & 0 & 0 & 0 & -2 & 0 & 0 & 0 & -12 \\
4 & 0 & 4 & -2 & 0 & -2 & 0 & 0 & -2 & -4 & 2 & 0 & -2 & 0 & 0 & -2 & -12 \\
0 & 0 & -4 & 2 & 0 & 0 & 0 & 0 & 2 & 4 & -2 & 0 & 0 & 0 & 0 & 2 & -12 \\
4 & 0 & -4 & 2 & 0 & -2 & 0 & 0 & 2 & 4 & -2 & 0 & -2 & 0 & 0 & 2 & -12 \\
0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 4 & 0 & 0 & -2 & 0 & 0 & 0 & -12 \\
0 & -4 & 0 & 0 & -1 & 0 & 1 & 2 & 0 & 0 & 0 & 1 & 0 & -1 & 2 & 0 & -12 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 12 \\
4 & 0 & 0 & 0 & 0 & -2 & 0 & 0 & 0 & 0 & 0 & 0 & -2 & 0 & 0 & 0 & 12 \\
4 & 0 & 4 & -2 & 0 & -2 & 0 & 0 & -2 & -4 & 2 & 0 & -2 & 0 & 0 & -2 & 12 \\
0 & 0 & -4 & 2 & 0 & 0 & 0 & 0 & 2 & 4 & -2 & 0 & 0 & 0 & 0 & 2 & 12 \\
4 & 0 & -4 & 2 & 0 & -2 & 0 & 0 & 2 & 4 & -2 & 0 & -2 & 0 & 0 & 2 & 12 \\
0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 4 & 0 & 0 & -2 & 0 & 0 & 0 & 12 \\
0 & -4 & 0 & 0 & -1 & 0 & 1 & 2 & 0 & 0 & 0 & 1 & 0 & -1 & 2 & 0 & 12 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}$$

It can be checked that $A^2 = 0$. Consider also the following two matrices B:

$$B = \frac{1}{12}\begin{pmatrix}
4 & 0 & 0 & 0 & 0 & -2 & 0 & 0 & 0 & 0 & 0 & 0 & -2 & 0 & 0 & 0 & 0 \\
0 & 0 & -4 & 2 & 0 & 0 & 0 & 0 & 2 & 4 & -2 & 0 & 0 & 0 & 0 & 2 & 0 \\
0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 4 & 0 & 0 & -2 & 0 & 0 & 0 & 0 \\
0 & -4 & 0 & 0 & -1 & 0 & 1 & 2 & 0 & 0 & 0 & 1 & 0 & -1 & 2 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 12
\end{pmatrix}$$

and C (shown here transposed to save space):

$$C^T = \begin{pmatrix}
3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
3 & 0 & 0 & 0 & 0 & 6 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix}$$

It can be verified that the product BC is the 5×5 identity matrix, that A and B have the same kernel, and that if we define $F(X) := X - (AX)^{*3}$ for $X \in \mathbb{C}^{17}$, we have that $f(x) = BF(Cx)$ for all $x \in \mathbb{C}^5$. This means that the mappings f and F are "paired" in the sense of [4].

It follows that the mapping F is a cubic-linear automorphism of $\mathbb{C}^{17}$ such that λF is not linearizable for any $\lambda \in \mathbb{C} \setminus \{0\}$, $|\lambda| \neq 1$.

We have implemented the procedure for finding A, B, C from f, that was described in [4], as the following routine written in the *Mathematica* programming language, version 3.0, by Wolfram Research Inc.

```
makePairing[cubicHomogeneous_Function]:=
  Module[{dimension,var,toCubeCombination,recombined,
      monomialList,combinationList,temp,comb,
      d0,b0,b,d,c,m,cm,dAnd0,a,answer},
dimension=Length[Last[cubicHomogeneous]];
var=Table[x[i],{i,dimension}];
SetAttributes[toCubeCombination,Listable];
toCubeCombination[a_+b_]:=toCubeCombination[a]+
  toCubeCombination[b];
toCubeCombination[0]:=0;
toCubeCombination[(num_.)*(a_)*(b_)*(c_)]:=
```

```
(num/24)*(a+b+c)^3+(num/24)*(a-b-c)^3-(num/24)*(a+b-c)^3-
     (num/24)*(a-b+c)^3/;
        NumberQ[num]&&!NumberQ[a]&&!NumberQ[b]&&!NumberQ[c];
toCubeCombination[(num_.)*(a_)*(b_)^2]:=
  (num/6)*(a+b)^3+(num/6)*(a-b)^3-(num/3)*a^3/;
        NumberQ[num]&&!NumberQ[a]&&!NumberQ[b]&&OrderedQ[{a,b}];
toCubeCombination[(num_.)*(a_)*(b_)^2]:=
  (num/6)*(a+b)^3-(num/6)*(b-a)^3-(num/3)*a^3/;
        NumberQ[num]&&!NumberQ[a]&&!NumberQ[b]&&!OrderedQ[{a,b}];
toCubeCombination[(num_.)*(a_)^3]:=
      num*a^3/;NumberQ[num]&&!NumberQ[a];
recombined=toCubeCombination[Evaluate[
      Expand[cubicHomogeneous@@var-var]]];
monomialList=Map[If[Head[#]===Plus,List@@#,{#}]&,
      recombined]//Flatten;
combinationList=
      Union[Select[monomialList,
            MatchQ[#1, (num_.)*(lin_)^3] & ]/.
            (num_.)*(lin_)^3:>lin/;NumberQ[num]];
temp=recombined/.
        Table[combinationList[[i]]^3->comb[i],
        {i,Length[combinationList]}];
    d0=Table[Coefficient[combinationList[[i]],x[j]],

{i,Length[combinationList]},{j,dimension}];
b0=-Table[
          Coefficient[temp[[i]],comb[j]],{i,dimension},{j,
            Length[combinationList]}];
For[i=0,Union[Flatten[Minors[b0,dimension]]]=={0},i++,
b0=Transpose[
          Join[Transpose[b0],
              {IdentityMatrix[dimension][[dimension - i]]}]];
d0=Join[d0,{Table[0,{dimension}]}]];
b=b0;
d=d0;
c= Module[{c,mat},
  mat=Array[c,{Length[b[[1]]],dimension}];
   mat/.
   Solve[b.mat==IdentityMatrix[dimension],
       Flatten[mat]][[1]]/.c[i_,j_]->0];
m= Transpose[NullSpace[b]];
m=m*LCM @@ Denominator[Union[Flatten[m]]];
cm=Transpose[Join[Transpose[c], Transpose[m]]];
dAnd0 =Transpose[
```

```
        Join[Transpose[d],
          Transpose[Table[0,{i,Length[d]},
          {j,Length[First[m]]}]]]];
a=dAnd0.Inverse[cm];
answer=(b.c==IdentityMatrix[dimension])&&(
          Union[Expand[cubicHomogeneous@@var-var+
              b.(a.c.var)^3]]=={0})&&(
          Union[Flatten[a.Transpose[NullSpace[b]]]]=={0})&&

(Union[Flatten[b.Transpose[NullSpace[a]]]]=={0});
If[answer,{a,b,c},Print["Something is wrong"]]];
```

The function `makePairing` takes a cubic-homogeneous function f from $\mathbb{C}^n$ to itself, with $n \geq 2$, and returns a triple of matrices A, B, C such that $\ker A = \ker B$, $BC = I_n$ and $f(x) = x - B(ACx)^{*3}$. If f has constant Jacobian determinant so has the mapping $X \to X - (AX)^{*3}$, and f is an automorphism if and only if F is (in the respective dimensions).

The routine has been tested only with the following three cubic-homogeneous examples, the first of which is the (0.1) above and the others are taken from [3] and [4]:

```
f = Function[{x1, x2, x3, x4, x5},
  {x1 + x2 * x5^2, x2 + x1^2 * x5 - x4 * x5^2, x3 + x2^2  *x5,
   x4 + 2 x1 * x2 * x5 - x3 * x5^2, x5}];
g = Function[{x1, x2, x3, x4},
  {x1 + (x3*x1 + x4 * x2) * x4,
   x2 - (x3*x1 + x4*x2) * x3, x3 + x4^3, x4}];
h = Function[{y1, y2, y3, y4, y5},
  {y1, y2, y3, y4, y5} + 3 * {0, 0,
    y1^2 * y2  +y1 * y2^2  +2 * y1 * y2 * y4 - 2 * y1^2 * y5,
    - y1^2 * y2 - y1 * y2^2  -2y1 * y2 * y3 - 2 * y1^2 * y5,
    - y2^2 * y3 - y2^2 * y4}];
```

We cannot guarantee that the algorithm will not run into bugs in other cases, but there is a built-in check that should warn if the results are not reliable. The command to give is simply `makePairing[f]`.

An electronic copy of a *Mathematica* notebook containing the routine should be available on the Web together with these proceedings.

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G_A-ACTIONS OBTAINED BY LOCAL SLICE CONSTRUCTIONS

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Abstract. Let k be a field of characteristic 0, and let B denote the polynomial ring in three variables over k. We describe local slice constructions, a procedure by which new locally nilpotent k-derivations of B can be obtained from given derivations of a certain type. Geometrically, a local slice construction corresponds to an elementary Cremona transformation of affine 3-space. In this way, one obtains the recently discovered rank-3 examples. In fact, every known locally nilpotent k-derivation of B can be from a partial derivative via a sequence of local slice constructions.

1 Introduction

In this article, we summarize the ideas and results found in [3], to which the reader is referred for details.

Let k be a field of characteristic zero, and let $k^{[n]}$ denote the polynomial ring in n variables over k. Recently, in [4], the first examples of locally nilpotent derivations of $k^{[n]}$ having maximal rank n were constructed. In an attempt to understand and generalize these examples, our present aim is to describe *local slice constructions*, a general procedure by which new locally nilpotent derivations can be constructed from given derivations of a certain type. The examples of [4] can be obtained from familiar derivations via local slice constructions, though they were not originally discovered in this way. In fact, we will see in what follows that every known locally nilpotent derivation of $k^{[3]}$ can be obtained from a partial derivative via a (finite) sequence of local slice constructions. Whether any others exist is the content of *Questions 1* below.

The following notation and definitions will be used. The word *derivation* will mean k-derivation. Let R be an integral k-domain, and let $R^{[n]}$ denote the polynomial ring in n variables over R. Let D be a derivation of R, and let $A = Ker(D)$, the kernel of D. D is *irreducible* if its image is contained in no proper principal ideal of R. D is *locally nilpotent* if, to each $f \in R$, there is an $n \geq 0$ such that $D^n f = 0$.

We say $s \in R$ is a *slice* for D if $Ds \in R^*$. (It is well known that, when D is locally nilpotent, A is factorially closed, and thus $R^* \subset A$.) The main fact concerning slices is the following.

Proposition 1 ([7], Proposition 2.1) *Suppose R is an integral k-domain, D is a locally nilpotent derivation of R having kernel A, and $s \in R$ is such that $Ds = 1$. Then $R = A[s] \cong A^{[1]}$, and D is given by $D = d/ds$.*

When D is locally nilpotent (and non-zero), we can find $r \in R$ such that $Dr \in A = Ker(D)$, but $Dr \neq 0$. If $Dr = f$, then D extends to a locally nilpotent derivation D_f on the localization R_f, and r becomes a slice for D_f. Moreover, the above theorem shows $R_f = A_f[r] = A_f^{[1]}$. This gives rise to the following.

Definition 1 Let R be an integral k-domain, and let D be a locally nilpotent derivation of R. An element $r \in R$ is called a *local slice* for D if $Dr \in Ker(D)/\{0\}$.

Of particular interest is the case $R = k^{[n]}$. In this case, for any locally nilpotent derivation D of R, the *rank* of D is defined to be the least integer $r \geq 0$ for which there exists a system of variables $(X_1, ..., X_n)$ of $k^{[n]}$ satisfying $k[X_{r+1}, ..., X_n] \subset Ker(D)$.

2 Local Slice Constructions

From now on, B will denote the polynomial ring $k[X,Y,Z] = k^{[3]}$. Given $f, g \in B$, define a derivation $D_{(f,g)}$ of B by

$$D_{(f,g)}(h) = \frac{\partial(f,g,h)}{\partial(X,Y,Z)} \ .$$

The following theorem of Miyanishi is required (c.f. [5]).

Theorem 1 *If D is any non-zero locally nilpotent derivation of $k^{[3]}$, then $Ker(D) \cong k^{[2]}$.*

Suppose D is a locally nilpotent derivation of B, with kernel $A = k[f,g]$. Then, up to A-multiples, $D = D_{(f,g)}$ (c.f. [2]). Let S denote the set $(k[f] - 0)$, and define

$$\Omega(f,g) = D^{-1}(gS) \ ,$$

a subset (possibly empty) of the set of local slices of D.

