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Nilpotent Jacobians

E.-M.G.M. Hubbers

$$F := \begin{pmatrix} -\mathbf{x}_1 \\ -\mathbf{x}_2 \\ -\mathbf{x}_3 \\ \vdots \\ -\mathbf{x}_n \end{pmatrix} + \begin{pmatrix} \mathbf{x}_3(\mathbf{x}_1 + \mathbf{x}_3\mathbf{x}_2)^2 \\ -(\mathbf{x}_1 + \mathbf{x}_3\mathbf{x}_2)^2 \\ \mathbf{0} \\ \vdots \\ \mathbf{0} \end{pmatrix}$$

Nijmegen 1998

Nilpotent Jacobians

een wetenschappelijke proeve op het gebied
van de Natuurwetenschappen, Wiskunde en Informatica

Proefschrift

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Preface

This thesis has not been written as a novel. Neither was it meant to become a best-seller about mathematics like ‘Der Zahlenteufel’ by Hans Magnus Enzensberger (cf. [23]). His book addresses everyone who is afraid of mathematics. I’m sorry, but my dissertation is not written that way. Someone who’s afraid of mathematics should not torture himself by reading this piece of work. My goal is not to interest as many readers as possible. The idea is to attract readers who are already interested in the subject of the Jacobian Conjecture and who are more or less familiar with the underlying mathematics. This thesis is written to provide those readers some answers to questions raised in the on-going saga of solving the Jacobian Conjecture. And yes, it becomes rather technical at several points. However in order to increase the readability at these places, (hopefully) clear examples are provided.

Now before each potential reader is scared away, let me point out the filtering aspect of chapter 1. A few things can happen while reading that chapter.

1. You are not really interested in the topic of the Jacobian Conjecture.
2. You are interested but you have difficulties understanding the mathematics in chapter 1.
3. You are interested in the subject and didn’t have too much trouble understanding the mathematics used.

My recommended actions in these situations are as follows.

- In situation 1, please do yourself a favour, stop reading this monograph and put it on a firm shelf.
- In situation 2, skip the main part and continue reading the conclusions, i.e. chapter 9. In case you find a conclusion interesting enough go back to the indicated chapter and try to understand this conclusion. Furthermore, it might be interesting to look at appendix B which is a description of the so-called Jacobian package, and check whether there are some procedures useful to your own research.

Perhaps it is also a good idea to read the paper [33], a transcript of a session that took place at a conference in Turin, 1997. Van den Essen brought the Jacobian Conjecture to trial. The conjecture was accused to be true. Van den Essen came up with evidence both pro and against the accusation. Finally the audience of mathematicians was asked to act as jury. This audience was not convinced

beyond reasonable doubt that the accusation was true. Therefore the Jacobian Conjecture was found 'not guilty' of being true.

- In situation 3 of course you are encouraged to read the whole dissertation! The mathematics used will not become much more difficult than in chapter 1. Only the amount of it increases rapidly in chapter 4 and chapter 5. Furthermore, keep in mind that there is light at the end of each tunnel, making it worth to go through it. After reaching the end of this transcript of my Ph.D. research you will have seen some remarkable results on polynomial mappings.

Naturally it is not up to me, the author, to claim that the results are interesting. However, the body of this thesis is based on a few papers. And the reactions I received already on those papers provide strong reasons to believe that the reader won't be disappointed spending time to understand this disquisition. In addition to this I think I'm entitled to say that any paper which solves a conjecture that has been around for over 35 years deserves some attention.

So please start reading chapter 1 and decide for yourself what to do with this thesis. But at least find out what the polynomial map on the cover implies.

Engelbert Hubbers
Nijmegen, May 1998

Chapter 1

The quest of the Jacobian Conjecture

Introduction

We start this chapter by presenting some pieces of the original article by Ott-Heinrich Keller, the paper which raised the question of the Jacobian Conjecture for the first time. After this we introduce some basic notions and of course the Jacobian Conjecture itself. The third section in this chapter gives an overview of the most important facts known about this conjecture. Finally the last section describes the new contributions of this thesis to this field of research.

1.1 Keller's Aussagen

In 1939 Ott-Heinrich Keller wrote the paper [59]. In this paper he investigates domains which do have a 'basis' (x_1, \dots, x_n) but do not have a linear basis, i.e. each element can be written as a polynomial $\sum \lambda_{i_1 \dots i_n} x_1^{i_1} \cdots x_n^{i_n}$. Keller's aim was to describe all possible bases of this kind and describe also by which transformations those bases can be mapped onto other bases of the same domain. In fact he only looks at *Ganze Cremona-Transformationen*, i.e. polynomial maps with coefficients in \mathbb{Z} . He gives five 'Aussagen' regarding these transformations:

1. (\overleftarrow{R}) : rational transformation in one direction,
2. (\overrightarrow{R}) : rational transformation in the opposite direction,
3. (\overleftarrow{G}) : algebraic transformation in one direction,
4. (\overrightarrow{G}) : algebraic transformation in the opposite direction and
5. (F) the 'Funktionaldeterminante' is ± 1 .

He uses these statements to check which combination is sufficient to claim that all of these five statements hold. Keller proves in this paper that each combination of four of these statements automatically implies the remaining statement. The next thing he

	statements	complete	counterexample
1	$\overline{(R)}$ $\overline{(R)}$ $\overline{(G)}$	no	(xy, y)
2	$\overline{(R)}$ $\overline{(R)}$ $\overline{(G)}$		
3	$\overline{(R)}$ $\overline{(G)}$ $\overline{(G)}$	no	(x^2, y)
4	$\overline{(R)}$ $\overline{(G)}$ $\overline{(G)}$		
5	$\overline{(R)}$ $\overline{(R)}$ (F)	no	$(x^2y, \frac{1}{x})$
6	$\overline{(R)}$ $\overline{(G)}$ (F)	?	
7	$\overline{(R)}$ $\overline{(G)}$ (F)		
8	$\overline{(R)}$ $\overline{(G)}$ (F)	no	$(2\sqrt{x}, \sqrt{x}y)$
9	$\overline{(R)}$ $\overline{(G)}$ (F)		
10	$\overline{(G)}$ $\overline{(G)}$ (F)	no	$(x + \sqrt{y}, y)$

Table 1.1: Keller's table

does is looking at the combinations of three of these statements and see whether such a combination gives a complete set of five true statements. This leads to the summary presented in table 1.1. As a consequence of the results for three statements it is easy to verify that two statements never imply the complete set of five. The cases which are equivalent by swapping domain and image are grouped together. From table 1.1 it follows that the only cases in which it is not known whether one can get a complete set of five true statements are case 6 and 7. In other words Keller's question is

Question 1.1

*Lassen Polynome mit der Funktionaldeterminante 1 sich stets durch Polynome umkehren?*¹

Also the comment Keller gives on this topic is worth recalling:

Mir scheint die Frage eine Untersuchung sehr zu lohnen, sie scheint jedoch bereits im ebenen Fall sehr schwierig zu sein.²

As anyone who is familiar with the research on this topic knows, Keller was completely right at this point.

1.2 Preliminaries

Let k be a commutative ring. Let $X = x_1, \dots, x_n$ and consequently let $k[X]$ denote the polynomial ring in the variables x_1, \dots, x_n .

¹Given polynomials $F_1, \dots, F_n \in \mathbb{Z}[X_1, \dots, X_n]$ with Jacobian determinant equal to 1, can each X_i be expressed as a polynomial in F_1, \dots, F_n ?

²'It seems to me that is certainly worthwhile to investigate this question. However it seems to be already very difficult in the plane.'

Definition 1.2

A *polynomial map* $F = (F_1, \dots, F_n) : k^n \rightarrow k^n$ is a map of the form

$$(x_1, \dots, x_n) \mapsto (F_1(x_1, \dots, x_n), \dots, F_n(x_1, \dots, x_n))$$

where each $F_i \in k[X]$.

Definition 1.3

A polynomial map F is called *invertible over k* if there exists a polynomial map $G \in k[X]^n$ such that

$$x_i = G_i(F_1, \dots, F_n) \tag{1.1}$$

for all i .

Note that even if k is a field $x = G(F(x))$ for all $x \in k^n$ is not the same as (1.1):

Example 1.4

Let $k = \mathbb{F}_3$. Let $F = G = (x - x^3) \in \mathbb{F}_3[x]$. Then $G(F(a)) = a$ for all $a \in \mathbb{F}_3$. However $G_1(F_1) = x - 2x^3 - x^9 \neq x$. Hence (1.1) does not hold for this G . In fact we can even prove that there exists no such G , or in other words we can prove that F is not invertible. Assume F is invertible with inverse G . Then $G = (a_1x + a_2x^2 + \dots + a_nx^n)$ where $a_i \in \mathbb{F}_3$ and $a_n \neq 0$. Then $G_1(F_1) = \text{lot}(x) - a_nx^{3n}$ where $\text{lot}(x)$ means lower order terms in x . Because G is the inverse of F we have that $G_1(F_1) = x$. Hence $a_n = 0$ which is a contradiction. Hence F is not invertible.

Remark 1.5

Note that if F is invertible then F needs to be injective.

Hence in example 1.4 we could have used a simpler argument to prove that F is not invertible: $F(0) = F(1) = 0$ and hence F is not injective.

If k is Noetherian and integral, invertible polynomial maps correspond bijectively with k -automorphisms of the ring $k[X]$ by the map:

$$F \rightarrow F^*$$

with

$$F^* : g \mapsto g(F_1, \dots, F_n)$$

Because of this equivalence we usually don't make any difference between these two. The group of k -automorphisms of $k[X]$ will be denoted by $\text{Aut}_k(k[X])$.

In this group $\text{Aut}_k(k[X])$ we have several subsets worth mentioning.

Definition 1.6

The map

$$E = (x_1, \dots, x_{i-1}, x_i + a, x_{i+1}, \dots, x_n)$$

where $a \in k[x_1, \dots, \hat{x}_i, \dots, x_n]$, is called an *elementary map*. (Here \hat{x}_i means omit x_i .) The inverse of E is given by:

$$E^{-1} = (x_1, \dots, x_{i-1}, x_i - a, x_{i+1}, \dots, x_n)$$

The set of elementary maps is denoted by $\mathfrak{e}_n(k)$.

This set $\mathfrak{E}_k(n)$ is not a subgroup of $\text{Aut}_k(k[X])$.

Definition 1.7

An invertible map $F : k^n \rightarrow k^n$ is called an *affine map* if $\deg_X(F_i) = 1$ for all i . The set of all affine maps is denoted by $\mathfrak{A}_n(k)$.

Note that $\mathfrak{A}_k(n)$ is a subgroup of $\text{Aut}_k(k[X])$. Furthermore if $F \in \mathfrak{A}_k(n)$ such that $F(0) = 0$ then there exists $\tilde{F} \in \text{Mat}_n(k)$ such that $F = \tilde{F}X$. Therefore we often do not distinguish between the map and the matrix.

Definition 1.8

The map F is called *tame* if it is a finite composition of elementary maps and affine maps. The set of tame maps is denoted by $\mathfrak{T}_n(k)$.

Definition 1.9

Let $F = (F_1, \dots, F_n)$ be a polynomial map such that $F_i = c_i x_i + H_i$ where $c_i \in k$ and $H_i \in k[X]$.

1. The map F is in (upper) triangular form if $H_i \in k[x_{i+1}, \dots, x_n]$ for all $1 \leq i \leq n - 1$ and $H_n \in k$. The subgroup of triangular maps is denoted by $\mathfrak{J}_n(k)$. These automorphisms are also called *de Jonquières automorphisms*.
2. F is linearly triangularisable if there exists $T \in \mathfrak{A}_n(k)$ such that $T^{-1}FT$ is in upper triangular form.

From now on we assume that k is a field. With the definitions above one can formulate this problem:

Problem 1.10 (Tame Generators Problem)

Is $\text{Aut}_k(k[X]) = \mathfrak{T}_n(k)$?

Or in other words: is every invertible polynomial map tame? If $n = 2$ this problem has an affirmative answer; it is known as the Jung-van der Kulk theorem. Actually Jung proved this theorem in 1942 for characteristic zero (cf. [57]) and about ten years later van der Kulk proved the theorem for positive characteristic (cf. [61]). If $n \geq 3$ this problem is still open. Nagata constructed in [75] his famous candidate counterexample:

$$\sigma = \begin{pmatrix} x - 2(xz + y^2)y - (xz + y^2)^2z \\ y + (xz + y^2)z \\ z \end{pmatrix}$$

Conjecture 1.11 (Nagata)

The map σ is not tame.

One easily verifies that σ is an automorphism.

In chapter 5 we'll get back to this problem of tame automorphisms. In this section we'll focus on the invertibility of polynomial maps. An important question is:

Question 1.12

How can we decide whether a given polynomial map $F : k^n \rightarrow k^n$ is invertible or not?

A necessary condition is easily found. Assume $F \in k[x_1]$ is invertible with inverse G . Hence by (1.1) $x_1 = G(F(x_1))$. Identifying $\text{Mat}_1(k[x_1]) = k[x_1]$ and taking derivatives using the chain rule one gets:

$$1 = G'(F(x_1)) \cdot F'(x_1) \quad (1.2)$$

In particular $F'(x_1)$ must be an invertible element in $k[x_1]$. But $(k[x_1])^* = k^*$. Hence we must have $F'(x_1) \in k^*$. If we want to extend this to higher dimensions we must have a more general notion of taking derivatives. Therefore we introduce the Jacobian.

Definition 1.13

Let $F : k^n \rightarrow k^n$ be a polynomial map. Then the matrix

$$J(F(X)) = \left(\frac{\partial}{\partial x_j} F_i \right)_{i,j \leq n}$$

is called the *Jacobian matrix* or *Jacobian* of F .

Note that $J(F(X)) \in \text{Mat}_n(k[X])$. We often write $JF(X)$ or even JF instead of $J(F(X))$. Using this Jacobian we get the multi-dimensional equivalent of (1.2):

$$I_n = JG(F(X)) \cdot JF(X) \quad (1.3)$$

Taking determinants we get the equation

$$1 = \det(JG(F(X))) \cdot \det(JF(X)) \quad (1.4)$$

And now we have a similar conclusion as above:

Theorem 1.14

If $F : k^n \rightarrow k^n$ is an invertible polynomial map then $\det(JF) \in k^$.*

If we note that Keller's 'Funktionaldeterminante' is nothing but the determinant of the Jacobian, it makes sense to give his name to a certain set of polynomial maps:

Definition 1.15

A polynomial map $F : k^n \rightarrow k^n$ is called a *Keller map* if $\det(JF) \in k^*$.

Now that we have seen that it is necessary that an invertible polynomial map is a Keller map, one can wonder whether this is also sufficient:

Question 1.16

Is each Keller map invertible?

For general fields k the answer is no.

Example 1.17

Consider $F : \mathbb{F}_p \rightarrow \mathbb{F}_p$ for some p prime where $F(x) = x - x^p$. Obviously $JF = (1)$ and hence $\det(JF) = 1 \in \mathbb{F}_p^*$. Hence F is a Keller map. However due to Fermat we know that $F(x) = 0$ for all $x \in \mathbb{F}_p$. Hence F is not injective. Hence F is a non-invertible Keller map.

So we must add the restriction that $\text{char}(k) = 0$.

Example 1.18

Now if we put $n = 1$ we see that $\det(JF) \in k^*$ implies $F'(x_1) \in k^*$ which implies on its turn that $F(x_1) = \lambda x_1 + \mu$ for $\lambda \in k^*$ and $\mu \in k$. Put $G := \lambda^{-1}x_1 + \lambda^{-1}\mu$ and we have found a polynomial inverse of F .

Hence if $n = 1$ and $\text{char}(k) = 0$ then question 1.16 has an affirmative answer. For $n \geq 2$ the answer to question 1.16 is not known. This question has become famous as the *Jacobian Conjecture*:

Conjecture 1.19 (Jacobian Conjecture, JC(k, n))

Let k be a field of characteristic zero. Let $F : k^n \rightarrow k^n$ be a Keller map. Then F is invertible.

We can generalise this conjecture by changing the assumptions on k .

Conjecture 1.20 (Generalised Jacobian Conjecture, JC(R, n))

Let R be a commutative ring contained in a \mathbb{Q} -algebra. Let $F \in R[X]^n$ be a Keller map. Then F is invertible over R .

Now if we put $R = \mathbb{Z}$ we get the original question 1.1 by Keller. Because it can be shown that question 1.1 also implies the Jacobian Conjecture, the Jacobian Conjecture is also known as Keller's Problem.

Obviously the Jacobian Conjecture can be found by taking $R = k$ where k is a field with $\text{char}(k) = 0$. More remarkable is the following fact:

Lemma 1.21

For any positive integer n , if the Jacobian Conjecture $JC(\mathbb{C}, n)$ is true then the Generalised Jacobian Conjecture $JC(R, n)$ is also true for any commutative ring R contained in a \mathbb{Q} -algebra.

The proof is based on the so-called *Lefschetz principle*. For an illustration of this principle see for instance [8] or [32].

Another interesting choice for the ring R is the field \mathbb{R} . In this case we get the so-called Real Jacobian Conjecture.

Conjecture 1.22 (Real Jacobian Conjecture)

If $F : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is a polynomial map with $\det(JF(x)) \in \mathbb{R}^$ for all $x \in \mathbb{R}^n$ then F is injective.*

Also in this case we have an interesting implication:

Lemma 1.23

If the Real Jacobian Conjecture is true then also the Jacobian Conjecture is true.

Proof. Let $F : \mathbb{C}^n \rightarrow \mathbb{C}^n$ be a polynomial map with $\det(JF) \in \mathbb{C}^*$. Define $\tilde{F} : \mathbb{R}^{2n} \rightarrow \mathbb{R}^{2n}$ by splitting the real and the imaginary part: $\tilde{F} := (\Re F_1, \Im F_1, \dots, \Re F_n, \Im F_n)$. Then $\det(J\tilde{F}) = |\det(JF)|^2 \in \mathbb{R}^*$. So if the Real Jacobian Conjecture holds, \tilde{F} and hence F is injective and invertible (see [17]). \square

However in 1994 Pinchuk found that conjecture 1.22 was false. In [79] he gives a counterexample in dimension 2:

Example 1.24

Define in $\mathbb{R}[x_1, x_2]$

$$\begin{aligned} t &:= x_1x_2 - 1 \\ h &:= t(x_1t + 1) \\ f &:= \frac{h+1}{x_1}(x_1t + 1)^2 \\ u &:= 170fh + 91h^2 + 195fh^2 + 69h^3 + 75h^3f + \frac{75}{4}h^4 \\ P &:= f + h \\ Q &:= -t^2 - 6th(h+1) - u \end{aligned}$$

then

$$F := (P, Q)$$

is a counterexample to the Real Jacobian Conjecture. One easily verifies that $\det(JF) = t^2 + (t + f(13 + 15h))^2 + f^2$ and this term is > 0 on \mathbb{R}^2 since it can only be zero if both t and f are zero. But if $t = 0$ then $f = \frac{1}{x_1} \neq 0$. And $F(1, 0) = F(-1, -2)$ which means that F is not injective.

The reader will have noticed that in the formulation of conjecture 1.22 we use ‘ F is injective’ instead of ‘ F is invertible’. In [2] Adjamagbo gives reasons for injective polynomial endomorphisms to be automorphisms.

In 1962 Białynicki-Birula and Rosenlicht proved in their paper [11] the following theorem.

Theorem 1.25

Let k be an algebraically closed field. Let $F : k^n \rightarrow k^n$ be an injective polynomial map. Then F is bijective.

We really need that k is algebraically closed:

Example 1.26

Let $F : \mathbb{Q} \rightarrow \mathbb{Q}$ be defined by $F(x) = x^3$. Clearly F is injective. However F is not surjective.

Furthermore Cynk and Rusek proved in [17] that

Theorem 1.27

Let F be as in theorem 1.25. Then the inverse of F is also a polynomial map.

Again we really need that k is algebraically closed:

Example 1.28

Let $F : \mathbb{R} \rightarrow \mathbb{R}$ be defined by $F(x) = x^3$. Now F is bijective. However the inverse of F is not a polynomial map.

An important lemma which can be used to decide whether a polynomial map is invertible deals with coordinate systems.

Definition 1.29

If $F = (F_1, \dots, F_n)$ is invertible then F_1, \dots, F_n is called a *coordinate system*. A polynomial f is called a *coordinate* if there exist F_2, \dots, F_n such that f, F_2, \dots, F_n is a coordinate system.

Now the lemma says:

Lemma 1.30

Let $F : k^n \rightarrow k^n$ be a polynomial map. Then F is invertible if and only if $k[x_1, \dots, x_n] = k[F_1, \dots, F_n]$.

Proof. If F is invertible with inverse $G = (G_1, \dots, G_n)$ then $x_i = G_i(F_1, \dots, F_n)$ for $1 \leq i \leq n$. Hence $x_i \in k[F_1, \dots, F_n]$ for all i . Now this means that $k[x_1, \dots, x_n] \subset k[F_1, \dots, F_n]$. Because also $k[x_1, \dots, x_n] \supset k[F_1, \dots, F_n]$, we have in fact equality: $k[x_1, \dots, x_n] = k[F_1, \dots, F_n]$.

Conversely, if $k[x_1, \dots, x_n] = k[F_1, \dots, F_n]$ then $x_i \in k[F_1, \dots, F_n]$ for all $1 \leq i \leq n$. Hence for each i there exists $G_i \in k[X]$ such that $x_i = G_i(F_1, \dots, F_n)$, which means that F is invertible. □

Note that lemma 1.30 more or less explains the name coordinate system.

Before we end this section by introducing some facts on derivations, we make a remark which deals with the characteristic. In example 1.17 we have seen that if we have $\text{char}(k) = p > 0$ then the Jacobian Conjecture does not hold. The reason for the trouble with this example lies in the fact that the characteristic divides the geometric degree of F . The geometric degree of F is the dimension of the field extension $\mathbb{F}_p(X) : \mathbb{F}_p(F)$. Notation $[\mathbb{F}_p(X) : \mathbb{F}_p(F)]$. In [1] Adjagbo describes a generalised formulation of the Jacobian Conjecture where he assumes that this does not occur.

Conjecture 1.31

Let k be a field with $\text{char}(k) = p > 0$. Let F be a Keller map. If $p \nmid [\mathbb{F}_p(X) : \mathbb{F}_p(F)]$ then F is invertible.

This conjecture is more or less backed up by the ‘First isomorphism theorem’ in [1, Theorem 2.3]. This theorem is a generalisation of a theorem by Formanek (cf. [41, Theorem 1]).

As promised we end this section with a few elementary notions on derivations. Let A be any ring.

Definition 1.32

A *derivation* on A is an additive map $D : A \rightarrow A$ satisfying Leibniz’ rule, i.e. satisfying

$$\begin{aligned} D(a + b) &= D(a) + D(b) \\ D(ab) &= D(a)b + aD(b) \end{aligned}$$

for all $a, b \in A$.

For instance if $A = \mathbb{C}[X]$ then the usual derivative $\frac{\partial}{\partial x_i}$ is a derivation on A .

An important tool for derivations is the exponent of a derivation. Let A be a \mathbb{Q} -algebra.

Definition 1.33

Let D be a derivation on A . Consider the formal power series ring $A[[T]]$ in one variable T over A . Extend D to a derivation \tilde{D} on $A[[T]]$ by the formula

$$\tilde{D} \left(\sum a_i T^i \right) := \sum D(a_i) T^i$$

Because this extension is quite clear we identify D with \tilde{D} . The *exponential map* associated to a derivation D is given by the formula:

$$\begin{aligned} \exp(TD) : A[[T]] &\rightarrow A[[T]] \\ g &\mapsto \sum_{p=0}^{\infty} \frac{1}{p!} D^p(g) T^p \end{aligned}$$

for all $g \in A[[T]]$.

It is not difficult to prove that

$$\frac{\partial}{\partial T} (\exp(TD)(a)) = \exp(TD)(D(a))$$

for all $a \in A$. This exponential map associated to a derivation D or, simply put, the exponent of D is important because:

Lemma 1.34

The map $\exp(TD) : A[[T]] \rightarrow A[[T]]$ is a ring automorphism.

For the proof see for instance [32].

Definition 1.33 contains an infinite summation. In order to reduce to finite summations we introduce the concept of locally nilpotent derivations.

Definition 1.35

Let D be a derivation on a ring A . D is a *locally nilpotent derivation* if and only if for each $a \in A$ there exists $n \in \mathbb{N}$ such that $D^n(a) = 0$.

Now for each $g \in A[[T]]$ the summation in definition 1.33 becomes finite for these locally nilpotent derivations. These locally nilpotent derivations become important in chapter 5.

1.3 State of the art

From now on we assume that $\text{char}(k) = 0$. The previous section was mainly used to introduce some definitions and conjectures. In this section we present the most important facts on the Jacobian Conjecture known at the moment. We split these facts

into two groups. The first group deals with the two-dimensional case. Here most theorems are based upon the degree of the polynomial map. The second group deals with reductions in higher dimensions, i.e. it presents some theorems of the form ‘if the Jacobian Conjecture holds for this specific class of polynomial maps’ then ‘the Jacobian Conjecture holds in general’.

1.3.1 Dimension $n = 2$

We have already seen in example 1.18 that if $n = 1$ the Jacobian Conjecture holds. For $n \geq 2$ it is still open in general. However there have been several achievements on special classes for $n = 2$.

For the rest of this section let $F = (F_1, F_2)$ and $m = \deg(F_1)$ and $n = \deg(F_2)$. In 1977 Baba and Nakai proved in [5] as a generalisation of a theorem by Magnus (cf. [64]) that

Theorem 1.36

Let F be a Keller map. If

- *n or m is prime,*
- *n or m equals 4 or*
- *$m > n$, $m = 2p$ for some odd prime p*

then F is invertible.

Appelgate and Onishi in [4] thought they had proved the following theorem. However Nagata in [76] showed that they had made a mistake and corrected the proof so theorem 1.36 was improved to this result:

Theorem 1.37

Let F be a Keller map. If n or m has at most two prime factors then F is invertible.

The last theorem of this kind is found by Moh in [74].

Theorem 1.38

Let F be a Keller map. If $m \leq 100$ and $n \leq 100$ then F is invertible.

Of course theorem 1.38 doesn't imply that the Jacobian Conjecture holds in general in dimension two. But it does imply that if one wants to find a two-dimensional counterexample, the degree must be over 100. A consequence is that it is almost impossible to write down a general candidate counterexample in dimension two and try to determine all relations between the general coefficients of the monomials. Let N be the number of monomials, which equals the number of coefficients in F . Then

$$N = 2 \sum_{i=1}^d i + 1 = (d + 1)^2 + d - 1$$

where $d = \deg(F)$. If $d \geq 101$ then $N \geq 10504$. This number is too high for a current computer algebra system like Maple release 5. In fact if we combine theorem 1.36 and theorem 1.38 we see that we can even take $d \geq 102$ which implies $N \geq 10710$.

A totally different approach was taken by Gwoździewicz more recently in 1993 (cf. [48]). He proves:

Theorem 1.39

If there exists one line $\ell \subset \mathbb{C}^2$ such that F is injective on this line ℓ , then F is invertible.

1.3.2 Higher dimensions

If $n \geq 3$ the situation becomes much more complex. Let $F = (F_1, \dots, F_n)$. In 1980 Wang came up with a theorem with bounds for the degrees (see [86]):

Theorem 1.40

Let F be a Keller map such that $\deg(F_i) \leq 2$ for all i . Then F is invertible.

It may seem that this theorem is not very strong: the condition $\deg(F_i) \leq 2$ seems too strong to get many good results. However the next theorem which was found by Bass, Connell and Wright and independently by Yagzhev (see [8]³ and [90]) puts this low bound for the degree in a different light. The important theorem is:

Theorem 1.41

If the Jacobian Conjecture holds for all $n \geq 2$ and all polynomial maps F with $\deg(F_i) \leq 3$ for all i , then the Jacobian Conjecture holds in general.

So now Wang's theorem implies that one 'only' has to prove the case of $\deg(F_i) = 3$. In fact we can even restrict to the case $\deg(F_i) = 3$ because in [8] theorem 1.41 is sharpened to

Theorem 1.42

If the Jacobian Conjecture holds for all $n \geq 2$ and all polynomial maps F of the form $F = X + H$ where H is cubic homogeneous, then the Jacobian Conjecture holds.

By saying that H is homogeneous of degree d we mean that each component H_i of H is homogeneous of degree d or $H_i = 0$. Naturally for cubic homogeneous we take $d = 3$.

In [89] Wright studied these cubic homogeneous maps. He proved:

Theorem 1.43

Let $F : k^3 \rightarrow k^3$ be a cubic homogeneous map such that $\det(JF) = 1$. Then there exists $T \in \mathfrak{A}_3(k)$ such that $T^{-1} \circ F \circ T$ is of the form

$$\begin{pmatrix} x_1 \\ x_2 - \frac{1}{3}x_1^3 \\ x_3 - x_1^2x_2 - ax_1x_2^2 - bx_2^3 \end{pmatrix} \tag{1.5}$$

for some $a, b \in k$.

³[8] is probably the reference most often used for papers on the subject of the Jacobian Conjecture.

Corollary 1.44

If $F \in k[x_1, x_2, x_3]^3$ is a cubic homogeneous Keller map then F is linearly triangularisable. In particular F is invertible. And hence the Jacobian Conjecture holds for this class of polynomial maps.

Inspired by this paper the author classified all cubic homogeneous Keller maps in dimension four. See [51] or theorem 3.2 on page 36. In dimension four the situation is less nice. There are several forms which are not linearly triangularisable. However all forms satisfied the Jacobian Conjecture. Hence one of the results of [51] was:

Corollary 1.45

The Jacobian Conjecture holds for all maps $F = X + H : k^n \rightarrow k^n$ where H is cubic homogeneous if $n \leq 4$.

In 1983 Drużkowski came up with a different reduction. In [20] he proved that it suffices to prove the Jacobian Conjecture for a special class of cubic homogeneous maps: the *cubic-linear maps*.

Theorem 1.46

If the Jacobian Conjecture holds for all polynomial maps of the form

$$F = X + \begin{pmatrix} \ell_1^3 \\ \vdots \\ \ell_n^3 \end{pmatrix}$$

where ℓ_i is a linear form in x_1, \dots, x_n , then the Jacobian Conjecture holds in general.

These linear forms are often described by matrices. Therefore we define a coefficient wise cube of a vector.

Definition 1.47

Given a vector $v \in k^n$, $v^{*3} = (v_1^3, \dots, v_n^3)$ or more general $v^{*r} = (v_1^r, \dots, v_n^r)$.

Obviously this means that an F as in theorem 1.46 can be written as $X + (AX)^{*3}$ for some $A \in \text{Mat}_n(k)$. We shall get back to this in chapter 6. More recently in 1993 Drużkowski proved in [22] the theorem:

Theorem 1.48

Let $k = \mathbb{C}$ or \mathbb{R} . If $F = X + (AX)^{*3} : k^n \rightarrow k^n$ is a polynomial map with $\det(JF) = 1$ such that $\text{rank}(A) \leq 2$ or $\text{corank}(A) \leq 2$, then F is a tame automorphism.

In chapter 8 we refine this theorem to theorem 8.3. We use this improved theorem to reduce the number of variables in a general cubic-linear map so that we are able to classify the cubic-linear maps in dimension five. Furthermore note that $A \in \text{Mat}_5(k)$ implies that either $\text{rank}(A) \leq 2$ or $\text{corank}(A) \leq 2$. Hence we have:

Corollary 1.49

The Jacobian Conjecture holds for cubic-linear Keller maps up to dimension five.

This corollary was improved by the author. Combining corollary 1.45, theorem 1.48 and theorem 1.50 below, he proved corollary 1.51.

Theorem 1.50

*Let $r \in \mathbb{N}$. If the Jacobian Conjecture holds for all $F = X + H : k^r \rightarrow k^r$ where H is cubic homogeneous, then for all $n \geq r$ and all $A \in \text{Mat}_n(k)$ such that $\text{rank}(A) = r$ the Jacobian Conjecture holds for the cubic-linear forms $G = X + (AX)^{*3}$.*

The proof is in [51]. Because this paper is not widely available the important results of this paper are recalled in [27].

Now combining as described above we get:

Corollary 1.51

The Jacobian Conjecture holds for cubic-linear Keller maps up to dimension seven.

This corollary holds because $A \in \text{Mat}_7(k)$ implies $\text{rank}(A) \leq 4$ or $\text{corank}(A) \leq 2$.

1.4 New contributions

In the previous two sections we have described what was already known at the time the author started working on this thesis. In this section the new results are explained.

Chapter 2 deals with the so-called Markus-Yamabe Conjecture (see conjecture 2.1). This conjecture has a strong bond with the Jacobian Conjecture. In fact it is also known under the name Global Asymptotic Stability Jacobian Conjecture. It was formulated by Markus and Yamabe in 1960 in an attempt to globalise the well-known stability result by Lyapunov. For certain cases they also provided the proof. In 1993 the two-dimensional case was proved by Feßler and independently by Gutierrez. The general case $n > 2$ remained unsolved. The Markus-Yamabe Conjecture is important for the research on dynamical systems. Furthermore it has a high impact on the Jacobian Conjecture: Martelli and Fournier proved that if the Markus-Yamabe Conjecture is true in the polynomial case, then the Jacobian Conjecture is true! However in 1995 the missing piece was found by collaboration of Cima, van den Essen, Gasull, Mañosas and the author. They found a simple polynomial counterexample for $n \geq 3$ to the Markus-Yamabe Conjecture. It is printed on the cover of this thesis. See also theorem 2.25.

In chapter 3 we answer the question how this counterexample to the Markus-Yamabe Conjecture was finally found after 35 years. In this chapter we present a new class of automorphisms H with nilpotent Jacobian: $\mathcal{H}_n(A)$. We show among other things that each $X + H$ with $H \in \mathcal{H}_n(A)$ is invertible and hence that the Jacobian Conjecture holds for all maps of this form $X + H$. See corollary 3.24. The power of this new class is that it defines a large class of invertible polynomial maps -all $F = X + H$ with $H \in \mathcal{H}_n(A)$ - ready to be used as a test case for problems concerning invertible polynomial maps. This chapter is based on [35] by van den Essen and the author.

In chapter 4 we introduce a syntactical description of the maps in $\mathcal{H}_n(A)$ by means of tuples of matrices and vectors. We call these tuples elements of $\mathcal{D}_n(A)$. We prove that there exists an ‘if and only if’ relation between tuples in $\mathcal{D}_n(A)$ and maps in

$\mathcal{H}_n(A)$. See theorem 4.5. The notion of $\mathcal{D}_n(A)$ makes it a lot easier to do some computations on maps in $\mathcal{H}_n(A)$. This chapter is based on [37] by van den Essen and the author.

In chapter 5 we use this $\mathcal{D}_n(A)$ -structure to connect the maps $F = X + H$ with $H \in \mathcal{H}_n(A)$ to locally nilpotent derivations. And we prove that such a map F can be written as a finite product of $\exp(D_i)$'s where each D_i is a locally nilpotent derivation. See theorem 5.12. Using this theorem in conjunction with a result by Smith, it is shown that each map $F = X + H$ with $H \in \mathcal{H}_n(A)$ is stably tame. See theorem 5.17. Furthermore we provide two methods to really compute the factorisations into tame automorphisms. This was a result of a collaboration with Wright (cf. [54]).

Chapter 6 is different. In a sense it can be seen as a turning point in this thesis. Chapter 3, chapter 4 and chapter 5 really belong to each other. Together they provide facts on a new class of polynomial maps. And chapter 2 can be seen as an application of this theory. Chapter 7 and chapter 8 both are chapters based on computations done with Maple. Chapter 6 uses a bit of both worlds. On one hand there is some *unplugged* mathematics about nilpotence, strong nilpotence, D-nilpotence, linearisations, Drużkowski matrices and a pairing between cubic homogeneous and cubic-linear maps. On the other hand there is also *plugged* mathematics, i.e. there are computations on Pinchuk's counterexample to the Real Jacobian Conjecture and on D-nilpotent matrices. The results presented in this section are mainly due to Berson, van den Essen, Gorni, Ivanenko, Meisters, Pinchuk, Tutaj, Zampieri and the author.

As stated before chapter 7 is a computational chapter. It provides a classification of quadratic homogeneous maps in dimension five under the assumption that a certain dependence problem holds. See problem 7.1 and theorem 7.10. Furthermore it is shown that these forms can all be reduced to two forms: one on triangular form and one which is not linearly triangularisable. See theorem 7.11.

Chapter 8 is also a computational chapter. In this chapter we give a complete classification of Drużkowski maps. See theorem 8.12. And we use this classification to complete the list of 5×5 matrices which are not cubic similar to each other. This list was initiated by Meisters. Compare his matrices on page 161 with the new ones on page 171.

Appendix A presents some details needed to provide a method of verification of some proofs for the chapters 7 and 8. These details cannot be omitted. However placing them in the running text would not increase the readability of it all. Therefore these details are put in an appendix.

Appendix B can be used as a short manual for the so-called Jacobian package: a collection of Maple procedures developed at Nijmegen. Without this package the computational research for this thesis would have been a lot harder.

Now for any reader who is still with us, mathematically spoken: enjoy reading the rest of this thesis!

Chapter 2

The Markus-Yamabe Conjecture unravelled

Introduction

One of the fields where the results of the research on polynomial maps can be applied is the field of dynamical systems. The first name that comes into mind when thinking about this field is the name Lyapunov. It was his thesis that started the work on stability of autonomous systems of differential equations. Since then several mathematicians have attempted to globalise Lyapunov's local results. One of these attempts was made in 1960 by Markus and Yamabe. They wrote the paper 'Global stability criteria for differential systems'. See [65]. This paper has become world famous because of the conjecture presented by the authors:

Conjecture 2.1 (Markus-Yamabe Conjecture, MYC(n))

Let F be a C^1 map from \mathbb{R}^n to \mathbb{R}^n such that for any $x \in \mathbb{R}^n$ all eigenvalues of $JF(x)$ have negative real part. If $F(p) = 0$, then p is a global attractor of the autonomous system $\dot{x} = F(x)$.

And in particular if $p = 0$, i.e. if $F(0) = 0$ then 0 is a global attractor of the differential system.

The link between this conjecture and the theory concerning polynomial automorphisms is given by the fact that if MYC(n) is true for polynomial vector fields, the Jacobian Conjecture would be true. Furthermore this conjecture is also known as the *Global Asymptotic Stability Jacobian Conjecture*.

The answer to MYC(1) is affirmative. Many mathematicians have studied this conjecture for $n \geq 2$. Our contribution is a polynomial counterexample for all $n \geq 3$. The next section gives a short review on the history of this Markus-Yamabe Conjecture. It is based on [32] and [39]. The third section shows the polynomial counterexample found by Cima, van den Essen, Gasull, Mañosas and the author. We also show some homogeneous counterexamples found by van den Essen.

2.1 A short history

2.1.1 Local and global attractors

Let $F : \mathbb{R}^n \rightarrow \mathbb{R}^n$ be a C^1 -vector field with $F(0) = 0$ and consider the system of ordinary differential equations

$$\begin{cases} \dot{x}_1(t) = F_1(x_1(t), \dots, x_n(t)) \\ \vdots \\ \dot{x}_n(t) = F_n(x_1(t), \dots, x_n(t)) \end{cases} \quad (2.1)$$

or abbreviated $\dot{x} = F(x(t))$. Obviously $x(t) = 0$ is a solution of (2.1). The origin is called locally asymptotic stable if for every $\varepsilon > 0$ there exists $\delta > 0$ such that if $\|x(0)\| < \delta$, then $\|x(t)\| < \varepsilon$ for all $t > 0$ and $\lim_{t \rightarrow \infty} x(t) = 0$. The set of $x_0 \in \mathbb{R}^n$ for which the solution satisfying $x(0) = x_0$ tends to the origin if t tends to infinity is called the basin of attraction. If this basin is the complete \mathbb{R}^n then the origin is a global attractor or globally stable.

A weak version of Lyapunov's theorem is

Theorem 2.2 (Lyapunov)

Let $F : \mathbb{R}^n \rightarrow \mathbb{R}^n$ be a C^1 -vector field such that $F(0) = 0$. If the real parts of all eigenvalues of $JF(0)$ are negative, then 0 is a local attractor of (2.1).

If we look at the formulation of theorem 2.2 and of conjecture 2.1 we see the similarities.

If we go back to 1949 Aizerman formulated a problem in [3] which has lead by Kalman's problem in 1957 in [58] to the Markus-Yamabe Conjecture. He considered the system

$$\begin{cases} \dot{x}(t) = Ax(t) + b\xi \\ \dot{\xi} = \varphi(\sigma) \\ \dot{\sigma} = c^*x(t) \end{cases} \quad (2.2)$$

where $A \in \text{Mat}_n(\mathbb{R})$, $b, c \in \text{Mat}_{n,1}(\mathbb{R})$, $x(t)$ an absolutely continuous n -dimensional vector function and $\varphi(\sigma)$ a continuous function is. Furthermore let (α, β) be any interval such that for all $\mu \in (\alpha, \beta)$ the system (2.2) with $\varphi(\sigma) = \mu\sigma$ is asymptotically stable. Now the Aizerman problem is:

Problem 2.3

Is the system (2.2) globally asymptotic stable whenever $\frac{\varphi(\sigma)}{\sigma} \in (\alpha, \beta)$ for all $\sigma \neq 0$?

In [80] Pliss gave a negative answer. He constructed an example of a system like (2.2) possessing a nontrivial periodic solution, with nonlinear φ .

In 1957 Kalman formulated a special case of this problem:

Problem 2.4

Is the system (2.2) globally asymptotically stable whenever the function φ is differentiable and $\varphi'(\sigma) \in (\alpha, \beta)$ for all σ ?

This problem has practical significance for physical reasons. The system describes the dynamics of an automated control system with one nonlinear block. For more details see for instance [6] and [43]. Kalman's problem is a special case of the Markus-Yamabe Conjecture.

Now back to the Markus-Yamabe Conjecture. A conjecture related -but not equivalent- to the MYC(n) is the conjecture known as the weak Markus-Yamabe Conjecture:

Conjecture 2.5

If $F : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is a C^1 -vector field such that for all $x \in \mathbb{R}^n$ the real parts of all eigenvalues of $JF(x)$ are negative then F is injective.

In 1963 it was Olech who showed that MYC(n) implies the weak MYC(n). In fact he even proved that MYC(2) is equivalent to the weak MYC(2). Cf. [77].

Theorem 2.6 (Olech)

Let $F : \mathbb{R}^n \rightarrow \mathbb{R}^n$ be a C^1 -vector field with $F(0) = 0$ and for all $x \in \mathbb{R}^n$ the real parts of all eigenvalues of $JF(x)$ are negative. Now if there exists constants $\varepsilon > 0$ and $r > 0$ such that $\|F(x)\| \geq r$ if $\|x\| \geq \varepsilon$, then 0 is a global attractor of (2.1).

Corollary 2.7

Weak MYC(2) implies MYC(2).

Proof. Since the real parts of all eigenvalues of $JF(x) < 0$ for all $x \in \mathbb{R}^2$ we have that $\det(JF(x)) > 0$. Hence by the local inverse function theorem it follows that F is a local homeomorphism. Hence there exists $\varepsilon > 0$ such that the open ball $\|x\| < \varepsilon$ is mapped bijectively by F to an open neighbourhood of $F(0) = 0 \in \mathbb{R}^2$. This open neighbourhood contains again an open ball $\|y\| < r$. Since F is injective it is not possible that there exists a point p outside the ball $\|x\| < \varepsilon$ such that $F(p)$ is in the ball $\|y\| < r$. Obviously this means: if $\|x\| \geq \varepsilon$ then $\|F(x)\| \geq r$. And hence we can apply theorem 2.6. □

As a result of this Olech proved that MYC(2) is true for all maps satisfying for all $x \in \mathbb{R}^2$:

$$\frac{\partial}{\partial x_1} F_1 \cdot \frac{\partial}{\partial x_2} F_2 \neq 0 \quad \text{or} \quad \frac{\partial}{\partial x_2} F_1 \cdot \frac{\partial}{\partial x_1} F_2 \neq 0$$

This was a generalisation of the results found by Hartman and found by Markus and Yamabe for $n = 2$. Hartman proved that 0 is a global attractor if we have the stronger assumption that for all $x \in \mathbb{R}^n$ the eigenvalues of the symmetric part of $JF(x)$ are all negative. In its turn this was a generalisation of a result by Krasowski (cf. [49], [60]). Markus and Yamabe showed in [65] that MYC(2) is true in the case that one of the four partial derivatives in $JF(x)$ vanishes for all $x \in \mathbb{R}^2$ or in other words: one of the two components of F depends only on one variable. For instance this is the case if (2.1) is a second order autonomous ODE.

Together Hartman and Olech found that by replacing the condition that all eigenvalues of $JF(x)$ are negative for all $x \in \mathbb{R}^n$ by the new condition

$$\max\{\lambda_j(x) + \lambda_k(x) \mid j < k\} < 0 \quad \forall x$$

where $\lambda_i(x)$ are the eigenvalues of the symmetric part of $JF(x)$, they could apply the method of [77] also in higher dimensions (cf. [50]). This new condition was needed because the original method was based on an essentially two-dimensional argument.

2.1.2 Relations between the Markus-Yamabe Conjecture and the Jacobian Conjecture

In 1982 Meisters brought new attention to the MYC(n) by discussing some of the relations between MYC(n) and the Jacobian Conjecture. See [66]. From the famous paper [8] by Bass, Connell and Wright, it is known that it suffices to prove the Jacobian Conjecture for maps of the form

$$F(x) = -x + H(x) \tag{2.3}$$

where H is a cubic homogeneous map. Furthermore $\det(JF(x)) \in \mathbb{R} \setminus \{0\}$ implies that $JF(x)$ is nilpotent for all x . In an unpublished manuscript Fournier and Martelli noticed that this condition implies that the eigenvalues of $JF(x) = -I_n + JH(x)$ are all equal to -1 . And hence a map F of this form satisfies the conditions in the MYC(n). Now assuming that the (weak) MYC(n) is true gives that such an F is injective. Viewing the complex polynomial map $F : \mathbb{C}^n \rightarrow \mathbb{C}^n$ as a real map $F : \mathbb{R}^{2n} \rightarrow \mathbb{R}^{2n}$ in the obvious way now gives the important relations

Corollary 2.8

If MYC(n) is true for all polynomial maps of the form (2.3), then the Jacobian Conjecture is true.

and

Corollary 2.9

If the weak MYC(n) is true for all $n \geq 2$ and all polynomial maps of the form (2.3) with JH nilpotent and H cubic homogeneous, then the Jacobian Conjecture is true.

Finally in 1988 Meisters and Olech proved MYC(2) for polynomial vector fields (cf. [72]). They showed that if F is a polynomial and

$$k := \max_{\mathcal{Y} \in \mathbb{R}^2} (\#F^{-1}(\mathcal{Y})) < \infty$$

then there exists $a \in \mathbb{R}^2$ such that $k = \#F^{-1}(a)$. Hence the map $x \rightarrow F(x) - a$ is bounded away from 0 in a neighbourhood of infinity and theorem 2.6 shows that every rest point $x = a$ with $x \in F^{-1}(a)$ is a global attractor. Now this means that there exists only one rest point and $k = 1$. And this implies F is injective.

As was already stated in the introduction the general MYC(2) has been solved in 1993. Independently by Gutierrez ([47]) and Feßler ([38] and [40]). Actually both solutions are solutions to the weak MYC(2). However corollary 2.7 shows that this also proves the ordinary MYC(2).

In 1988 Barabanov published the paper [6] containing ideas to construct counterexamples in the C^1 -class for $n \geq 4$ to the Kalman conjecture, which is a special case of

the MYC(n), and the MYC(n) itself. In 1994 such a counterexample was constructed by Bernat and Llibre. Their example was even an analytic one.

In 1994 Glutsuk also published a proof of the complete MYC(2). See for this proof [44].

However MYC(3) and MYC(n) for $n \geq 4$ in the polynomial case, remained completely open.

2.1.3 The LaSalle Problem

Strongly connected to the MYC(n) was the so-called Discrete Markus-Yamabe Problem as it was brought to our attention by Cima, Gasull and Mañosas in 1995 (cf. [16]). However further research learned that this problem had already been formulated by LaSalle in 1976. See [84].

Problem 2.10 (LaSalle Problem, DMYP(n))

Let $F : \mathbb{R}^n \rightarrow \mathbb{R}^n$ be a C^1 -vector field with $F(0) = 0$ and for all $x \in \mathbb{R}^n$ all eigenvalues of $JF(x)$ have absolute value less than one. Is it true that 0 is a global attractor of F ?

Or in other words: if $k \rightarrow \infty$ does $F^k(x) \rightarrow 0$? The answer to the DMYP(1) is obviously affirmative. The answer to DMYP(2) is negative in general: Szlenk has produced an example of a rational map $F : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ for which DMYP(2) is false (see [16, Theorem D]). Unfortunately he died in 1995. Szlenk worked with Cima, Gasull and Mañosas in Barcelona. After his rational counterexample they restricted their research to polynomial vector fields. In [16] they show that DMYP(2) has a positive answer in this polynomial case. Their proof is based on the useful notice that if $n = 2$ any map F , such that all eigenvalues of $JF(x)$ have absolute value less than one for all x , has a unique fixed point. For higher dimensions it is not known whether this is true or not. This observation has lead Cima, Gasull and Mañosas to formulate the Fixed Point Conjecture:

Conjecture 2.11 (FPC(n))

Let $F : \mathbb{R}^n \rightarrow \mathbb{R}^n$ be a polynomial map such that $JF(x)$ has all its eigenvalues with modulus less than one at each $x \in \mathbb{R}^n$. Then F has a unique fixed point.

Using standard linear algebra and considering the real and the imaginary part of the components of F it can be verified that the complex version of FPC(n) (where \mathbb{R} is replaced by \mathbb{C}) is equivalent with the real version. An important result obtained by Cima, Gasull and Mañosas is the relation between FPC(n) and JC(n):

Theorem 2.12

The Jacobian Conjecture is equivalent to the Fixed Point Conjecture.

For the proof we refer to their paper [16].

Remark 2.13

Note that this theorem is not formulated as: JC(n) is equivalent with FPC(n). Otherwise the fact that FPC(2) is true would have implied a proof for the two-dimensional Jacobian Conjecture. By looking at the proof given in [16] one sees why this is not the case.

2.1.4 Finding the missing pieces

So far what we have seen is that for $n = 2$ both the Markus-Yamabe Conjecture and the LaSalle Problem are true for polynomial vector fields. The end of 1995 showed that this is not the case for larger dimensions.

November 1995, Cima came to visit our department in Nijmegen. This was only very recently after van den Essen and the author had found, inspired by a paper of Deng, a class of polynomial counterexamples to the LaSalle Problem for $n \geq 4$. Since Cima and her colleagues from Barcelona were preparing the manuscript [16], this class of examples came right on time. So far they had only been looking at examples with a periodic orbit, like the two-dimensional example by Szlenk. Our example stems from a different class. It doesn't have periodic orbit. It has a solution which tends to infinity if t tends to infinity. And for the Discrete Markus-Yamabe Problem to have an affirmative answer it should have gone to 0.

During Cima's visit to Nijmegen, we tried to prove that we could use the counterexample to the LaSalle Problem or the Discrete Markus-Yamabe Problem to construct a counterexample to the real Markus-Yamabe Conjecture also. Numeric computations gave us the impression that we had to be right. So we have to find m such that the following system has a solution which tends to infinity if t tends to infinity.

$$\begin{aligned}
 & \begin{cases} x_1'(t) = -x_1(t) + x_4(t)d^2(t) \\ x_2'(t) = -x_2(t) - x_3(t)d^2(t) \\ x_3'(t) = -x_3(t) + x_4^m(t) \\ x_4'(t) = -x_4(t) \end{cases} \quad (2.4) \\
 \Rightarrow & \begin{cases} x_1'(t) = -x_1(t) + x_4(t)d^2(t) \\ x_2'(t) = -x_2(t) - x_3(t)d^2(t) \\ x_3'(t) = -x_3(t) + x_4^m(t) \\ x_4(t) = c_4 e^{-t} \end{cases} \\
 \Rightarrow & \begin{cases} x_1'(t) = -x_1(t) + x_4(t)d^2(t) \\ x_2'(t) = -x_2(t) - x_3(t)d^2(t) \\ x_3(t) = \frac{1}{1-m}c_4^m e^{-mt} + c_3 e^{-t} \end{cases} \\
 \Rightarrow & \begin{cases} x_1'(t) = -x_1(t) + x_4(t)d^2(t) \\ d(t) = -(x_3(t)x_1'(t) + x_4(t)x_2'(t)) \\ x_3(t) = \frac{1}{1-m}c_4^m e^{-mt} + c_3 e^{-t} \end{cases} \\
 \Rightarrow & \begin{cases} x_1'(t) = -x_1(t) + x_4(t)d^2(t) \\ d(t) = -(x_3(t)(-x_1(t) + x_4(t)d^2(t)) + x_4(t)(-x_2(t) - x_3(t)d^2(t))) \\ x_3(t) = \frac{1}{1-m}c_4^m e^{-mt} + c_3 e^{-t} \end{cases} \\
 \Rightarrow & \begin{cases} x_1'(t) = -x_1(t) + c_4 e^{-t} d^2(t) \\ d'(t) = -2d(t) + x_4^m(t)x_1(t) \\ x_3(t) = \frac{1}{1-m}c_4^m e^{-mt} + c_3 e^{-t} \end{cases} \\
 \Rightarrow & \begin{cases} x_1'(t) = -x_1(t) + c_4 e^{-t} d^2(t) \\ x_1(t) = c_4^{-m} e^{mt} (d'(t) + 2d(t)) \\ x_3(t) = \frac{1}{1-m}c_4^m e^{-mt} + c_3 e^{-t} \end{cases}
 \end{aligned}$$

Obviously $c_4 \neq 0$. Otherwise any solution would tend to 0 for sure. Substituting the second equation in the first equation gives that $d(t)$ must satisfy

$$\begin{aligned}
& mc_4^{-m}e^{mt}(d'(t) + 2d(t)) + c_4^{-m}e^{mt}(d''(t) + 2d'(t)) = \\
& -c_4^{-m}e^{mt}(d'(t) + 2d(t)) + c_4e^{-t}d^2(t) \\
\Rightarrow & c_4^{-m}e^{mt}(d''(t) + (m+2)d'(t) + 2md(t)) = \\
& -c_4^{-m}e^{mt}(d'(t) + 2d(t)) + c_4e^{-t}d^2(t) \\
\Rightarrow & c_4^{-m}e^{mt}(d''(t) + (m+3)d'(t) + (2m+2)d(t)) = c_4e^{-t}d^2(t) \\
\Rightarrow & d''(t) + (m+3)d'(t) + (2m+2)d(t) = c_4^{m+1}e^{-(m+1)t}d^2(t) \quad (2.5)
\end{aligned}$$

Since equation (2.5) is of a type which normally doesn't permit finding an exact solution we stopped at this point. Due to the fact that Cima, Gasull and Mañosas know much more about dynamical systems, they would look into the original four-dimensional system. Within two weeks we received e-mail that by simply trying a solution of the form $a_ie^{b_it}$ for $m = 3$, they had found a solution which tends to infinity if t tends to infinity. And hence a polynomial counterexample to the MYC(n) was found for $n \geq 4$. Within a week this four-dimensional example was modified to a three-dimensional counterexample.

Furthermore like we found the continuous counterexample in dimension four by starting with the discrete counterexample in dimension four, we were able to find a discrete counterexample in dimension three by starting with the three-dimensional continuous counterexample.

So at the end of 1995 both the Markus-Yamabe Conjecture and the LaSalle Problem were completely solved. A summary of these results is shown in table 2.1.

	$n = 1$	$n = 2$	$n \geq 3$
LaSalle Problem (polynomial)	true ^a	true ^b	false ^c
LaSalle Problem (C^1)	true ^a	false ^d	false ^b
Markus-Yamabe Conjecture (polynomial)	true ^e	true ^f	false ^c
Markus-Yamabe Conjecture (C^1)	true ^e	true ^g	false ^c

^a1976: LaSalle

^b1995: Cima, Gasull and Mañosas

^c1995: Cima, van den Essen, Gasull, Hubbers and Mañosas

^d1995: Szlenk

^e1960: Markus and Yamabe

^f1988: Meisters and Olech

^g1993: Feßler, Gutierrez and 1994: Glutsuk

Table 2.1: Solutions to MYC and DMYP

For more information about the history of the Markus-Yamabe Conjecture see [71].

2.2 The counterexamples

In this section we present the actual counterexamples mentioned in the previous section.

2.2.1 C^1 MYC(4)

In 1994 Bernat and Llibre were able to construct an analytic counterexample to MYC(4). They were inspired by Barabanov's paper [6] in which some ideas were presented to build such counterexamples. Bernat and Llibre constructed a system that has a non-constant periodic solution. See [9].

Example 2.14

Consider the system:

$$\begin{cases} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -x_4 \\ \dot{x}_3 &= x_1 - 2x_4 - \frac{9131}{900}\psi(x_4) \\ \dot{x}_4 &= x_1 + x_3 - x_4 - \frac{1837}{180}\psi(x_4) \end{cases} \quad (2.6)$$

There exists a C^1 function ψ such that system (2.6) is a system with a solution with a non-constant periodic orbit. In fact ψ can be chosen from C^r for any $r \geq 1$, from C^∞ or even analytic.

2.2.2 C^1 DMYP(n) for $n \geq 2$

While working in Barcelona with Cima, Gasull and Mañosas, Szlenk came up with the following example.

Example 2.15

Let $F : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ be defined as

$$F(x, y) = \left(-\frac{ky^3}{1+x^2+y^2}, \frac{kx^3}{1+x^2+y^2} \right)$$

where $k \in (1, \frac{2}{3}\sqrt{3})$. This map F satisfies the following properties:

1. Set $p = (x, y) \in \mathbb{R}^2$ and let λ be an eigenvalue of $JF(p)$. If $xy = 0$ then $\lambda = 0$. Otherwise $\lambda \notin \mathbb{R}$ and $|\lambda| < \frac{\sqrt{3}k}{2}$.
2. $F^4((\frac{1}{\sqrt{k-1}}, 0)) = (\frac{1}{\sqrt{k-1}}, 0)$.
3. F is injective.

It verifies the hypothesis of the LaSalle Problem, but obviously 0 is not a global attractor. Hence it is a counterexample in dimension two.

Cima, Gasull and Mañosas modified this example to get a counterexample which is a diffeomorphism. Furthermore it was enlarged to arbitrary dimensions in a natural way.

Example 2.16

Let $G_a : \mathbb{R}^n \rightarrow \mathbb{R}^n$ be defined as

$$G_a(x_1, \dots, x_n) = \left(-\frac{kx_2^3}{1 + x_1^2 + x_2^2} - ax_1, \frac{kx_1^3}{1 + x_1^2 + x_2^2} - ax_2, cx_3, \dots, cx_n \right)$$

where $|c| < 1$ and $k \in (1, \frac{2}{3}\sqrt{3})$. Then if a is small enough, the map G_a is a global diffeomorphism from \mathbb{R}^n into itself which satisfies the following properties:

1. For all $x \in \mathbb{R}^n$ and for all λ eigenvalue of $(DG_a)(x)$, $|\lambda| < 1$.
2. $G_a(0) = 0$ and for $p = (\frac{1}{\sqrt{k-1}}, 0, \dots, 0)$ one has $G_a^4(p) = p$.

For the proofs of the statements in these two examples, we refer to [16].

2.2.3 Polynomial DMYP(n) for $n \geq 4$

November 1995, van den Essen and the author came up with a polynomial counterexample for $n \geq 4$ to the LaSalle Problem. See the paper [36]. We found this example after reading the preprint [18] by Deng. In this paper he presents a theorem which was very useful in proving our claim of having a counterexample. Later on it turned out that this theorem had already been published by Rosay and Rudin in [81].

Consider the polynomial ring $\mathbb{R}[x] := \mathbb{R}[x_1, \dots, x_n]$.

Example 2.17

Define $d(x) := x_3x_1 + x_4x_2 \in \mathbb{R}[x]$. Let F be the polynomial automorphism

$$F = (x_1 + x_4d(x)^2, x_2 - x_3d(x)^2, x_3 + x_4^m, x_4, \dots, x_n)$$

Theorem 2.18

If F is as in example 2.17, then for each $0 < \lambda < 1$, λF is a counterexample to the Discrete Markus-Yamabe Problem alias the LaSalle Problem. More precisely if $0 < \lambda < 1$ and $a \in \mathbb{R}$ such that $a\lambda > 1$, then the first component of $(\lambda F)^k(a, a, \dots, a)$ tends to infinity if k tends to infinity.

Definition 2.19

For each $\lambda > 0$ and $a > 0$ put $(\lambda F)^k(a) := (\lambda F)^k(a, a, \dots, a)$ and denote the first component of this vector by $f_k(\lambda, a)$. So

$$f_k(\lambda, a) := ((\lambda F)^k(a))_1,$$

for all $k \geq 1$. Furthermore put

$$d_k(\lambda, a) := d((\lambda F)^k(a)),$$

for all $k \geq 1$.

Lemma 2.20

1. $d(\lambda F(x)) = \lambda^2[x_4^{m+1}d(x)^2 + d(x) + x_4^m x_1]$
2. $d_{k+1}(\lambda, a) \geq \lambda^2(\lambda^k a)^{m+1}(d_k(\lambda, a))^2$, for all $k \geq 1$.
3. $f_{k+1}(\lambda, a) \geq \lambda^{k+1}a(d_k(\lambda, a))^2$, for all $k \geq 1$.