Proposition 2 *Given* $r \in \Omega(f,g)$, $\Omega(f,g) = \{ r' \in B \mid S^{-1}A[r'] = S^{-1}A[r] \}$

Suppose $\Omega(f,g)$ is not empty. Let $\bar{B}$ denote the domain B modulo (g), and let $\bar{D}$ denote the induced locally nilpotent derivation on $\bar{B}$. Then for every $r \in \Omega(f,g)$, $\bar{r} \in Ker(\bar{D})$. Since $Ker(\bar{D})$ is the algebraic closure in $\bar{B}$ of $\bar{A} = k[\bar{f}] \cong k^{[1]}$, there exists $\varphi \in k[f][r]$ such that $\varphi(r) \in (g)$. If we choose φ to be of minimal r-degree in $k[f][r]$, and irreducible as an element of $k[f,r] = k^{[2]}$, then φ is unique up to non-zero constant multiples.

Define $h = \varphi(r)/g$, and let $\Delta(f,g,r)$ (or simply Δ) denote the derivation on B defined by $D_{(f,h)}$. (Up to non-zero constant multiples, Δ is uniquely determined by f, g, and r.) The crux of the matter is the following fact.

Theorem 2 *If* $K = k(f,h)$ *and* $B_K = K[X,Y,Z]$, *then* $K[r] = B_K$. *Consequently,* Δ *is locally nilpotent.*

We say Δ is the derivation obtained from D via the *local slice construction* using f, g, and r. Note that, if $Ker\Delta = k[f,h]$, then $r \in \Omega(f,h)$, so we can carry out a further local slice construction. However, since $g = (1/h)\varphi(r)$, this simply results in reversing the process: $\Delta(f,h,r) = D$. To continue the process inductively, we may, by *Proposition 2* above, replace r with any r' for which $S^{-1}A[r'] = S^{-1}A[r]$.

It may also happen that the original derivation, D, admits a local slice r such that $Dr = fg$. Then $\Delta r = -fh$, so both $\Omega(f,h)$ and $\Omega(h,f)$ contain r. Thus, to continue the process inductively, we may also use $\Delta(h,f,r)$ instead of $\Delta(f,h,r)$.

In order to determine the kernel of Δ, the following criterion is quite useful.

Proposition 3 (Kernel Criterion) *Suppose* $a,b \in B = k[X,Y,Z]$ *are such that* $\delta := D_{(a,b)}$ *is locally nilpotent and non-zero. Then the following are equivalent.*

(i) $k[a,b] = Ker(\delta)$

(ii) δ *is irreducible, and* $Ker(\delta) \subset k(a,b)$

Proof. The implication $(i) \Rightarrow (ii)$ follows from [2], *Corollary 2.5.* Conversely, assume (ii) holds. By *Theorem 1* above, there exist $u,v \in B$ such that $Ker(\delta) = k[u,v] \cong k^{[2]}$. It follows that

$$\delta = D_{(a,b)} = \frac{\partial(a,b)}{\partial(u,v)} \cdot D_{(u,v)} \ .$$

Since δ is irreducible, $\frac{\partial(a,b)}{\partial(u,v)} \in k^*$, i.e., (a,b) is a "Jacobian pair" for $k[u,v]$. Since $k(a,b) = k(u,v)$, the inclusion $k[a,b] \hookrightarrow k[u,v]$ is birational. It is well known that the Jacobian Conjecture is true in the birational case, and we thus conclude $k[a,b] = k[u,v]$.[1] ■

[1] The same reasoning yields yet another equivalent formulation of the two-dimensional Jacobian Conjecture: Given $a,b \in B$, if $D_{(a,b)}$ is irreducible, then $Ker\, D_{(a,b)} = k[a,b]$.

Remark 1 The reader will probably have noticed that, geometrically, passage from $k[f,g,r] \cong k^{[3]}$ to $k[f,h,r] \cong k^{[3]}$ corresponds to a birational transformation of k^3 of a particularly elementary sort. Thus, algebraic passage from D to Δ via a local slice construction may be thought of geometrically as a sequence blow-ups of 3-space, followed by a sequence of blow-downs.

3 Rank Three Examples

The examples discussed in this section are homogeneous in the standard sense (as maps of B). These tend to be easier to work with. It should be noted, however, that local slice constructions can be used to construct rank-3 examples which are weighted-homogeneous, or which are not homogeneous in any (non-trivial) grading of B.

Given any locally nilpotent derivation D of B, D is homogeneous iff there exist homogeneous polynomials $f,g \in B$ such that $Ker(D) = k[f,g]$ [8]. In this case, we say D is homogeneous of *type* (e_1,e_2), where $e_1 = \min\{\deg f, \deg g\}$, and $e_2 = \max\{\deg f, \deg g\}$. As noted in [4], the rank of D is 3 iff $e_1 > 1$. Following the appearance of the $(2,5)$ example in [4], Daigle gave the following beautiful geometric characterization of the homogeneous locally nilpotent derivations of B. ($\mathbf{P}^2$ denotes the projective plane over k.)

Theorem 3 ([1]) *Let f_1 and f_2 be homogeneous elements of $k^{[3]}$, and let C_1 and C_2 be the corresponding projective plane curves which they define. The following are equivalent.*

- *There exists a locally nilpotent derivation D of $k^{[3]}$ such that $Ker(D) = k[f_1,f_2]$.*

- *$\mathbf{P}^2$ minus $(C_1 \cup C_2)$ is isomorphic to $\mathbf{P}^2$ minus 2 lines.*

In other words, f_1 and f_2 define a locally nilpotent derivation precisely when there exists a plane Cremona transformation which is an isomorphism away from C_1 and C_2, and which transforms C_1 and C_2 into a pair of lines.

In order to construct examples, let D be the rank-2 locally nilpotent (linear) derivation on B defined by $DX = 0$, $DY = X$, and $DZ = 2Y$. Then D is homogeneous, and the kernel of D is $k[X,F]$, where $F = XZ - Y^2$. Since $D(FY) = XF$, we see that neither $\Omega(X,F)$ nor $\Omega(F,X)$ is empty.

3.1 Examples of Type (2,4n+1)

Given $n \in \mathbf{Z}^+$, set $r_n = (X^{2n+1} + F^nY) \in \Omega(F,X)$, and let $\Delta_n = \Delta(F,X,r_n)$. By *Theorem 2*, Δ_n is locally nilpotent. Now $F \equiv (-Y^2)$ and $r_n \equiv (-1)^nY^{2n+1}$ modulo X, so the minimal polynomial we need is $\varphi(r_n) = F^{2n+1} + r_n^2$. Therefore, $G_n := \varphi(r_n)/X = (ZF^{2n} + 2X^{2n}F^nY + X^{4n+1})$ is homogeneous of degree $(4n+1)$.

As shown in [3], Δ_n is irreducible. *Theorem 2* shows $Ker(\Delta_n) \subset k(F,G_n)$. By the *Kernel Criterion*, it follows that $Ker(\Delta_n) = k[F,G_n]$. Therefore Δ_n is of type $(2,4n+1)$, and is consequently of rank 3. Note that when $n = 1$, we obtain the $(2,5)$ example first discussed in [4].

The set of points fixed by the $\mathbf{G}_a$-action on $\mathbf{A}^3$ induced by Δ_n is precisely the set of points where $\Delta_n X$, $\Delta_n Y$, and $\Delta_n Z$ vanish

simultaneously, and it is easy to check that this set is the line defined by $X = Y = 0$. Every other orbit is a line, i.e., isomorphic to $\mathbf{G}_a \cong \mathbf{A}^1$. Let $\pi : \mathbf{A}^3 \to \mathbf{A}^2$ be the morphism induced by the inclusion $A \hookrightarrow B$ $(A = k[F, G_n])$. Then the fiber over the point $(a, b) \in \mathbf{A}^2$ is defined by the ideal $(f - a, g - b)$ in B, and each fiber is a union of orbits. The fiber over the origin $(0, 0)$ is the line of fixed points. If neither a nor b is 0, then the fiber over (a, b) is a single (coordinate) line in $\mathbf{A}^3$. The most interesting fibers lie over points $(0, b)$ and $(a, 0)$ for $a \neq 0$ and $b \neq 0$. Over $(0, b)$, $b \neq 0$, the fiber consists of $4n + 1$ (coordinate) lines lying on the surface defined by F. And over $(a, 0)$, $a \neq 0$, the fiber consists of two (coordinate) lines lying on the surface defined by G_n.

3.2 Examples of Fibonacci Type

We again use D as above, and fix $r = X^3 + FY$ $(F = XZ - Y^2)$. Inductively, define functions

$$\begin{aligned} H_1 &= X \\ H_2 &= F \\ H_{n+1} &= \frac{1}{H_{n-1}}(H_n^3 + r^{a_n}) \qquad (a_n = \deg H_n) \ . \end{aligned}$$

As shown in [3], each H_n *is* a polynomial. Observe that $a_{n+1} = 3a_n - a_{n-1}$, giving every other element of the Fibonacci sequence. (These degrees increase rather quickly, e.g., a_{23} already exceeds one billion.) Let $\delta_n = D_{(H_n, H_{n+1})}$ for $n \geq 1$. It turns out that, for each $n \geq 2$, the derivation δ_n is irreducible and locally nilpotent of rank three and type (a_n, a_{n+1}), having kernel $k[H_n, H_{n+1}]$. To prove this, it is shown that $\delta_n = \Delta(H_n, H_{n-1}, r)$ for each such n.

4 The Graph of Kernels

We are actually interested in subrings A of B which occur as the kernel of some locally nilpotent derivation, rather than in any specific derivation of which A is the kernel. With this in mind, we will, in this section, rephrase some of our terms, results and questions in the language of graphs.

Define Γ to be the graph such that vert(Γ) is the set of all kernels of non-zero locally nilpotent k-derivations of B, and such that two vertices A and A' are connected by an edge iff there exist derivations D, D' of B with $Ker(D) = A$, $Ker(D') = A'$, and D' is obtained from D by means of a local slice construction. Let A_0 denote the vertex $k[X, Y]$, corresponding to the partial derivative $(\partial/\partial Z)$.

The group $GA_3(k)$ of k-automorphisms of B acts on Γ by conjugation, and we let $\mathcal{G}$ denote the quotient graph. (Observe that, by Rentschler's Theorem [6], the corresponding quotient graph in dimension 2 consists of a *single* vertex, namely, that corresponding to the partial derivative.) Let $\mathcal{G}_0$ denote the connected component of

A_0 in $\mathcal{G}$. The following terminology will be used. (We do not distinguish between $A \in \text{vert}(\Gamma)$ and its equivalence class in $\text{vert}(\mathcal{G})$.)

1. The *rank* of a vertex A is the rank of any derivation D on B with $Ker(D) = A$.
2. A vertex A is *homogeneous* if there exists a homogeneous derivation D of B with $Ker(D) = A$. The *type* of a homogeneous vertex is the type of the corresponding derivation. (It is possible that more than one vertex could be associated with a given type.)
3. A vertex A is *free* if there exists a locally nilpotent derivation D on B with $Ker(D) = A$ and $(im\, D) = (1)$.

Note that A_0 is the unique vertex of $\mathcal{G}$ of rank one. Moreover, the results of [3] show the following.

Proposition 4 *$\mathcal{G}_0$ contains every rank-two vertex of $\mathcal{G}$.*

We close with some questions.

Question 1 *Is $\mathcal{G}$ connected ?*

Question 2 *Does every homogeneous vertex of $\mathcal{G}$ lie in $\mathcal{G}_0$?*

Question 3 *Does $\mathcal{G}_0$ contain a free vertex other than A_0 ?*

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CUBIC SIMILARITY IN DIMENSION FIVE

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Dedicated to Gary Meisters

Abstract. In this paper we classify all Drużkowski maps $F = X + (AX)^{*3}$ from $\mathbb{C}^5$ to $\mathbb{C}^5$ for which $J((AX)^{*3})$ is nilpotent. With this classification of maps we obtain the complete set of representatives of Meisters' cubic similarity relation in dimension five. This paper is a summary of the very large paper [3].

1 Introduction

The first time I got interested in the subject of cubic similarity was back in 1993. It was in the middle of June, a few days before I would go on a four week holiday to Moscow. In order to get credits for a class on polynomial mappings by Arno van den Essen, I had been working with two fellow students on linear cubic homogeneous maps. When we handed in our paper with the final results, Arno reminded us that the next day some American would give a talk about cubic linear maps. Naturally I went there and I listened to a very nice talk by Gary Meisters. One of the most impressive points in this talk was the point where he was showing some slides containing a list of matrices which turned out to be the representatives of the cubic similarity relation in dimension three, four and five. Though Gary already showed 19 matrices in dimension five, he was pretty sure that this list was not complete yet ...