Proof. the first claim is easy to verify. Consequently, since all monomials in $d(\lambda F(x))$ have positive coefficients, we get

$$\begin{aligned} d_{k+1}(\lambda, a) &= d((\lambda F)(\lambda F)^k(a)) \\ &\geq \lambda^2((\lambda F)^k(a))_4^{m+1}d((\lambda F)^k(a))^2 \\ &= \lambda^2(\lambda^k a)^{m+1}(d_k(\lambda, a))^2 \end{aligned}$$

since the fourth component of $(\lambda F)^k(a)$ equals $\lambda^k a$. This proves the second claim. Finally

$$\begin{aligned} f_{k+1}(\lambda, a) &= (\lambda F)_1((\lambda F)^k(a)) \\ &\geq \lambda((\lambda F)^k(a))_4 d((\lambda F)^k(a)) \end{aligned}$$

(using that $(\lambda F)_1 = \lambda x_4 d(x)^2 + \lambda x_1$). So $f_{k+1}(\lambda, a) \geq \lambda^{k+1}a(d_k(\lambda, a))^2$, which proves the last claim. \square

Lemma 2.21

We have:

$$\begin{aligned} f_k(\lambda, a) &\geq \lambda^{p_k} a^{p_k + (2m+1)(k-1) + 4} \\ d_k(\lambda, a) &\geq \lambda^{p_k + m(k-1) + 1} a^{p_k + (2m+1)(k-1) + m + 4} \end{aligned}$$

for all $k \geq 1$, where $p_1 = 1$ and $p_{k+1} = 2p_k + (2m+1)(k-1) + 4$ for all $k \geq 1$.

Proof. Use induction on k . Details are left to the reader. \square

Proof of theorem 2.18. From the estimation of $f_k(\lambda, a)$ in lemma 2.21 it follows immediately that $\lim_{k \rightarrow \infty} f_k(\lambda, a) = \infty$ if $\lambda a > 1$. Furthermore one easily verifies that $\lambda F = \lambda X + H$ with JH nilpotent. So for all $x \in \mathbb{R}^n$ the eigenvalues of $JF(x)$ are equal to λ . \square

Corollary 2.22

Let $m = 5$ and $0 < \lambda < 1$. Put $\tilde{F} := \lambda F \lambda^{-1}$. Then $\tilde{F} = X + H$ with H homogeneous of degree 5 and JH is nilpotent. However 0 is not a global attractor of $\tilde{F} \circ \lambda (= \lambda F)$.

2.2.4 Polynomial MYC(n) for $n \geq 4$

This counterexample is basically the same as example 2.17. We take the specific $m = 4$ and $-X$ in this case. See [15].

Theorem 2.23

Let F be the polynomial automorphism

$$F = (-x_1 + x_4 d(x)^2, -x_2 - x_3 d(x)^2, -x_3 + x_4^4, -x_4, \dots, -x_n)$$

Then F is a counterexample to the Markus-Yamabe Conjecture.

Proof. All we have to do is show that there exists a solution of the system $\dot{x} = F(x)$ which tends to infinity if t tends to infinity. Take

$$\begin{cases} x_1(t) &= 490e^{9t} \\ x_2(t) &= \frac{700}{3}e^{6t} \\ x_3(t) &= -\frac{1}{3}e^{-4t} \\ x_4(t) &= e^{-t} \\ \vdots & \\ x_n(t) &= e^{-t} \end{cases}$$

Obviously this goes to infinity if t goes to infinity. □

2.2.5 Polynomial MYC(n) for $n \geq 3$

This example was found by changing F from theorem 2.23 a little bit. Instead of taking $d(x) = x_3x_1 + x_4x_2$ we now take $d(x) = x_1 + x_3x_2$.

Example 2.24

Let F be the polynomial automorphism

$$F(x_1, \dots, x_n) = (-x_1 + x_3 d(x)^2, -x_2 - d(x)^2, -x_3, \dots, -x_n)$$

Theorem 2.25

F in example 2.24 is a counterexample to the Markus-Yamabe Conjecture.

Proof. Again we have to find a solution which doesn't go to 0 if t tends to infinity. Take

$$\begin{cases} x_1(t) &= 18e^t \\ x_2(t) &= -12e^{2t} \\ x_3(t) &= e^{-t} \\ \vdots & \\ x_n(t) &= e^{-t} \end{cases}$$

It is easy to verify that this solution displays the desired behaviour if $t \rightarrow \infty$. □

2.2.6 Polynomial DMYP(n) for $n \geq 3$

Again let $d(x) = x_1 + x_3x_2$.

Example 2.26

Let F be the polynomial automorphism

$$F(x_1, \dots, x_n) = \left(\frac{1}{2}x_1 + x_3d(x)^2, \frac{1}{2}x_2 - d(x)^2, \frac{1}{2}x_3, \dots, \frac{1}{2}x_n \right)$$

Theorem 2.27

F in example 2.26 is a counterexample to the Discrete Markus-Yamabe Problem.

Proof. If we show that there exists an initial condition $x^{(0)}$ such that the sequence $x^{(n+1)} = F(x^{(n)})$ tends to infinity if t tends to infinity, we are done. One easily verifies that for all $x \in \mathbb{R}^n$ the eigenvalues of $JF(x)$ are equal to $\frac{1}{2}$. Now take $x^{(0)} = (\frac{147}{32}, -\frac{63}{32}, 1, 0, \dots, 0)$. By induction one verifies that this gives a sequence such that

$$x^{(n)} = \left(\frac{147}{32} \cdot 2^n, -\frac{63}{32} \cdot 2^{2n}, \left(\frac{1}{2}\right)^n, 0, \dots, 0 \right)$$

And this clearly goes to infinity if t goes to infinity. □

2.2.7 Homogeneous counterexamples to MYC(n) for $n \geq 5$

The counterexamples to the MYC given above are not homogeneous. Here we present counterexamples found by van den Essen that are quadratic homogeneous or cubic homogeneous. Or better: counterexamples of the form $F = -X + Q$ and $F = -X + H$ where Q is quadratic homogeneous and H is cubic homogeneous.

Example 2.28

Let $n \geq 5$. Let $Q := (x_2x_5, x_1^2 - x_4x_5, x_2^2, 2x_1x_2 - x_3x_5, 0, \dots, 0)$ and $F = -X + Q$. Then F is a counterexample to the Markus-Yamabe Conjecture. We can prove this by presenting the solution

$$x(t) = (30e^t, 60e^{2t}, 720e^{4t}, 720e^{3t}, e^{-t}, \dots, e^{-t})$$

of the system $\dot{x} = F(x)$. This solution clearly tends to infinity if t tends to infinity.

Example 2.29

Let $n \geq 5$. Let Q be as in example 2.28. Let $H = x_5Q$. Then H is cubic homogeneous. Now $F = -X + H$ is a counterexample to the Markus-Yamabe Conjecture. The solution

$$x(t) = (120e^{3t}, 480e^{5t}, 23040e^{9t}, 11520e^{7t}, e^{-t}, \dots, e^{-t})$$

is a solution of $\dot{x} = F(x)$ that goes to infinity if t goes to infinity.

We will explain later on how these examples were found.

Chapter 3

A new class of polynomial automorphisms

Introduction

Section 2.2 has shown some remarkable polynomial counterexamples. After seeing those simple examples naturally the big question arises: how did we come up with these examples. In this chapter we provide the answer to this question. All polynomial examples in section 2.2 are elements of a new, large class of polynomial automorphisms, denoted by $\mathcal{H}_n(A)$, which was defined by van den Essen and the author in [35]. Furthermore we describe some of the very nice properties of this class. During this chapter X denotes the sequence x_1, \dots, x_n .

3.1 A dependence problem

From [8] it is well known that it suffices to prove the Jacobian Conjecture for cubic homogeneous polynomial maps, i.e. maps of the form

$$F = X + H : \mathbb{C}^n \rightarrow \mathbb{C}^n$$

where $H = (H_1, \dots, H_n)$ and each H_i is either zero or a homogeneous polynomial map of degree three. In this case the Jacobian condition $\det(JF) \in \mathbb{C}^*$ is equivalent to JH is nilpotent. Hence understanding nilpotent Jacobian matrices is very important in the study of the Jacobian Conjecture. The study of cubic homogeneous maps $F = X + H$ where JH is nilpotent was initiated by Wright. In his paper [89] he showed (cf theorem 1.43):

Theorem 3.1

Let k be a field with $\text{char}(k) = 0$. Let $F = X + H : k^3 \rightarrow k^3$ be a cubic homogeneous polynomial map such that JH is nilpotent. Then there exists $T \in \text{Gl}_3(k)$ such that

$$T^{-1}HT = (0, h_2(x_1), h_3(x_1, x_2))$$

This article was the starting point for the author's masters thesis (cf. [51]). By a tedious computer search the author succeeded in classifying the cubic homogeneous maps in dimension four (with nilpotent Jacobian that is).

Theorem 3.2

Let k be a field with $\text{char}(k) = 0$. Let $F = X - H$ be a cubic homogeneous polynomial map in dimension four, such that $\det(JF) = 1$. Then there exists $T \in \mathfrak{A}_4(k)$ with $T^{-1} \circ F \circ T$ being one of the following forms:

1.
$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \quad -a_4x_1^3 - b_4x_1^2x_2 - c_4x_1^2x_3 - e_4x_1x_2^2 - f_4x_1x_2x_3 \\ \quad -h_4x_1x_3^2 - k_4x_2^3 - l_4x_2^2x_3 - n_4x_2x_3^2 - q_4x_3^3 \end{pmatrix}$$
2.
$$\begin{pmatrix} x_1 \\ x_2 \quad -\frac{1}{3}x_1^3 - h_2x_1x_3^2 - q_2x_3^3 \\ x_3 \\ x_4 \quad -x_1^2x_3 - h_4x_1x_3^2 - q_4x_3^3 \end{pmatrix}$$
3.
$$\begin{pmatrix} x_1 \\ x_2 \quad -\frac{1}{3}x_1^3 - c_1x_1^2x_4 + 3c_1x_1x_2x_3 - \frac{16q_4c_1^2 - r_4^2}{48c_1^2}x_1x_3^2 - \frac{1}{2}r_4x_1x_3x_4 \\ \quad + \frac{3}{4}r_4x_2x_3^2 - \frac{r_4q_4}{12c_1}x_3^3 - \frac{r_4^2}{16c_1}x_3^2x_4 \\ x_3 \\ x_4 \quad -x_1^2x_3 + \frac{r_4}{4c_1}x_1x_3^2 - 3c_1x_1x_3x_4 + 9c_1x_2x_3^2 - q_4x_3^3 - \frac{3}{4}r_4x_3^2x_4 \end{pmatrix}$$
4.
$$\begin{pmatrix} x_1 \\ x_2 \quad -\frac{1}{3}x_1^3 \\ x_3 \quad -x_1^2x_2 - e_3x_1x_2^2 - k_3x_2^3 \\ x_4 \quad -e_4x_1x_2^2 - k_4x_2^3 \end{pmatrix}$$
5.
$$\begin{pmatrix} x_1 \\ x_2 \quad -\frac{1}{3}x_1^3 + i_3x_1x_2x_4 - j_2x_1x_4^2 + s_3x_2x_4^2 + i_3^2x_3x_4^2 - t_2x_4^3 \\ x_3 \quad -x_1^2x_2 - \frac{2s_3}{i_3}x_1x_2x_4 - i_3x_1x_3x_4 - j_3x_1x_4^2 - \frac{s_3^2}{i_3^2}x_2x_4^2 \\ \quad -s_3x_3x_4^2 - t_3x_4^3 \\ x_4 \end{pmatrix}$$
6.
$$\begin{pmatrix} x_1 \\ x_2 \quad -\frac{1}{3}x_1^3 - j_2x_1x_4^2 - t_2x_4^3 \\ x_3 \quad -x_1^2x_2 - e_3x_1x_2^2 - g_3x_1x_2x_4 - j_3x_1x_4^2 - k_3x_2^3 - m_3x_2^2x_4 \\ \quad -p_3x_2x_4^2 - t_3x_4^3 \\ x_4 \end{pmatrix}$$

$$7. \begin{pmatrix} x_1 \\ x_2 - \frac{1}{3}x_1^3 \\ x_3 - x_1^2x_2 - e_3x_1x_2^2 - k_3x_2^3 \\ x_4 - x_1^2x_3 - e_4x_1x_2^2 - f_4x_1x_2x_3 - h_4x_1x_3^2 - k_4x_2^3 - l_4x_2^2x_3 \\ -n_4x_2x_3^2 - q_4x_3^3 \end{pmatrix}$$

$$8. \begin{pmatrix} x_1 \\ x_2 - \frac{1}{3}x_1^3 \\ x_3 - x_1^2x_2 - e_3x_1x_2^2 + g_4x_1x_2x_3 - k_3x_2^3 + m_4x_2^2x_3 + g_4^2x_2^2x_4 \\ x_4 - x_1^2x_3 - e_4x_1x_2^2 - \frac{2m_4}{g_4}x_1x_2x_3 - g_4x_1x_2x_4 - k_4x_2^3 \\ -\frac{m_4^2}{g_4^2}x_2^2x_3 - m_4x_2^2x_4 \end{pmatrix}$$

Wright showed in particular that if $n = 3$ and JH is nilpotent, then $F = X + H$ is linearly triangularisable. The author's classification for $n = 4$ shows that this is not the case if $n = 4$. (Form 3, 5 and 8 are not linearly triangularisable.) However it turns out that the rows of the Jacobian matrices are linearly dependent over k (or equivalently that H_1, H_2, H_3 and H_4 are linearly dependent over k). This notion has led to the formulation of the dependence problem:

Problem 3.3 (Dependence Problem, DP)

Let $d \in \mathbb{N}$, $d \geq 1$ and $H = (H_1, \dots, H_n) \in k[x_1, \dots, x_n]^n$ be homogeneous of degree d such that JH is nilpotent. Does it follow that H_1, \dots, H_n are linearly dependent over k ?

If we omit the assumption that H is homogeneous we get a more general question:

Problem 3.4 (Generalised Dependence Problem, GDP)

Let $H = (H_1, \dots, H_n) \in k[x_1, \dots, x_n]^n$ with JH nilpotent. Are the rows of JH linearly dependent?

This generalised formulation is defined in terms of dependent rows of JH instead of dependent components of H . It is not difficult to show that these two dependency formulations are equivalent if the H_i 's have a common zero. (Note that this common zero is necessarily. Put $H = (x_1, x_1 + 1)$ then H_1 and H_2 are linearly independent, however the rows of JH are exactly the same.)

A more important question is why we disregard the fact that H is homogeneous of degree d in the generalised version. This is based on the following reason. Let H be as in DP, hence $H \in k[x_1, \dots, x_n]^n$ homogeneous of degree d such that JH is nilpotent. If we assume that DP holds, this means that we can find suitable $T \in \text{Gl}_n(k)$ such that we can replace H by $T^{-1}HT$ where $H_n = 0$. Obviously this means that

$$JH = J_{x_1, \dots, x_n}H = \begin{pmatrix} J_{x_1, \dots, x_{n-1}}(H_1, \dots, H_{n-1}) & * \\ 0 & 0 \end{pmatrix}$$

The assumption JH is nilpotent implies that the submatrix $J_{x_1, \dots, x_{n-1}}(H_1, \dots, H_{n-1})$ is also nilpotent. If we regard the components H_1, \dots, H_{n-1} as polynomials in $k(x_n)[x_1, \dots, x_{n-1}]$ we get a situation that cries for an induction argument. However

the problem is that the polynomials H_1, \dots, H_{n-1} need not be homogeneous polynomials in x_1, \dots, x_{n-1} . Therefore we formulated the GDP. Skipping the demand of homogeneity has the benefit that we can use induction arguments.

This GDP has also appeared in the papers [69], [78] and [16]. In these papers it was formulated as a conjecture. That this is in fact a very important problem is made clear by the next theorem:

Theorem 3.5

The Generalised Dependence Problem implies the Jacobian Conjecture.

Proof. The proof is by induction on n . From [8] it is known that it suffices to prove the Jacobian Conjecture for maps F such that $JF = I_n + N$ where N is nilpotent. It is known that JC(1) holds. Now suppose JC(n) holds for certain n . Take $F : \mathbb{C}^{n+1} \rightarrow \mathbb{C}^{n+1}$ such that $JF = I_{n+1} + N$ with N nilpotent. GDP now tells us that there exists some $T \in \text{Gl}_{n+1}(\mathbb{C})$ such that

$$T^{-1}FT = X + \begin{pmatrix} H'_1(x_1, \dots, x_{n+1}) \\ \vdots \\ H'_n(x_1, \dots, x_{n+1}) \\ 0 \end{pmatrix}$$

Obviously $\det(T^{-1}FT) = 1$. Now define for each $a \in \mathbb{C}$ the polynomial map

$$F_a := T^{-1}FT(x_1, \dots, x_n, a)$$

Then also $\det(F_a) = 1$. So F_a is an n -dimensional map with $\det(F_a) \in \mathbb{C}^*$ and hence we can apply the induction hypothesis to claim that F_a is injective for all $a \in \mathbb{C}$. Now this implies that $T^{-1}FT$ and F are injective. Which of course means that JC($n + 1$) holds. \square

From [51] we already know that GDP has an affirmative answer for $n \leq 2$ in case k is a unique factorisation domain. Van den Essen found in 1996 a class of examples showing that the answer for $n \geq 3$ is negative (cf. [30]). In fact the GDP has been solved completely. We split this solution into two parts. In order to prove the positive part we need a lemma.

Lemma 3.6

Let $f \in k[X] \setminus k$. Let r be the dimension of the k -vectorspace generated by all partial derivatives $\partial_i f$. Then $r \geq 1$ and there exists $T \in \text{Gl}_n(k)$ such that $g := f(TX)$ only depends on x_1, \dots, x_r and $\partial_1 g, \dots, \partial_r g$ are linearly independent over $k[g]$.

Proof. Since f is not a constant, obviously $r \geq 1$. There exists $T \in \text{Gl}_n(k)$ such that $\partial_1 f(TX), \dots, \partial_r f(TX)$ are linearly independent and $\partial_j f(TX) = 0$ for $j > r$. Let $g = f(TX)$. Suppose $\sum_{i=1}^r a_i(g) \partial_i g = 0$ for some $a_i(g) \in k[g]$. Without loss of generality we may assume that $\text{gcd}(a_1(T), \dots, a_r(T)) = 1$ in $k[T]$. Hence there exists i such that $a_i(0) \neq 0$. Now write $a_i = a_i(0) - T\tilde{a}_i$ with $\tilde{a}_i \in k[T]$. From this we deduce that $\sum_{i=1}^r a_i(0) \partial_i g = f \sum_{i=1}^r \tilde{a}_i(g) \partial_i(g)$. Looking at the X -degree on both the left and the right hand side shows that the left hand side must be 0. But this cannot be since $\partial_1(g), \dots, \partial_r(g)$ are linearly independent over k and at least one $a_i(0)$ is non-zero. \square

Proposition 3.7

If JH is nilpotent and $\text{rank}(JH) \leq 1$, then the rows of JH are linearly dependent over k .

Proof. The case $\text{rank}(JH) = 0$ is clear. Therefore assume $\text{rank}(JH) = 1$. Let K denote the quotient field of $k[H_1, \dots, H_n]$. Let A be the integral closure of $k[H_1, \dots, H_n]$ in K . Then A is Noetherian and $\dim(A) = 1$. Hence A is a Dedekind ring. Furthermore A is contained in $k[X]$, because if $a \in A$ then a is integral over $k[X]$ and since K is contained in $k(X)$ it follows that $a \in k[X]$, since the unique factorisation domain $k[X]$ is integrally closed. So A is a Dedekind ring contained in $k[X]$. Now by Zak's theorem (see [91]) we deduce that $A = k[f]$ for some $f \in k[X]$. In particular every H_i is a polynomial in f , say $H_i = h_i(f)$ for some $h_i \in k[T]$. In particular $\text{rank}(JH) = 1$ implies $f \notin k$. Now by lemma 3.6 there exists $T \in \text{Gl}_n(k)$ and $r \geq 1$ such that $g := f(TX)$ satisfies $\partial_1 g, \dots, \partial_r g$ are linearly independent over $k[g]$ and g only depends on x_1, \dots, x_r . Now put $Q := T^{-1}HT$. Then

$$Q = T^{-1} \begin{pmatrix} h_1(g) \\ \vdots \\ h_n(g) \end{pmatrix} = \begin{pmatrix} q_1(g) \\ \vdots \\ q_n(g) \end{pmatrix}$$

for some $q_i(g) \in k[g]$. Now observe that

$$JQ = T^{-1}JH(TX)T$$

is also nilpotent Since $\text{Tr}(JQ) = 0$,

$$\sum_{i=1}^r q'_i(g) \partial_i(g) = 0$$

From the fact that $\partial_1 g, \dots, \partial_r g$ are linearly independent over $k[g]$ we deduce that $q'_i(g) \in k$ for all $1 \leq i \leq r$. So the first r rows of JQ are zero and hence the rows of JQ are linearly dependent over k . Consequently the rows of JH are linearly dependent over k . \square

Proposition 3.8

Let $r \geq 2$. Let $n \geq r + 1$. Then there exists a polynomial map $H : k^n \rightarrow k^n$ such that JH is nilpotent, $\text{rank}(JH) = r$ and the rows of JH are linearly independent over k .

Proof. The proof of this proposition follows directly from example 3.9. \square

Example 3.9

Let $r \geq 2$, $n \geq r + 1$ and $\alpha(x_1) \in k[x_1]$ such that $\deg_{x_1}(\alpha(x_1)) = r$. Put $f := x_2 - \alpha(x_1)$ and $H = (H_1, \dots, H_n)$ defined by

$$H_i := \begin{cases} f & i = 1 \\ x_{i+1} + \frac{(-1)^i}{(i-1)!} \alpha^{(i-1)}(x_1) f^{i-1} & 2 \leq i \leq r \\ \frac{(-1)^{r+1}}{r!} \alpha^{(r)} f^r & i = r + 1 \\ f^{i-1} & r + 2 \leq i \leq n \end{cases}$$

Note that if $n = r + 1$ this means that $\{i \mid r + 2 \leq i \leq n\} = \emptyset$. Furthermore JH is nilpotent and $\text{rank}(JH) = r$ but the rows of JH are linearly independent over k .

Observe that the last $n - (r + 1)$ columns of JH are zero. This implies that for the nilpotence and the rank of JH we can restrict to $J_{x_1, \dots, x_{r+1}}H$, the $(r + 1) \times (r + 1)$ upper left submatrix of JH . Let's call this submatrix J . We prove the nilpotence of J by showing that $J^{r+1}e_i = 0$ for all $1 \leq i \leq r + 1$ where e_i are the unit vectors in k^{r+1} . Since the i -th column of J is equal to e_{i-1} for $i = 3, \dots, r + 1$, one gets that $Je_i = e_{i-1}$ for $i = 3, \dots, r + 1$. Let $\partial = \frac{\partial}{\partial x_1}$ during this example. Then we have

$$\begin{aligned}
 J_{i,1} &= \begin{cases} -\partial\alpha(x_1) & i = 1 \\ \frac{(-1)^i}{(i-1)!}\partial^i\alpha(x_1)f^{i-1} + \frac{(-1)^{i-1}}{(i-2)!}\partial^{i-1}\alpha(x_1)f^{i-2}\partial\alpha(x_1) & 2 \leq i \leq r + 1 \end{cases} \\
 J_{i,2} &= \begin{cases} 1 & i = 1 \\ \frac{(-1)^i}{(i-2)!}\partial^{i-1}\alpha(x_1)f^{i-2} & 2 \leq i \leq r + 1 \end{cases} \\
 J_{i,k} &= \begin{cases} 1 & k > 2, k = i + 1 \\ 0 & k > 2, k \neq i + 1 \end{cases}
 \end{aligned}$$

Now we can compute J^2e_2 . Obviously Je_2 gives the second column of J , so for $2 \leq i \leq r + 1$ we get

$$\begin{aligned}
 (J^2e_2)_i &= \sum_{k=1}^{r+1} J_{i,k}J_{k,2} \\
 &= J_{i,1}J_{1,2} + J_{i,2}J_{2,2} + J_{i,i+1}J_{i+1,2} \\
 &= \left(\frac{(-1)^i}{(i-1)!}\partial^i\alpha(x_1)f^{i-1} + \frac{(-1)^{i-1}}{(i-2)!}\partial^{i-1}\alpha(x_1)f^{i-2}\partial\alpha(x_1) \right) \cdot 1 + \\
 &\quad \left(\frac{(-1)^i}{(i-2)!}\partial^{i-1}\alpha(x_1)f^{i-2} \right) \cdot \partial\alpha(x_1) + \\
 &\quad 1 \cdot \left(\frac{(-1)^{i+1}}{(i-1)!}\partial^i\alpha(x_1)f^{i-1} \right) \\
 &= 0
 \end{aligned}$$

Obviously, $(J^2e_2)_1 = -\partial\alpha(x_1) + \partial\alpha(x_1) = 0$. So $J^2e_2 = 0$. Combined with $Je_i = e_{i-1}$ for $i = 3, \dots, r + 1$ it now follows that $J^{r+1}e_i = 0$ for $2 \leq i \leq r + 1$. We only have to look at $J^{r+1}e_1$. One can verify¹ that

$$J^{r+1}e_1 = \left(\frac{(-1)^{r+1}}{r!}\partial^{r+1}\alpha(x_1)f^r, 0, \dots, 0 \right)$$

However $\deg_{x_1}(\alpha(x_1)) = r$ hence $\partial^{r+1}\alpha(x_1) = 0$. And hence we have seen that $J^{r+1}e_i = 0$ for $i = 1, \dots, r + 1$. Hence $J^{r+1} = 0$, which of course means that J is nilpotent.

¹It is easy to check that this works given certain r and n , however it is difficult to write it down in a general way. The problem is that $J^r e_1$ still contains a lot of 'rubbish' in the last r coefficients, which suddenly vanishes after multiplying the last time.

The second claim is that $\text{rank}(J) = r$. This follows from two observations. First, J is nilpotent and hence $\det(J) = 0$ and hence $\text{rank}(J) \leq r$. Second, the image of J contains the vectors Je_2, e_2, \dots, e_r and note that $Je_2 = e_1 + \sum_{i=2}^{r+1} c_i e_i$ so surely we have r independent vectors in the image of J . Hence $\text{rank}(J) \geq r$.

We prove the last claim by showing that (H_1, \dots, H_n) are linearly independent over k . Let

$$\sum_{i=1}^n \lambda_i H_i = 0$$

for certain $\lambda_i \in k$. Clearly $\lambda_2 = \dots = \lambda_r = 0$ since x_i only appears in H_i . So

$$\lambda_1 H_1 + \sum_{i=r+1}^n \lambda_i H_i = 0$$

However for each H_i in this expression we have that $\deg_{x_2}(H_i) = i$. So we can conclude that also $\lambda_1 = \lambda_{r+1} = \dots = \lambda_n = 0$. This proves the last claim.

Theorem 3.10

The Generalised Dependence Problem is true for $n \leq 2$ and false for $n \geq 3$.

Proof. Combine proposition 3.7 and proposition 3.8. □

3.2 The class $\mathcal{H}_n(A)$

The class introduced in this section is based on the question:

Question 3.11

What if the Generalised Dependence Problem is true?

As stated before, this part of the theory was build long before the counterexample against this problem was found.

The assumption that GDP is true has lead to a fairly general class of polynomial automorphisms. In fact, the main trick in the definition is based on the power of the ‘for all’ clause in its formulation. Throughout this section let A denote an arbitrary commutative ring. Let $\mathcal{N}_n(A)$ denote the set of all polynomial maps $H \in A[X]^n$ such that JH is nilpotent. In this section we often view $A[X]^n$ as $\text{Mat}_{n,1}(A[X])$. Normally we do not make explicit distinction between the usual vectors or these matrices. Now we can introduce the main definition of this section:

Definition 3.12

For all commutative rings A we define:

- $\mathcal{H}_1(A) = A$ and for $n \geq 2$
- $H \in \mathcal{H}_n(A)$ if and only if there exist
 1. $T \in \text{Mat}_n(A)$,

2. $c \in A^n$ and
3. $H_\star \in \mathcal{H}_{n-1}(A[x_n])$

such that

$$H = \text{Adj}(T) \left(\begin{array}{c} H_\star \\ 0 \end{array} \right) \Big|_{TX} + c \quad (3.1)$$

where $\text{Adj}(T)$ denotes the adjoint matrix of T and $|_{TX}$ the ‘evaluation at the vector TX ’.

The idea behind this definition is the GDP. From this problem it follows that maps with nilpotent Jacobian, have linearly dependent rows over A . Hence we can make a change of coordinates in order to get a 0 in the last component of the map. Since in general T^{-1} need not exist, we use $\text{Adj}(T)$ to arrange this change of coordinates. In order to simplify notations we abbreviate (3.1) by $H = H_\star[T, c]$ or by $H = H_\star[T]$ in case $c = 0$. In particular the clause ‘ $H = H_\star[T, c]$ ’ means that there exist such T, c and H_\star .

We start by presenting the ‘basic’ example of this class.

Example 3.13

Let $H = (H_1, H_2) \in A[x_1, x_2]^2$. Then $H \in \mathcal{H}_2(A)$ if and only if there exist $T = \begin{pmatrix} t_1 & t_2 \\ a_1 & a_2 \end{pmatrix} \in \text{Mat}_2(A)$, $c_1, c_2 \in A$ and $f(x_2) \in \mathcal{H}_1(A[x_2]) = A[x_2]$ such that

$$\begin{pmatrix} H_1 \\ H_2 \end{pmatrix} = \begin{pmatrix} a_2 & -t_2 \\ -a_1 & t_1 \end{pmatrix} \begin{pmatrix} f(x_2) \\ 0 \end{pmatrix} \Big|_{\begin{pmatrix} t_1 x_1 + t_2 x_2 \\ a_1 x_1 + a_2 x_2 \end{pmatrix}} + \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$$

In other words: if and only if H_1 and H_2 are of the form

$$\begin{aligned} H_1 &= a_2 f(a_1 x_1 + a_2 x_2) + c_1 \\ H_2 &= -a_1 f(a_1 x_1 + a_2 x_2) + c_2 \end{aligned}$$

for some $a_1, a_2, c_1, c_2 \in A$ and $f \in A[x_2]$. In the rest of this thesis we do not write $f \in A[x_2]$ but $f \in A[Y]$. (Or if Y is already used, $f(T) \in A[T]$.) We do this to emphasise the fact that f is a polynomial in one variable Y . In this example we used x_2 only to show that it was the second row of TX we have to substitute in f .