Right after I returned from Moscow I started working on my Master's Thesis on cubic homogeneous maps. During the work done for this thesis I found that Gary's list in dimension four was complete.

After I started as a PhD-student in Nijmegen, I got back to this subject in the spring of 1996. And this time I was able to solve the dimension five case.

The reason that two years had passed after finishing my Master's Thesis and starting with the final research to the five dimensional case, was the complexity of this case. It was only in 1996 that we rediscovered the paper [1] by Drużkowski. In conjuction with a theorem from my Master's Thesis [2], we now were able to reduce the general case to the triangular cubic linear case.

2 Reduction to triangular matrices

We start with a few basic definitions.

Definition 1 Let $a = (a_1, \ldots, a_n) \in \mathbb{C}^n$. Then $a^{*3} := (a_1^3, \ldots, a_n^3)$.

Definition 2 Let A be a linear matrix over $\mathbb{C}$. Then the map $F = X + (AX)^{*3}$ is called *cubic-linear* or *in Drużkowski form.*

Note that in some other papers such a cubic linear map F is called Drużkowski map only if $\det(JF) = 1$.

Definition 3 Let $F = X + (AX)^{*3}$ and $G = X + (BX)^{*3}$ be two polynomial automorphisms in Drużkowski form. Then the matrices $A, B \in \mathrm{Mat}_{n,n}(\mathbb{C})$ are called *cubic similar* ($A \overset{3}{\sim} B$) if there exists a linear invertible polynomial map T with $T^{-1}FT = G$.

The idea behind this definition is that it is rather special that if T is a linear invertible map and F is a Drużkowski form one has that $T^{-1}FT$ is again on Drużkowski form and therefore this property deserves a name.

Definition 3 is in terms of maps. For computational use however it is often preferable to work in terms of matrices.

Lemma 1 *Let $F = X + (AX)^{*3}$ and $G = X + (BX)^{*3}$ be two polynomial maps on Drużkowski form. Then $A \overset{3}{\sim} B$ if and only if there exists $T \in \mathrm{Gl}_n(\mathbb{C})$ with $(ATX)^{*3} = T(BX)^{*3}$.*

Proof. The following statements can be read from top to bottom or the other way round. In either case each statement is equivalent with the next one in the sequence.

- $A \overset{3}{\sim} B$.
- There exists an invertible map T with $T^{-1}FT = G$.
- There exists an invertible map T with $T^{-1}(TX + (ATX)^{*3}) = X + (BX)^{*3}$.
- There exists an invertible map T with $X + T^{-1}(ATX)^{*3} = X + (BX)^{*3}$.

- There exists an invertible map T with $T^{-1}(ATX)^{*3} = (BX)^{*3}$.
- There exists an invertible matrix T with $T^{-1}(ATX)^{*3} = (BX)^{*3}$.
- There exists an invertible matrix T with $(ATX)^{*3} = T(BX)^{*3}$.

This proves the lemma. □

From [2] we know that

Theorem 1 *Let $r \in \mathbb{N}$. If the Jacobian Conjecture holds for every polynomial map $F : \mathbb{C}^r \to \mathbb{C}^r$ where F has the special form*

$$F = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_r \end{pmatrix} + \begin{pmatrix} H_1(x_1, \dots, x_r) \\ H_2(x_1, \dots, x_r) \\ \vdots \\ H_r(x_1, \dots, x_r) \end{pmatrix}$$

with $H_i = 0$ or $\deg(H_i) = 3$ (H_i homogeneous for all $i \in \{1, \dots, r\}$) then for all $n \geq r$ and all $A \in \mathrm{Mat}_{n,n}(\mathbb{C})$ the Jacobian Conjecture holds for all Drużkowski forms

$$G = X + (AX)^{*3}$$

with $\mathrm{rank}(A) = r$ and $X = (x_1, \dots, x_n)$.

Before we present our main reduction theorem we show a few lemmas, which we will need for the proof of this main theorem. The proofs can be found in [3]. The first two lemmas are proved purely theoretically. For the third and the fourth lemma we had to do some computations to solve the corresponding systems of equations.

Lemma 2 *Let $F = X + (AX)^{*3}$ with $A \in \mathrm{Mat}_{5,5}(\mathbb{C})$ and $J((AX)^{*3})$ is nilpotent. Then there exists linear invertible T such that $T^{-1}FT = X + (BX)^{*3}$ where the last row of B is a null row.*

Lemma 3 *Assume $\mathrm{rank}(A) = 2$. By lemma 2 we have that the last row is equal to zero. Now if we write*

$$A = \begin{pmatrix} a_1 & a_2 & a_3 & a_4 & a_5 \\ b_1 & b_2 & b_3 & b_4 & b_5 \\ c_1 & c_2 & c_3 & c_4 & c_5 \\ d_1 & d_2 & d_3 & d_4 & d_5 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} & & & & a_5 \\ & & A' & & b_5 \\ & & & & c_5 \\ & & & & d_5 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

and we consider the Drużkowski form $X' + (A'X')^{*3}$ (where $X' = (x_1, \dots, x_4)$) we may assume that A' equals

$$\begin{pmatrix} a_1 & a_2 & a_3 & a_4 \\ \lambda_2 a_1 & \lambda_2 a_2 & \lambda_2 a_3 & \lambda_2 a_4 \\ \lambda_3 a_1 & \lambda_3 a_2 & \lambda_3 a_3 & \lambda_3 a_4 \\ \lambda_4 a_1 & \lambda_4 a_2 & \lambda_4 a_3 & \lambda_4 a_4 \end{pmatrix} \text{ or } \begin{pmatrix} a_1 & a_2 & a_3 & a_4 \\ b_1 & b_2 & b_3 & b_4 \\ \lambda a_1 + \mu b_1 & \lambda a_2 + \mu b_2 & \lambda a_3 + \mu b_3 & \lambda a_4 + \mu b_4 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

and $d_5 = 0$.

Lemma 4 *Let A and A' be as in lemma 3. Assume*

$$A' = \begin{pmatrix} a_1 & a_2 & a_3 & a_4 \\ \lambda_2 a_1 & \lambda_2 a_2 & \lambda_2 a_3 & \lambda_2 a_4 \\ \lambda_3 a_1 & \lambda_3 a_2 & \lambda_3 a_3 & \lambda_3 a_4 \\ \lambda_4 a_1 & \lambda_4 a_2 & \lambda_4 a_3 & \lambda_4 a_4 \end{pmatrix}$$

Then there exists a linear invertible map $T \in \mathbb{C}[X]$ and $B \in \mathrm{Gl}_5(k)$ such that $T^{-1} \circ (X + (AX)^{*3}) \circ T = X + (BX)^{*3}$ with B is upper triangular with null diagonal.

Lemma 5 *Let A and A' be as in lemma 3. Assume*

$$A' = \begin{pmatrix} a_1 & a_2 & a_3 & a_4 \\ b_1 & b_2 & b_3 & b_4 \\ \lambda a_1 + \mu b_1 & \lambda a_2 + \mu b_2 & \lambda a_3 + \mu b_3 & \lambda a_4 + \mu b_4 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Then there exists a linear invertible map $T \in \mathbb{C}[X]$ and $B \in \mathrm{Gl}_5(k)$ such that $T^{-1} \circ (X + (AX)^{*3}) \circ T = X + (BX)^{*3}$ with B is upper triangular with null diagonal.

After these technical lemmas we can finally give the main reduction theorem, which is an improvement of [1, Theorem 2.1] for the case $n = 5$.

Theorem 2 *If a polynomial map $F = X + (AX)^{*3} : \mathbb{C}^5 \to \mathbb{C}^5$ has $\det(JF) = 1$ and $\mathrm{rank}(A) < 3$ or $\mathrm{corank}(A) < 3$, then there exists an invertible linear map L such that $L \circ F \circ L^{-1} = X + (BX)^{*3}$, with B is upper triangular with null diagonal.*

Proof. Though the original theorem in Drużkowski's paper [1] only claims that F is a tame automorphism, we can almost copy the proof as it is presented in that paper. Simply because in three of the four cases it is shown that LFL^{-1} has the desired form (and hence F is tame).

- $\mathrm{rank}(A) = 1$. The proof is exactly the same as in [1].
- $\mathrm{corank}(A) = 1$. From lemma 2 it follows that we are always in case (i) of Drużkowski's paper.
- $\mathrm{corank}(A) = 2$. From lemma 2 it now follows that we are always in case (iii) of Drużkowski's paper.
- $\mathrm{rank}(A) = 2$. This is the only part where Drużkowski doesn't show that F can be transformed to the desired form. To prove this case we use the lemmas 3, 4 and 5.

□

Since we are working in dimension five, we have that either $\mathrm{rank}(A) < 3$ or $\mathrm{corank}(A) < 3$ and hence:

Corollary 1 *Let $F = X + (AX)^{*3} : \mathbb{C}^5 \to \mathbb{C}^5$ such that $\det(JF) = 1$ then there exists an invertible linear map L such that $L \circ F \circ L^{-1} = X + (BX)^{*3}$, with B is upper triangular with null diagonal.*

3 Meisters' representatives

In [5] Meisters presents a list of seventeen mutually inequivalent matrices with respect to the cubic similarity relation in dimension five. The names of these matrices are based on the following notions.

- A J indicates that the matrix is on Jordan normal form.
- An N indicates that it is a nilpotent matrix which is not on Jordan normal form, but does not need parameters in it.
- A P indicates that it is a nilpotent matrix which contains parameters which cannot be reduced to a single complex number.
- The first number is the rank of the matrix.
- The second number is the nilpotence index of $J((AX)^{*3}$, where A is the matrix.
- The small letters at the end are used as an index.
- For some P matrices an extra integer is appended to show the number of parameters in it.

In [4] it is shown that the rank and the nilpotence index as mentioned above are *invariants* with respect to the cubic similarity relation. Therefore it makes sense to use these figures to assign proper names to the matrices. Note that the nilpotence index of the matrix A itself is *not* an invariant. In fact A does not need to be nilpotent at all.

$$\underset{J(1,2)}{\begin{pmatrix} 0&1&0&0&0\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0 \end{pmatrix}} \quad \underset{J(2,2)}{\begin{pmatrix} 0&1&0&0&0\\0&0&0&0&0\\0&0&0&1&0\\0&0&0&0&0\\0&0&0&0&0 \end{pmatrix}} \quad \underset{J(2,3)}{\begin{pmatrix} 0&1&0&0&0\\0&0&1&0&0\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0 \end{pmatrix}}$$

$$\underset{N(2,3a)}{\begin{pmatrix} 0&0&1&0&0\\0&0&1&1&0\\0&0&0&1&0\\0&0&0&0&0\\0&0&0&0&0 \end{pmatrix}} \quad \underset{J(3,3)}{\begin{pmatrix} 0&1&0&0&0\\0&0&1&0&0\\0&0&0&0&0\\0&0&0&0&1\\0&0&0&0&0 \end{pmatrix}} \quad \underset{J(3,4)}{\begin{pmatrix} 0&1&0&0&0\\0&0&1&0&0\\0&0&0&1&0\\0&0&0&0&0\\0&0&0&0&0 \end{pmatrix}}$$

$$\underset{N(3,3a)}{\begin{pmatrix} 0&0&1&0&0\\0&0&1&1&0\\0&0&0&1&1\\0&0&0&0&0\\0&0&0&0&0 \end{pmatrix}} \quad \underset{N(3,4a)}{\begin{pmatrix} 0&1&1&0&0\\0&0&1&0&0\\0&0&0&1&0\\0&0&0&0&0\\0&0&0&0&0 \end{pmatrix}} \quad \underset{N(3,4b)}{\begin{pmatrix} 0&0&1&0&0\\0&0&1&1&0\\0&0&0&1&0\\0&0&0&0&1\\0&0&0&0&0 \end{pmatrix}}$$

$$\underset{N(3,4c)}{\begin{pmatrix} 0&0&1&0&0\\0&0&1&1&0\\0&0&0&1&1\\0&0&0&0&1\\0&0&0&0&0 \end{pmatrix}} \quad \underset{J(4,5)}{\begin{pmatrix} 0&1&0&0&0\\0&0&1&0&0\\0&0&0&1&0\\0&0&0&0&1\\0&0&0&0&0 \end{pmatrix}} \quad \underset{N(4,5a)}{\begin{pmatrix} 0&1&1&0&0\\0&0&1&0&0\\0&0&0&1&0\\0&0&0&0&1\\0&0&0&0&0 \end{pmatrix}}$$

$$\begin{pmatrix} 0&1&0&0&0\\ 0&0&1&1&0\\ 0&0&0&1&0\\ 0&0&0&0&1\\ 0&0&0&0&0 \end{pmatrix} \quad \begin{pmatrix} 0&1&1&0&0\\ 0&0&1&1&0\\ 0&0&0&1&0\\ 0&0&0&0&1\\ 0&0&0&0&0 \end{pmatrix} \quad \begin{pmatrix} 0&1&0&1&0\\ 0&0&1&0&0\\ 0&0&0&1&0\\ 0&0&0&0&1\\ 0&0&0&0&0 \end{pmatrix}$$

$N(4,5b)$ $N(4,5c)$ $N(4,5d)$

$$\begin{pmatrix} 0&1&1&0&0\\ 0&0&1&a&0\\ 0&0&0&1&0\\ 0&0&0&0&1\\ 0&0&0&0&0 \end{pmatrix} \quad \begin{pmatrix} 0&1&1&b&0\\ 0&0&1&a&0\\ 0&0&0&1&0\\ 0&0&0&0&1\\ 0&0&0&0&0 \end{pmatrix}$$

$P(4,5c)$ $P(4,5c2)$

Remark 1 Note the following points:

- $P(4,5c)$ is not called $P(4,5a)$, which should be natural if one uses the small letter just as an index as with the N-matrices. However in this case the c is used because $P(4,5c)_{|a=1} = N(4,5c)$, where $P(4,5c)_{|a=1}$ means substitute $a=1$ in $P(4,5c)$.
- Note also that $P(4,5c)_{|a=0} = N(4,5a)$. Hence we add the restriction that $a \notin \{0,1\}$ for $P(4,5c)$.
- $P(4,5c)_{|a=a_1} \overset{3}{\not\sim} P(4,5c)_{|a=a_2}$ if $a_1 \neq a_2$.
- $P(4,5c2)_{|b=0} = P(4,5c)$, hence we add the restriction $b \neq 0$ for $P(4,5c2)$. Note that there are no restrictions on the a in $P(4,5c2)$.