Remark 3.14

It was already shown in [51, Theorem 3.1] that if A is a \mathbb{Q} -algebra and a unique factorisation domain then each $H \in A[x_1, x_2]$ with JH is nilpotent, can be written as in example 3.13. Hence in that case we have $\mathcal{H}_2(A) = \mathcal{N}_2(A)$.

In [13] Campbell has given a different description of this class:

Definition 3.15

Let $A(X) = LX + C$ and $B(X) = MX + D$ be affine maps with L the classical adjoint matrix of M . Hence $ML = \det(M)I_n = aI_n$ ($a \in A$). Given $F = X + H$ with unipotent

Jacobian matrix, consider the new map $G = X + A \circ H \circ B$. From $G(X) = X + L(H(MX + D)) + C$ it follows that $J(G) = I + L(J(H) \circ B)M = I + N$ where N is nilpotent. ($N^k = L(J(H) \circ B)ML(J(H) \circ B)M \cdots L(J(H) \circ B)M = a^{k-1}L(J(H) \circ B)M$.) The class of maps containing all triangularisable maps and closed under the given construction is not a group but it contains an inverse of any map in it.

3.3 Properties of $\mathcal{H}_n(A)$

The class defined in the previous section has some remarkable properties. For instance the Jacobian Conjecture turns out to be true for all $F = X + H$ with $H \in \mathcal{H}_n(A)$. In this section we present some of these properties.

The first observation is that the new class is a subset of the class of maps with nilpotent Jacobian.

Theorem 3.16

For all commutative rings A and all $n \geq 1$ we have $\mathcal{H}_n(A) \subset \mathcal{N}_n(A)$.

Proof. By induction on n . The case $n = 1$ is clear. So assume $n \geq 2$. Let $H \in \mathcal{H}_n(A)$ and $H = H_\star[T, c]$. Then

$$JH = JH_\star[T, c] = \text{Adj}(T) \begin{pmatrix} (J_{x_1, \dots, x_{n-1}} H_\star)_{|TX} & * \\ 0 & 0 \end{pmatrix}_{|T} \quad (3.2)$$

The induction hypothesis states that $J_{x_1, \dots, x_{n-1}} H_\star$ is nilpotent. Hence so is $(J_{x_1, \dots, x_{n-1}} H_\star)_{|TX}$. Equation (3.2) now shows that $JH_\star[T, c]$ is nilpotent. Hence JH is nilpotent. So indeed $\mathcal{H}_n(A) \subset \mathcal{N}_n(A)$ for each commutative ring A . \square

A useful lemma to manipulate the elements of $\mathcal{H}_n(A)$ is given by:

Lemma 3.17

Let $H \in \mathcal{H}_n(A)$, $r \in A$, $c \in A^n$. Then

1. $rH + c \in \mathcal{H}_n(A)$.
2. If $S \in \text{Mat}_n(A)$ then $\text{Adj}(S)H_{|SX} \in \mathcal{H}_n(A)$.
3. If $\varphi : A \rightarrow S$ is a ring homomorphism then $\varphi(H) \in \mathcal{H}_n(S)$ where $\varphi(H)$ is obtained by applying φ to the coefficients of H .

Proof. The first and the third claim follow readily by induction on n . Therefore only the second claim remains to be proved. So let $S \in \text{Mat}_n(A)$ and $H \in \mathcal{H}_n(A)$. Then according to definition 3.12 we get

$$\begin{aligned} \text{Adj}(S)H_{|SX} &= \text{Adj}(S) \left(\text{Adj}(T) \begin{pmatrix} H_\star \\ 0 \end{pmatrix}_{|TX} + c \right)_{|SX} \\ &= \text{Adj}(S) \text{Adj}(T) \left(\begin{pmatrix} H_\star \\ 0 \end{pmatrix}_{|TX} \right)_{|SX} + c \end{aligned}$$

$$= \text{Adj}(TS) \left(\begin{array}{c} H_\star \\ 0 \end{array} \right)_{|(TS)X} + c$$

and here we are back in the form of equation (3.1). \square

Corollary 3.18

Let $A[Y]$ be the polynomial ring in one variable over A . Let $a \in A$. If $H = (H_1, \dots, H_n) \in \mathcal{H}_n(A[Y])$, then $H(Y = a) \in \mathcal{H}_n(A)$.

Proof. Apply the third claim of lemma 3.17 to the substitution homomorphism $\varphi : A[Y] \rightarrow A$ sending Y to a . \square

An addition to the previous lemma was found by the author and Wright in [54]:

Lemma 3.19

Let $H(x_1, \dots, x_n) \in \mathcal{H}_n(A)$, $d \in A$ and $a = (a_1, \dots, a_n) \in A^n$. Then also $H(x_1 = dx_1 + a_1, \dots, x_n = dx_n + a_n) \in \mathcal{H}_n(A)$.

Proof. Induction on n . If $n = 1$ we get $H(x_1) \in \mathcal{H}_1(A) = A$ and hence $H(x_1)$ is a constant and hence also $H(x_1 = dx_1 + a_1) \in \mathcal{H}_1(A)$. Now assume $n \geq 2$. Let $H = H_\star[T, c]$. We want to show that $H(x_1 = dx_1 + a_1, \dots, x_n = dx_n + a_n) \in \mathcal{H}_n(A)$. Note that we can write this map as the composition of three maps: H , a translation over a , Tr_a and a multiplication with d , D_d : $H \circ Tr_a \circ D_d$. Hence we get:

$$\begin{aligned} H(x_1 = dx_1 + a_1, \dots, x_n = dx_n + a_n) &= \\ &= H \circ Tr_a \circ D_d \\ &= \left(Tr_c \circ \text{Adj}(T) \left(\begin{array}{c} H_\star \\ 0 \end{array} \right) \circ T \right) \circ Tr_a \circ D_d \\ &= Tr_c \circ \text{Adj}(T) \left(\begin{array}{c} H_\star \\ 0 \end{array} \right) \circ Tr_{\tilde{a}} \circ T \circ D_d \quad (\tilde{a} := Ta) \\ &= Tr_c \circ \text{Adj}(T) \left(\begin{array}{c} H_\star \\ 0 \end{array} \right) \circ Tr_{\tilde{a}} \circ D_d \circ T \\ &= Tr_c \circ \text{Adj}(T) \left(\begin{array}{c} H_\star(x_1 = dx_1 + \tilde{a}_1, \dots, x_n = dx_n + \tilde{a}_n) \\ 0 \end{array} \right) \circ T \end{aligned}$$

Now we are done if we can show that $H_\star(x_1 = dx_1 + \tilde{a}_1, \dots, x_n = dx_n + \tilde{a}_n) \in \mathcal{H}_{n-1}(A[x_n])$. Note that x_1, \dots, x_{n-1} are variables and x_n is a constant in $\mathcal{H}_{n-1}(A[x_n])$. By this notion we can apply lemma 3.19 with the substitution homomorphism

$$\begin{aligned} A[x_n] &\rightarrow A[x_n] \\ x_n &\mapsto dx_n + \tilde{a}_n \end{aligned}$$

to see that $\hat{H} = H_*(x_n = dx_n + \tilde{a}_n) \in \mathcal{H}_{n-1}(A[x_n])$. And now we can apply the induction hypothesis on \hat{H} and the ring $A[x_n]$ to get that $\hat{H}(x_1 = dx_1 + \tilde{a}_1, \dots, x_{n-1} = dx_{n-1} + \tilde{a}_{n-1}) \in \mathcal{H}_{n-1}(A[x_n])$. And hence

$$\begin{aligned} H_*(x_1 = dx_1 + \tilde{a}_1, \dots, x_n = dx_n + \tilde{a}_n) &= \\ &= \hat{H}(x_1 = dx_1 + \tilde{a}_1, \dots, x_{n-1} = dx_{n-1} + \tilde{a}_{n-1}) \in \mathcal{H}_{n-1}(A[x_n]). \end{aligned}$$

This proves the lemma. □

This proof shows the use of the ‘for all’ clause in the definition. At the highest level we have that A is a commutative ring. If we use definition 3.12 one time we get a new ring: $A[x_n]$. However this ring is again commutative, so the definition can be applied again with this new commutative ring $A[x_n]$ instead of the original A .

We need another lemma before we can prove the more interesting theorems.

Lemma 3.20

If $H \in \mathcal{H}_n(A)$ and $a \in A$ then $(X + aH)^{-1} = X + aG$ with $G \in \mathcal{H}_n(A)$.

Proof. By induction on n . If $n = 1$ then $H \in \mathcal{H}_1(A)$ means $H \in A$. And hence

$$\begin{aligned} (X + aH)^{-1} &= X - aH \\ &= X + a(-H) \end{aligned}$$

So take $G = -H \in A = \mathcal{H}_1(A)$ and we are done.

Now assume $n \geq 2$. Let $H \in \mathcal{H}_n(A)$ and $H = H_*[T, c]$. Then there exist T with $\det(T) = d$, H_* and c such that

$$H = T^{-1} \left(\begin{array}{c} dH_* \\ 0 \end{array} \right)_{|TX} + c$$

and hence

$$\begin{aligned} X + aH &= X + T^{-1} \left(\begin{array}{c} adH_* \\ 0 \end{array} \right)_{|TX} + ac \\ &= Tr_{ac} \circ T^{-1} \left(X + \left(\begin{array}{c} adH_* \\ 0 \end{array} \right)_{|TX} \right) \end{aligned}$$

Consequently:

$$\begin{aligned} (X + aH)^{-1} &= \left(Tr_{ac} \circ T^{-1} \left(X + \left(\begin{array}{c} adH_* \\ 0 \end{array} \right)_{|TX} \right) \right)^{-1} \\ &\quad [\text{with } X' = (x_1, \dots, x_{n-1})] \\ &= \left(T^{-1} \left(\begin{array}{c} (X' + adH_*)^{-1} \\ x_n \end{array} \right)_{|TX} \right) \circ Tr_{-ac} \\ &\quad [\text{with induction and } G_* \in \mathcal{H}_{n-1}(A[X_n])] \end{aligned}$$

$$\begin{aligned}
&= \left(T^{-1} \begin{pmatrix} X' + adG_{\star} \\ X_n \end{pmatrix} \Big|_{TX} \right) \circ Tr_{-ac} \\
&= \left(X + T^{-1} \begin{pmatrix} adG_{\star} \\ 0 \end{pmatrix} \Big|_{TX} \right) \circ Tr_{-ac} \\
&= \left(X + \text{Adj}(T) \begin{pmatrix} aG_{\star} \\ 0 \end{pmatrix} \Big|_{TX} \right) \circ Tr_{-ac} \\
&= \left(X + a \text{Adj}(T) \begin{pmatrix} G_{\star} \\ 0 \end{pmatrix} \Big|_{TX} \right) \circ Tr_{-ac} \\
&\quad [\text{with } \hat{G} \in \mathcal{H}_n(A)] \\
&= (X + a\hat{G}) \circ Tr_{-ac} \\
&= X - ac + a\hat{G}(X - ac) \\
&= X + a(\hat{G}(X - ac) - c)
\end{aligned}$$

Now apply lemma 3.19 to \hat{G} and note that $\hat{G}(X - ac) \in \mathcal{H}_n(A)$ since $\hat{G} \in \mathcal{H}_n(A)$. Hence take $G = \hat{G}(X - ac) - c \in \mathcal{H}_n(A)$. The conclusion is that we have

$$\begin{aligned}
(X + aH)^{-1} &= X + a(\hat{G}(X - ac) - c) \\
&= X + aG
\end{aligned}$$

with $G \in \mathcal{H}_n(A)$. □

This lemma is the basis for this important theorem:

Theorem 3.21

Let $F = X + H$ with $H \in \mathcal{H}_n(A)$. Then $F^{-1} = X + G$ with $G \in \mathcal{H}_n(A)$.

Proof. Apply lemma 3.20 with $a = 1$. □

Theorem 3.21 presents a very important property of this class. The impact lies in the fact that we can construct arbitrary complex examples by means of definition 3.12, but still we know automatically that these examples are invertible. Or even better: we still know that their inverse mappings are also polynomial mappings.

The missing piece to connect the class $\mathcal{H}_n(A)$ with the Jacobian Conjecture is the value of $\det(JF)$ if $F = X + H$ with $H \in \mathcal{H}_n(A)$. Obviously, $\det(JF) \in A^*$, otherwise F cannot have a polynomial inverse. However we can even prove the stronger fact that $\det(JF) = 1$. We prove this by introducing so-called universal rings. Let

$$A[T_{i,j}] := A[T_{i,j}; 1 \leq i, j \leq n] \tag{3.3}$$

be the polynomial ring in n^2 indeterminates over A . Put $T_u := (T_{i,j})$. Put $d := \det(T_u)$ and consider the ring $S := A[T_{i,j}][d^{-1}]$.

Lemma 3.22

Take the definitions as above. Then d is not a zero-divisor in $A[T_{i,j}]$.

Proof. By induction on n . The case $n = 1$ is clear. So assume $n \geq 2$. Put $d_{n-1} := \det(T_{i,j})_{1 \leq i, j \leq n-1}$. Define $A_* := A[T_{n,i}, T_{i,n}; 1 \leq i \leq n-1]$ and $B := A_*[T_{i,j}; 1 \leq i, j \leq n-1]$. Consequently $B = A[T_{i,j}; (i, j) \neq (n, n)]$. Developing d with respect to the n -th column of T_u gives

$$d = d_{n-1}T_{n,n} + b$$

for some $b \in B$. In particular b does not contain any $T_{n,n}$. The assumption that d is a zero-divisor in $A[T_{i,j}]$ leads to the fact that there exists an element $g \in A[T_{i,j}]$, $g \neq 0$, such that $dg = 0$. If we develop g after powers of $T_{n,n}$ we get $g = g_m T_{n,n}^m + \dots + g_0$ for some $m \geq 0$, $g_m \neq 0$ and $g_i \in B$ for all i . Now if we develop $dg = 0$ after powers of $T_{n,n}$ all coefficients of $T_{n,n}^i$ must be 0, so in particular the coefficient of $T_{n,n}^{m+1}$, $d_{n-1}g_m$, equals 0. However, applying the induction hypothesis to A_* we get that d_{n-1} is no zero-divisor in $A_*[T_{i,j}; 1 \leq i, j \leq n-1]$. Hence $g_m = 0$, which contradicts the fact that $g_m \neq 0$. Therefore we cannot maintain the assumption that d is a zero-divisor in $A[T_{i,j}]$, which proves the lemma. \square

A direct consequence of this lemma is that $A[T_{i,j}] \subset S$. Now we can prove the announced theorem:

Theorem 3.23

Let $H \in \mathcal{H}_n(A)$ and $F = X + H$. Then $\det(JF) = 1$.

Proof. As usual the proof is by induction on n . Also as usual the case $n = 1$ is clear. Hence assume $n \geq 2$ and let $H = H_*[T]$ for some $T = (t_{i,j}) \in \text{Mat}_n(A)$ and $H_* \in \mathcal{H}_{n-1}(A[x_n])$.²

Now let S and T_u be as above. Put $S_0 = A[T_{i,j}]$. By lemma 3.22 we know that $S_0 \subset S$. By lemma 3.19 we can view H_* as an element of $\mathcal{H}_{n-1}(S_0[x_n]) \subset \mathcal{H}_{n-1}(S[x_n])$. We define the universal H_u and F_u :

$$\begin{aligned} H_u &:= T_u^{-1} \begin{pmatrix} dH_* \\ 0 \end{pmatrix} \Big|_{T_u X} \\ F_u &:= X + H_u \end{aligned}$$

If we take the determinant of the Jacobian we get:

$$\begin{aligned} \det(JF_u) &= \det(J(X + H_u)) \\ &= \det \left(T_u^{-1} J \left(X + \begin{pmatrix} dH_* \\ 0 \end{pmatrix} \right) \Big|_{T_u X} T_u \right) \\ &= \det (J_{x_1, \dots, x_{n-1}}(X' + dH_*)) \Big|_{T_u X} \end{aligned}$$

where $X' = (x_1, \dots, x_{n-1})$. However the induction hypothesis in conjunction with the fact that $dH_* \in \mathcal{H}_{n-1}(S[x_n])$ implies that

$$\det (J_{x_1, \dots, x_{n-1}}(X' + dH_*)) \Big|_{T_u X} = 1$$

Hence also $\det(JF_u) = 1$. Finally making the substitutions $T_{i,j} \rightarrow t_{i,j}$ we find that $\det(JF) = 1$. \square

²Since the c has no effect on the Jacobian of F , we may assume $c = 0$ without loss of generality.

An immediate consequence of theorem 3.21 and theorem 3.23 is:

Corollary 3.24

Let $F = X + H$ with $H \in \mathcal{H}_n(A)$. Then the Jacobian Conjecture holds for F .

The next theorem we present may not be as important as corollary 3.24 but probably even more spectacular. It is a generalisation of an observation originally made by Meisters. In [67] he noticed that for certain examples this property holds. Before we give the theorem we need a lemma.

Lemma 3.25

Let $H = (H_1, \dots, H_n) \in A[X]^n$. Assume $H_n = c_n \in A$. Now define for each $1 \leq i \leq n - 1$ $H_{i0} := H_i(x_n = c_n)$. Put $H_0 := (H_{10}, \dots, H_{(n-1)0}) \in A[x_1, \dots, x_{n-1}]^{n-1}$. Then for all $p \geq 2$:

$$H^p = \begin{pmatrix} H_0^{p-1}(H_1, \dots, H_{n-1}) \\ c_n \end{pmatrix}$$

Proof. By induction on p . First note that for all $1 \leq i \leq n - 1$

$$H_i(H_1, \dots, H_n) = H_{i0}(H_1, \dots, H_{n-1}) \tag{3.4}$$

which proves the case if $p = 2$. Now assume $p \geq 3$. Then by the induction hypothesis we get

$$H^p = H^{p-1} \circ H = \begin{pmatrix} H_0^{p-1}(H_1, \dots, H_{n-1}) \\ c_n \end{pmatrix} \circ H$$

Applying (3.4) now gives

$$\begin{aligned} H^p &= \begin{pmatrix} H_0^{p-1}(H_{10}(H_1, \dots, H_n), \dots, H_{(n-1)0}(H_1, \dots, H_n)) \\ c_n \end{pmatrix} \\ &= \begin{pmatrix} H_0^p(H_1, \dots, H_{n-1}) \\ c_n \end{pmatrix} \end{aligned}$$

which proves the lemma. □

Theorem 3.26

Let $H \in \mathcal{H}_n(A)$. Then $H^n := H \circ \dots \circ H \in A^n$ for all $n \geq 2$.

Proof. Let $H = H_\star[T, c]$ for some $c \in A^n$, $T = (t_{i,j}) \in \text{Mat}_n(A)$ and $H_\star \in \mathcal{H}_{n-1}(A[x_n])$. Like in the proof of theorem 3.23 we consider the universal matrix T_u , the ring $S := A[T_{i,j}][d^{-1}]$ where $d = \det(T_u)$ and the universal $H_u \in S[X]^n$. We can restrict to prove that $H_u^n \in S^n$, because this means $H_u^n \in S^n \cap A[T_{i,j}][X]^n = A[T_{i,j}]^n$. And making the substitutions $T_{i,j} \rightarrow t_{i,j}$ gives $H^n \in A^n$ as desired. In order to prove this claim, observe that:

$$\begin{aligned} H_u &= T_u^{-1} \begin{pmatrix} dH_\star \\ 0 \end{pmatrix} \Big|_{T_u X} + T_u^{-1}(T_u c) \\ &= T_u^{-1} \left(\begin{pmatrix} dH_\star \\ 0 \end{pmatrix} + T_u c \right) \Big|_{T_u X} \\ &= T_u^{-1} \begin{pmatrix} \widetilde{H}_\star \\ a \end{pmatrix} \Big|_{T_u X} \end{aligned}$$

where $\begin{pmatrix} \widetilde{H}_\star \\ a \end{pmatrix} = \begin{pmatrix} H_\star \\ 0 \end{pmatrix} + T_u c$. Note that $a \in S$ and $\widetilde{H}_\star \in \mathcal{H}_{n-1}(S[x_n])$. Clearly

$$H_u^n = T_u^{-1} \begin{pmatrix} \widetilde{H}_\star \\ a \end{pmatrix}^n \Big|_{T_u X}$$

So it suffices to show that $\begin{pmatrix} \widetilde{H}_\star \\ a \end{pmatrix}^n \in S^n$. Therefore we may assume that $H = (H_1, \dots, H_n) \in A[X]^n$ and $H_\star = (H_1, \dots, H_{n-1}) \in \mathcal{H}_{n-1}(A[x_n])$ and $H_n \in A$. Like in lemma 3.25 write c_n instead of H_n , write $H_{i0} = H_i(x_n = c_n)$ for $1 \leq i \leq n-1$ and $H_0 = (H_{10}, \dots, H_{(n-1)0})$. Then lemma 3.25 gives

$$H^n = \begin{pmatrix} H_0^{n-1}(H_1, \dots, H_{n-1}) \\ c_n \end{pmatrix} \quad (3.5)$$

Furthermore corollary 3.18 shows that $H_0 \in \mathcal{H}_{n-1}(A)$. So if $n = 2$ then $H^2 \in A^2$. Finally if $n \geq 3$ we can apply the induction hypothesis on H_0 , giving that $H_0^{n-1} \in A^{n-1}$, whence $H^n \in A^n$ by (3.5). \square

Corollary 3.27

Let $H \in \mathcal{H}_n(A)$. If $H(0) = 0$ then $H^n = 0$.

Proof. If $H \in \mathcal{H}_n(A)$ such that $H(0) = 0$ then $H = H_\star[T]$, i.e. $c = 0$. So in particular $c_n = 0$. Therefore by (3.5) we have

$$H^n = \begin{pmatrix} H_0^{n-1}(H_1, \dots, H_{n-1}) \\ 0 \end{pmatrix}$$

However since $c = 0$ we have $H_0(0) = 0$ also. And now we can apply the induction hypothesis to claim that $H^n = 0$. \square

3.4 Examples

In this section we show how the polynomial examples given before can all be seen as elements of $\mathcal{H}_n(A)$ for certain n and certain A .

Example 3.28

Let $F = X + H$ be as in example 2.17. Take $A = \mathbb{R}[x_3, x_4]$.³ Then

$$(H_1, H_2) = (x_4(x_3x_1 + x_4x_2)^2, -x_3(x_3x_1 + x_4x_2)^2) \in \mathcal{H}_2(\mathbb{R}[x_4][x_3])$$

The fact that this statement is true immediately follows from example 3.13: take $a_1 = x_3$, $a_2 = x_4$, $c_1 = 0$, $c_2 = 0$ and $f(Y) = Y^2$. Consequently we have that $(H_1, H_2, 0) \in \mathcal{H}_3(\mathbb{R}[x_4])$ and hence $(H_1, H_2, x_4^m) \in \mathcal{H}_3(\mathbb{R}[x_4])$. However this implies that $(H_1, H_2, x_4^m, 0) \in \mathcal{H}_4(\mathbb{R})$. And obviously $(H_1, H_2, x_4^m, 0, \dots, 0) \in \mathcal{H}_n(\mathbb{R})$. So we have shown that $F = X + H$ with $H \in \mathcal{H}_n(\mathbb{R})$.

³Usually we write this in the form $\mathbb{R}[x_4][x_3]$, because this shows the way the ring can be unravelled by the definition of our class.

Example 3.29

The polynomial counterexample to the Markus-Yamabe Conjecture for $n \geq 4$ (theorem 2.23) is almost the same as the example above. Here we have to take $m = 4$. With the same steps as in example 3.28 we can show that $F = -X + H$ with $H \in \mathcal{H}_n(\mathbb{R})$.

Example 3.30

The 3-dimensional counterexample from example 2.24 is slightly different. In the notation of example 3.13 take $a_1 = 1$, $a_2 = x_3$, $c_1 = 0$, $c_2 = 0$ and $f(Y) = Y^2$. Then $(H_1, H_2) \in \mathcal{H}_2(\mathbb{R}[x_3])$. Consequently $(H_1, H_2, 0) \in \mathcal{H}_3(\mathbb{R})$ and $(H_1, H_2, 0, \dots, 0) \in \mathcal{H}_n(\mathbb{R})$. So here we have that $F = -X + H$ with $H \in \mathcal{H}_n(\mathbb{R})$. And in the same way we can show that the polynomial counterexample to DMYP(n) for $n \geq 3$ is $F = \frac{1}{2}X + H$ with $H \in \mathcal{H}_n(\mathbb{R})$.

Example 3.31

In this example we show that the eight mappings in theorem 3.2 are also of the form $F = X + H$ with $H \in \mathcal{H}_4(\mathbb{C})$. Note: in this example the names a_1 , a_2 , c_1 and c_2 may appear with different meanings. They can appear as the values in example 3.13, but they can also appear as a constant already appearing in the formulation of theorem 3.2. We assume this won't be a problem for the reader.

1. Before we can see that this map is in $\mathcal{H}_4(\mathbb{C})$ we conjugate with the permutation map $P_{(14)}$ to swap the first and the fourth row:

$$F := X + \begin{pmatrix} -a_4x_4^3 - b_4x_4^2x_2 - c_4x_4^2x_3 - e_4x_4x_2^2 - f_4x_4x_2x_3 \\ -h_4x_4x_3^2 - k_4x_2^3 - l_4x_2^2x_3 - n_4x_2x_3^2 - q_4x_3^3 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

Now take $a_1 = 0$, $a_2 = 1$, $c_1 = -h_4x_4x_3^2 - c_4x_4^2x_3 - q_4x_3^3$, $c_2 = 0$ and

$$f(Y) = -b_4x_4^2Y - e_4x_4Y^2 - f_4x_4Yx_3 - k_4Y^3 - l_4Y^2x_3 - n_4Yx_3^2$$

Note that $f(Y) \in \mathbb{C}[x_4][x_3][Y]$ and $c_1 \in \mathbb{C}[x_4][x_3]$. Now let

$$\begin{aligned} & \begin{pmatrix} H_1 \\ H_2 \end{pmatrix} \\ &= \begin{pmatrix} a_2f(a_1x_1 + a_2x_2) + c_1 \\ -a_1f(a_1x_1 + a_2x_2) + c_2 \end{pmatrix} \\ &= \begin{pmatrix} f(x_2) + c_1 \\ 0 \end{pmatrix} \\ &= \begin{pmatrix} -b_4x_4^2x_2 - e_4x_4x_2^2 - f_4x_4x_2x_3 - k_4x_2^3 \\ -l_4x_2^2x_3 - n_4x_2x_3^2 - h_4x_4x_3^2 - c_4x_4^2x_3 - q_4x_3^3 \\ 0 \end{pmatrix} \end{aligned}$$

Then $(H_1, H_2) \in \mathcal{H}_2(\mathbb{C}[x_4][x_3])$. And clearly $(H_1, H_2, 0) \in \mathcal{H}_3(\mathbb{C}[x_4])$. Now it follows that $(H_1 - a_4x_4^3, H_2, 0) \in \mathcal{H}_3(\mathbb{C}[x_4])$. And finally $(H_1 - a_4x_4^3, H_2, 0, 0) \in \mathcal{H}_4(\mathbb{C})$.

2. For the second map we conjugate again with $P_{(14)}$:

$$F := X + \begin{pmatrix} -x_3x_4^2 - h_4x_4x_3^2 - q_4x_3^3 \\ -\frac{1}{3}x_4^3 - h_2x_4x_3^2 - q_2x_3^3 \\ 0 \\ 0 \end{pmatrix}$$

Take $a_1 = 0, a_2 = 0, c_1 = -x_3x_4^2 - h_4x_4x_3^2 - q_4x_3^3, c_2 = -h_2x_4x_3^2 - q_2x_3^3$ and $f(Y) = 0$. Now let

$$\begin{aligned} \begin{pmatrix} H_1 \\ H_2 \end{pmatrix} &= \begin{pmatrix} a_2f(a_1x_1 + a_2x_2) + c_1 \\ -a_1f(a_1x_1 + a_2x_2) + c_2 \end{pmatrix} \\ &= \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} \\ &= \begin{pmatrix} -x_3x_4^2 - h_4x_4x_3^2 - q_4x_3^3 \\ -h_2x_4x_3^2 - q_2x_3^3 \end{pmatrix} \end{aligned}$$

Obviously $(H_1, H_2) \in \mathcal{H}_2(\mathbb{C}[x_4][x_3])$. Furthermore $(H_1, H_2 - \frac{1}{3}x_4^3, 0) \in \mathcal{H}_3(\mathbb{C}[x_4])$. And finally $(H_1, H_2 - \frac{1}{3}x_4^3, 0, 0) \in \mathcal{H}_4(\mathbb{C})$.

3. Conjugate with $P_{(134)}$. We get

$$F = X + \begin{pmatrix} -x_3^2x_4 + \frac{1}{4}\frac{r_4x_3x_4^2}{c_1} - 3c_1x_1x_3x_4 + 9c_1x_2x_4^2 \\ -q_4x_4^3 - \frac{3}{4}r_4x_4^2x_1 \\ -\frac{1}{3}x_3^3 - c_1x_3^2x_1 + 3c_1x_3x_2x_4 - \frac{1}{3}x_3x_4^2q_4 + \frac{1}{48}\frac{x_3x_4^2r_4^2}{c_1^2} \\ -\frac{1}{2}r_4x_1x_3x_4 + \frac{3}{4}r_4x_2x_4^2 - \frac{1}{12}\frac{r_4q_4x_4^3}{c_1} - \frac{1}{16}\frac{r_4^2x_4^2x_1}{c_1} \\ 0 \\ 0 \end{pmatrix}$$

Take $a_1 = -4c_1x_3 - r_4x_4$ and $a_2 = 12c_1x_4$. Put $c_1 = \frac{1}{4}\frac{x_4x_3(-4c_1x_3+r_4x_4)}{c_1}$ and put $c_2 = -\frac{1}{48}\frac{x_3(16x_3^2c_1^2+16x_4^2q_4c_1^2-r_4^2x_4^2)}{c_1^2}$. Take $f(Y) = \frac{1}{16}\frac{Y}{c_1}$. Let

$$\begin{aligned} &\begin{pmatrix} H_1 \\ H_2 \end{pmatrix} \\ &= \begin{pmatrix} 12c_1x_4f((-4c_1x_3 - r_4x_4)x_1 + 12c_1x_4x_2) + c_1 \\ -(-4c_1x_3 - r_4x_4)f((-4c_1x_3 - r_4x_4)x_1 + 12c_1x_4x_2) + c_2 \end{pmatrix} \\ &= \begin{pmatrix} -x_3^2x_4 + \frac{1}{4}\frac{r_4x_3x_4^2}{c_1} - 3c_1x_1x_3x_4 + 9c_1x_2x_4^2 - \frac{3}{4}r_4x_4^2x_1 \\ -\frac{1}{3}x_3^3 - c_1x_3^2x_1 + 3c_1x_3x_2x_4 - \frac{1}{3}x_3x_4^2q_4 + \frac{1}{48}\frac{x_3x_4^2r_4^2}{c_1^2} \\ -\frac{1}{2}r_4x_1x_3x_4 + \frac{3}{4}r_4x_2x_4^2 - \frac{1}{16}\frac{r_4^2x_4^2x_1}{c_1} \end{pmatrix} \end{aligned}$$

Now $(H_1, H_2) \in \mathcal{H}_2(\mathbb{C}[x_4][x_3])$. Hence $(H_1 - q_4x_4^3, H_2 - \frac{1}{12}\frac{r_4q_4x_4^3}{c_1}, 0) \in \mathcal{H}_3(\mathbb{C}[x_4])$. Add the last row: $(H_1 - q_4x_4^3, H_2 - \frac{1}{12}\frac{r_4q_4x_4^3}{c_1}, 0, 0) \in \mathcal{H}_4(\mathbb{C})$.