4 Classification of Drużkowski maps

Theorem 2 gives us the reduction we need. It means that the most general Drużkowski map $X + (AX)^{*3}$ in dimension five is given by the matrix A:

$$\begin{pmatrix} 0 & a_2 & a_3 & a_4 & a_5\\ 0&0&b_3&b_4&b_5\\ 0&0&0&c_4&c_5\\ 0&0&0&0&d_5\\ 0&0&0&0&0 \end{pmatrix} \tag{4.1}$$

Independent of the ten parameters in it this matrix is strong nilpotent. So this matrix is in fact on its own a description of all Drużkowski maps with $J((AX)^{*3})$ is nilpotent. However since our final goal is finding representatives with respect to the cubic similarity relation, it makes sense to split the general case into the five possible values for the nilpotence index of the associated Jacobian matrix. As was noted before, this nilpotence index is invariant under cubic similarity.

Using this observation we compute $J((AX)^{*3})^n$ for $n = 1, \ldots, 5$ and assume that the resulting matrix is the null matrix. Obviously for $n = 1$ this means that A equals the null matrix, which gives the identity map $I_5 : \mathbb{C}^5 \to \mathbb{C}^5$. Therefore we only consider the cases with nilpotence index ≥ 2.

4.1 Nilpotence index two

Assuming $J((AX)^{*3})^2 = 0$ gives a system of 119 equations in the ten parameters. Solving this system gives fifteen solutions. In figure 1 we show the tree along which we found these solutions. One starts at the top with the complete system. One solves a few simple equations. Normally this gives a few possible partial solutions. Each arrow presents such a solution. And each solution may imply some assumptions on the parameters. After substituting these partial solutions one gets new reduced systems of equations. And at this point the process is repeated. Hence each arrow represents some assumptions; these are listed at the bottom. Furthermore, the boxed numbers in the tree correspond to the numbered matrices given below.

The fifteen solutions are presented by their corresponding matrices. For each matrix the rank is listed together with the assumptions used in the process to find them as mentioned above. Naturally, assumptions of the form $b_3 = 0$ are not shown since they are already used in the matrix and hence b_3 does not appear in the matrix anymore.

1. $$\begin{pmatrix} 0 & 0 & 0 & 0 & a_5 \\ 0 & 0 & 0 & 0 & b_5 \\ 0 & 0 & 0 & 0 & c_5 \\ 0 & 0 & 0 & 0 & d_5 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$
rank 1.

2. $$\begin{pmatrix} 0 & a_2 & a_3 & a_4 & a_5 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$
rank 1, $a_2 \neq 0$.

3. $$\begin{pmatrix} 0 & 0 & 0 & a_4 & a_5 \\ 0 & 0 & 0 & 0 & b_5 \\ 0 & 0 & 0 & 0 & c_5 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$
rank 2, $a_4 \neq 0$.

4. $$\begin{pmatrix} 0 & 0 & a_3 & a_4 & a_5 \\ 0 & 0 & 0 & 0 & b_5 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$
rank 2, $a_3 \neq 0$.

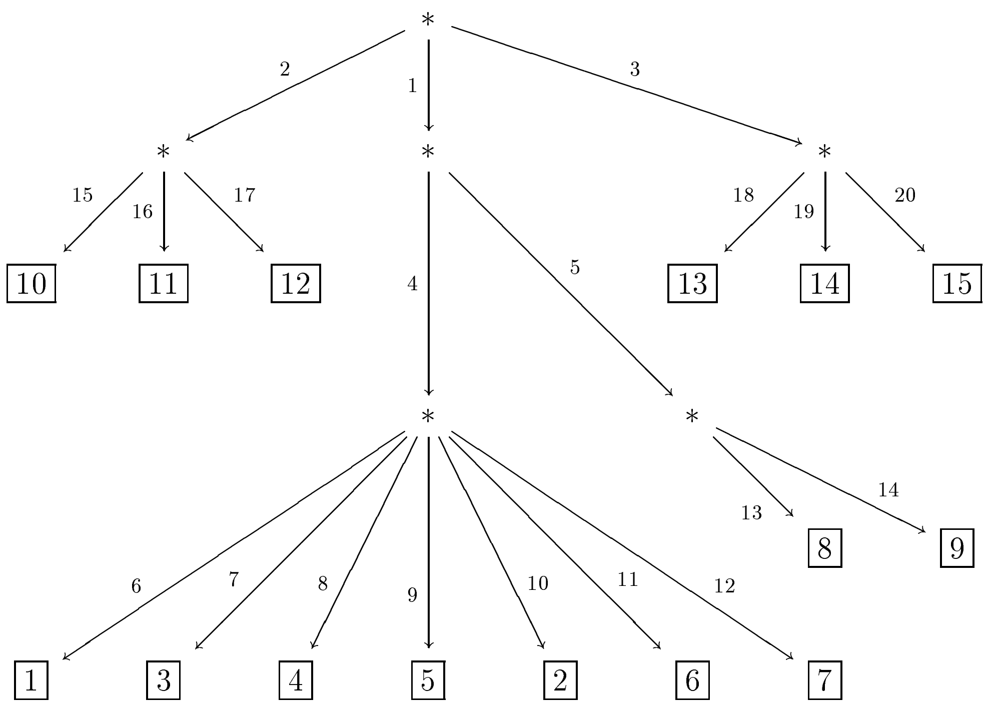
5. $$\begin{pmatrix} 0 & 0 & a_3 & -\frac{a_3 c_5^3}{d_5^3} & a_5 \\ 0 & 0 & 0 & 0 & b_5 \\ 0 & 0 & 0 & 0 & c_5 \\ 0 & 0 & 0 & 0 & d_5 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$
rank 2, $a_3 \neq 0$, $d_5 \neq 0$.

6. $$\begin{pmatrix} 0 & a_2 & -\frac{a_2 b_5^3}{c_5^3} & a_4 & a_5 \\ 0 & 0 & 0 & 0 & b_5 \\ 0 & 0 & 0 & 0 & c_5 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$
rank 2, $a_2 \neq 0$, $c_5 \neq 0$.

7. $$\begin{pmatrix} 0 & a_2 & a_3 & \frac{-a_2 b_5^3 - a_3 c_5^3}{d_5^3} & a_5 \\ 0 & 0 & 0 & 0 & b_5 \\ 0 & 0 & 0 & 0 & c_5 \\ 0 & 0 & 0 & 0 & d_5 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$
rank 2, $a_2 \neq 0$, $d_5 \neq 0$.

8. $$\begin{pmatrix} 0 & 0 & 0 & a_4 & a_5 \\ 0 & 0 & 0 & b_4 & b_5 \\ 0 & 0 & 0 & 0 & c_5 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$
rank 2, $b_4 \neq 0$.

9. $$\begin{pmatrix} 0 & 0 & a_3 & a_4 & a_5 \\ 0 & 0 & 0 & b_4 & b_5 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$
rank 2, $a_3 \neq 0$, $b_4 \neq 0$.



1:$b_3=0, c_4=0$

2:$b_3=0, c_4\neq0, d_5=0$

3:$b_3\neq0, c_4=0, a_2=0$

4:$b_3=0, c_4=0, b_4=0$

5:$b_3=0, c_4=0, b_4\neq0, a_2=0, d_5=0$

6:$b_3=0, c_4=0, b_4=0, a_2=0, a_3=0, a_4=0$

7:$b_3=0, c_4=0, b_4=0, a_2=0, a_3=0, a_4\neq0, d_5=0$

8:$b_3=0, c_4=0, b_4=0, a_2=0, a_3\neq0, d_5=0, c_5=0$

9:$b_3=0, c_4=0, b_4=0, a_2=0, a_3, d_5\neq0$

10:$b_3=0, c_4=0, b_4=0, a_2\neq0, c_5=0, b_5=0, d_5=0$

11:$b_3=0, c_4=0, b_4=0, a_2, c_5\neq0, d_5=0$

12:$b_3=0, c_4=0, b_4=0, a_2, d_5\neq0$

13:$b_3=0, c_4=0, b_4\neq0, a_3=0, a_2=0, d_5=0$

14:$b_3=0, c_4=0, b_4, a_3\neq0, a_2=0, d_5=0, c_5=0$

15:$b_3=0, c_4\neq0, a_2=0, a_3=0, d_5=0$

16:$b_3=0, c_4, a_2\neq0, a_3=0, b_5=0, b_4=0, d_5=0$

17:$b_3=0, c_4, a_2, b_4\neq0, d_5=0$

18:$b_3\neq0, d_5=0, c_5=0, c_4=0, a_2=0$

19:$b_3, d_5\neq0, a_3=0, a_4=0, c_4=0, a_2=0$

20:$b_3, d_5, a_3\neq0, c_4=0, a_2=0$

Figure 1. Solution tree for nilpotence index two

10. $\begin{pmatrix} 0 & 0 & 0 & a_4 & a_5 \\ 0 & 0 & 0 & b_4 & b_5 \\ 0 & 0 & 0 & c_4 & c_5 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$ rank 2, $c_4 \neq 0$.

11. $\begin{pmatrix} 0 & a_2 & 0 & a_4 & a_5 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & c_4 & c_5 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$ rank 2, $a_2 \neq 0$, $c_4 \neq 0$.

12. $$\begin{pmatrix} 0 & a_2 & -\frac{a_2 b_4^3}{c_4^3} & a_4 & a_5 \\ 0 & 0 & 0 & b_4 & \frac{b_4 c_5}{c_4} \\ 0 & 0 & 0 & c_4 & c_5 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$
rank 2, $a_2 \neq 0$, $b_4 \neq 0$, $c_4 \neq 0$.

13. $$\begin{pmatrix} 0 & 0 & a_3 & a_4 & a_5 \\ 0 & 0 & b_3 & b_4 & b_5 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$
rank 2, $b_3 \neq 0$.

14. $$\begin{pmatrix} 0 & 0 & 0 & 0 & a_5 \\ 0 & 0 & b_3 & -\frac{b_3 c_5^3}{d_5^3} & b_5 \\ 0 & 0 & 0 & 0 & c_5 \\ 0 & 0 & 0 & 0 & d_5 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$
rank 2, $b_3 \neq 0$, $d_5 \neq 0$.

15. $$\begin{pmatrix} 0 & 0 & a_3 & -\frac{a_3 c_5^3}{d_5^3} & a_5 \\ 0 & 0 & b_3 & -\frac{b_3 c_5^3}{d_5^3} & b_5 \\ 0 & 0 & 0 & 0 & c_5 \\ 0 & 0 & 0 & 0 & d_5 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$
rank 2, $a_3 \neq 0$, $b_3 \neq 0$, $d_5 \neq 0$.

4.2 Nilpotence index three

In this case we have a system of 123 equations. Solving this system gives ten solutions. Ordered by rank these solutions are:

16. $$\begin{pmatrix} 0 & a_2 & a_3 & a_4 & a_5 \\ 0 & 0 & b_3 & b_4 & b_5 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$
rank 2, $a_2 \neq 0$, $b_3 \neq 0$.

17. $$\begin{pmatrix} 0 & 0 & 0 & a_4 & a_5 \\ 0 & 0 & 0 & b_4 & b_5 \\ 0 & 0 & 0 & c_4 & c_5 \\ 0 & 0 & 0 & 0 & d_5 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$
rank 2, $d_5 \neq 0$.

18. $$\begin{pmatrix} 0 & a_2 & a_3 & a_4 & a_5 \\ 0 & 0 & 0 & 0 & b_5 \\ 0 & 0 & 0 & 0 & c_5 \\ 0 & 0 & 0 & 0 & d_5 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$
rank 2, $a_3 \neq 0$, $d_5 \neq 0$.

19. $$\begin{pmatrix} 0 & a_2 & a_3 & a_4 & a_5 \\ 0 & 0 & 0 & b_4 & b_5 \\ 0 & 0 & 0 & c_4 & c_5 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$
rank 3.

20. $$\begin{pmatrix} 0 & a_2 & 0 & a_4 & a_5 \\ 0 & 0 & 0 & 0 & b_5 \\ 0 & 0 & 0 & c_4 & c_5 \\ 0 & 0 & 0 & 0 & d_5 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$
rank 3, $a_2 \neq 0$, $d_5 \neq 0$.

21. $$\begin{pmatrix} 0 & 0 & a_3 & a_4 & a_5 \\ 0 & 0 & 0 & b_4 & b_5 \\ 0 & 0 & 0 & 0 & c_5 \\ 0 & 0 & 0 & 0 & d_5 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$
rank 3, $a_3 \neq 0$, $b_4 \neq 0$, $d_5 \neq 0$.

22. $$\begin{pmatrix} 0 & -\frac{a_3 c_4^3}{b_4^3} & a_3 & a_4 & a_5 \\ 0 & 0 & 0 & b_4 & \frac{b_4 c_5}{c_4} \\ 0 & 0 & 0 & c_4 & c_5 \\ 0 & 0 & 0 & 0 & d_5 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$
rank 3, $a_3 \neq 0$, $b_4 \neq 0$, $c_4 \neq 0$, $d_5 \neq 0$.

23. $\begin{pmatrix} 0 & 0 & a_3 & a_4 & a_5 \\ 0 & 0 & b_3 & b_4 & b_5 \\ 0 & 0 & 0 & 0 & c_5 \\ 0 & 0 & 0 & 0 & d_5 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$

rank 3, $b_3 \neq 0$.

24. $\begin{pmatrix} 0 & a_2 & a_3 & a_4 & a_5 \\ 0 & 0 & b_3 & -\frac{b_3 c_5^3}{d_5^3} & b_5 \\ 0 & 0 & 0 & 0 & c_5 \\ 0 & 0 & 0 & 0 & d_5 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$

rank 3, $a_2 \neq 0$, $d_5 \neq 0$.

25. $\begin{pmatrix} 0 & 0 & a_3 & a_4 & a_5 \\ 0 & 0 & b_3 & b_4 & b_5 \\ 0 & 0 & 0 & c_4 & c_5 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$

rank 3, $b_3 \neq 0$, $c_4 \neq 0$.

In [3] one can find the solution tree corresponding to these solutions.

4.3 Nilpotence index four

Here we have a system of 56 equations. There are only four solutions:

26. $\begin{pmatrix} 0 & 0 & a_3 & a_4 & a_5 \\ 0 & 0 & b_3 & b_4 & b_5 \\ 0 & 0 & 0 & c_4 & c_5 \\ 0 & 0 & 0 & 0 & d_5 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$

rank 3.

27. $\begin{pmatrix} 0 & a_2 & a_3 & a_4 & a_5 \\ 0 & 0 & 0 & b_4 & b_5 \\ 0 & 0 & 0 & c_4 & c_5 \\ 0 & 0 & 0 & 0 & d_5 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$

rank 3, $a_2 \neq 0$.

28. $\begin{pmatrix} 0 & a_2 & a_3 & a_4 & a_5 \\ 0 & 0 & b_3 & b_4 & b_5 \\ 0 & 0 & 0 & 0 & c_5 \\ 0 & 0 & 0 & 0 & d_5 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$

rank 3, $a_2 \neq 0$, $b_3 \neq 0$.

29. $\begin{pmatrix} 0 & a_2 & a_3 & a_4 & a_5 \\ 0 & 0 & b_3 & b_4 & b_5 \\ 0 & 0 & 0 & c_4 & c_5 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$

rank 3, $a_2 \neq 0$, $b_3 \neq 0$, $c_4 \neq 0$.

The solution tree is quite simple. It can be found in [3].

4.4 Nilpotence index five

Finally the last case gives one solution since all matrices of the form (4.1) are nilpotent.

30. $\begin{pmatrix} 0 & a_2 & a_3 & a_4 & a_5 \\ 0 & 0 & b_3 & b_4 & b_5 \\ 0 & 0 & 0 & c_4 & c_5 \\ 0 & 0 & 0 & 0 & d_5 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$, rank 4, $a_2 \neq 0$, $b_3 \neq 0$, $c_4 \neq 0$, $d_5 \neq 0$.

5 Cubic similarity reduction

The basic result of the previous section is that we can reformulate corollary 1 to:

Corollary 2 *Let* $F = X + (AX)^{*3} : \mathbb{C}^5 \to \mathbb{C}^5$ *such that* $\det(JF) = 1$ *then there exists an invertible linear map* L *such that* $L \circ F \circ L^{-1} = X + (BX)^{*3}$*, where* B *is the null matrix or* B *is one of the thirty matrices presented in section 4.*

The next thing we have to do is check whether these maps are cubic similar to the matrices of section 3. In order to find these relations we use the fact that the rank is an invariant of this matrix. At this point it is more practical to use the rank as an invariant than the nilpotence index of the corresponding jacobian. This is because we have to make some assumptions on the parameters still appearing in the matrices of the previous section, and mostly the effect on the rank of these assumptions are more clearly than the effects on the nilpotence index.

The basic approach taken is:

1. Try to reduce A to cases already known by use of permutaion matrices.
2. Take a general linear map T containing parameters.
3. Compute B where B is given by $X + (BX)^{*3} = T^{-1} \circ (X + (AX)^{*3}) \circ T$.
4. Compare B with the already known representatives.
5. Guess which one of those can be identified with B. (Call this matrix M.)
6. Solve $B = M$ in the variables of T.
7. If this system has no solution:
   - Guess another M.
   - If all representatives have been tried, one probably has found a matrix which is not equivalent to the known representatives.
   - Reduce A as much as possible to M', i.e. solve $B_{i,j} = 0$ or $B_{i,j} = 1$ for as many entries $B_{i,j}$ as possible.
   - Prove that the new M' is indeed not cubic similar to all the old representatives of the same rank.
8. If this system has at least one solution:
   - Try to simplify the solution(s) by setting free parameters equal to zero or to one in case they cannot be set to zero.
   - Check if this T implies some new assumptions on the original parameters in the matrices in order to have that T is invertible.
     - If it does not, you have found that $A \overset{3}{\sim} M$ in general.
     - If it does, assume these assumptions don't hold and apply this information to reduce A to A' and repeat the complete process on A'.

In [3] this process is described for each of the thirty matrices. Here we will show one example.

Example 1 Consider $F = X + (AX)^{*3}$ where

$$A = \begin{pmatrix} 0 & 0 & 0 & a_4 & a_5 \\ 0 & 0 & 0 & b_4 & b_5 \\ 0 & 0 & 0 & 0 & c_5 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

We already know that $b_4 \neq 0$. If we compute $T_1^{-1}FT_1 = X + (BX)^{*3}$ for a general map T_1 and try to solve the cases $B = J(2,2)$, $B = J(2,3)$ and $B = N(2,3a)$ we don't get any solution at all. So most probably we have found a new representative. If we try to reduce this B, we see that we can find T_1 such that $B_{1,4} = 1$, $B_{2,4} = 1$, $B_{2,5} = 1$ and $B_{3,5} = 1$ and all other $B_{i,j} = 0$. We call this M'. Looking carefully at the definition of cubic similarity shows that this M' is indeed not cubic similar to the known representatives with rank two. We call this new representative $N(2,2a)$. The T_1 we have used is

$$\left(\frac{(b_5\,a_4 - b_4\,a_5)^3\ x_1}{{b_4}^3}, \frac{(b_5\,a_4 - b_4\,a_5)^3\ x_2}{{a_4}^3}, {c_5}^3\,x_3\ , \frac{(b_5\,a_4 - b_4\,a_5)\ x_4}{b_4\,a_4} - \frac{a_5\,x_5}{a_4}, x_5 \right)$$

If we look at this T_1 we see that it is invertible only if $a_4 \neq 0$, $c_5 \neq 0$ and $b_5a_4 - b_4a_5 \neq 0$. (We already know that $b_4 \neq 0$.)

Now assume that $a_4 \neq 0$ and $c_5 \neq 0$ but $b_5a_4 - b_4a_5 = 0$ and start the process again. After taking a new T_2 and compute $T_2^{-1}FT_2$, we get a matrix B that can be identified with $J(2,2)$. Solving this system yields that T_2 is

$$\left(x_5 + {a_4}^3\,x_3, {b_4}^3\,x_3, x_1, x_4 - \frac{b_5\,x_2}{b_4\,c_5}, \frac{x_2}{c_5} \right)$$

Looking at T_2 we note that we don't need any new assumptions. From T_1 it already follows that we have to look at the cases where $a_4 = 0$ and $c_5 = 0$.

Now assume $a_4 \neq 0$ and $b_5a_4 - b_4a_5 \neq 0$ but $c_5 = 0$. In this case the map T_3 gives $T_3^{-1}FT_3$ which is cubic similar to $J(2,2)$ where T_3 is given by

$$\left(-\frac{(b_5\,a_4 - b_4\,a_5)^3\ x_1}{{b_4}^3}, -\frac{(b_5\,a_4 - b_4\,a_5)^3\ x_3}{{a_4}^3}, x_5, \frac{a_5\,x_4}{a_4} - \frac{x_2\,b_5}{b_4}, x_2 - x_4 \right)$$

Note that this map T_3 does not imply any new assumptions.

Now assume $a_4 \neq 0$ but $b_5a_4 - b_4a_5 = 0$ and $c_5 = 0$. We can immediately skip this case since it gives a matrix A with $\mathrm{rank}(A) = 1$.

So the next case is $a_4 = 0$. In order to remain in a rank two case we must have that either $a_5 \neq 0$ or $c_5 \neq 0$. We may assume $c_5 \neq 0$ since a simple permutation $P = (x_3, x_2, x_1, x_4, x_5)$ swaps the first and third row. So now we can use T_4 is

$$\left(x_5 + {a_5}^3\,x_3, {b_4}^3\,x_1, {c_5}^3\,x_3, x_2 - \frac{b_5\,x_4}{b_4}, x_4 \right)$$

to get that $T_4^{-1}FT_4$ is cubic similar to $J(2,2)$.

And with this last case we have solved the case for this matrix completely, since T_4 does not imply any new assumptions.