4. Conjugate with $P_{(14)(23)}$. We get:

$$F := X + \begin{pmatrix} -e_4x_4x_3^2 - k_4x_3^3 \\ -x_4^2x_3 - e_3x_4x_3^2 - k_3x_3^3 \\ -\frac{1}{3}x_4^3 \\ 0 \end{pmatrix}$$

Now take $a_1 = 0$, $a_2 = 1$, $c_1 = -e_4x_4x_3^2 - k_4x_3^3$, $c_2 = -x_4^2x_3 - e_3x_4x_3^2 - k_3x_3^3$ and $f(Y) = 0$. Let

$$\begin{aligned} \begin{pmatrix} H_1 \\ H_2 \end{pmatrix} &= \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} \\ &= \begin{pmatrix} -e_4x_4x_3^2 - k_4x_3^3 \\ -x_4^2x_3 - e_3x_4x_3^2 - k_3x_3^3 \end{pmatrix} \end{aligned}$$

And now we can show the usual sequence again: first $(H_1, H_2) \in \mathcal{H}_4(\mathbb{C}[x_4][x_3])$; then $(H_1, H_2, -\frac{1}{3}x_4^3) \in \mathcal{H}_3(\mathbb{C}[x_4])$ and finally $(H_1, H_2, -\frac{1}{3}x_4^3, 0) \in \mathcal{H}_4(\mathbb{C})$.

5. Conjugate with $P_{(13)}$. This gives

$$F := X + \begin{pmatrix} -x_3^2x_2 - 2\frac{s_3x_3x_2x_4}{i_3} - i_3x_1x_3x_4 - j_3x_3x_4^2 - \frac{s_3^2x_2x_4^2}{i_3^2} \\ -s_3x_1x_4^2 - t_3x_4^3 \\ -\frac{1}{3}x_3^3 + i_3x_3x_2x_4 - j_2x_3x_4^2 + s_3x_2x_4^2 + i_3^2x_1x_4^2 \\ -t_2x_4^3 \\ 0 \\ 0 \end{pmatrix}$$

Take $a_1 = i_3^2x_4$, $a_2 = s_3x_4 + i_3x_3$, $c_1 = -j_3x_3x_4^2$, $c_2 = -\frac{1}{3}x_3^3 - j_2x_3x_4^2$ and $f(Y) = -\frac{Y}{i_3^2}$. Let

$$\begin{aligned} \begin{pmatrix} H_1 \\ H_2 \end{pmatrix} &= \begin{pmatrix} (s_3x_4 + i_3x_3)f(i_3^2x_4x_1 + (s_3x_4 + i_3x_3)x_2) + c_1 \\ -i_3^2x_4f(i_3^2x_4x_1 + (s_3x_4 + i_3x_3)x_2) + c_2 \end{pmatrix} \\ &= \begin{pmatrix} -s_3x_1x_4^2 - \frac{s_3^2x_4^2x_2}{i_3^2} - 2\frac{s_3x_3x_2x_4}{i_3} \\ -i_3x_1x_3x_4 - x_3^2x_2 - j_3x_3x_4^2 \\ i_3^2x_1x_4^2 + s_3x_2x_4^2 + i_3x_3x_2x_4 - \frac{1}{3}x_3^3 \\ -j_2x_3x_4^2 \end{pmatrix} \end{aligned}$$

The usual sequence: $(H_1, H_2) \in \mathcal{H}_2(\mathbb{C}[x_4][x_3])$; $(H_1 - t_3x_4^3, H_2 - t_2x_4^3, 0) \in \mathcal{H}_3(\mathbb{C}[x_4])$ and hence we get the desired result $(H_1 - t_3x_4^3, H_2 - t_2x_4^3, 0, 0) \in \mathcal{H}_4(\mathbb{C})$.

6. Conjugate with $P_{(13)}$. This gives the map

$$F := X + \begin{pmatrix} -x_3^2x_2 - e_3x_3x_2^2 - g_3x_3x_2x_4 - j_3x_3x_4^2 - k_3x_2^3 \\ -m_3x_2^2x_4 - p_3x_2x_4^2 - t_3x_4^3 \\ -\frac{1}{3}x_3^3 - j_2x_3x_4^2 - t_2x_4^3 \\ 0 \\ 0 \end{pmatrix}$$

Take $a_1 = 0, a_2 = 1, c_1 = -j_3x_3x_4^2, c_2 = -\frac{1}{3}x_3^3 - j_2x_3x_4^2$ and $f(Y) = -x_3^2Y - e_3x_3Y^2 - g_3x_3Yx_4 - k_3Y^3 - m_3Y^2x_4 - p_3Yx_4^2$. Now let

$$\begin{aligned} \begin{pmatrix} H_1 \\ H_2 \end{pmatrix} &= \begin{pmatrix} f(x_2) + c_1 \\ c_2 \end{pmatrix} \\ &= \begin{pmatrix} -x_3^2x_2 - e_3x_3x_2^2 - g_3x_3x_2x_4 - k_3x_2^3 - m_3x_2^2x_4 \\ -p_3x_2x_4^2 - j_3x_3x_4^2 \\ -\frac{1}{3}x_3^3 - j_2x_3x_4^2 \end{pmatrix} \end{aligned}$$

Then $(H_1, H_2) \in \mathcal{H}_2(\mathbb{C}[x_4][x_3]), (H_1 - t_3x_4^3, H_2 - t_2x_4^3, 0) \in \mathcal{H}_3(\mathbb{C}[x_4])$. Finally $(H_1 - t_3x_4^3, H_2 - t_2x_4^3, 0, 0) \in \mathcal{H}_3(\mathbb{C})$.

7. Conjugate with $P_{(14)(23)}$. This gives

$$F := X + \begin{pmatrix} -x_4^2x_2 - e_4x_4x_3^2 - f_4x_4x_3x_2 - h_4x_4x_2^2 - k_4x_3^3 \\ -l_4x_3^2x_2 - n_4x_3x_2^2 - q_4x_2^3 \\ -x_4^2x_3 - e_3x_4x_3^2 - k_3x_3^3 \\ -\frac{1}{3}x_4^3 \\ 0 \end{pmatrix}$$

Take $a_1 = 0, a_2 = 1, c_1 = [-e_4x_4x_3^2 - k_4x_3^3, c_2 = -x_4^2x_3 - e_3x_4x_3^2 - k_3x_3^3$ and $f(Y) = -x_4^2Y - f_4x_4x_3Y - h_4x_4Y^2 - l_4x_3^2Y - n_4x_3Y^2 - q_4Y^3$. Now let

$$\begin{aligned} \begin{pmatrix} H_1 \\ H_2 \end{pmatrix} &= \begin{pmatrix} f(x_2) + c_1 \\ c_2 \end{pmatrix} \\ &= \begin{pmatrix} -x_4^2x_2 - e_4x_4x_3^2 - f_4x_4x_3x_2 - h_4x_4x_2^2 - k_4x_3^3 \\ -l_4x_3^2x_2 - n_4x_3x_2^2 - q_4x_2^3 \\ -x_4^2x_3 - e_3x_4x_3^2 - k_3x_3^3 \end{pmatrix} \end{aligned}$$

First we get $(H_1, H_2) \in \mathcal{H}_2(\mathbb{C}[x_4][x_3])$; then we get $(H_1, H_2, -\frac{1}{3}x_4^3) \in \mathcal{H}_3(\mathbb{C}[x_4])$ and finally we get $(H_1, H_2, -\frac{1}{3}x_4^3, 0) \in \mathcal{H}_4(\mathbb{C})$.

8. Conjugate with $P_{(1423)}$. We get the map

$$F := X + \begin{pmatrix} -x_4^2x_3 - e_3x_4x_3^2 + g_4x_1x_4x_3 - k_3x_3^3 + m_4x_3^2x_1 \\ + g_4^2x_3^2x_2 \\ -x_4^2x_1 - e_4x_4x_3^2 - 2\frac{m_4x_4x_3x_1}{g_4} - g_4x_3x_4x_2 - k_4x_3^3 \\ -\frac{m_4^2x_3^2x_1}{g_4^2} - m_4x_3^2x_2 \\ -\frac{1}{3}x_4^3 \\ 0 \end{pmatrix}$$

Take $a_1 = g_4x_4 + m_4x_3, a_2 = x_3g_4^2, c_1 = -x_4^2x_3 - e_3x_4x_3^2 - k_3x_3^3, c_2 = -e_4x_4x_3^2 - k_4x_3^3$ and $f(Y) = \frac{Y}{g_4^2}$. Let

$$\begin{pmatrix} H_1 \\ H_2 \end{pmatrix} = \begin{pmatrix} x_3g_4^2f((g_4x_4 + m_4x_3)x_1 + x_3g_4^2x_2) + c_1 \\ -(g_4x_4 + m_4x_3)f((g_4x_4 + m_4x_3)x_1 + x_3g_4^2x_2) + c_2 \end{pmatrix}$$

$$= \begin{pmatrix} -x_4^2 x_3 - e_3 x_4 x_3^2 + g_4 x_1 x_4 x_3 - k_3 x_3^3 \\ + m_4 x_3^2 x_1 + g_4^2 x_3^2 x_2 \\ -x_4^2 x_1 - e_4 x_4 x_3^2 - 2 \frac{m_4 x_4 x_3 x_1}{g_4} - g_4 x_3 x_4 x_2 \\ -k_4 x_3^3 - \frac{m_4^2 x_3^2 x_1}{g_4^2} - m_4 x_3^2 x_2 \end{pmatrix}$$

First $(H_1, H_2) \in \mathcal{H}_2(\mathbb{C}[x_4][x_3])$; then $(H_1, H_2, -\frac{1}{3}x_4^3) \in \mathcal{H}_3(\mathbb{C}[x_4])$ and finally we get $(H_1, H_2, -\frac{1}{3}x_4^3, 0) \in \mathcal{H}_4(\mathbb{C})$.

As one can see in these examples, finding the c_1 and c_2 is not a problem. Finding the a_1 and a_2 is equivalent with finding the dependency relations between the rows. Finally take the remaining part as $f(Y)$.

3.5 Saturation of $\mathcal{H}_n(A)$ leads to $\overline{\mathcal{H}_n(A)}$

In section 3.3 we proved the theorem that stated $\mathcal{H}_n(A) \subset \mathcal{N}_n(A)$. (Theorem 3.16.) Naturally the big question is:

Question 3.32

Is $\mathcal{H}_n(A) = \mathcal{N}_n(A)$?

In [51] a partial answer was already given. Even though the question had not been asked at that time. In that paper it is shown that:

Theorem 3.33

Let A be a unique factorisation domain with characteristic zero. Then $\mathcal{H}_2(A) = \mathcal{N}_2(A)$.

Proof. The proof is divided into four parts.

1. First assume that $A = k$ is a field. Then the result is proved in [8].
2. Now let A be a unique factorisation domain and let $H = (H_1, H_2) \in \mathcal{N}_2(A)$. Then $H \in \mathcal{N}_2(K)$ where K is the quotient field of A . So by the first case there exist $g(Y) \in K[Y]$ with $g(0) = 0$ and $v_1, v_2, d_1, d_2 \in K$ such that

$$\begin{aligned} H_1 &= v_2 g(v_1 x_1 + v_2 x_2) + d_1 \\ H_2 &= -v_1 g(v_1 x_1 + v_2 x_2) + d_2 \end{aligned}$$

(see example 3.13). So clearing denominators we get: there exist $a \in A$, $a \neq 0$, $f(Y) \in A[Y]$ with $f(0) = 0$ and $\mu_1, \mu_2, c_1, c_2 \in A$ such that

$$\begin{aligned} aH_1 &= \mu_2 g(\mu_1 x_1 + \mu_2 x_2) + c_1 \\ aH_2 &= -\mu_1 g(\mu_1 x_1 + \mu_2 x_2) + c_2 \end{aligned} \tag{3.6}$$

Substituting $x_1 = x_2 = 0$ in (3.6) we obtain that $c_1 = a\tilde{c}_1$ and $c_2 = a\tilde{c}_2$ for some $\tilde{c}_1, \tilde{c}_2 \in A$. So replacing H_i by $H_i - \tilde{c}_i$ we may assume that $c_1 = c_2 = 0$.

3. Now we show that we may assume that $\gcd(\mu_1, \mu_2) = 1$: therefore let $\mu = \widetilde{\mu}_1 d$, $\mu_2 = \widetilde{\mu}_2 d$ where $d = \gcd(\mu_1, \mu_2)$. So $\gcd(\widetilde{\mu}_1, \widetilde{\mu}_2) = 1$ and $\mu_i f(\mu_1 x_1 + \mu_2 x_2) = \widetilde{\mu}_i d f(d(\widetilde{\mu}_1 x_1 + \widetilde{\mu}_2 x_2))$. Hence if we put $\widetilde{f}(Y) = d f(dY)$ we get

$$\mu_i f(\mu_1 x_1 + \mu_2 x_2) = \widetilde{f}(\widetilde{\mu}_1 x_1 + \widetilde{\mu}_2 x_2).$$

4. Consequently we may assume that $\gcd(\mu_1, \mu_2) = 1$. Write $f = \sum_{i=1}^N f_i Y^i$, with $f_i \in A$. From (3.6) we may assume that $\gcd(a, f_1, \dots, f_N) = 1$.

Claim: a is a unit in A (and hence we are done).

Suppose that p is a prime factor of a . Then (3.6) implies that p divides $f(\mu_1 x_1 + \mu_2 x_2)$ (since $\gcd(\mu_1, \mu_2) = 1$). So in particular p divides both $f(\mu_1 x_1)$ and $f(\mu_2 x_2)$, so p divides $f_i \mu_1^i$ and $f_i \mu_2^i$ for all $i \geq 1$ and hence p divides f_i for all $i \geq 1$ which contradicts $\gcd(a, f_1, \dots, f_N) = 1$. So a is a unit. □

Unfortunately even for $n = 2$ we really need the assumption that A is a unique factorisation domain. The next example will show a domain A with $\mathcal{H}_2(A) \not\subseteq \mathcal{N}_2(A)$.

Example 3.34

Let A denotes the domain $\mathbb{Z}[X, Y, Z]/(X^2 + YZ)$. Let $H_1 = c_1 x_1 + c_2 x_2, H_2 = d_1 x_1 + d_2 x_2$ in $A[x_1, x_2]$ where $c_1 = \overline{X}, c_2 = \overline{Y}, d_1 = \overline{Z}$ and $d_2 = -\overline{X}$. Then

1. $H = (H_1, H_2) \in \mathcal{N}_2(A)$.
2. $H \notin \mathcal{H}_2(A)$.
3. $\overline{Y}H \in \mathcal{H}_2(A)$.

These claims can easily be proved:

1. $JH = \begin{pmatrix} \overline{X} & \overline{Y} \\ \overline{Z} & -\overline{X} \end{pmatrix}$. Since $\text{Tr}(JH) = 0$ and $\det(JH) = -(\overline{X}^2 + \overline{Y}\overline{Z}) = 0$ we deduce that $H \in \mathcal{N}_2(A)$.
2. Suppose $H \in \mathcal{H}_2(A)$. Then by example 3.13 there exist $a_1, a_2 \in A$ and $f \in A[T]$ with $f(0) = 0$ such that

$$\begin{aligned} H_1 &= a_2 f(a_1 x_1 + a_2 x_2) \\ H_2 &= -a_1 f(a_1 x_1 + a_2 x_2) \end{aligned}$$

Now since both $\deg(H_1) = \deg(H_2) = 1$ we deduce that $f(T) = bT$ for some $b \in A \setminus \{0\}$. Consequently $\overline{X} = ba_1 a_2$ and $\overline{Y} = ba_2^2$. Let $A_1, A_2, B \in \mathbb{Z}[X, Y, Z]$ such that $a_1 = \overline{A_1}, a_2 = \overline{A_2}$ and $b = \overline{B}$. Then multiplying \overline{X} by a_2 and \overline{Y} by a_1 we obtain $a_2 \overline{X} = a_1 \overline{Y}$, i.e. $A_2 X - A_1 Y = c(X^2 + YZ)$ for some $c \in \mathbb{Z}[X, Y, Z]$. Consequently $X(A_2 - cX) = Y(A_1 + cZ)$. So $A_2 - cX = dY$ for some $d \in \mathbb{Z}[X, Y, Z]$ and hence $A_1 + cZ = dX$. Summarising

$$A_1 = dX - cZ \text{ and } A_2 = cX + dY$$

with $c, d \in \mathbb{Z}[X, Y, Z]$. Consequently the equation $\overline{X} = ba_1 a_2$, i.e. $X - BA_1 A_2 \in (X^2 + YZ)$, implies $X \in (X, Y, Z)^2$, a contradiction. So $H \notin \mathcal{H}_2(A)$.

3. $\bar{Y}H = \begin{pmatrix} \bar{Y}\bar{X}x_1 + \bar{Y}^2x_2 \\ \bar{Y}\bar{Z}x_1 - \bar{Y}\bar{X}x_2 \end{pmatrix}$. Since $\bar{Y}\bar{Z} = -\bar{X}^2$, we see that we can take $a_1 = \bar{X}$, $a_2 = \bar{Y}$ and $f(T) = T$ to get the desired form of example 3.13.

The third claim in example 3.34 shows that the fact that $H \notin \mathcal{H}_2(A)$ can be fixed by multiplying with \bar{Y} , an element of A . If we restrict to domains instead of rings this means that $\mathcal{H}_n(A)$ is not saturated with respect to $A \setminus \{0\}$. However if one looks at $\mathcal{N}_n(A)$, with A a domain, one sees that $\mathcal{N}_n(A)$ is in fact saturated with respect to $A \setminus \{0\}$. So if we want a new class of polynomial automorphisms that resembles the class $\mathcal{N}_n(A)$, at least this saturation aspect should be included. Therefore we modified definition 3.12 slightly in order to cover this saturation point. Because this saturation can be seen as some sort of closure of the class $\mathcal{H}_n(A)$ we denote this new class by $\overline{\mathcal{H}_n(A)}$.

Definition 3.35

For all commutative domains A we define:

- $\overline{\mathcal{H}_1(A)} = A$ and for $n \geq 2$
- $H \in \overline{\mathcal{H}_n(A)}$ if and only if there exist
 1. $T \in \text{Mat}_n(A)$,
 2. $c \in A^n$,
 3. $r \in A \setminus \{0\}$ and
 4. $H_* \in \overline{\mathcal{H}_{n-1}(A[x_n])}$

such that

$$rH = \text{Adj}(T) \begin{pmatrix} H_* \\ 0 \end{pmatrix}_{|_{TX}} + c \quad (3.7)$$

Example 3.36

Note that the H from example 3.34 is an element of $\overline{\mathcal{H}_2(A)}$. This is the impact of the third claim in the original example.

If we look at the properties of $\overline{\mathcal{H}_n(A)}$ we see that this class has much in common with $\mathcal{H}_n(A)$. (As was to be expected of course.)

Theorem 3.37

For all commutative domains A and $n \geq 1$ we have $\overline{\mathcal{H}_n(A)} \subset \mathcal{N}_n(A)$.

Proof. The proof is basically the same as the proof of theorem 3.16. Following this proof we find that $J(rH)$ is nilpotent. However this directly means that also JH is nilpotent. \square

Obviously the following question is raised after seeing this theorem:

Question 3.38

Is $\overline{\mathcal{H}_n(A)} = \mathcal{N}_n(A)$?

Before we can provide an answer to this question, we need to show some more properties of the class $\overline{\mathcal{H}_n(A)}$.

Lemma 3.39

Let $H \in \overline{\mathcal{H}_n(A)}$, $q \in A$, $d \in A^n$. Then

1. $qH + c \in \overline{\mathcal{H}_n(A)}$.
2. If $S \in \text{Mat}_n(A)$ then $\text{Adj}(S)H|_{SX} \in \overline{\mathcal{H}_n(A)}$.
3. If $\varphi : A \rightarrow S$ is a ring homomorphism then $\varphi(H) \in \overline{\mathcal{H}_n(S)}$ where $\varphi(H)$ is obtained by applying φ to the coefficients of H .

Proof. If $H \in \overline{\mathcal{H}_n(A)}$ then

$$rH = \text{Adj}(T) \begin{pmatrix} H_\star \\ 0 \end{pmatrix} \Big|_{TX} + \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$$

for certain $r \in A \setminus \{0\}$, $T \in \text{Mat}_n(A)$, $c \in A^2$ and $H_\star \in \overline{\mathcal{H}_{n-1}(A[x_n])}$. Obviously

$$\begin{aligned} r(qH + d) &= qrH + rd \\ &= q \text{Adj}(T) \begin{pmatrix} H_\star \\ 0 \end{pmatrix} \Big|_{TX} + \begin{pmatrix} qc_1 \\ qc_2 \end{pmatrix} + \begin{pmatrix} rd_1 \\ rd_2 \end{pmatrix} \\ &= \text{Adj}(T) \begin{pmatrix} qH_\star \\ 0 \end{pmatrix} \Big|_{TX} + \begin{pmatrix} qc_1 + rd_1 \\ qc_2 + rd_2 \end{pmatrix} \end{aligned}$$

which is of the desired form. and proves the first claim. In the same way also the other two claims can be proved. \square

Lemma 3.40

Let $H(x_1, \dots, x_n) \in \overline{\mathcal{H}_n(A)}$, $d \in A$ and $a = (a_1, \dots, a_n) \in A^n$. Then also $H(x_1 = dx_1 + a_1, \dots, x_n = dx_n + a_n) \in \overline{\mathcal{H}_n(A)}$.

Proof. Induction on n . If $n = 1$ we get $H(x_1) \in \overline{\mathcal{H}_1(A)} = A$ and hence $H(x_1)$ is a constant and hence also $H(x_1 = dx_1 + a_1) \in \overline{\mathcal{H}_1(A)}$. Now assume $n \geq 2$. Let $rH = H_\star[T, c]$. We want to show that $H(x_1 = dx_1 + a_1, \dots, x_n = dx_n + a_n) \in \overline{\mathcal{H}_n(A)}$.

We'll show that $rH(x_1 = dx_1 + a_1, \dots, x_n = dx_n + a_n) = \text{Adj}(\tilde{T}) \begin{pmatrix} \widetilde{H}_\star \\ 0 \end{pmatrix} \Big|_{\tilde{T}} + \tilde{c}$ for

some $\tilde{T} \in \text{Mat}_n(A)$, $\tilde{c} \in A^n$ and $\widetilde{H}_\star \in \overline{\mathcal{H}_{n-1}(A[x_n])}$. Like in the proof of lemma 3.19 we split H into three parts: $H \circ Tr_a \circ D_d$. Copying the scheme of lemma 3.19 we get:

$$\begin{aligned} rH(x_1 = dx_1 + a_1, \dots, x_n = dx_n + a_n) &= \\ &= D_r \circ H \circ Tr_a \circ D_d \\ &= \left(Tr_c \circ \text{Adj}(T) \begin{pmatrix} H_\star \\ 0 \end{pmatrix} \circ T \right) \circ Tr_a \circ D_d \end{aligned}$$

$$\begin{aligned}
&= \text{Tr}_c \circ \text{Adj}(T) \begin{pmatrix} H_\star \\ 0 \end{pmatrix} \circ \text{Tr}_{\tilde{a}} \circ T \circ D_d \quad (\tilde{a} := Ta) \\
&= \text{Tr}_c \circ \text{Adj}(T) \begin{pmatrix} H_\star \\ 0 \end{pmatrix} \circ \text{Tr}_{\tilde{a}} \circ D_d \circ T \\
&= \text{Tr}_c \circ \text{Adj}(T) \begin{pmatrix} H_\star(x_1 = dx_1 + \tilde{a}_1, \dots, x_n = dx_n + \tilde{a}_n) \\ 0 \end{pmatrix} \circ T
\end{aligned}$$

Now take $\tilde{T} = T$ and $\tilde{c} = c$. Hence again we are done if we can show that $H_\star(x_1 = dx_1 + \tilde{a}_1, \dots, x_n = dx_n + \tilde{a}_n) \in \overline{\mathcal{H}_{n-1}(A[x_n])}$, because this would imply that we can take $\widetilde{H}_\star = H_\star$. Fortunately, this goes in exactly the same way as it was shown in the proof of lemma 3.19. \square

Using the last two lemmas we can prove the important theorem:

Theorem 3.41

Let $F = X + H$ with $H \in \overline{\mathcal{H}_n(A)}$.

1. Then $\det(JF) = 1$ and
2. F is invertible over A .
3. $H^n \in A^n$ for all $n \geq 1$.

The proofs are based on their counterparts in the $\mathcal{H}_n(A)$ -theorem. Only here we have to use some localisations as well.

Another property of $\overline{\mathcal{H}_n(A)}$ that is of use to provide an answer to the questions 3.32 and 3.38 is presented in the next lemma.

Lemma 3.42

Let A be a domain, $n \geq 1$ and $H \in \overline{\mathcal{H}_n(A)}$ such that $H(0) = 0$. Then the H_i 's are linearly dependent over A .

Proof. Assume $H \neq 0$. By definition there exist some $r \in A \setminus \{0\}$, $T \in \text{Mat}_n(A)$, $c \in A^n$ and $H_\star \in \overline{\mathcal{H}_{n-1}(A[x_n])}$ such that

$$rH = \text{Adj}(T) \begin{pmatrix} H_\star \\ 0 \end{pmatrix}_{|TX} + c \quad (3.8)$$

Since $H(0) = 0$, $T \neq 0$. (Otherwise $rH = c = 0$ and hence $H = 0$.) Multiplying (3.8) from the left by T gives:

$$rTH = \det(T) \begin{pmatrix} H_\star \\ 0 \end{pmatrix}_{|TX} + Tc \quad (3.9)$$

Now we have to look at two cases:

1. If $\det(T) = 0$ we get $rTH = Tc$ and hence $TH = 0$ because $H(0) = 0$. Now because we assumed that $T \neq 0$, we have that at least one row of T is not completely zero. And this means that we have a dependency relation on the H_i 's over A in this row.
2. If $\det(T) \neq 0$, certainly the last row of T must be non-zero. Looking at the n -th component of the vectors in (3.9) now gives $r(TH)_n = (Tc)_n$. Again by using $H(0) = 0$ we find that there exists a dependency relation on the H_i 's over A .

□

Finally we are able to reveal the answer to the two questions in this section. Now if we take another look at the inclusions:

$$\mathcal{H}_n(A) \subset \overline{\mathcal{H}_n(A)} \subset \mathcal{N}_n(A) \quad (3.10)$$

we know by example 3.34 that the first inclusion can be strict. The second can be strict also, as follows from the next theorem.

Theorem 3.43

Let A be any \mathbb{Q} -algebra. Then $\overline{\mathcal{H}_n(A)} \subsetneq \mathcal{N}_n(A)$ for all $n \geq 3$.

Proof. Put $f := x_2 - x_1^2$ and $H = (H_1, \dots, H_n)$ defined by

$$H_i := \begin{cases} f & i = 1 \\ x_3 + 2x_1f & i = 2 \\ -f^2 & i = 3 \\ f^{i-1} & 4 \leq i \leq n \end{cases}$$

This is a special case of example 3.9 with $r = 2$. And in example 3.9 we have seen that:

1. JH is nilpotent but
2. the H_i 's are linearly independent.

So $H \in \mathcal{N}_n(A)$, but $H \notin \overline{\mathcal{H}_n(A)}$. Otherwise there would be a contradiction with lemma 3.42. □

Chapter 4

$\mathcal{D}_n(A)$ for $\mathcal{H}_n(A)$

Introduction

In this chapter we expand the theory developed in the previous chapter with the notion of $\mathcal{D}_n(A)$. From definition 3.12 and the examples from section 3.4 it follows that the class $\mathcal{H}_n(A)$ is pretty large, but also that it is not always easy to show that a certain polynomial map is or is not an element of $\mathcal{H}_n(A)$. The expansion we describe in this chapter provides a way to describe the elements of $\mathcal{H}_n(A)$ directly by a sequence of matrices and vectors. As a bonus this notion of $\mathcal{D}_n(A)$ yields a method to describe each $F = X + H$ with $H \in \mathcal{H}_n(A)$ as a finite product of $\exp(D_i)$'s where each D_i is a locally nilpotent derivation satisfying $D_i^2(x_j) = 0$ for $j = 1, \dots, n$. This will be shown in chapter 5. This chapter is based on the paper [37].

4.1 Notation

Throughout this chapter A denotes an arbitrary commutative ring and X denotes the sequence x_1, \dots, x_n . Hence $A[X]$ denotes the polynomial ring in n variables over A . Furthermore if $G = (G_1, \dots, G_n) \in A[X]^n$ and $S = (S_{ij}(X)) \in \text{Mat}_{p,q}(A[X])$ then $S(G)$ or $S|_G$ denotes the matrix S evaluated at the vector G , i.e. the matrix $(S_{ij}(x_k = G_k))_{i,j}$ for $k = 1, \dots, n$. In particular if $F \in A[X]^n$, which can be treated as $\text{Mat}_{n,1}(A[X])$, the composition of the polynomial maps F and G , denoted $F \circ G$, is equal to $F(G)$. We emphasise this composition because we use it in the theory on $\mathcal{D}_n(A)$. We define a new matrix multiplication Δ , which uses this composition and will be used alongside the usual matrix multiplication.

Definition 4.1

Let $S, T \in \text{Mat}_n(A[X])$. Then

$$S \Delta T := S(TX)T$$

We must read this as: S evaluated at the vector TX times the matrix T . Hence the type of this Δ -multiplication is $\text{Mat}_n(A[X]) \times \text{Mat}_n(A[X]) \rightarrow \text{Mat}_n(A[X])$. In order to increase the readability we sometimes write $S|_{TX}T$. This multiplication has the following properties:

Lemma 4.2

The multiplication Δ

1. is associative
2. but not commutative.