6 New representatives

Examining all thirty matrices from section 4 in a similar way as in example 1 completely classifies the Drużkowski maps in dimension five with respect to the cubic similarity relation. This tedious process gives the following new representatives:

$$\begin{pmatrix} 0&0&0&1&0\\ 0&0&0&1&1\\ 0&0&0&0&1\\ 0&0&0&0&0\\ 0&0&0&0&0 \end{pmatrix} \quad \begin{pmatrix} 0&0&0&1&0\\ 0&0&0&1&1\\ 0&0&0&1&-1\\ 0&0&0&0&1\\ 0&0&0&0&0 \end{pmatrix} \quad \begin{pmatrix} 0&1&1&0&0\\ 0&0&0&1&0\\ 0&0&0&0&1\\ 0&0&0&0&0\\ 0&0&0&0&0 \end{pmatrix}$$

$N(2,2a)$ $N(2,3b)$ $N(3,3b)$

$$\begin{pmatrix} 0&0&1&0&1\\ 0&0&1&0&0\\ 0&0&0&1&0\\ 0&0&0&0&1\\ 0&0&0&0&0 \end{pmatrix} \quad \begin{pmatrix} 0&0&1&0&0\\ 0&0&0&1&1\\ 0&0&0&1&0\\ 0&0&0&0&1\\ 0&0&0&0&0 \end{pmatrix} \quad \begin{pmatrix} 0&0&1&1&0\\ 0&0&0&1&1\\ 0&0&0&1&0\\ 0&0&0&0&1\\ 0&0&0&0&0 \end{pmatrix}$$

$N(3,4e)$ $N(3,4f)$ $N(3,4g)$

$$\begin{pmatrix} 0&0&1&1&1\\ 0&0&1&0&0\\ 0&0&0&1&0\\ 0&0&0&0&1\\ 0&0&0&0&0 \end{pmatrix} \quad \begin{pmatrix} 0&1&1&0&0\\ 0&0&0&1&1\\ 0&0&0&1&0\\ 0&0&0&0&1\\ 0&0&0&0&0 \end{pmatrix} \quad \begin{pmatrix} 0&1&0&1&0\\ 0&0&0&1&1\\ 0&0&0&1&0\\ 0&0&0&0&1\\ 0&0&0&0&0 \end{pmatrix}$$

$N(3,4h)$ $N(3,4i)$ $N(3,4j)$

$$\begin{pmatrix} 0&0&1&a&0\\ 0&0&1&1&0\\ 0&0&0&1&0\\ 0&0&0&0&1\\ 0&0&0&0&0 \end{pmatrix} \quad \begin{pmatrix} 0&0&1&0&a\\ 0&0&1&1&0\\ 0&0&0&1&0\\ 0&0&0&0&1\\ 0&0&0&0&0 \end{pmatrix} \quad \begin{pmatrix} 0&0&1&1&0\\ 0&0&0&1&a\\ 0&0&0&1&0\\ 0&0&0&0&1\\ 0&0&0&0&0 \end{pmatrix}$$

$P(3,4a)$ $P(3,4c)$ $P(3,4g)$

$$\begin{pmatrix} 0&0&1&1&a\\ 0&0&1&0&0\\ 0&0&0&1&0\\ 0&0&0&0&1\\ 0&0&0&0&0 \end{pmatrix} \quad \begin{pmatrix} 0&1&a&0&0\\ 0&0&0&1&1\\ 0&0&0&1&0\\ 0&0&0&0&1\\ 0&0&0&0&0 \end{pmatrix} \quad \begin{pmatrix} 0&1&0&1&0\\ 0&0&0&1&a\\ 0&0&0&1&0\\ 0&0&0&0&1\\ 0&0&0&0&0 \end{pmatrix}$$

$P(3,4h)$ $P(3,4i)$ $P(3,4j)$

$$\begin{pmatrix} 0&0&1&a&b\\ 0&0&1&1&0\\ 0&0&0&1&0\\ 0&0&0&0&1\\ 0&0&0&0&0 \end{pmatrix} \quad \begin{pmatrix} 0&1&a&1&0\\ 0&0&0&1&b\\ 0&0&0&1&0\\ 0&0&0&0&1\\ 0&0&0&0&0 \end{pmatrix} \quad \begin{pmatrix} 0&1&0&1&0\\ 0&0&1&1&0\\ 0&0&0&1&0\\ 0&0&0&0&1\\ 0&0&0&0&0 \end{pmatrix}$$

$P(3,4a2)$ $P(3,4j2)$ $N(4,5e)$

$$\begin{pmatrix} 0&1&0&1&0\\ 0&0&1&a&0\\ 0&0&0&1&0\\ 0&0&0&0&1\\ 0&0&0&0&0 \end{pmatrix}$$

$P(4,5e)$

Remark 2 Similar to remark 1 we note the following:

- In $N(2,3b)$ the -1 seems a bit strange: why isn't it $P(2,3a)$ with a parameter a on the place of the -1? The answer is in fact pretty simple. As long as $a \notin \{0,1\}$, $P(2,3a) \overset{3}{\sim} N(2,3b)$. Furthermore $P(2,3a)_{|a=0} \overset{3}{\sim} P(2,3a)_{|a=1} \overset{3}{\sim} N(2,3a)$. So independent of the value of the parameter a, $P(2,3a)$ can be reduced to a matrix with no parameters left in it. So there's no need to add a P-matrix.
- $P(3,4a)_{|a=1} \overset{3}{\sim} N(3,4a)$ and $P(3,4a)_{|a=0} = N(3,4b)$.
- $P(3,4c)_{|a=1} \overset{3}{\sim} N(3,4c)$ and $P(3,4c)_{|a=0} = N(3,4b)$.
- $P(3,4g)_{|a=1} = N(3,4g)$ and $P(3,4g)_{|a=0} \overset{3}{\sim} N(3,4a)$.
- $P(3,4h)_{|a=1} = N(3,4h)$ and $P(3,4h)_{|a=0} \overset{3}{\sim} N(3,4b)$.
- $P(3,4i)_{|a=1} = N(3,4i)$ and $P(3,4i)_{|a=0} \overset{3}{\sim} N(3,4a)$.
- $P(3,4j)_{|a=1} = N(3,4j)$ and $P(3,4j)_{|a=0} \overset{3}{\sim} N(3,4a)$.
- $P(3,4a2)_{|a=0} = P(3,4c)$ and $P(3,4a2)_{|b=0} = P(3,4a)$, hence $P(3,4c2)$ would have been a correct name also.
- $P(3,4j2)_{|a=0} = P(3,4j)$. Furthermore we have $P(3,4j2)_{|b=0,a=-1} \overset{3}{\sim} N(3,3a)$ and $P(3,4j2)_{|b=0,a\neq 0,a\neq -1} \overset{3}{\sim} N(3,4a)$.
- $P(4,5e)_{|a=1} = N(4,5e)$ and $P(4,5e)_{|a=0} = N(4,5d)$.
- So we add for $P(3,4a)$, $P(3,4c)$, $P(3,4g)$, $P(3,4h)$, $P(3,4i)$, $P(3,4j)$ and $P(4,5e)$ the restriction that $a \notin \{0,1\}$. For $P(3,4a2)$ and $P(3,4j2)$ we add $a,b \neq 0$.

The final claim in this paper is that the seventeen matrices by Meisters in section 3 together with the nineteen matrices in this section give a complete family of inequivalent matrices with respect to Meisters' cubic similarity relation in dimension five. Unfortunately in dimension five the amount of work compared to the work in dimension four has increased enormously. Therefore it doesn't look very promising to start with research on the dimension six case. Especially if one bares in mind that the five-dimensional case only worked out because of the strong reduction theorem 2 and the fact that we don't have such a theorem in dimension six. The problem for this theorem is that we now have the possibility that $\text{rank}(A) = \text{corank}(A) = 3$ and we cannot use Drużkowski theorem anymore. But nevertheless, even with an equivalent reduction theorem in dimension six, it would most probably still be too complex to compute.

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THE KING OF THE TALKING FROGS AND POLYNOMIAL AUTOMORPHISMS

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In honour of Gary Meisters on the occassion of his retirement

1 Preface

Maybe you don't understand the first half of the title, so let me first explain this to you.
In April 1992 my wife Sandra, our daughter Raissa (then five years old) and I visited Gary and Mary Ellen in Lincoln. During the very pleasant stay at their house Raissa found out that Gary talked very much. On the airport of Lincoln, waiting for our flight to Holland Raissa suddenly said to Gary: "you are the King of the talking frogs!" As you can imagine Gary was very pleased with this name. And to show her his appreciation he has sent her later several frogs, varying from very small to very large!
The name **King** of the **talking** frogs is, in my opinion, a very good characterization of Gary. In an always **humoristic** style he discusses mathematics and other things of life with everyone who wants to listen and more importantly he spreads his ideas

around.
In this way he makes many people aware what is going on in our field. His well-known very long list of "polynomial mapping papers" is a beautiful example of his "missionary" work. Also his recent idea to award prizes for interesting questions has shown to be effective (the final solution of the Markus-Yamabe Conjecture has its origin in Gary's \$100 prize which he announced at the Curaçao Conference in July 1994: I will come back to this point in the next sections).
Finally I like to mention another important characteristic of Gary, namely that he always comes up with many questions and is not afraid to make conjectures!
Let me conclude this short preface by giving my interpretation of his initials G.H.M. namely. He is a

Great Humoristic Mathematician

2 A short survey of Gary's work on polynomial automorphisms

The first paper of Gary I could track back concerning polynomial automorphisms is his 1982 paper in [30], in which he discusses both Jacobian problems: the Jacobian Problem from algebraic Geometry and the Jacobian Conjecture from differential equations, also known as the Markus-Yamabe Conjecture.
For the sake of completeness I will recall briefly both conjectures (see [2, 9, 12, 27, 29]).

Conjecture 1 (Jacobian Conjecture, 1939) *Let $F : \mathbb{C}^n \to \mathbb{C}^n$ be a polynomial map with det $JF \in \mathbb{C}^*$, then F is invertible.*

Conjecture 2 (Markus-Yamabe Conjecture, 1960) *Let $F : \mathbb{R}^n \to \mathbb{R}^n$ be a C^1-vector field satisfying the Markus-Yamabe Assumption (MYA) i.e.*

(MYA) For all $x \in \mathbb{R}^n$ the real parts of all eigenvalues of $JF(x)$ are negative

then 0 is a global attractor of the system

$$\dot{x}(t) = F(x(t))$$

i.e. every solution of this system converges to 0 if t tends to infinity.

Gary's 1982-paper mentioned above is "classical" by now: it is very clearly written and describes the state of the art around 1982 concerning both conjectures. The paper is full of questions and nice examples. It should be read by everyone interested in polynomial automorphisms and related topics, in particular various connections with differential equations are given. In this paper he already introduces the notion of **Polynomial flows**, or as Gary likes it better **Polyflows**.

2.1 Polynomial flows

Let $F : \mathbb{R}^n \to \mathbb{R}^n$ be a C^1-vector field and consider the system of ordinary differential equations

$$\dot{x}(t) = F(x(t))$$

The unique solution which at $t = 0$ has a given value $x_0 \in \mathbb{R}^n$ is denoted by $x(t, x_0)$. The system above is called a **Polynomial flow system** and F a **Polynomial flow vector field** if and only if the map

$$x_0 \to x(t, x_0)$$

is a polynomial map. In other words, the solution depends polynomially on the initial condition. In fact the map above is a polynomial automorphism for every t where the solution is defined.
In [3] Bass and Meisters study such systems and amongst other things they show that a polynomial flow vector field F is automatically polynomial (this seems obvious, but it is not!), the solutions are complete i.e. are defined for all $t \in \mathbb{R}$ and that there exist an integer d and real analytic functions $a_\alpha : \mathbb{R} \to \mathbb{R}^n$ such that

$$x(t, x_0) = \sum_{|\alpha| \leq d} a_\alpha(t) {x_0}^\alpha$$

Using these results one can prove a very nice characterization of polynomial vector fields, due to Coomes and Zurkowski [6], namely that F is a polynomial flow vectorfield if and only if the associated derivation $D := \sum F_i \partial_i$ is **locally finite** on $\mathbb{R}[X] := \mathbb{R}[X_1, \ldots, X_n]$ (i.e. for every element $g \in \mathbb{R}[X]$ there exists an integer m such that

$$\deg D^q(g) \leq m$$

for all $q \in \mathbb{N}$. Furthermore in [3] Bass and Meisters give a complete classification of all polynomial vector fields in dimension two (see also the papers [10] and [40] for alternative proofs).
Polynomial flow vector fields in dimension ≥ 3 remain still to be understood.

2.2 Polynomial flows and the Jacobian Conjecture

In [35] Meisters and Olech related polynomial flows to the Jacobian Conjecture, namely let $F : \mathbb{R}^n \to \mathbb{R}^n$ be a polynomial map with $\det JF = 1$. For each $v \in \mathbb{R}^n$ and each $x_0 \in \mathbb{R}^n$, consider the system

$$\dot{x}(t) = JF(x)^{-1} v, \; x(0) = x_0$$

Then Meisters and Olech proved

Proposition 1 *The Jacobian Conjecture is equivalent to the following statement: For each $v \in \mathbb{R}^n$ the solution $x(t, x_0, v)$ of the system above depends polynomially on both x_0 and t.*

This result was also obtained by Adjamagbo in 1986, but remained unpublished; in fact it was this result which Adjamagbo explained to me in June 1986 when I heard about the Jacobian Conjecture for the first time.
In [10] I used this polyflow result to give an inversion formula for polynomial automorphisms. Later I realised that this result can be easily extended to arbitray $\mathbb{Q}$-algebras and we get the following result (the proof is left to the reader: just use the formal inverse function theorem and Taylor expansion)

Theorem 1 (Inversion Formula) *Let R be any commutative $\mathbb{Q}$-algebra and $F \in Aut_R R[X_1, \ldots, X_n]$ with inverse $G = (G_1, \ldots, G_n)$, then*

$$G_{i(d)} = \frac{1}{d!} D^d(X_i)_{|X=0}$$

for all $d \geq 1$, all i.

Here $_{|X=0}$ means substitute $X_1 = 0, \ldots, X_n = 0$, $G_{i(d)}$ is the homogeneous component of degree d of G_i and finally D is the derivation

$$D = \sum Y_i \frac{\partial}{\partial F_i}$$

on the polynomial ring $R[X_1, \ldots, X_n, Y_1, \ldots, Y_n]$.