Proof. For the associativity we have if $P, Q, R \in \text{Mat}_n(A[X])$:

$$\begin{aligned}(P \Delta Q) \Delta R &= (P(X) \Delta Q(X)) \Delta R(X) \\ &= (P(X) |_{Q(X) \cdot X} \cdot Q(X)) \Delta R(X) \\ &= (P(Q(X) \cdot X) \cdot Q(X)) \Delta R(X) \\ &= (P(Q(X)X) \cdot Q(X)) |_{R(X) \cdot X} \cdot R(X) \\ &= P(Q(R(X) \cdot X) \cdot R(X) \cdot X) \cdot Q(R(X) \cdot X) \cdot R(X) \\ &= P(X) |_{Q(R(X) \cdot X) \cdot R(X) \cdot X} \cdot Q(R(X) \cdot X) \cdot R(X) \\ &= P(X) \Delta (Q(R(X) \cdot X) \cdot R(X)) \\ &= P(X) \Delta (Q(X) |_{R(X) \cdot X} \cdot R(X)) \\ &= P(X) \Delta (Q(X) \Delta R(X)) \\ &= P \Delta (Q \Delta R)\end{aligned}$$

And hence Δ is associative. The non commutativity is easily shown. For instance take $S \in \text{Mat}_n(A) \subset \text{Mat}_n(A[X])$ and $T \in \text{Mat}_n(A[X])$. In this case we get $S \Delta T = S \cdot T$, the normal matrix multiplication, which is not commutative. \square

Because this multiplication is associative we can safely write

$$S_1 \Delta S_2 \Delta \cdots \Delta S_n$$

where each $S_i \in \text{Mat}_n(A[X])$.

In the rest of this chapter we need a lot of coercions. In order to connect the $p \times p$ matrices or p -dimensional vectors, where $p \leq n$, we must extend them to $n \times n$ matrices and n -dimensional vectors. Therefore we define $\tilde{\cdot}^n$ as the coercion operator. Let $T \in \text{Mat}_p(A[X])$ and $c \in A[X]^p$, then

$$\tilde{T}^n = \begin{pmatrix} T & 0 \\ 0 & I_{n-p} \end{pmatrix} \in \text{Mat}_n(A[X])$$

obtained by adding the $(n - p) \times (n - p)$ identity matrix and

$$\tilde{c}^n = \begin{pmatrix} c \\ 0 \\ \vdots \\ 0 \end{pmatrix} \in A[X]^n$$

is obtained by adding $n - p$ zeros. Because these coercions are so trivial, we usually omit the superscript n and sometimes even omit the $\tilde{\cdot}$; if in the rest of this chapter dimension problems between matrices and vectors occur, always apply this coercion operator.

4.2 Definition of $\mathcal{D}_n(A)$

After describing these preliminaries we finally can present the definition of $\mathcal{D}_n(A)$.

Definition 4.3

Let A be a commutative ring. Then

- $\mathcal{D}_1(A)$ is the set of 1-tuples (c_1) with $c_1 \in A$.
- for $n \geq 2$ $\mathcal{D}_n(A)$ is the set of $(2n - 1)$ -tuples

$$(T, c) := (T_2, \dots, T_n, c_1, \dots, c_n) \quad (4.1)$$

where

1. $T_n \in \text{Mat}_n(A)$,
2. $T_i \in \text{Mat}_n(A[x_{i+1}, \dots, x_n])$ for $2 \leq i \leq n - 1$,
3. $c_n \in A^n$ and
4. $c_i \in A[x_{i+1}, \dots, x_n]^i$ for $2 \leq i \leq n - 1$.

By simply omitting the largest matrix and vector, we get a natural map (for $n \geq 2$; the tuples of $\mathcal{D}_1(A)$ do not even contain matrices) $\pi : \mathcal{D}_n(A) \rightarrow \mathcal{D}_{n-1}(A[x_n])$ defined by

$$\begin{aligned} n = 2 & \quad \pi : (T_2, c_1, c_2) \mapsto (c_1) \\ n \geq 3 & \quad \pi : (T_2, \dots, T_n, c_1, \dots, c_n) \mapsto (T_2, \dots, T_{n-1}, c_1, \dots, c_{n-1}) \end{aligned}$$

In order to decrease the amount of parentheses and increase the readability we usually write $\pi(T, c)$ instead of $\pi((T, c))$.

The link between elements of $\mathcal{D}_n(A)$ and polynomial mappings is made by the map $E_n : \mathcal{D}_n(A) \rightarrow A[X]^n$.

Definition 4.4

Let $n \geq 1$. The map

$$E_n : \mathcal{D}_n(A) \rightarrow A[X]^n$$

is defined by

- $E_1((c)) = c$ for all $(c) \in \mathcal{D}_1(A)$
- $E_n((T, c)) = c_n + \sum_{p=0}^{n-2} E_{n,p}((T, c))$ for $n \geq 2$ where
 - $E_{n,0}((T, c)) = \text{Adj}(T_n) \widetilde{c_{n-1}|_{T_n X}}$ for all $(T, c) \in \mathcal{D}_n(A)$
 - and for $n \geq 3$ and $1 \leq p \leq n - 2$ by

$$E_{n,p}((T, c)) = \text{Adj}(T_n) \left(\begin{array}{c} E_{n-1,p-1}(\pi(T, c)) \\ 0 \end{array} \right) \Big|_{T_n X}$$

Again instead of $E_n((T, c))$ or $E_{n,p}((T, c))$ we usually write $E_n(T, c)$ and $E_{n,p}(T, c)$. Furthermore, from now on we will write $(T, c) \in \mathcal{D}_1(A)$, even though this automatically means that $(T, c) = (c_1)$ with $c_1 \in A$. We do this because we don't want to make the distinction every time for $n = 1$ and $n \geq 2$, where the $n = 1$ is a trivial case. Furthermore, most important theorems are formulated only for $n \geq 2$.

Now based on this definition we can build a bridge between this $\mathcal{D}_n(A)$ and the $\mathcal{H}_n(A)$ of the previous chapter. The link is given by

Theorem 4.5

Let $n \geq 1$. Let $H \in A[X]^n$. Then $H \in \mathcal{H}_n(A)$ if and only if there exists $(T, c) \in \mathcal{D}_n(A)$ such that

$$H = E_n(T, c)$$

Proof. The case $n = 1$ is obvious. Let $H = (H_1)$ then

$$\begin{aligned} (H_1) \in \mathcal{H}_n(A) &\Leftrightarrow H_1 \in A \\ &\Leftrightarrow (H_1) \in \mathcal{D}_1(A) \end{aligned}$$

Now let $n = 2$. Because the $n = 1$ case is a lot different from $n = 2$ we check $n = 2$ completely and do not use induction until $n \geq 3$. By definition 3.12 $H \in \mathcal{H}_n(A)$ if and only if

$$H = \text{Adj}(T) \begin{pmatrix} H_\star \\ 0 \end{pmatrix}_{|TX} + c$$

for some $T \in \text{Mat}_2(A)$, $c \in A^2$ and $H_\star \in \mathcal{H}_1(A[x_2]) = A[x_2]$. Now take the tuple (T, H_\star, c) . Note $(T, H_\star, c) \in \mathcal{D}_2(A)$. Now compute $E_2(T, H_\star, c)$.

$$\begin{aligned} E_2(T, H_\star, c) &= c + \sum_{p=0}^0 E_{2,0}(T, H_\star, c) \\ &= c + E_{2,0}(T, H_\star, c) \\ &= c + \text{Adj}(T) \begin{pmatrix} H_\star \\ 0 \end{pmatrix}_{|TX} \\ &= H \end{aligned}$$

Which proves one direction. Of course if one takes arbitrary $(T, c) = (T_2, c_1, c_2) \in \mathcal{D}_2(A)$, one gets that $E_2(T, c) = c + \text{Adj}(T) \begin{pmatrix} c_1 \\ 0 \end{pmatrix}_{|TX} \in \mathcal{H}_2(A)$ because $c_1 \in A[x_2] = \mathcal{H}_1(A[x_2])$, $T \in \text{Mat}_2(A)$ and $c \in A^2$. This proves the other direction if $n = 2$.

Now assume $n \geq 3$. Then

$$H = \text{Adj}(T_n) \begin{pmatrix} H_\star \\ 0 \end{pmatrix}_{|T_n X} + c_n$$

where $T_n \in \text{Mat}_n(A)$, $c_n \in A^n$ and $H_\star \in \mathcal{H}_{n-1}(A[x_n])$. Now by the induction hypothesis on H_\star we have that

$$H_\star = E_{n-1}(T, c)$$

for some $(T, c) \in \mathcal{D}_{n-1}(A[x_n])$. Now put $(T', c') = (T, T_n, c, c_n)$. Observe that $(T', c') \in \mathcal{D}_n(A)$ and $\pi(T', c') = (T, c)$. So

$$\begin{aligned}
H &= \text{Adj}(T_n) \left(\begin{array}{c} H_\star \\ 0 \end{array} \right)_{|T_n X} + c_n \\
&= \text{Adj}(T_n) \left(\begin{array}{c} E_{n-1}(T, c) \\ 0 \end{array} \right)_{|T_n X} + c_n \\
&= \text{Adj}(T_n) \left(\begin{array}{c} c_{n-1} + \sum_{p=0}^{n-3} E_{n-1,p}(T, c) \\ 0 \end{array} \right)_{|T_n X} + c_n \\
&= \sum_{p=0}^{n-3} \text{Adj}(T_n) \left(\begin{array}{c} E_{n-1,p}(T, c) \\ 0 \end{array} \right)_{|T_n X} + \text{Adj}(T_n) \left(\begin{array}{c} c_{n-1} \\ 0 \end{array} \right)_{|T_n X} + c_n \\
&= \sum_{p=0}^{n-3} \text{Adj}(T_n) \left(\begin{array}{c} E_{n-1,p}(\pi(T', c')) \\ 0 \end{array} \right)_{|T_n X} + E_{n,0}(T', c') + c_n \\
&= \sum_{p=0}^{n-3} E_{n,p+1}(T', c') + E_{n,0}(T', c') + c_n \\
&= \sum_{p=1}^{n-2} E_{n,p}(T', c') + E_{n,0}(T', c') + c_n \\
&= \sum_{p=0}^{n-2} E_{n,p}(T', c') + c_n \\
&= E_n(T', c')
\end{aligned}$$

So if $H \in \mathcal{H}_n(A)$ take $(T', c') \in \mathcal{D}_n(A)$. And in reverse if $(T', c') \in \mathcal{D}_n(A)$, then $\pi(T', c') \in \mathcal{D}_{n-1}(A[x_n])$ and by induction there exists $H_\star \in \mathcal{H}_{n-1}(A[x_n])$ with $H_\star = E_{n-1}(\pi(T', c'))$. By definition $\text{Adj}(T_n) \left(\begin{array}{c} H_\star \\ 0 \end{array} \right)_{|T_n X} + c_n \in \mathcal{H}_n(A)$. And this completes the proof. \square

We complete this section by giving a theorem which makes it easier to compute the individual $E_{n,p}(T, c)$ for $(T, c) \in \mathcal{D}_n(A)$ and proper p .

Theorem 4.6

Let $n \geq 2, 0 \leq p \leq n - 2$ and $(T, c) \in \mathcal{D}_n(A)$. Then

$$E_{n,p}(T, c) = \text{Adj} \left(\tilde{T}_{n-p} \Delta \cdots \Delta \tilde{T}_{n-1} \Delta T_n \right) \tilde{c}_{n-p-1} |_{(\tilde{T}_{n-p} \Delta \cdots \Delta \tilde{T}_{n-1} \Delta T_n) X}$$

Proof. By induction on p . The case $p = 0$ is obvious. So let $p \geq 1$. Then

$$\begin{aligned}
&E_{n,p}(T, c) \\
&= \text{Adj}(T_n) \left(\begin{array}{c} E_{n-1,p-1}(\pi(T, c)) \\ 0 \end{array} \right)_{|T_n X}
\end{aligned}$$

$$\begin{aligned}
&= \text{(by the induction hypothesis)} \\
&\quad \text{Adj}(T_n) \left[\text{Adj} \left(\tilde{T}_{n-p} \Delta \cdots \Delta \tilde{T}_{n-1} \right) \tilde{c}_{n-p-1} \Big|_{(\tilde{T}_{n-p} \Delta \cdots \Delta \tilde{T}_{n-1})X} \right]_{T_n X} \\
&= \text{Adj}(T_n) \left[\text{Adj} \left(\tilde{T}_{n-p} \Delta \cdots \Delta \tilde{T}_{n-1} \right)_{T_n X} \left(\tilde{c}_{n-p-1} \Big|_{(\tilde{T}_{n-p} \Delta \cdots \Delta \tilde{T}_{n-1})X} \right)_{T_n X} \right] \\
&= \text{Adj}(T_n) \left[\text{Adj} \left((\tilde{T}_{n-p} \Delta \cdots \Delta \tilde{T}_{n-1})_{T_n X} \right) \left(\tilde{c}_{n-p-1} \Big|_{(\tilde{T}_{n-p} \Delta \cdots \Delta \tilde{T}_{n-1})X} \right)_{T_n X} \right] \\
&= \text{Adj}(T_n) \left[\text{Adj} \left((\tilde{T}_{n-p} \Delta \cdots \Delta \tilde{T}_{n-1})_{T_n X} \right) \left(\tilde{c}_{n-p-1} \Big|_{(\tilde{T}_{n-p} \Delta \cdots \Delta \tilde{T}_{n-1})_{T_n X} T_n X} \right) \right] \\
&= \text{Adj} \left((\tilde{T}_{n-p} \Delta \cdots \Delta \tilde{T}_{n-1})_{T_n X} \cdot T_n \right) \left(\tilde{c}_{n-p-1} \Big|_{((\tilde{T}_{n-p} \Delta \cdots \Delta \tilde{T}_{n-1})_{T_n X} T_n)X} \right) \\
&= \text{Adj} \left(\tilde{T}_{n-p} \Delta \cdots \Delta \tilde{T}_{n-1} \Delta T_n \right) \tilde{c}_{n-p-1} \Big|_{(\tilde{T}_{n-p} \Delta \cdots \Delta \tilde{T}_{n-1} \Delta T_n)X}
\end{aligned}$$

□

Remark 4.7

Note that this means that we do not need the complete $(T, c) \in \mathcal{D}_n(A)$ to compute $E_{n,p}(T, c)$ for given p , but only the tuple $(T_{n-p}, \dots, T_n, c_{n-p-1})$. We shall use this property in chapter 5.

4.3 Examples

Example 4.8

Let $F = X + H$ where $H = (x_4(x_3x_1 + x_4x_2)^2, -x_3(x_3x_1 + x_4x_2)^2, x_4^m, 0)$. See examples 2.17 and 3.28. We have seen that $H \in \mathcal{H}_4(\mathbb{R})$. According to theorem 4.5 there must exist $T \in \mathcal{D}_4(\mathbb{R})$ such that $H = E_4(T, c)$. We have to find a tuple $(T, c) \in \mathcal{D}_4(\mathbb{R})$ which represents H . How do we do that? First we determine c_4 , the constant terms: $c_4 := H(0)$. Then put $H' := H - c_4$ and try to find a linear T such that $\text{Adj}(T)H'T$ had the last row equal to zero. Because H' already has a zero on the last row, we can take the trivial $T_4 := I_4$. Next step is determine the constants of $\mathbb{R}[x_4]$, hence substitute $x_1 = x_2 = x_3 = 0$. This gives c_3 . Now we have to find again a linear dependence over $\mathbb{R}[x_4]$ which is in this case again trivial. This method is repeated until we have a two-dimensional map. The result in this example is:

$$T = \left(\left(\begin{array}{cc} 1 & 0 \\ x_3 & x_4 \end{array} \right), \left(\begin{array}{ccc} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array} \right), \left(\begin{array}{cccc} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array} \right) \right)$$

and the tuple of vectors

$$c = \left(\left(\begin{array}{c} x_2^2 \\ 0 \\ 0 \end{array} \right), \left(\begin{array}{c} 0 \\ 0 \end{array} \right), \left(\begin{array}{c} 0 \\ 0 \\ x_4^m \end{array} \right), \left(\begin{array}{c} 0 \\ 0 \\ 0 \\ 0 \end{array} \right) \right)$$

Now by definition 4.3 the combined tuple $(T, c) \in \mathcal{D}_4(\mathbb{R})$. Remains to show that $H = E_4(T, c)$. Therefore we start with computing $E_{4,0}(T, c)$, $E_{4,1}(T, c)$ and $E_{4,2}(T, c)$.

$$\begin{aligned}
E_{4,0}(T, c) &= \text{Adj}(T_4) \widetilde{c}_{3|T_4 X} \\
&= \begin{pmatrix} 0 \\ 0 \\ x_4^m \\ 0 \end{pmatrix} \\
E_{4,1}(T, c) &= \text{Adj}(\widetilde{T}_3 \Delta T_4) \widetilde{c}_{2|(\widetilde{T}_3 \Delta T_4) X} \\
&= \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \\
E_{4,2}(T, c) &= \text{Adj}(\widetilde{T}_2 \Delta \widetilde{T}_3 \Delta T_4) \widetilde{c}_{1|(\widetilde{T}_2 \Delta \widetilde{T}_3 \Delta T_4) X} \\
&= \text{Adj} \left(\begin{pmatrix} 1 & 0 \\ x_3 & x_4 \end{pmatrix} \right) \begin{pmatrix} x_2^2 \\ 0 \\ 0 \\ 0 \end{pmatrix} \Big| \begin{pmatrix} 1 & 0 \\ x_3 & x_4 \end{pmatrix} X \\
&= \begin{pmatrix} x_4 & 0 \\ -x_3 & 1 \end{pmatrix} \begin{pmatrix} x_2^2 \\ 0 \\ 0 \\ 0 \end{pmatrix} \Big| \begin{pmatrix} x_1 \\ x_3 x_1 + x_4 x_2 \end{pmatrix} \\
&= \begin{pmatrix} x_4(x_3 x_1 + x_4 x_2)^2 \\ -x_3(x_3 x_1 + x_4 x_2)^2 \\ 0 \\ 0 \end{pmatrix} \\
E_4(T, c) &= c_4 + E_{4,0}(T, c) + E_{4,1}(T, c) + E_{4,2}(T, c) \\
&= \begin{pmatrix} x_4(x_3 x_1 + x_4 x_2)^2 \\ -x_3(x_3 x_1 + x_4 x_2)^2 \\ x_4^m \\ 0 \end{pmatrix}
\end{aligned}$$

So indeed $H = E_4(T, c)$.

Example 4.9

In this example we consider the last item of example 3.31. Let $F = X + H$ where

$$H = \begin{pmatrix} -x_4^2x_3 - e_3x_4x_3^2 + g_4x_1x_4x_3 - k_3x_3^3 + m_4x_3^2x_1 \\ + g_4^2x_3^2x_2 \\ -x_4^2x_1 - e_4x_4x_3^2 - 2\frac{m_4x_4x_3x_1}{g_4} - g_4x_3x_4x_2 - k_4x_3^3 \\ -\frac{m_4^2x_3^2x_1}{g_4^2} - m_4x_3^2x_2 \\ -\frac{1}{3}x_4^3 \\ 0 \end{pmatrix}$$

Now consider the following element (T, c) of $\mathcal{D}_4(\mathbb{C})$ where

$$T = \left(\begin{pmatrix} 1 & 0 \\ g_4x_4 + m_4x_3 & x_3g_4^2 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \right)$$

and

$$c = \left(\begin{pmatrix} \frac{x_2}{g_4} \end{pmatrix}, \begin{pmatrix} -x_4^2x_3 - e_3x_4x_3^2 - k_3x_3^3 \\ -e_4x_4x_3^2 - k_4x_3^3 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ -\frac{1}{3}x_4^3 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \right)$$

Because T_3 and T_4 are identity matrices

$$\tilde{T}_2\tilde{T}_3\tilde{T}_4 = \tilde{T}_2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ g_4x_4 + m_4x_3 & x_3g_4^2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

and hence

$$\text{Adj}(\tilde{T}_2\tilde{T}_3\tilde{T}_4) = \begin{pmatrix} x_3g_4^2 & 0 & 0 & 0 \\ -g_4x_4 - m_4x_3 & 1 & 0 & 0 \\ 0 & 0 & x_3g_4^2 & 0 \\ 0 & 0 & 0 & x_3g_4^2 \end{pmatrix}$$

With this observation it becomes easy to compute the $E_{4,p}$'s using theorem 4.6:

$$E_{4,0}(T, c) = \begin{pmatrix} 0 \\ 0 \\ -\frac{1}{3}x_4^3 \\ 0 \end{pmatrix}$$

$$E_{4,1}(T, c) = \begin{pmatrix} -x_4^2x_3 - e_3x_4x_3^2 - k_3x_3^3 \\ -e_4x_4x_3^2 - k_4x_3^3 \\ 0 \\ 0 \end{pmatrix}$$

$$E_{4,2}(T, c) = \begin{pmatrix} x_3 g_4^2 \left(\frac{x_1 x_4}{g_4} + \frac{x_1 m_4 x_3}{g_4^2} + x_3 x_2 \right) \\ (-g_4 x_4 - m_4 x_3) \left(\frac{x_1 x_4}{g_4} + \frac{x_1 m_4 x_3}{g_4^2} + x_3 x_2 \right) \\ 0 \\ 0 \end{pmatrix}$$

Now because $c_4 = 0$,

$$\sum_{p=0}^2 E_{4,p}(T, c) = \begin{pmatrix} -x_4^2 x_3 - e_3 x_4 x_3^2 - k_3 x_3^3 + x_3 g_4^2 \left(\frac{x_1 x_4}{g_4} + \frac{x_1 m_4 x_3}{g_4^2} + x_3 x_2 \right) \\ -e_4 x_4 x_3^2 - k_4 x_3^3 + (-g_4 x_4 - m_4 x_3) \left(\frac{x_1 x_4}{g_4} + \frac{x_1 m_4 x_3}{g_4^2} + x_3 x_2 \right) \\ -\frac{1}{3} x_4^3 \\ 0 \end{pmatrix}$$

And expanding this map gives exactly H .

Both of these previous examples are basically two-dimensional examples. This is due to the fact that T_3 and T_4 are identity matrices. The next example shows a bit more complicated example.

Example 4.10

Consider the tuple (T, c) where

$$T = \left(\begin{pmatrix} 3x_3 + 4 & 9x_3 + 1 \\ 3x_3 + 5 & 5x_3 + 5 \end{pmatrix}, \begin{pmatrix} 6 & 0 & 9 \\ 4 & 6 & 8 \\ 4 & 9 & 6 \end{pmatrix} \right)$$

$$c = \left((7 + 9x_2 + 2x_3), \begin{pmatrix} 4x_3 + 2 \\ 6x_3 + 4 \end{pmatrix}, \begin{pmatrix} 5 \\ 2 \\ 8 \end{pmatrix} \right)$$

Note that $(T, c) \in \mathcal{D}_3(\mathbb{C})$. Note also that this tuple is of low degree: the x_i 's appear with degree 1. The definition of $\mathcal{D}_n(A)$ doesn't specify any restraints on this degree. Hence we can still regard this tuple as a fairly simple example. However if we compute

$H = E_3(T, c)$, we get

$$\left(\begin{array}{l} -3838 - 278406 x_1 - 192051 x_2 - 470259 x_3 - 19941066 x_1 x_2^2 \\ - 5088528 x_1 x_3 - 32995269 x_2^2 x_3 - 2517966 x_2^2 - 9182484 x_3^3 \\ - 1575216 x_1^2 - 4663332 x_1 x_2 - 4088556 x_3^2 - 7811478 x_3 x_2 \\ - 31659012 x_3^2 x_2 - 17451288 x_1 x_3^2 - 9251010 x_2^3 \\ - 11025072 x_1^2 x_3 - 12243312 x_1^2 x_2 - 2314656 x_1^3 \\ - 39470976 x_1 x_2 x_3 \\ 298 + 19568 x_1 + 14328 x_2 + 32952 x_3 + 1885680 x_1 x_2^2 + 412800 x_1 x_3 \\ + 3120120 x_2^2 x_3 + 200880 x_2^2 + 868320 x_3^3 + 128000 x_1^2 \\ + 377280 x_1 x_2 + 331200 x_3^2 + 630720 x_3 x_2 + 2993760 x_3^2 x_2 \\ + 1650240 x_1 x_3^2 + 874800 x_2^3 + 1042560 x_1^2 x_3 + 1157760 x_1^2 x_2 \\ + 218880 x_1^3 + 3732480 x_1 x_2 x_3 \\ 2126 + 156252 x_1 + 106542 x_2 + 264078 x_3 + 10465524 x_1 x_2^2 \\ + 2773152 x_1 x_3 + 17316666 x_2^2 x_3 + 1377324 x_2^2 + 4819176 x_3^3 \\ + 858144 x_1^2 + 2542968 x_1 x_2 + 2228904 x_3^2 + 4261572 x_3 x_2 \\ + 16615368 x_3^2 x_2 + 9158832 x_1 x_3^2 + 4855140 x_2^3 + 5786208 x_1^2 x_3 \\ + 6425568 x_1^2 x_2 + 1214784 x_1^3 + 20715264 x_1 x_2 x_3 \end{array} \right)$$

And this is not exactly what normally comes into mind when thinking about simple examples. Now if one computes the Jacobian matrix of H one gets the dreadful matrix of figure 4.1 on page 77. However if we compute the determinant of this matrix we get $\det(JH) = 0$. Just like it should be. Furthermore, this matrix is nilpotent: $JH^3 = 0$. Now if we put $F = X + H$ then theorem 3.21 claims that F is invertible and $F^{-1} = X + G$ for some $G \in \mathcal{H}_3(\mathbb{C})$. Using Maple one finds F^{-1} hence F is indeed invertible. Computing $G = F^{-1} - X$:

$$\left(\begin{array}{l} -1322095737593 + 562925050 x_1 x_2 x_3 - 8046577845 x_3^2 \\ + 120636840 x_3^2 x_2 + 845635080 x_1 x_3^2 + 656691210 x_1^2 x_2 \\ + 40152875 x_2^2 x_3 + 1972984475 x_1^2 x_3 + 93682230 x_1 x_2^2 \\ + 416803408445 x_1 + 59461646615 x_2 + 178645437465 x_3 \\ - 37547516280 x_1 x_3 - 5356517610 x_3 x_2 - 43801725275 x_1^2 \\ - 12497480980 x_1 x_2 - 891441665 x_2^2 + 120815055 x_3^3 \\ + 1534415890 x_1^3 + 4454830 x_2^3 \\ 547301492640 - 222071350 x_1 x_2 x_3 + 3226259745 x_3^2 - 47590680 x_3^2 x_2 \\ - 333599160 x_1 x_3^2 - 259061670 x_1^2 x_2 - 15840125 x_2^2 x_3 \\ - 778333325 x_1^2 x_3 - 36957210 x_1 x_2^2 - 169821056471 x_1 \\ - 24226331033 x_2 - 72786830019 x_3 + 15054593970 x_1 x_3 \\ + 2147659500 x_3 x_2 + 17562193985 x_1^2 + 5010777220 x_1 x_2 \\ + 357413195 x_2^2 - 47660985 x_3^3 - 605320030 x_1^3 - 1757410 x_2^3 \\ 2902456223481 - 1239468000 x_1 x_2 x_3 + 17699928390 x_3^2 \\ - 265622400 x_3^2 x_2 - 1861948800 x_1 x_3^2 - 1445925600 x_1^2 x_2 \\ - 88410000 x_2^2 x_3 - 4344186000 x_1^2 x_3 - 206272800 x_1 x_2^2 \\ - 915934267548 x_1 - 130668398424 x_2 - 392577077412 x_3 \\ + 82592673330 x_1 x_3 + 11782654590 x_3 x_2 + 96349960980 x_1^2 \\ + 27490529880 x_1 x_2 + 1960892820 x_2^2 - 266014800 x_3^3 \\ - 3378530400 x_1^3 - 9808800 x_2^3 \end{array} \right)$$

In fact the claim in theorem 3.21 is stronger. The theorem also implies that $G \in \mathcal{H}_3(\mathbb{C})$. We prove that this is correct by specifying $(T, c) \in \mathcal{D}_3(\mathbb{C})$ such that $E_3(T, c) = G$. Take the tuple (T', c') where

$$\begin{aligned} T' &= \left(\left(\begin{array}{cc} 1 & 0 \\ 43x_3 - 332 & 109x_3 - 802 \end{array} \right), \left(\begin{array}{ccc} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 7 & 1 & 3 \end{array} \right) \right) \\ c'_1 &= (-405x_2) \\ c'_2 &= \left(\begin{array}{c} 3262006485x_3^3 - 72419200605x_3^2 + 535936312395x_3 \\ -1286846595x_3^3 + 29036337705x_3^2 - 218360490057x_3 \end{array} \right) \\ c'_3 &= \left(\begin{array}{c} -1322095737593 \\ 547301492640 \\ 2902456223481 \end{array} \right) \end{aligned}$$

Using a computer one easily shows that $G = E_3(T, c)$ and hence by theorem 4.5 $G \in \mathcal{H}_3(\mathbb{C})$.

Now if one looks at the total degree of both H and G one sees that for both maps the total degree equals 3. However in the $(T, c) \in \mathcal{D}_3(\mathbb{C})$ corresponding to H , the x_i 's only appear with degree 1, whereas the tuple $(T', c') \in \mathcal{D}_3(\mathbb{C})$ has higher degree: $\deg_{x_3}(c_2) = 3$. Therefore the question whether there exists a tuple $(T'', c'') \in \mathcal{D}_3(\mathbb{C})$ with more or less the same structure as (T', c') is raised. In particular an affirmative answer would imply that the description by the tuples in $\mathcal{D}_3(\mathbb{C})$ is not unique! This observation leads to the more general question

Question 4.11

Do there exist $(T, c) \in \mathcal{D}_n(A)$ and $(T', c') \in \mathcal{D}_n(A)$ such that

1. $(T, c) \neq (T', c')$ and
2. $E_n(T, c) = E_n(T', c')$?

Now applying the same technique on H used to find (T', c') corresponding to G in example 4.10, we find the tuple $(T'', c'') \in \mathcal{D}_3(\mathbb{C})$ where

$$\begin{aligned} T'' &= \left(\left(\begin{array}{cc} 1 & 0 \\ -64 + 387x_3 & 194 + 981x_3 \end{array} \right), \left(\begin{array}{ccc} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 7 & 1 & 3 \end{array} \right) \right) \\ c''_1 &= (5x_2) \\ c''_2 &= \left(\begin{array}{c} 1456785x_3^3 + 964980x_3^2 + 141915x_3 \\ -574695x_3^3 - 171990x_3^2 + 43017x_3 \end{array} \right) \\ c''_3 &= \left(\begin{array}{c} 1453 \\ 724 \\ -3609 \end{array} \right) \end{aligned}$$

such that $H = E_3(T'', c'')$. Hence the answer to the general question 4.11 is affirmative:

Corollary 4.12

Description of $\mathcal{H}_n(A)$ by means of $\mathcal{D}_n(A)$ is not unique.

As a matter of fact we could have seen this already in the first example of a $\mathcal{D}_n(A)$ structure: look at example 4.8. There we have the tuple $(T, c) \in \mathcal{D}_4(\mathbb{R})$ with

$$T = \left(\begin{pmatrix} 1 & 0 \\ x_3 & x_4 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \right)$$

$$c = \left(\begin{pmatrix} x_2^2 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ x_4^m \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \right)$$

Now if we take (T', c') where

$$T' = \left(\begin{pmatrix} 1 & 0 \\ x_3 & x_4 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \right)$$

$$c' = \left(\begin{pmatrix} x_2^2 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 3 + x_4^m \end{pmatrix}, \begin{pmatrix} -1 \\ -2 \\ -3 \\ 0 \end{pmatrix} \right)$$

Obviously also $(T', c') \in \mathcal{D}_4(\mathbb{R})$. One easily verifies that $E_{4,2}(T, c) = E_{4,2}(T', c')$ and that $E_{4,1}(T, c) = E_{4,1}(T', c')$. Furthermore

$$E_{4,0}(T', c') = \begin{pmatrix} 1 \\ 2 \\ 3 + x_4^m \\ 0 \end{pmatrix} = E_{4,0}(T, c) + \begin{pmatrix} 1 \\ 2 \\ 3 \\ 0 \end{pmatrix}$$

Now

$$\begin{aligned} E_4(T', c') &= c'_4 + E_{4,0}(T', c') + E_{4,1}(T', c') + E_{4,2}(T', c') \\ &= \begin{pmatrix} -1 \\ -2 \\ -3 \\ 0 \end{pmatrix} + E_{4,0}(T, c) + \begin{pmatrix} 1 \\ 2 \\ 3 \\ 0 \end{pmatrix} + E_{4,1}(T, c) + E_{4,2}(T, c) \\ &= c_4 + E_{4,0}(T, c) + E_{4,1}(T, c) + E_{4,2}(T, c) \\ &= E_4(T, c) \end{aligned}$$

The trick applied here to get from (T, c) to (T', c') makes use of the fact that the constants in c_4 , which are elements of \mathbb{R} , are also constants in c_3 which are elements of

$\mathbb{R}[x_4]$. In fact this is also what happened in example 4.10. The difference is that here we didn't only shift between c_3 and c_2 , but also between c_1 and c_2 . This has as a side-effect that there is not such a trivial relation between (T, c) and (T'', c'') as was in the case above. The constant x_3 in $\mathbb{R}[x_3]$ is subject to changes by substitution of T_2X in c_1 ; the real constants 1, 2 and 3 above are not changed by substitution of T_4X in c_3 .

4.4 Group-like behaviour

Let $\mathcal{F}_n(A) = \{X + H | H \in \mathcal{H}_n(A)\}$. In theorem 3.21 it was shown that every element $F \in \mathcal{F}_n(A)$ has an inverse in this same class. Obviously $0 \in \mathcal{H}_n(A)$ hence $X \in \mathcal{F}_n(A)$. Taking the composition as operator, this means that $\mathcal{F}_n(A)$ is almost a group. If we can show that it is closed under this composition, it is a group in the usual sense. Unfortunately this is not the case as we shall show in the next example.

Example 4.13

Consider the tuples $(T, c), (T', c') \in \mathcal{D}_2(\mathbb{C})$

$$\begin{aligned} (T, c) &= \left(\begin{pmatrix} 6 & 1 \\ 7 & 9 \end{pmatrix}, (6x_2 + 9), \begin{pmatrix} 6 \\ 8 \end{pmatrix} \right) \\ (T', c') &= \left(\begin{pmatrix} 3 & 1 \\ 4 & 6 \end{pmatrix}, (x_2 + 3), \begin{pmatrix} 3 \\ 2 \end{pmatrix} \right) \end{aligned}$$

Computing $H = E_2(T, c)$ and $H' = E_2(T', c')$ gives

$$H = \begin{pmatrix} 87 + 378x_1 + 486x_2 \\ -55 - 294x_1 - 378x_2 \end{pmatrix}, \quad H' = \begin{pmatrix} 21 + 24x_1 + 36x_2 \\ -10 - 16x_1 - 24x_2 \end{pmatrix}$$

Now if we compose the maps $X + H$ and $X + H'$ we get

$$FF = (X + H) \circ (X + H') = \begin{pmatrix} 1966x_1 + 1237 + 1161x_2 \\ -2684x_2 - 4545x_1 - 2838 \end{pmatrix}$$

And now the claim is that $FF \notin \mathcal{F}_2(\mathbb{C})$. We prove it using example 3.13. According to example 3.13 $HH \in \mathcal{H}_2(\mathbb{C})$ if and only if there exist $a_1, a_2, c_1, c_2 \in \mathbb{C}$ and $f(Y) \in \mathbb{C}[Y]$ such that

$$HH = \begin{pmatrix} a_2f(a_1x_1 + a_2x_2) + c_1 \\ -a_1f(a_1x_1 + a_2x_2) + c_2 \end{pmatrix} \quad (4.2)$$

Because $\deg_{x_1, x_2}(HH) = 1$, obviously $\deg_Y(f(Y)) \leq 1$. So put $f := kY + l$ for $k, l \in \mathbb{C}$. Substitution of this f in equation (4.2) and collection of the coefficients of the x_i gives the equations:

$$\begin{aligned} 0 &= -c_2 - 2838 + a_1l \\ 0 &= 1237 - a_2l - c_1 \\ 0 &= 1965 - a_1ka_2 \\ 0 &= a_1^2k - 4545 \\ 0 &= 1161 - a_2^2k \\ 0 &= -2685 + a_1ka_2 \end{aligned}$$

Obviously $a_2 \neq 0$. Hence $k = \frac{1161}{a_2^2}$. Substitution of k now gives the equations:

$$0 = -c_2 - 2838 + a_1 l$$

$$0 = 1237 - a_2 l - c_1$$

$$0 = 1965 - 1161 \frac{a_1}{a_2}$$

$$0 = -2685 + 1161 \frac{a_1}{a_2}$$

$$0 = 1161 \frac{a_1^2}{a_2^2} - 4545$$

Solving the third equation gives $a_1 = \frac{655}{387} a_2$. Substitution of this partial solution in the fourth equation gives:

$$0 = -720$$

And hence we cannot find a_1, a_2, c_1, c_2 and f as desired. Which means that $HH \notin \mathcal{H}_2(\mathbb{C})$ and hence $FF \notin \mathcal{F}_2(\mathbb{C})$.

Corollary 4.14

The set $\mathcal{F}_n(A)$ is not closed under composition.

In fact we could have noted this also by looking at $\mathcal{N}_2(\mathbb{C})$. Both JH and JH' are nilpotent and hence both elements of $\mathcal{N}_2(\mathbb{C})$. However

$$JHH = \begin{pmatrix} 1965 & 1161 \\ -4545 & -2685 \end{pmatrix} \quad \text{and} \quad JHH^2 = \begin{pmatrix} -1415520 & -835920 \\ 3272400 & 1932480 \end{pmatrix}$$

and clearly JHH is not nilpotent. Hence $HH \notin \mathcal{N}_2(\mathbb{C})$. And we have seen that $\mathcal{H}_2(\mathbb{C}) = \mathcal{N}_2(\mathbb{C})$. See theorem 3.33.

Naturally, the question arises,

Question 4.15

Does $\mathcal{D}_n(A)$ behave like a group?

Because of the different dimensions at all slots in the elements of $\mathcal{D}_n(A)$, the only natural operation on the tuples seems to be a component wise addition. Obviously $(T, c) + (T', c') \in \mathcal{D}_n(A)$. Furthermore the tuple consisting of 0-matrices and 0-vectors has the function of a unit element. And the inverse of (T, c) can be taken by $(-T, -c)$. So it is possible to put a group structure on $\mathcal{D}_n(A)$. However the big question is what the consequences of this group structure are with respect to the corresponding elements of $\mathcal{H}_n(A)$. Of course, one would hope that if $X + E_n(T, c) = F$ then $X + E_n(-T, -c) = F^{-1}$. However this is not true as the following example shows.

Example 4.16

Let $(T, c), (-T, -c) \in \mathcal{D}_2(\mathbb{C})$ where

$$(T, c) = \left(\left(\begin{pmatrix} 5 & 7 \\ 7 & 5 \end{pmatrix}, (4x_2^2 + 6x_2 + 3), \begin{pmatrix} 7 \\ 2 \end{pmatrix} \right) \right)$$

$$(-T, -c) = \left(\left(\begin{pmatrix} -5 & -7 \\ -7 & -5 \end{pmatrix}, (-4x_2^2 - 6x_2 - 3), \begin{pmatrix} -7 \\ -2 \end{pmatrix} \right) \right)$$

then (with $F = X + E_2(T, c)$ and $F' = X + E_2(-T, -c)$)

$$\begin{aligned} F &= \begin{pmatrix} 211x_1 + 22 + 980x_1^2 + 1400x_1x_2 + 500x_2^2 + 150x_2 \\ -209x_2 - 19 - 1372x_1^2 - 1960x_1x_2 - 700x_2^2 - 294x_1 \end{pmatrix} \\ F' &= \begin{pmatrix} -209x_1 + 8 + 980x_1^2 + 1400x_1x_2 + 500x_2^2 - 150x_2 \\ 211x_2 - 23 - 1372x_1^2 - 1960x_1x_2 - 700x_2^2 + 294x_1 \end{pmatrix} \end{aligned}$$

If $F' = F^{-1}$ then $F \circ F' = X$. However $F \circ F'$ equals

$$\begin{pmatrix} -16519x_1 - 11800x_2 + 1960x_1^2 + 2800x_1x_2 + 1000x_2^2 + 67880 \\ 23128x_1 + 16521x_2 - 2744x_1^2 - 3920x_1x_2 - 1400x_2^2 - 95032 \end{pmatrix}$$

So this example shows that there exists a group structure on $\mathcal{D}_n(A)$, but unfortunately there is no clear relation between the inverse in $\mathcal{D}_n(A)$ and the inverse in $\mathcal{F}_n(A)$. However in some cases it is easy to prove that there exists a simple connection between a $\mathcal{D}_n(A)$ -structure for F and the tuple for F^{-1} . Therefore we define a new operation on $\mathcal{D}_n(A)$.

Definition 4.17

Let $(T, c), (T', c') \in \mathcal{D}_n(A)$. Then

$$(T, c) \mp (T', c') = (T_2 - T'_2, \dots, T_n - T'_n, c_1 + c'_1, \dots, c_n + c'_n)$$

Note that $\mathcal{D}_n(A)$ with this operation \mp is not a group: though $(0, 0)$ behaves as a unit element and the inverse of (T, c) is given by $(T, -c)$, there is a problem with the associativity. However if we compare the inverse with respect to \mp in $\mathcal{D}_n(A)$ with the inverse with respect to \circ in $\mathcal{F}_n(A)$, we see that sometimes there exists a trivial link between these two inverses.

Proposition 4.18

Let $(T, c) \in \mathcal{D}_n(A)$ such that $\tilde{T}_i \in \text{Mat}_n(A)$ for $2 \leq i \leq n$ and $\tilde{c}_i \in A^n$ for $1 \leq i \leq n$ and $F = X + E_n(T, c)$. Then $F^{-1} = X + E_n(T, -c)$.

Proof. If $n = 1$ everything is clear. So assume $n \geq 2$. Now

$$\begin{aligned} F^{-1} &= (X + E_n(T, c))^{-1} \\ &= \left(X + c_n + \sum_{p=0}^{n-2} E_{n,p}(T, c) \right)^{-1} \\ &= \left(X + c_n + \sum_{p=0}^{n-2} \text{Adj}(\tilde{T}_{n-p\Delta} \cdots \Delta \tilde{T}_{n-1\Delta} T_n) \tilde{c}_{n-p-1} \mid_{(\tilde{T}_{n-p\Delta} \cdots \Delta \tilde{T}_{n-1\Delta} T_n)X} \right)^{-1} \\ &= \left(X + c_n + \sum_{p=0}^{n-2} \text{Adj}(\tilde{T}_{n-p\Delta} \cdots \Delta \tilde{T}_{n-1\Delta} T_n) \tilde{c}_{n-p-1} \right)^{-1} \end{aligned}$$

$$\begin{aligned}
&= X - c_n - \sum_{p=0}^{n-2} \text{Adj}(\tilde{T}_{n-p} \Delta \cdots \Delta \tilde{T}_{n-1} \Delta T_n) \tilde{c}_{n-p-1} \\
&= X - c_n + \sum_{p=0}^{n-2} \text{Adj}(\tilde{T}_{n-p} \Delta \cdots \Delta \tilde{T}_{n-1} \Delta T_n) (-\tilde{c}_{n-p-1}) |_{(\tilde{T}_{n-p} \Delta \cdots \Delta \tilde{T}_{n-1} \Delta T_n)X} \\
&= X - c_n + \sum_{p=0}^{n-2} E_{n,p}(T, -c) \\
&= X + E_n(T, -c)
\end{aligned}$$

□

Unfortunately this only holds in this trivial case. As soon as we have some x_i appearing in (T, c) this doesn't necessarily hold, as the next example will show.

Example 4.19

Let $(T, c) \in \mathcal{D}_2(\mathbb{C})$ as in example 4.16. Let $F = X + E_2(T, c)$ and $F' = X + E_2(T, -c)$. Then

$$F' = \begin{pmatrix} -209x_1 - 22 - 980x_1^2 - 1400x_1x_2 - 500x_2^2 - 150x_2 \\ 211x_2 + 19 + 1372x_1^2 + 1960x_1x_2 + 700x_2^2 + 294x_1 \end{pmatrix}$$

Composing gives:

$$F \circ F' = \begin{pmatrix} -16519x_1 - 11800x_2 + 67850 \\ 23128x_1 + 16521x_2 - 94990 \end{pmatrix}$$

which clearly implies that $F' \neq F^{-1}$.

In chapter 5 we will come back to this point and show that given $F = X + E_n(T, c)$ for some $(T, c) \in \mathcal{D}_n(A)$, \mp may not give the right tuple $(T, -c)$ such that $F^{-1} = X + E_n(T, -c)$, but it does provide a very good key to compute F^{-1} .

-278406 - 19941066 x_2^2 - 5088528 x_3 - 3150432 x_1 - 4663332 x_2 - 17451288 x_3^2 - 22050144 x_1x_3 - 24486624 x_1x_2 - 6943968 x_1^2 - 39470976 x_3x_2	-192051 - 39882132 x_1x_2 - 65990538 x_3x_2 - 5035932 x_2 - 4663332 x_1 - 7811478 x_3 - 31659012 x_3^2 - 27753030 x_2^2 - 12243312 x_1^2 - 39470976 x_1x_3	-470259 - 5088528 x_1 - 32995269 x_2^2 - 27547452 x_3^2 - 8177112 x_3 - 7811478 x_2 - 63318024 x_3x_2 - 34902576 x_1x_3 - 11025072 x_1^2 - 39470976 x_1x_2
19568 + 1885680 x_2^2 + 412800 x_3 + 256000 x_1 + 377280 x_2 + 1650240 x_3^2 + 2085120 x_1x_3 + 2315520 x_1x_2 + 656640 x_1^2 + 3732480 x_3x_2	14328 + 3771360 x_1x_2 + 6240240 x_3x_2 + 401760 x_2 + 377280 x_1 + 630720 x_3 + 2993760 x_3^2 + 2624400 x_2^2 + 1157760 x_1^2 + 3732480 x_1x_3	32952 + 412800 x_1 + 3120120 x_2^2 + 2604960 x_3^2 + 662400 x_3 + 630720 x_2 + 5987520 x_3x_2 + 3300480 x_1x_3 + 1042560 x_1^2 + 3732480 x_1x_2
156252 + 10465524 x_2^2 + 2773152 x_3 + 1716288 x_1 + 2542968 x_2 + 9158832 x_3^2 + 11572416 x_1x_3 + 12851136 x_1x_2 + 3644352 x_1^2 + 20715264 x_3x_2	106542 + 20931048 x_1x_2 + 34633332 x_3x_2 + 2754648 x_2 + 2542968 x_1 + 4261572 x_3 + 16615368 x_3^2 + 14565420 x_2^2 + 6425568 x_1^2 + 20715264 x_1x_3	264078 + 2773152 x_1 + 17316666 x_2^2 + 14457528 x_3^2 + 4457808 x_3 + 4261572 x_2 + 33230736 x_3x_2 + 18317664 x_1x_3 + 5786208 x_1^2 + 20715264 x_1x_2

Figure 4.1: JH where H as in example 4.10.

Chapter 5

Stably tame automorphisms

Introduction

In chapter 4 we introduced the $\mathcal{D}_n(A)$ -structure as a method to describe mappings $F = X + H$ with $H \in \mathcal{H}_n(A)$ in a short, explicit manner. In this chapter we go beyond this point and show that it is also useful from a theoretical point of view. We use the $\mathcal{D}_n(A)$ -structure to prove that $F = X + H$ with $H \in \mathcal{H}_n(A)$ can be written as a finite product of $\exp(D_i)$'s where each D_i is a locally nilpotent derivation. In order to find these derivations we start this chapter by describing a link between the $(\mathcal{H}_n(A), \mathcal{D}_n(A))$ -tuple on one side and derivations on the other side.

5.1 Nice derivations

The main aim of this section is to prove that for given $(T, c) \in \mathcal{D}_n(A)$ there exists a locally nilpotent derivation D such that $X + E_{n,p}(T, c) = \exp(D)$ for each $0 \leq p \leq n-2$. In fact we shall show that $D^2(x_i) = 0$ for $1 \leq i \leq n$. In order to accomplish this, we introduce the notions of so-called nice derivations. Therefore we have to generalise some notions of chapter 3 to arbitrary finitely generated A -algebras.

So let $B := A[x_1, \dots, x_n]$ be a finitely generated A -algebra and D a subset of $\text{Der}_A(B)$. By B^D we denote the set of all $b \in B$ such that $d(b) = 0$ for all $d \in D$.

Definition 5.1

Let $D \subset \text{Der}_A(B)$ a finite subset and $\tau \in \text{Der}_A(B)$.

1. The derivation τ is *derived from D in at most one step* if τ is of the form

$$\tau = \sum_{d \in D} b_d d \quad (5.1)$$

where $b_d \in B^D$ for all $d \in D$.

2. Let $m \geq 2$. The derivation τ is *derived from D in at most m steps* if there exists a sequence of finite subsets

$$D = D_0, D_1, D_2, \dots, D_m$$

of $\text{Der}_A(B)$ such that $\tau \in D_m$ and all elements of D_i are derived from D_{i-1} in at most one step, for all $1 \leq i \leq m$.

3. Let $m \geq 2$. If τ is derived from D in at most m steps and $d_1 d_2(x_i) = 0$ for all $d_1, d_2 \in D$ and all i , then τ is called *nice of order $\leq m$* , with respect to x_1, \dots, x_n and D .

This definition enables us to prove

Proposition 5.2

Let $d_1, d_2 \in D$ such that $d_1 d_2(x_i) = 0$ for all x_i . Let $D = D_0, \dots, D_m$ be subsets of $\text{Der}_A(B)$ such that all elements of D_i are derived from D_{i-1} in at most one step. Then $d_1 d_2(x_i) = 0$ for all x_i and all $d_1, d_2 \in D_m$.

Proof. By induction on m . If $m = 0$ then $D_0 = D$, hence the statement is true. Now assume $m \geq 2$ and $d_1, d_2 \in D_m$. Then

$$\begin{aligned} d_1 &= \sum_{d \in D_{m-1}} b_d d \\ d_2 &= \sum_{e \in D_{m-1}} c_e e \end{aligned}$$

for $b_d, c_e \in B^{D_{m-1}}$. Now

$$\begin{aligned} d_1 d_2(x_i) &= d_1 \left(\sum_{e \in D_{m-1}} c_e e(x_i) \right) \\ &= \sum_{d \in D_{m-1}} b_d d \left(\sum_{e \in D_{m-1}} c_e e(x_i) \right) \\ &= \sum_{d \in D_{m-1}} \sum_{e \in D_{m-1}} b_d d(c_e e(x_i)) \\ &= \sum_{d \in D_{m-1}} \sum_{e \in D_{m-1}} b_d d(c_e) e(x_i) + b_d c_e d(e(x_i)) \end{aligned}$$

Note that $c_e \in B^{D_{m-1}}$ and $d \in D_{m-1}$. Hence $d(c_e) = 0$. By induction we have that $d(e(x_i)) = de(x_i) = 0$ for all i and $d, e \in D_{m-1}$. Hence $d_1 d_2(x_i) = 0$ for all i and $d_1, d_2 \in D_m$. \square

A trivial consequence of this proposition is:

Corollary 5.3

Let $\tau \in \text{Der}_A(B)$ such that τ is nice of order m with respect to x_1, \dots, x_n and D . Then $\tau^2(x_i) = 0$ for $1 \leq i \leq n$. And hence τ is locally nilpotent.

We demonstrate these aspects on the so-called Winkelmann derivation. See [87].

Example 5.4

Let $\partial_i = \frac{\partial}{\partial x_i}$ and $\tau = (1 + x_4x_2 - x_5x_3)\partial_1 + x_5\partial_2 + x_4\partial_3$, a derivation on $B := A[x_1, x_2, x_3, x_4, x_5]$. Let $D = \{\partial_1, \partial_2, \partial_3\}$. Then τ is nice of order two with respect to x_1, x_2, x_3, x_4, x_5 and D . We prove this claim by presenting a sequence of finite subsets of $\text{Der}_A(B)$,

$$D = D_0, D_1, D_2$$

Take

$$\begin{aligned} D_1 &:= \{\partial_1, x_5\partial_2 + x_4\partial_3\} \\ D_2 &:= \{\tau\} \end{aligned}$$

Note that in definition 5.1 it is not demanded that the set D_i of this sequence is a subset of D_{i+1} . We only have to show is that each D_i is a finite subset of $\text{Der}_A(B)$ and that each element of D_i can be derived from D_{i-1} in at most one step. Obviously $\{1, x_4, x_5\} \subset B^D$, hence ∂_1 and $x_5\partial_2 + x_4\partial_3$ are derived from D in one step. It is also clear that $\{1 + x_4x_2 - x_5x_3, x_4, x_5\} \subset B^{D_1}$. Hence τ is derived from D_1 in one step. Obviously we have $d_1d_2(x_i) = 0$ for all $d_1, d_2 \in D$, hence τ is nice of order two with respect to x_1, x_2, x_3, x_4, x_5 and D . According to corollary 5.3 it should hold that $\tau^2(x_i) = 0$ for all x_i . Let's check:

	τ	τ^2
x_1	$1 + x_4x_2 - x_5x_3$	$x_5x_4 - x_4x_5 = 0$
x_2	x_5	0
x_3	x_4	0
x_4	0	0
x_5	0	0

And of course the corollary really works.

Now let $B := A[x_1, \dots, x_n]$ be a finitely generated A -algebra. Consider the ring homomorphism $\varphi : A[X_1, \dots, X_n] \rightarrow B$ defined by $\varphi(X_i) = x_i$ for all i . For each $p, q \geq 1$ consider the natural extension

$$\varphi : \text{Mat}_{p,q}(A[X_1, \dots, X_n]) \rightarrow \text{Mat}_{p,q}(B).$$

Then for each $(T, c) \in \mathcal{D}_n(A)$ we define

$$E_{n,p}(T, c)(x) := \varphi(E_{n,p}(T, c)) \in B^n.$$

Now let $(\partial_1, \dots, \partial_n)$ be an n -tuple of A -derivations of B . To each vector $b = (b_1, \dots, b_n)^t \in B^n$ we associate the following A -derivation of B :

$$D(b; \partial_1, \dots, \partial_n) := b_1\partial_1 + \dots + b_n\partial_n$$

Before we present the next lemma we introduce some more notation.

$$\begin{pmatrix} x'_1 \\ \vdots \\ x'_n \end{pmatrix} := T_n \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$$

$$\begin{aligned} \begin{pmatrix} \partial'_1 \\ \vdots \\ \partial'_n \end{pmatrix} &:= (\text{Adj}(T_n))^t \begin{pmatrix} \partial_1 \\ \vdots \\ \partial_n \end{pmatrix} \\ X'_{n-1} &:= \begin{pmatrix} x'_1 \\ \vdots \\ x'_{n-1} \end{pmatrix} \\ \rho(T, c) &:= \pi(T(X_n = x'_n), c(X_n = x'_n)) \in \mathcal{D}_n(A[x'_n]) \end{aligned}$$

Lemma 5.5

Let $n \geq 3$ and $1 \leq p \leq n - 2$. Let $(T, c) \in \mathcal{D}_n(A)$. Then

$$D(E_{n,p}(T, c)(X); \partial_1, \dots, \partial_n) = D(E_{n-1,p-1}(\rho(T, c))(X'_{n-1}); \partial'_1, \dots, \partial'_{n-1}).$$

Proof.

$$\begin{aligned} &D(E_{n,p}(T, c)(x); \partial_1, \dots, \partial_n) \\ &= \sum_{i=1}^n (E_{n,p}(T, c)(x))_i \partial_i \\ &= (E_{n,p}(T, c)(x))^t \begin{pmatrix} \partial_1 \\ \vdots \\ \partial_n \end{pmatrix} \\ &= \left(\text{Adj}(T_n) \begin{pmatrix} E_{n-1,p-1}(\pi(T, c)) \\ 0 \end{pmatrix} \Big|_{T_n X} \right)^t \begin{pmatrix} \partial_1 \\ \vdots \\ \partial_n \end{pmatrix} \\ &= \left((E_{n-1,p-1}(\pi(T, c)) \Big|_{T_n X})^t, 0 \right) (\text{Adj}(T_n))^t \begin{pmatrix} \partial_1 \\ \vdots \\ \partial_n \end{pmatrix} \\ &= \left((E_{n-1,p-1}(\rho(T, c))(X'_{n-1}))^t, 0 \right) \begin{pmatrix} \partial'_1 \\ \vdots \\ \partial'_n \end{pmatrix} \\ &= \left((E_{n-1,p-1}(\rho(T, c))(X'_{n-1})) \right)^t \begin{pmatrix} \partial'_1 \\ \vdots \\ \partial'_{n-1} \end{pmatrix} \\ &= D(E_{n-1,p-1}(\rho(T, c))(X'_{n-1}); \partial'_1, \dots, \partial'_{n-1}) \end{aligned}$$

□

In addition to this lemma we prove

Lemma 5.6

Let $a \in A$ and let $\partial_1, \dots, \partial_n$ be A -derivations of B such that $\partial_i(x_j) = a\delta_{ij}$ for all i, j . Then

$$\partial'_i(x'_j) = a \det(T_n) \delta_{ij}$$

for all i, j .

Proof. Denote the i -th column of $\text{Adj}(T_n)$ by $(t_{1,i}^*, \dots, t_{n,i}^*)^t$ and the j -th row of T_n by $(t_{j,1}, \dots, t_{j,n})$. Then

$$\begin{aligned} \partial'_i(x'_j) &= \left(\sum_{s=1}^n t_{s,i}^* \partial_s \right) \left(\sum_{s=1}^n t_{j,s} x_s \right) \\ &= \sum_{s=1}^n a t_{s,i}^* t_{j,s} \\ &= a (T_n \text{Adj}(T_n))_{j,i} \\ &= a \det(T_n) \delta_{i,j} \end{aligned}$$

□

These lemmas can now be used to prove that the derivations associated with the polynomial mappings $E_{n,p}(T, c)$ are in fact nice derivations.

Theorem 5.7

Let $\partial_1, \dots, \partial_n$ be A -derivations on $A[x_1, \dots, x_n]$ such that there exists an element $a \in A$ such that $\partial_i(x_j) = a\delta_{ij}$ for all i, j . Let $(T, c) \in \mathcal{D}_n(A)$. Then the A -derivation $d := D(E_{n,p}(T, c)(x); \partial_1, \dots, \partial_n)$ is nice with respect to x_1, \dots, x_n and $D_0 := \{\partial_1, \dots, \partial_n\}$, for all $n \geq 2$ and all $0 \leq p \leq n - 2$.

Proof. The proof is split into three parts.

1. The hypotheses on the ∂_i imply that $dd'(x_i) = 0$ for all $d, d' \in D_0$ and all i .
2. First we consider the case $p = 0$. Then

$$E_{n,0}(T, c) = \text{Adj}(T_n) \widetilde{c_{n-1}|_{T_n X}}$$

So

$$d = (\widetilde{c_{n-1}|_{T_n X}})^t (\text{Adj}(T_n))^t \begin{pmatrix} \partial_1 \\ \vdots \\ \partial_n \end{pmatrix}$$

Write $\widetilde{c_{n-1}}^t = (\gamma_1(x_n), \dots, \gamma_{n-1}(x_n), 0)$. Then the definition of x'_n and the ∂'_j imply that

$$\begin{aligned} d &= (\gamma_1(x'_n), \dots, \gamma_{n-1}(x'_n), 0) (\partial'_1, \dots, \partial'_n)^t \\ &= \sum_{i=1}^{n-1} \gamma_i(x'_n) \partial'_i \end{aligned} \tag{5.2}$$

Put $D_1 := \{\partial'_1, \dots, \partial'_{n-1}\}$ and observe that $D_1 \subset \text{Der}_A(B)$ and that each element of D_1 is derived from D_0 in at most one step. Finally since $\partial'_i(x'_n) = 0$ for all $1 \leq i \leq n - 1$ (by lemma 5.6) we get that $\gamma_i(x'_n) \in B^{D_1}$ for all $1 \leq i \leq n - 1$. So (5.2) implies that d is derived from D_1 in at most one step. Consequently d is derived from D_0 in at most two steps. So d is nice with respect to x_1, \dots, x_n and D_0 by case 1.

3. Now we prove the theorem by induction on n . If $n = 2$, then $p = 0$ and we are in case 2. So let $n \geq 3$. By case 2 we may assume that $p \geq 1$. Then by lemma 3.17 we have

$$d = D(E_{n-1,p-1}(\rho(T, c))(X'_{n-1}); \partial'_1, \dots, \partial'_{n-1})$$

with $\rho(T, c) \in \mathcal{D}_{n-1}(A[x'_n])$. By lemma 5.6 we can apply the induction hypothesis to the ring $A[x'_n]$ and the $(n-1)$ -tuple of $A[x'_n]$ -derivations $\partial'_1, \dots, \partial'_{n-1}$ on the $A[x'_n]$ -algebra $B' := A[x'_n][x'_1, \dots, x'_{n-1}]$. So the $A[x'_n]$ -derivation d on B' is nice with respect to $D'_0 := \{\partial'_1, \dots, \partial'_{n-1}\}$ and x'_1, \dots, x'_{n-1} . So there exists a sequence

$$D'_0, D'_1, \dots, D'_m$$

of finite subsets of $\text{Der}_{A[x'_n]}(B')$ such that $d \in D'_m$ and D'_i is derived from D'_{i-1} in at most one step for all $1 \leq i \leq m$. Now observe that $D'_0 \subset \text{Der}_A(B)$ and that $B' \subset B$ since by definition obviously $x'_i \in B$ for all i . Consequently if d' is an $A[x'_n]$ -derivation of B' derived from D'_0 in at most one step, then $d' \in \text{Der}_A(B)$. Hence $D'_1 \subset \text{Der}_A(B)$. Arguing in a similar way we conclude by induction on i that $D'_i \subset \text{Der}_A(B)$ for all $0 \leq i \leq m$. Since as remarked in case 2 above, all elements of D'_0 ($= D_1$ in case 2) are derived from D_0 in at most one step we deduce that d is derived from D_0 in at most $m + 1$ steps. Just define $D_i := D'_{i-1}$ for all $1 \leq i \leq m + 1$. Hence d is nice with respect to x_1, \dots, x_n and D_0 by case 1.