2.3 The solution of the 2-dimensional Markus-Yamabe Conjecture for polynomial vector fields

In 1987 there was the first break-through concerning the Markus-Yamabe conjecture: Meisters and Olech established the 2-dimensional Markus-Yamabe Conjecture for polynomial vector field in the plane (see [34]). (A maybe less known result is that in [33] Meisters and Olech showed that for polynomial flow vector fields in any dimension the Markus-Yamabe Conjecture is true!) The proof given in [34] is based on two results

1. A result of Olech ([37]) stating that to show that the Markus-Yamabe Conjecture in the plane is true, it suffices to show that the Markus-Yamabe assumption implies that $F : \mathbb{R}^2 \to \mathbb{R}^2$ is injective.
2. If det $JF(x)$ is non-zero for all $x \in \mathbb{R}^2$, then the number of elements in the fibre $F^{-1}(x)$ is bounded by a constant N which does not depend on x.

(This result was later generalised by the author in [11] to arbitrary dimension and one year later improved in [1]).
In 1993 the general C^1-case of the two dimensional Markus-Yamabe Conjecture was proved independently by Fessler and Gutierrez (in [19] and [24] respectively). In 1994 another proof was given by Glutsuk in [20].

2.4 Strong nilpotence

In 1991 Meisters and Olech invented a new notion: **strong nilpotence**. In their paper [36] they studied invertible quadratic homogeneous polynomial maps $X+Q : \mathbb{R}^n \to \mathbb{R}^n$ i.e. $Q = (Q_1, \ldots, Q_n)$ and each Q_i is a homogeneous polynomial map of degree 2. The Jacobian matrix JQ is called **strongly nilpotent** if for every n-tuple of vectors $v_1, \ldots, v_n$ in $\mathbb{R}^n$ we have

$$JQ(v_1) \cdots JQ(v_n) = 0.$$

They introduced this notion because they observed that the expressions describing formulas for the inverse of the map $X+Q$ could be simplified if strong nilpotence holds. In the paper [36] it is shown that if $n \leq 4$ nilpotence of JQ (which is equivalent to $X+Q$ is invertible) is indeed equivalent to strong nilpotence. However if $n \geq 5$ counterexamples to this equivalence are given.

In [15] Hubbers and the author generalise the notion of strong nilpotence to arbitrary polynomial maps $H : k^n \to k^n$, where k is any field of characteristic zero. If k is an infinite field the definition agrees with the one given above. The main result of [15] is

Theorem 2 *Let $H : k^n \to k^n$ be a polynomial map. Then JH is strongly nilpotent if and only if there exists $T \in Gl_n(k)$ such that $T^{-1} \cdot JH \cdot T$ is an uppertriangular matrix with zeros on the main diagonal.*

2.5 Power Similarity

Another notion invented by Meisters is **power similarity**. He invented it in order to study the so-called Drużkowski-forms, or cubic-linear mappings.

Consider two matrices $A, B \in M_n(\mathbb{C})$ and their respective Drużkowski forms

$$F_A(X) := X + (AX)^3 \text{ and } F_B(X) := X + (BX)^3$$

Then A and B are called **power-3-similar** (or for short **power similar**) if there exists $T \in Gl_n(k)$ such that

$$F_B = T^{-1} \cdot F_A \cdot T$$

or equivalently

$$(BX)^3 = T^{-1}(ATX)^3$$

i.e.

$$T(BX)^3 = (ATX)^3$$

In the paper [31] a complete set of representatives for power similarity in dimension 3 is given as well as a list of 6 representatives fot the case $n = 4$. It was later shown by Hubbers in [25] that this list is complete! The case $n = 5$ was started by Meisters and completed by Hubbers in [26]. We refer to Hubbers' paper in this proceedings.

3 The DMZ-Conjecture and the solution of the Markus-Yamabe Conjecture

In this section I will describe how a conjecture due to Deng, Meisters and Zampieri has led to the final solution of the Markus-Yamabe Conjecture.
The story started in 1992 when David Wright proved the following result (see [38]).

Proposition 2 *Let $F = (X_1+H_1, X_2+H_2, X_3+H_3)$ where each H_i is homogeneous of degree three or $H_i = 0$. If det $JF = 1$, then there exists $T \in Gl_n(\mathbb{C})$ such that*

$$T^{-1} \cdot F \cdot T = (X_1 + a(X_2, X_3), X_2 + b(X_3), X_3)$$

In particular F is invertible.

In februari 1994 Engelbert Hubbers in his Masters thesis ([25]) completely classified the cubic homogeneous case in dimension four. His result is:

Theorem 3 (Hubbers, 1994) *Let $F = X - H$ be a cubic homogeneous polynomial map in dimension four, such that $\det(JF) = 1$. Then there exists some $T \in GL_4(\mathbb{C})$ with $T^{-1} \circ F \circ T$ being one of the following forms:*

1. $$\begin{pmatrix} x_1 & \\ x_2 & \\ x_3 & \\ x_4 & -a_4x_1^3 - b_4x_1^2x_2 - c_4x_1^2x_3 - e_4x_1x_2^2 - f_4x_1x_2x_3 \\ & -h_4x_1x_3^2 - k_4x_2^3 - l_4x_2^2x_3 - n_4x_2x_3^2 - q_4x_3^3 \end{pmatrix}$$

2. $$\begin{pmatrix} x_1 & \\ x_2 & -\frac{1}{3}x_1^3 - h_2x_1x_3^2 - q_2x_3^3 \\ x_3 & \\ x_4 & -x_1^2x_3 - h_4x_1x_3^2 - q_4x_3^3 \end{pmatrix}$$

3. $$\begin{pmatrix} x_1 & \\ x_2 & -\frac{1}{3}x_1^3 - c_1x_1^2x_4 + 3c_1x_1x_2x_3 - \frac{16q_4c_1^2-r_4^2}{48c_1^2}x_1x_3^2 - \frac{1}{2}r_4x_1x_3x_4 \\ & +\frac{3}{4}r_4x_2x_3^2 - \frac{r_4q_4}{12c_1}x_3^3 - \frac{r_4^2}{16c_1}x_3^2x_4 \\ x_3 & \\ x_4 & -x_1^2x_3 + \frac{r_4}{4c_1}x_1x_3^2 - 3c_1x_1x_3x_4 + 9c_1x_2x_3^2 - q_4x_3^3 - \frac{3}{4}r_4x_3^2x_4 \end{pmatrix}$$

4. $$\begin{pmatrix} x_1 & \\ x_2 & -\frac{1}{3}x_1^3 \\ x_3 & -x_1^2x_2 - e_3x_1x_2^2 - k_3x_2^3 \\ x_4 & -e_4x_1x_2^2 - k_4x_2^3 \end{pmatrix}$$

5. $$\begin{pmatrix} x_1 & \\ x_2 & -\frac{1}{3}x_1^3 + i_3x_1x_2x_4 - j_2x_1x_4^2 + s_3x_2x_4^2 + i_3^2x_3x_4^2 - t_2x_4^3 \\ x_3 & -x_1^2x_2 - \frac{2s_3}{i_3}x_1x_2x_4 - i_3x_1x_3x_4 - j_3x_1x_4^2 - \frac{s_3^2}{i_3^2}x_2x_4^2 \\ & -s_3x_3x_4^2 - t_3x_4^3 \\ x_4 & \end{pmatrix}$$

$$6.\ \begin{pmatrix} x_1 \\ x_2 & -\frac{1}{3}x_1^3 - j_2x_1x_4^2 - t_2x_4^3 \\ x_3 & -x_1^2x_2 - e_3x_1x_2^2 - g_3x_1x_2x_4 - j_3x_1x_4^2 - k_3x_2^3 - m_3x_2^2x_4 \\ & -p_3x_2x_4^2 - t_3x_4^3 \\ x_4 \end{pmatrix}$$

$$7.\ \begin{pmatrix} x_1 \\ x_2 & -\frac{1}{3}x_1^3 \\ x_3 & -x_1^2x_2 - e_3x_1x_2^2 - k_3x_2^3 \\ x_4 & -x_1^2x_3 - e_4x_1x_2^2 - f_4x_1x_2x_3 - h_4x_1x_3^2 - k_4x_2^3 - l_4x_2^2x_3 \\ & -n_4x_2x_3^2 - q_4x_3^3 \end{pmatrix}$$

$$8.\ \begin{pmatrix} x_1 \\ x_2 & -\frac{1}{3}x_1^3 \\ x_3 & -x_1^2x_2 - e_3x_1x_2^2 + g_4x_1x_2x_3 - k_3x_2^3 + m_4x_2^2x_3 + g_4^2x_2^2x_4 \\ x_4 & -x_1^2x_3 - e_4x_1x_2^2 - \frac{2m_4}{g_4}x_1x_2x_3 - g_4x_1x_2x_4 - k_4x_2^3 \\ & -\frac{m_4^2}{g_4^2}x_2^2x_3 - m_4x_2^2x_4 \end{pmatrix}$$

If we look at this result we make an astonishing discovery:
at least one of the H_i is zero! Equivalently, the hypothesis $\det JF = 1$, which is equivalent to JH is nilpotent, implies that $H_1, \ldots, H_4$ are **linearly dependent over** k**!** This leads to

Problem 1 (Dependence Problem, DP) Let k be a field of characteristic zero. Let $H = (H_1, \ldots, H_n)$, with each H_i homogeneous of degree 3 or $H_i = 0$. Does the hypothesis JH is nilpotent imply that the H_i are linearly dependent over k?

This problem is still **open** for all $n \geq 5$. In my opinion it is the most important open problem related to the Jacobian Conjecture.
The assumption that an even stronger version of the dependence problem would be true, has led to the discovery of a large class of polynomial automorphisms over any commutative ring A. (see [16, 17] for more details).
Without going into any detail let us say the following: let A be a commutative ring and $n \in \mathbb{N}$. We define a subset $H_n(A)$ of $A[X]^n$ with the property that if $H \in H_n(A)$ then JH is nilpotent. Furthermore the corresponding map $F := X + H$ is a polynomial automorphism over A, in fact F is stably tame! The elements of $H_2(A)$ are of the following form:
Let $H = (H_1, H_2) \in A[X_1, X_2]^2$. Then $H \in H_2(A)$ if and only if

$$\begin{aligned} H_1 &= a_2 f(a_1X_1 + a_2X_2) + c_1 \\ H_2 &= -a_1 f(a_1X_1 + a_2X_2) + c_2 \end{aligned}$$

for some a_i, c_i in A and some $f(T) \in A[T]$ with $f(0) = 0$.

Then in March 1994 Deng, Meisters and Zampieri proposed a new attack to the Jacobian Conjecture, inspired by an old result of Poincare and Siegel.
They conjecture the following (see [8]).

Conjecture 3 (DMZ-Conjecture) *Let $F : \mathbb{C}^n \to \mathbb{C}^n$ be a polynomial map of the form $F = X + H$, where each H_i is homogeneous of some degree $d \geq 2$ and $\det JF = 1$. Then for all $s \in \mathbb{C}$, $|s|$ sufficiently large, sF is* ***global analytic linearisable*** *i.e. there exists an analytic isomorphism $h_s : \mathbb{C}^n \to \mathbb{C}^n$ such that*

$$h_s{}^{-1} \circ sF \circ h_s = sX.$$

Because of the result of Poincare and Siegel mentioned above one knows that h_s exists locally in a neighborhood of 0 and that h_s is unique if one assumes that $h_s(0) = 0$ and $Jh_s(0) = I_n$. Furthermore, if DMZ is true then so is the Jacobian Conjecture: namely by a classical result of Bass, Connell, Wright and Yagzhev (see [2] resp. [39]) it suffices to prove the Jacobian Conjecture for cubic homogeneous polynomial maps. Now if $h_s{}^{-1} \circ sF \circ h_s = sX$ for some non-zero $s \in \mathbb{C}$ it follows that sF and hence F is injective, which by another classical result implies that F is invertible.

In [8] the authors proved that $h_s{}^{-1}$ is entire, but could not prove it for h_s. Meisters was very sceptical about the new approach and decided to do some computer experiments about the structure of h_s in case $F = X+H$ with H cubic homogeneous. **To his own surprise** he found that in all the cases he computed h_s was **much nicer** as expected: all h_s were **polynomial automorphisms!** His sceptism changed into optimism and he formulated

Conjecture 4 (Meisters' Linearisation Conjecture, MLC) *Let $F = X + H$ be cubic homogeneous with JH nilpotent (or equivalently $\det JF = 1$). Then for every $s \in \mathbb{C}^*$ (except a finite number of roots of unity) there exists an invertible* ***polynomial*** *map $h_s : \mathbb{C}^n \to \mathbb{C}^n$ such that*

$$h_s{}^{-1} \circ sF \circ h_s = sX.$$

Meisters formulated this conjecture at the Curaçao Conference in July 1994, (see [12]), where he offered a \$100 reward for the first person to find a counterexample to his conjecture.
Some weeks later Hubbers and I found the following partial confirmation of MLC (see [15]).

Proposition 3 *Let $H : \mathbb{C}^n \to \mathbb{C}^n$ be a polynomial map with JH* ***strongly nilpotent****. Then $F = X + H$ satisfies MLC, and hence F is invertible.*

Combining this with Wrights result mentioned above we get

Corollary 1 *MLC is true if $n \leq 3$.*

Inspired by these results my aim was to prove MLC for all F of the form $F = X+H$ with $H \in H_n(\mathbb{C})$. However in september 1994 I found a counterexample in $H_n(\mathbb{C})$ for all $n \geq 4$ (see [14]), namely

Theorem 4 *Let $n \geq 4$. Put $d := X_3X_1 + X_4X_2$, then*

$$F = (X_1 + X_4d, X_2 - X_3d, X_3 + {X_4}^3, X_4, \ldots, X_n)$$

is a counterexample to MLC.

Now immediately the question was raised: is this F also a counterexample to the DMZ-Conjecture?
In July 1995 Gorni and Zampieri showed in [22] that the answer is **no!** They showed that for all complex s with $|s| \neq 1$ the map $h_s : \mathbb{C}^n \to \mathbb{C}^n$ is global analytic linearisable. So DMZ and hence the approach to the Jacobian Conjecture remained open.
In August 1995 I also received a preprint ([7]) of Bo Deng in which he showed that F is not a counterexample to the DMZ-Conjecture. His proof was based on

Lemma 1 *Let$F : \mathbb{C}^n \to \mathbb{C}^n$ be a polynomial automorphism with $F(0) = 0$ of the form $F = X + H$ with JH nilpotent. Let $0 < |s| < 1$. Then sF is global analytic linearisable if and only if 0 is a global attractor of sF (i.e. for every $x \in \mathbb{C}^n$ we have that $(sF)^m(x)$ tends to 0 if m tends to infinity).*

When I saw this lemma I realised that we now had a way to investigate the DMZ-Conjecture: so I said to Engelbert, just take some complicated F of the form $F = X + H$ with $H \in H_5(\mathbb{C})$, take some $0 < s < 1$ and check if 0 is an attractor of sF by iterating sF in some arbitrary points, and then start proving!
I was convinced that for all $F = X + H$ with $H \in H_n(\mathbb{C})$ 0 would be a global attractor of sF if $|s| < 1$. In fact in 1976 LaSalle had made the following stronger conjecture, see ([28]), which was reinvented (independently) in 1994 by Cima, Gasull and Mañosas. They called it the Discrete Markus-Yamabe Problem (see [5])

Conjecture 5 (LaSalle Conjecture (Discrete Markus-Yamabe Problem)) *Let $F : \mathbb{R}^n \to \mathbb{R}^n$ be a polynomial map with $F(0) = 0$ and such that for all $x \in \mathbb{R}^n$ all eigenvalues of $JF(x)$ are smaller than 1 in absolute value. Then 0 is a global attractor of F.*

So Hubbers started to look at some complicated examples in dimension 5. A few days later he came to me with the following example

Example 1 Let $s = (1/100)$ and $F = (X_1 + 3X_1{X_2}^2 + {X_4}^4 + {X_1}^3 - 3{X_4}^2X_1{X_2}^2 - 3{X_4}^2X_1{X_2}^2 - 2{X_1}^3{X_4}^3 - 3{X_4}^3{X_1}^2X_2 - X_4{X_2}^3 - {X_4}^4{X_1}^3 + 3{X_1}^2X_4X_2 + 2X_4{X_1}^3 - {X_4}^3 + {X_2}^3 + 3{X_1}^2X_2,\ -3{X_4}^2X_1{X_2}^2 - 6{X_4}^3{X_1}^2X_2 - 3{X_4}^4{X_1}^2X_2 - 3{X_4}^3X_1{X_2}^2 - X_2 + 3X_1X_4{X_2}^2 - {X_4}^3 + 3X_1{X_2}^2 - {X_4}^5{X_1}^3 - {X_4}^2{X_2}^3 + {X_2}^3 + 6{X_1}^2X_2X_4 - 3{X_4}^4{X_1}^3 + {X_1}^3 - 2{X_1}^3{X_4}^3 + 3{X_1}^2X_2 + 2{X_4}^2{X_1}^3 + 3{X_1}^3X_4 + {X_4}^5,\ 3{X_4}^2X_1{X_2}^2 + 3{X_4}^2{X_1}^2X_2 + 3{X_4}^3{X_1}^2X_2 + X_3 - 2X_1{X_2}^2 + X_4{X_2}^3 - {X_4}^4 - {X_2}^3 - {X_1}^2X_4X_2 + {X_4}^4{X_1}^3 + 2{X_1}^3{X_4}^3 - {X_1}^2X_2 + {X_4}^2{X_1}^3,\ -X_4,\ X_5)$.

If $v := (0, 0, 0, 3.6314, 0)$, then computer calculations indicate that

$$\lim_{m\to\infty} (sF)^m(v) = 0.$$

Furthermore if $w := (0, 0, 0, 3.6315, 0)$, then computations indicate that

$$\lim_{m\to\infty} (sF)^m(w) = \infty.$$

The next day I went to Poland for two weeks. On the airport I started to think about this phenomenon. Hubbers' example was far to complicated to prove anything, so I had to find an easier example.
Then I remembered that both in the Gorni-Zampieri paper and the Deng-paper they **essentially used** that both X_1 and X_2 appeared **linearly** in the F giving the counterexample to MLC. So I had to find a "better" automorphism F in which both X_1 and X_2 do not appear linearly. Having the whole class $H_4(\mathbb{C})$ at my disposal I simply took the simplest possibility

$$F = (X_1 + X_4 d^2, X_2 - X_3 d^2, X_3 + {X_4}^3, X_4)$$

which is of the form $X + H$ with $H \in H_4(\mathbb{C})$. So F is an automorphism and since JH is nilpotent all eigenvalues of $J(sF) = sI + s(JH)$ are equal to s. So if $0 < s < 1$ and if the LaSalle Conjecture is true then 0 should be a global attractor! However during my visit at Torun I showed that for all $a \in \mathbb{R}$ and all $0 < s < 1$ such that $as > 1$ we have that $(sF)^m(a, a, \ldots, a)$ tends to infinity if m tends to infinity! So sF gave a counterexample to the LaSalle Conjecture. By also considering real numbers s with $s > 1$ and considering $(sF)^{-1}$ we finally got (see [18])

Theorem 5 (van den Essen, Hubbers) *Let $n \geq 4$, $m \geq 1$. Put $d := X_3X_1 + X_4X_2$ and*

$$F = (X_1 + X_4 d^2, X_2 - X_3 d^2, X_3 + {X_4}^m, X_4, \ldots, X_n)$$

Then

1. *For all $0 < s < 1$ sF gives a counterexample to the LaSalle Conjecture.*
2. *$F = (X_1 + X_4 d^2, X_2 - X_3 d^2, X_3 + X_4^5, X_4, \ldots, X_n)$ gives a counterexample to the DMZ-Conjecture.*

After having found the above counterexamples to the Dicrete Markus-Yamabe Problem, it was natural to ask: does the following system give a counterexample to the Markus-Yamabe Conjecture?

$$\begin{aligned}
\dot{x}_1(t) &= -x_1 + x_4 d(x)^2 \\
\dot{x}_2(t) &= -x_2 - x_3 d(x)^2 \\
\dot{x}_3(t) &= -x_3 + {x_4}^m \\
\dot{x}_4(t) &= -x_4
\end{aligned}$$

More precisely, does there exist solutions which tend to infinity if t tends to infinity? (remember that according MYC all solutions should tend to 0!)

So we conjectured that such a solution should exist!
Again we asked the computer for help, however this time the computer could not help us at all.
In the second week of November 1995 Anna Cima visited Nijmegen. I described her our 4-dimensional candidate counterexamples. About one week after she left I received an email confirming that there do exist solutions which tend to infinity! One week later Cima, Gasull and Mañosas where able to modify the 4-dimensional counterexample into a 3-dimensional counterample. So we finally obtained the following result (see [4])

Theorem 6 (Cima, van den Essen, Gasull, Hubbers, Mañosas) *Let* $n \geq 3$, $d(X) := X_3 X_1 + X_2$ *and* $F : \mathbb{R}^n \to \mathbb{R}^n$ *given by*

$$F(x_1, \ldots, x_n) = (-x_1 + d(x)^2, -x_2 - x_3 d(x)^2, -x_3, \ldots, -x_n).$$

Then F *gives a counterexample to MYC. More precisely*

$$x_1(t) = 12e^{2t},\ x_2(t) = -18e^t,\ x_3(t) = \cdots = x_n(t) = e^{-t}$$

is a solution of $\dot{x}(t) = F(x)$ *which tends to infinity if* t *tends to infinity!*

Remark 1 After we sent out our counterexample to several people we received, by email, a preprint of Glutsuk, ([21]), in which he constructs a C^1-counterexample to the Markus-Yamabe Conjecture in dimension three.

4 Meisters' Cubic Linear Linearisation Conjecture and new counterexamples to the Markus-Yamabe Conjecture

In the previous section we saw how the MYC was completely solved. However the story of the DMZ-Conjecture was not finished yet. There would still be the possibility that the DMZ-Conjecture is true for all Drużkowski forms, leaving open a proof for the Jacobian Conjecture. This lead Meisters to the following conjecture (see [32]).

Conjecture 6 (Meisters' Cubic Linear Linearisation Conjecture, CLLC) *If* F *is of the form* $(X_1 + {L_1}^3, \ldots, X_n + {L_n}^3)$ *with* $\det JF = 1$ *and each* L_i *is a linear form, then for all* $|s| \neq 1$ *there exists* $h_s : \mathbb{C}^n \to \mathbb{C}^n$, *a global analytic automorphism such that*

$${h_s}^{-1} \circ sF \circ h_s = sX$$

This time Gary offered a \$200 reward for a counterexample!

The problem here is that it is hard to find cubic linear forms which are invertible: in fact in the literature there was only one interesting example in dimension 16, due to

Drużkowski (see [9]). So Hubbers and I tried to prove that it did not satisfy DMZ. Simultaneously Gorni and Zampieri were looking at the same example and tried to prove that it does satify DMZ. Therefore they developped a very elegant theory of **pairing between Cubic Linear forms and Cubic Homogeneous forms** i.e. to every cubic linear map F they associated a cubic homogeneous map f (in less variables) and to every cubic homogeneous map f a cubic linear map F (in more variables) in such a way that one of them satisfies DMZ if and only if the other one does! (see [23])
They calculated the cubic homogeneous map f associated to the 16-dimensional cubic linear example F mentioned above and found that f was even polynomially linearisable, hence the same holds for F. So this F is not a counterexample to DMZ, so the approach to prove the Jacobian Conjecture via CLLC remained open!

However in November 1996 I found a 5-dimensional counterexample to DMZ which was cubic homogeneous. So using the Gorni-Zampieri theory this finally leads to a 17-dimensional counterexample to CLLC, which kills this approach to prove the Jacobian Conjecture! More precisely we get the following result (see [14])

Theorem 7 *Let $n \geq 5$ and $F : \mathbb{C}^n \to \mathbb{C}^n$ be defined by*

$$\begin{aligned} F &= (X_1 + X_2 {X_5}^2, X_2 + {X_1}^2 X_5 - X_4 {X_5}^2, \\ &\quad X_3 + {X_2}^2 X_5, X_4 + 2X_1X_2X_5 - X_3{X_5}^2, X_5, \ldots, X_n) \end{aligned}$$

Then F is invertible and for every non-zero $s \in \mathbb{C}$ with $|s| \neq 1$ the map sF is not global analytic linearisable.

Corollary 2 *There exists a counterexample in dimension 17 to Meisters' cubic linear linearisation conjecture!*

To conclude this section we will show that the cubic homogeneous map described above can also be used to give both **cubic** as well as **quadratic homogeneous counterexamples to MYC**. Namely let

$$Q = (X_2X_5, {X_1}^2 - X_4X_5, {X_2}^2, 2X_1X_2 - X_3X_5, 0, \ldots, 0)$$

Then we have (see [14])

Theorem 8 *Let $n \geq 5$ and $F = -X + Q$ (resp. $F = -X + X_5 Q$). Then F gives a counterexample to MYC. More precisely*

$$x(t) = (30e^t, 60e^{2t}, 720e^{4t}, 720e^{3t}, e^{-t}, \ldots, e^{-t})$$

(resp.

$$x(t) = (120e^{3t}, 480e^{5t}, 23040e^{9t}, 11520e^{7t}, e^{-t}, \ldots, e^{-t}))$$

is a solution of $\dot{x}(t) = F(x)$ which tends to infinity if t tends to infinity.

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