This concludes the proof. □

An important consequence of this theorem yields:

Corollary 5.8

Let $(T, c) \in \mathcal{D}_n(A)$. Let $n \geq 2$ and $0 \leq p \leq n - 2$. Put

$$D := D\left(E_{n,p}(T, c); \frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n}\right).$$

Then D is nice with respect to x_1, \dots, x_n and $\{\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n}\}$. Furthermore we have $\exp(D) = X + E_{n,p}(T, c)$ and the inverse map is given by $\exp(-D) = X - E_{n,p}(T, c)$.

Proof. The first part is an immediate consequence of theorem 5.7. Furthermore $D^2(x_i) = 0$ by corollary 5.3. So $\exp(D)(X) = X + E_{n,p}(T, c)(X)$ and the inverse map is given by $\exp(-D)(X) = X - E_{n,p}(T, c)(X)$. □

In particular we have achieved our goal as mentioned at the start of this section.

5.2 $\mathcal{H}_n(A)$ implies stably tameness

5.2.1 Exponents of locally nilpotent derivations

In chapter 4 we have seen that we can write each $H \in \mathcal{H}_n(A)$ as $E_n(T, c)$ for some $(T, c) \in \mathcal{D}_n(A)$. By definition of $E_n(T, c)$ this means that H can be written as a sum

of $E_{n,p}(T, c)$'s. In section 5.1 we have seen that each $X + E_{n,p}(T, c)$ can be written as $\exp(D)$ for some locally nilpotent derivation. In this section we are going to combine these properties. More precisely, we are going to show that each F where $F = X + H$ and $H \in \mathcal{H}_n(A)$ can be written as a finite product of $\exp(D_i)$'s where the D_i 's are locally nilpotent derivations. See theorem 5.12. And as a consequence we show that such polynomial maps are stably tame.

Before we are able to prove the announced theorem, we must do some preliminary work.

Lemma 5.9

Let $n \geq 3$, $0 \leq p \leq n - 2$, $0 \leq j \leq p$ and $(T, c) \in \mathcal{D}_n(A)$. Put $f := X + \sum_{q=p}^{n-2} E_{n,q}(T, c)$. Then

$$[(\tilde{T}_{n-p+j} \Delta \cdots \Delta \tilde{T}_{n-1} \Delta T_n) f]_i = [(\tilde{T}_{n-p+j} \Delta \cdots \Delta \tilde{T}_{n-1} \Delta T_n) X]_i$$

for all $i \geq n - p + j$.

Proof. Put $U := \tilde{T}_{n-p+j} \Delta \cdots \Delta \tilde{T}_{n-1} \Delta T_n$. It suffices to show that for each $q \geq p$

$$[UE_{n,q}(T, c)]_i = 0 \tag{5.3}$$

for all $i \geq n - p + j$. So let $q \geq p$, then $q \geq p - j$. The proof is split into two cases.

1. We first treat the case that $q = p - j$. Then $j = 0$ and $q = p$. Consequently $U = \tilde{T}_{n-p} \Delta \cdots \Delta \tilde{T}_{n-1} \Delta T_n$, $E_{n,q}(T, c) = E_{n,p}(T, c)$ and hence by theorem 4.5

$$\begin{aligned} UE_{n,q}(T, c) &= U \text{Adj}(U) \widetilde{c_{n-p-1}|_{UX}} \\ &= \det(U) \widetilde{c_{n-p-1}|_{UX}} \end{aligned}$$

Since the last $p + 1$ coordinates of $\widetilde{c_{n-p-1}}$ are zero, we obtain that

$$[UE_{n,q}(T, c)]_i = 0$$

for all $i \geq n - p$, which proves the case that $q = p - j$.

2. Now assume that $q \geq p - j + 1$. So $n - q \leq n - p + j - 1$. Put $V := \widetilde{T_{n-q} \Delta \cdots \Delta T_{n-p+j-1}}$. Then by theorem 4.5 we can write

$$\begin{aligned} E_{n,q}(T, c) &= \text{Adj}(V \Delta U) \widetilde{c_{n-q-1}|_{(V \Delta U)X}} \\ &= \text{Adj}(V|_{UX} \cdot U) \widetilde{c_{n-q-1}|_{(V \Delta U)X}} \\ &= \text{Adj}(U) \text{Adj}(V|_{UX}) \widetilde{c_{n-q-1}|_{(V \Delta U)X}} \end{aligned}$$

Consequently

$$UE_{n,q}(T, c) = \det(U) \text{Adj}(V|_{UX}) \widetilde{c_{n-q-1}|_{(V \Delta U)X}} \tag{5.4}$$

Note that V , hence $V|_{UX}$, is of the form \tilde{B} for some $B \in M_{n-p+j-1}(A[X])$. Furthermore $(\widetilde{c_{n-q-1}})_i = 0$ if $i \geq n - q$ which implies that

$$\left(\widetilde{c_{n-q-1}|_{(V \Delta U)X}} \right)_i = 0$$

if $i \geq n - p + j$ (since $n - p + j > n - q$). Now the desired result (5.3) follows from (5.4).

□

Corollary 5.10

Notations as in lemma 5.9. Then

$$\left(\widetilde{T}_{n-(p-j)\Delta} \cdots \Delta \widetilde{T}_{n-1\Delta} T_n\right)(f) = \widetilde{T}_{n-(p-j)\Delta} \cdots \Delta \widetilde{T}_{n-1\Delta} T_n.$$

Proof. By induction on $N := p - j$. If $N = 0$ the result is clear. So let $N \geq 1$. Then

$$\begin{aligned} & \left(\widetilde{T}_{n-(p-j)\Delta} \cdots \Delta \widetilde{T}_{n-1\Delta} T_n\right)(f) \\ &= \widetilde{T}_{n-(p-j)} \Big|_{(T_{n-(p-j)+1\Delta} \cdots \Delta \widetilde{T}_{n-1\Delta} T_n)(f)f} (T_{n-(p-j)+1\Delta} \cdots \Delta \widetilde{T}_{n-1\Delta} T_n)(f) \\ &= \widetilde{T}_{n-(p-j)} \Big|_{(T_{n-(p-j)+1\Delta} \cdots \Delta \widetilde{T}_{n-1\Delta} T_n)f} (T_{n-(p-j)+1\Delta} \cdots \Delta \widetilde{T}_{n-1\Delta} T_n) \end{aligned}$$

by the induction hypothesis. Finally observe that the matrix elements of \widetilde{T}_{n-p+j} depend only on $x_{n-p+j+1}, \dots, x_n$. The result follows immediately from lemma 5.9 (with $j + 1$ instead of j). □

Lemma 5.11

Let $n \geq 3$, $0 \leq p \leq n - 3$ and $(T, c) \in \mathcal{D}_n(A)$. Then

$$\exp(-D(E_{n,p}(T, c))) \circ \left(X + \sum_{q=p}^{n-2} E_{n,q}(T, c)\right) = X + \sum_{q=p+1}^{n-2} E_{n,q}(T, c).$$

Proof. Put $G := \exp(-D(E_{n,p}(T, c)))$. So $G = X - E_{n,p}(T, c)$ (by corollary 5.8). Hence if we put

$$U := \widetilde{T}_{n-p\Delta} \cdots \Delta \widetilde{T}_{n-1\Delta} T_n$$

then by theorem 4.5 we get

$$G = X - \text{Adj}(U) \widetilde{c}_{n-p-1|UX}$$

So if we put

$$f := X + \sum_{q=p}^{n-2} E_{n,q}(T, c)$$

then

$$G \circ f = f - \text{Adj}(U(f)) \widetilde{c}_{n-p-1|U(f)f}$$

Since $U(f) = U$ (by corollary 5.10 above with $j = 0$) we get

$$G \circ f = f - \text{Adj}(U) \widetilde{c}_{n-p-1|Uf}$$

Now observe that each component of \widetilde{c}_{n-p-1} belongs to $A[x_{n-p}, \dots, x_n]$ and that for each $i \geq n - p$ $(Uf)_i = (UX)_i$ by lemma 5.9. So $\widetilde{c}_{n-p-1|Uf} = \widetilde{c}_{n-p-1|UX}$ and hence

$$\begin{aligned} G \circ f &= f - \text{Adj}(U) \widetilde{c}_{n-p-1|UX} \\ &= f - E_{n,p}(T, c) \end{aligned}$$

by theorem 4.5. □

Now we have enough tools to prove the main theorem of this section.

Theorem 5.12

Let $n \geq 2$. Let $F = X + H$, where $H = E_n(T, c)$, for some $(T, c) \in \mathcal{D}_n(A)$. Then

$$F = \exp\left(D\left(c_n; \frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n}\right)\right) \prod_{p=0}^{n-2} \exp\left(D\left(E_{n,p}(T, c); \frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n}\right)\right).$$

Proof. Observe that

$$\exp\left(-D\left(c_n; \frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n}\right)\right) \circ F = \sum_{p=0}^{n-2} E_{n,p}(T, c).$$

So the case $n = 2$ follows from corollary 5.8. Hence we may assume that $n \geq 3$. Now theorem 5.12 follows directly from lemma 5.11 and corollary 5.8. \square

5.2.2 Stably tameness

Theorem 5.12 yields a direct connection with the theory about stably tame automorphisms. In particular we solve the stably tame generators conjecture for the $\mathcal{H}_n(A)$ -class. Let us recall this conjecture (it was already mentioned in [7], [21], [24], [27] and [56]):

Conjecture 5.13

For every invertible polynomial map $F : k^n \rightarrow k^n$ over a field k there exist t_1, \dots, t_m such that

$$F^{[m]} = (F, t_1, \dots, t_m) : k^{n+m} \rightarrow k^{n+m}$$

is tame, i.e. F is stably tame.

In order to find a solution to this conjecture for the $\mathcal{H}_n(A)$ -class we use a well known result by Martha Smith in [85], which connects maps of the form $\exp(aD)$ where D is a locally nilpotent derivation and $a \in \ker(D)$, to tame automorphisms:

Lemma 5.14

Let D be a locally nilpotent derivation of $A[X]$. Let $a \in \ker(D)$. Extend D to $A[X][t]$ by setting $D(t) = 0$. Note that tD is locally nilpotent. Define $\rho \in \text{Aut}_A A[X][t]$ by $\rho(x_i) = x_i, i = 1, \dots, n$ and $\rho(t) = t + a$. Then

$$(\exp(aD), t) = \rho^{-1} \exp(-tD) \rho \exp(tD).$$

Corollary 5.15

Let D and a be as in lemma 5.14. If D is conjugate by a tame automorphism to a triangular derivation, then $(\exp(aD), t)$ is tame.

We can use this lemma and corollary to find that $X + E_{n,p}(T, c)$ is stably tame where $(T, c) \in \mathcal{D}_n(A)$. In fact we prove a stronger fact: we show that any nice derivation of a certain class gives a stably tame polynomial map.

Lemma 5.16

Let $D := \{\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n}\}$. Let τ be a nice derivation of order m with respect to x_1, \dots, x_n and D on $A[X]$. Then $\exp(a\tau)$ is stably tame for all $a \in \ker(\tau)$.

Proof. We use induction on m . Consider the case $m = 1$. Then $\tau = \sum_{d \in D} b_d d$ with $b_d \in A[X]^D = \cap_{d \in D} \ker(d) = A$. And hence $\tau(x_i) \in A$ and clearly τ is on triangular form. So now we can apply corollary 5.15 and find that $\exp(a\tau)$ is stably tame.

Now consider the case $m > 1$. We may assume that for all nice derivations $\sigma \in \text{Der}_A(A[X])$ of order $m - 1$ with respect to D and x_1, \dots, x_n and for any commutative ring A we have that $\exp(a\sigma)$ is stably tame for all $a \in \ker(\sigma)$. Let τ be nice of order m . Define ρ and extend τ to $A[X][t]$ as in lemma 5.14 (in fact we extend all derivations of D_i to $A[X][t]$ in this way). Now from

$$(\exp(a\tau), t) = \rho^{-1} \exp(-t\tau) \rho \exp(t\tau)$$

it follows it suffices to see that $\exp(t\tau)$ is stably tame. Now we see that $t\tau = \sum_{d \in D_{m-1}} t b_d d$ with $t b_d \in A[X][t]^{D_{m-1}}$. However, from this it follows that

$$\begin{aligned} \exp(t\tau) &= \exp\left(\sum_{d \in D_{m-1}} t b_d d\right) \\ &= \prod_{d \in D_{m-1}} \exp(t b_d d) \end{aligned}$$

This last equation follows from theorem 4.6. Obviously it suffices to prove that each $\exp(t b_d d)$ is stably tame to conclude that $\exp(t\tau)$ is stably tame. Note that d is a nice derivation of order $m - 1$ and $t b_d \in \ker(d)$. Hence we can apply the induction hypothesis to the ring $A[t]$ and find that $\exp(t\tau)$ is stably tame and hence $\exp(a\tau)$ is stably tame. \square

Now that the real work has been done, we can present the main theorem of this section:

Theorem 5.17

Let $F = X + H$ with $H \in \mathcal{H}_n(A)$. Then F is stably tame.

Proof. Looking at theorem 5.12 we see that each $F = X + H$ with $H \in \mathcal{H}_n(A)$ can be written as the product of a finite number of $\exp(a_i D_i)$'s where each D_i is a nice derivation with respect to x_1, \dots, x_n and $\{\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n}\}$ and $a_i \in \ker(D_i)$. Applying lemma 5.16 n times gives us the desired result: F is stably tame. \square

Hence the stably tame generators conjecture is true for each F with $F = X + H$ where $H \in \mathcal{H}_n(A)$.

5.3 What happens in $\mathcal{H}_2(A)$?

Theorem 5.17 shows that $F = X + H$ with $H \in \mathcal{H}_n(A)$ is stably tame. Naturally one could think one step further and ask the question:

Question 5.18

Let $F = X + H$ with $H \in \mathcal{H}_n(A)$. Is F tame?

Since this is a strong claim, the answer is most likely negative. In fact Nagata already presented an example in 1972 in [75], which is in fact an example in $\mathcal{H}_2(A)$ which is not tame.

The proof is based on the theorem below due to van der Kulk in [61].

Theorem 5.19

Let k be a field. $\text{Aut}_k(k[x_1, x_2])$ is the amalgamated product of the affine automorphisms $\mathfrak{A}_2(k)$ and the de Jonquières automorphisms $\mathfrak{J}_2(k)$.

Let $\mathfrak{J}\mathfrak{A}_2(k)$ be the group generated by $\mathfrak{J}_2(k)$ and $\mathfrak{A}_2(k)$

Example 5.20 (Nagata)

Let A be an integral domain which is not a principal ideal. Let $a, b \in A$ such that $aA + bA$ is not a principal ideal. Let $X = x_1, x_2$. Let $\tau : A[x_1, x_2] \rightarrow A[x_1, x_2]$ be the A -homomorphism defined by

$$\tau \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_1 + bf(ax_1 + bx_2) \\ x_2 - af(ax_1 + bx_2) \end{pmatrix}$$

where $f \in A[Y]$ and $\deg(f) \geq 2$. Then $\tau \in \text{Aut}_2(k)$ but $\tau \notin \mathfrak{J}\mathfrak{A}_2(k)$.

The first part of this claim is easy. Note that $\tau - X \in \mathcal{H}_2(A)$ and hence $\tau \in \text{Aut}_2(A)$. The second claim requires some more work. Suppose τ is tame. Since $d := \deg(f) \geq 2$ the homogeneous component of degree d of τ_1 is $br(ax_1 + bx_2)^d$, where r is the coefficient of Y^d in f . It is $-ar(ax_1 + bx_2)^d$ for τ_2 . Now by a result of Furter in [42] we deduce that there exists

$$C := \begin{pmatrix} p_1 & p_2 \\ p_3 & p_4 \end{pmatrix} \in \text{Gl}_2(A)$$

with $c := \det(C)$, such that

$$p_3br(ax_1 + bx_2)^d + p_4(-a)r(ax_1 + bx_2)^d = 0$$

i.e. $p_3b - p_4a = 0$. Because $p_3(-p_2) + p_4p_1 = c \in A^*$ we get

$$\begin{pmatrix} -p_2 & p_1 \\ b & -a \end{pmatrix} \begin{pmatrix} p_3 \\ p_4 \end{pmatrix} = \begin{pmatrix} c \\ 0 \end{pmatrix}$$

Now Cramer's rule says: $p_3 = -ca$ and $p_4 = -cb$. Substitution in c gives:

$$\begin{aligned} c &= p_3(-p_2) + p_4p_1 \\ &= -ca(-p_2) + (-cb)p_1 \\ &= p_2ca - p_1cb \end{aligned}$$

Because A is a domain this means:

$$1 = p_2a - p_1b$$

And this means that $1 \in (a, b)$ which is a contradiction. Hence τ not tame, i.e. $\tau \notin \mathfrak{J}\mathfrak{A}_2(A)$

The actual example Nagata described in [75] was τ^{-1} instead of τ .

So the conclusion of this section is that even if $n = 2$ it is not necessarily true that $F = X + H$ with $H \in \mathcal{H}_n(A)$ implies that F is tame.

5.4 Finding the factorisations

If we look back at theorem 5.17 we see that there is no upper bound specified for the number of variables one needs to add to get a tame automorphism. In this section we describe two algorithms to find the precise factorisations into tame automorphisms. Both of these methods were found by David Wright and the author. The first -quick- method was found during the authors visit to Washington University, May 1997. The second -stronger- method was found during the process of rewriting the paper [54].

In the descriptions of these methods we often omit the tildes formally needed to get the right dimensions and simply write T or c . Furthermore if we recall theorem 4.6 and remark 4.7, we see that we do not need the complete tuple (T, c) in order to compute $E_{n,p}(T, c)$: only $(T_{n-p}, \dots, T_n, c_{n-p-1})$ is needed. Therefore we introduce a small modification of the definition of $E_{n,p}(T, c)$ in definition 4.4.

Definition 5.21

We define

$$E_{n,p}(T; c_{n-p-1}) := E_{n,p}(T', c') \quad (5.5)$$

for some $(T', c') \in \mathcal{D}_n(A)$ where

$$(T', c') = (T'_2, \dots, T'_{n-p-1}, T_{n-p}, \dots, T_n, c'_1, \dots, c'_{n-p-2}, c_{n-p-1}, c'_{n-p}, \dots, c'_n)$$

So the semicolon shows that we use a 'stripped' version of $(T, c) \in \mathcal{D}_n(A)$. We make further abuse of the notation by still saying $(T; c) \in \mathcal{D}_n(A)$.

Remark 5.22

In [14] Cheng and Wang give an algorithm to write two-dimensional polynomial maps in characteristic zero as a product of linear and triangular ones.

5.4.1 The quick method

The approach we present in this section only deals with $\mathcal{H}_n(A)$. It acts on the level of polynomial maps; no $\mathcal{D}_n(A)$ or derivations are used. This method provides an explicit recipe to find tame R and S such that $R \circ F^{\lfloor \frac{n(n-1)}{2} \rfloor} \circ S$ is tame.

We start by looking at the two-dimensional case.

Example 5.23

Let $F = X + H$ with $H \in \mathcal{H}_2(A)$. From [35] it follows that F can be written as

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \text{Adj}(T) \begin{pmatrix} f \\ 0 \end{pmatrix}_{|_{TX}} + c = \begin{pmatrix} x_1 + a_2 f(a_1 x_1 + a_2 x_2) + c_1 \\ x_2 - a_1 f(a_1 x_1 + a_2 x_2) + c_2 \end{pmatrix}$$

where $T = \begin{pmatrix} 1 & 0 \\ a_1 & a_2 \end{pmatrix} \in M_2(A)$, $c = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} \in A^2$ and $f \in A[x_2]$. Now extend this F to $F^{[1]} := (F_1, F_2, x_3)$. Define $P := (x_1, x_2, x_3 + f(a_1x_1 + a_2x_2))$. Then

$$F^{[1]} \circ P = \begin{pmatrix} x_1 + a_2f(a_1x_1 + a_2x_2) + c_1 \\ x_2 - a_1f(a_1x_1 + a_2x_2) + c_2 \\ x_3 + f(a_1x_1 + a_2x_2) \end{pmatrix}$$

Define also $Q := (x_1 - c_1, x_2 - c_2, x_3)$ and $R := (x_1 - a_2x_3, x_2 + a_1x_3, x_3)$. Then

$$R \circ Q \circ F^{[1]} \circ P = \begin{pmatrix} x_1 - a_2x_3 \\ x_2 + a_1x_3 \\ x_3 + f(a_1x_1 + a_2x_2) \end{pmatrix}$$

And hence

$$\begin{aligned} R \circ Q \circ F^{[1]} \circ P \circ R^{-1} &= \begin{pmatrix} x_1 \\ x_2 \\ x_3 + f(a_1(x_1 + a_2x_3) + a_2(x_2 - a_1x_3)) \end{pmatrix} \\ &= \begin{pmatrix} x_1 \\ x_2 \\ x_3 + f(a_1x_1 + a_2x_2) \end{pmatrix} \end{aligned}$$

which is on triangular form and hence tame.

Remark 5.24

Note that this means that the map in example 5.20 can be reduced to a tame map by adding only one variable.

If we take another look at the definition of P , Q and R , we see that we can describe these maps directly in terms of T , c and f :

$$P = \begin{pmatrix} X \\ x_3 + f(TX) \end{pmatrix}, \quad Q = \begin{pmatrix} X - c \\ x_3 \end{pmatrix}, \quad R = \begin{pmatrix} X - \text{Adj}(T) \begin{pmatrix} x_3 \\ 0 \end{pmatrix} \\ x_3 \end{pmatrix}$$

And it is exactly this idea that gives us the quick method to factor into tame automorphisms. Note that $Q \circ F^{[1]} = R^{-1} \circ P \circ R \circ P^{-1}$, a commutator.

The main theorem in this section uses lemma 3.19. In fact this lemma was discovered only in conjunction with this theorem.

Theorem 5.25

Let $F = X + H$ with $H \in \mathcal{H}_n(A)$. Then there exist tame automorphisms U and V of $A[x_1, \dots, x_n, y_1, \dots, y_{n-1}]$ such that $U \circ F^{[n-1]} \circ V$ is of the form $(X, Y + H')$ where $H' \in \mathcal{H}_{n-1}(A[x_1, \dots, x_n])$ (with respect to the variables y_1, \dots, y_{n-1}).

Proof. Let

$$H = \text{Adj}(T) \begin{pmatrix} H_\star \\ 0 \end{pmatrix} \Big|_{TX} + c$$

with $T \in M_n(A)$, $c \in A^n$ and $H_\star \in \mathcal{H}_{n-1}(A[x_n])$ (with respect to the variables x_1, \dots, x_n). Put $d = \det(T)$ and $TX = (L_1, \dots, L_n)$. Define

$$P := \begin{pmatrix} X \\ Y + H_\star(TX) \end{pmatrix}, \quad Q := \begin{pmatrix} X - c \\ Y \end{pmatrix}, \quad R := \begin{pmatrix} X - \text{Adj}(T) \begin{pmatrix} Y \\ 0 \end{pmatrix} \\ Y \end{pmatrix}$$

$$U := R \circ Q, \quad V := P \circ R^{-1}$$

Then

$$U \circ F^{[n-1]} \circ V = \begin{pmatrix} X \\ Y + H_\star \left(TX + d \begin{pmatrix} Y \\ 0 \end{pmatrix} \right) \end{pmatrix}$$

And if we can show that $H' := H_\star(TX + d(Y, 0)) \in \mathcal{H}_{n-1}(A[x_1, \dots, x_n])$ with respect to the variables y_1, \dots, y_{n-1} , this theorem is proved. To do this, note that $H_\star(x_1, \dots, x_n) \in \mathcal{H}_{n-1}(A[x_n])$ with respect to the variables x_1, \dots, x_{n-1} . In particular x_n is not a variable but a scalar. So obviously

$$\begin{aligned} A[x_n][x_1, \dots, x_{n-1}] &\rightarrow A[x_n][y_1, \dots, y_{n-1}] \\ x_n &\mapsto x_n \\ x_i &\mapsto y_i \quad \text{for } i < n \end{aligned}$$

shows that $H_\star(y_1, \dots, y_{n-1}) \in \mathcal{H}_{n-1}(A[x_n])$ with respect to the variables y_1, \dots, y_{n-1} .

Now consider the homomorphism $\varphi : A[x_n] \rightarrow A[x_1, \dots, x_n]$ with $\varphi(x_n) = L_n$. Apply [35, Lemma 2.1] and see that $\hat{H} := H_\star(y_1, \dots, y_{n-1}, L_n) \in \mathcal{H}_{n-1}(A[x_1, \dots, x_n])$ with respect to the variables y_1, \dots, y_{n-1} . Finally apply lemma 3.19 to $\hat{H}(y_1, \dots, y_{n-1})$ and the ring $A[x_1, \dots, x_n]$ to conclude that

$$\begin{aligned} H' &= H_\star(TX + d(Y, 0)) \\ &= H_\star(dy_1 + L_1, \dots, dy_{n-1} + L_{n-1}, L_n) \\ &= \hat{H}(y_1, \dots, y_{n-1})(dy_1 + L_1, \dots, dy_{n-1} + L_{n-1}) \\ &\in \mathcal{H}_{n-1}(A[x_1, \dots, x_n]) \end{aligned}$$

with respect to the variables y_1, \dots, y_{n-1} . This completes the proof. \square

Corollary 5.26

Let F be as in theorem 5.25. Then $F\left[\frac{n(n-1)}{2}\right]$ is tame.

Proof. With induction on n . If $n = 1$ everything is clear. So assume $n \geq 2$. By theorem 5.25 we have that there exist tame automorphisms U and V such that

$$G = U \circ F^{[n-1]} \circ V = (X, Y + H')$$

with $H' \in \mathcal{H}_{n-1}(A[x_1, \dots, x_n])$. Now by induction we know that

$$G\left[\frac{(n-1)(n-2)}{2}\right]$$

is tame and hence

$$F\left[n-1+\frac{(n-1)(n-2)}{2}\right]$$

is tame. Obviously $n-1+\frac{(n-1)(n-2)}{2}=\frac{n(n-1)}{2}$, which proves the corollary. \square

We end this section by applying this quick method to the map of example 4.8 where we take $m=3$. Although this is basically a simple example, the computations are already pretty complicated. Hence all computations are done using Maple. In order to save some space, we display the vectors horizontally.

Example 5.27

Let F and $(T, c) \in \mathcal{D}_4(\mathbb{C})$ be as in example 4.8 with $m=3$. Then by theorem 5.25 we can define:

$$R_1 := (x_1, x_2, x_3, x_4, x_5, x_6, x_7)$$

$$R_2 := (x_1 - x_5, x_2 - x_6, x_3 - x_7, x_4, x_5, x_6, x_7)$$

$$S_1 := \left(x_1, x_2, x_3, x_4, x_5 + x_4 (x_3 x_1 + x_4 x_2)^2, \right. \\ \left. x_6 - x_3 (x_3 x_1 + x_4 x_2)^2, x_7 + x_4^3 \right)$$

$$S_2 := (x_1 + x_5, x_2 + x_6, x_3 + x_7, x_4, x_5, x_6, x_7)$$

such that

$$R_2 \circ R_1 \circ F^{[3]} \circ S_1 \circ S_2 = \\ \left(x_1, x_2, x_3, x_4, \right. \\ \left. x_5 + x_4 (x_7 x_1 + x_3 x_1 + x_6 x_4 + x_4 x_2 + x_5 x_7 + x_5 x_3)^2, \right. \\ \left. x_6 - (x_3 + x_7) (x_7 x_1 + x_3 x_1 + x_6 x_4 + x_4 x_2 + x_5 x_7 + x_5 x_3)^2, \right. \\ \left. x_7 + x_4^3 \right)$$

Now if we restrict ourselves to the last three components, we can find a describing tuple $(T', c') \in \mathcal{D}_3(\mathbb{C}[x_1, x_2, x_3, x_4])$ for this three-dimensional map:

$$T' := \left(\left(\begin{array}{cc} 1 & 0 \\ x_3 + x_7 & x_4 \end{array} \right), \left(\begin{array}{ccc} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array} \right) \right)$$

$$c' := \left((x_6 (2 x_4 x_2 + 2 x_3 x_1 + 2 x_7 x_1 + x_6)), \right. \\ \left(x_7 x_4 x_1 (2 x_3 x_1 + x_7 x_1 + 2 x_4 x_2), -x_7 (4 x_3 x_1 x_4 x_2 \right. \\ \left. + 2 x_7 x_4 x_1 x_2 + 3 x_3^2 x_1^2 + 3 x_7 x_3 x_1^2 + x_4^2 x_2^2 + x_7^2 x_1^2) \right), \\ \left. (x_4 (x_3 x_1 + x_4 x_2)^2, -x_3 (x_3 x_1 + x_4 x_2)^2, x_4^3) \right)$$

And with this structure we can define

$$R_3 := \left(x_1, x_2, x_3, x_4, x_5 - x_4 (x_3 x_1 + x_4 x_2)^2, \right. \\ \left. x_6 + x_3 (x_3 x_1 + x_4 x_2)^2, x_7 - x_4^3, x_8, x_9 \right) \\ R_4 := (x_1, x_2, x_3, x_4, x_5 - x_8, x_6 - x_9, x_7, x_8, x_9)$$

$$S_3 := \left(x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8 + x_4 (x_5 x_3 + x_5 x_7 + x_6 x_4 + x_7 x_1) \right. \\ \left. (x_5 x_3 + x_5 x_7 + x_7 x_1 + 2 x_3 x_1 + x_6 x_4 + 2 x_4 x_2), \right. \\ x_9 - 2 x_4 x_3^2 x_1 x_6 - 2 x_4 x_3^2 x_5 x_2 - 2 x_4 x_3^2 x_5 x_6 \\ - 4 x_4 x_3 x_7 x_1 x_2 - 4 x_4 x_3 x_7 x_1 x_6 - 4 x_4 x_3 x_7 x_5 x_2 \\ - 4 x_4 x_3 x_7 x_5 x_6 - 2 x_4 x_7^2 x_1 x_2 - 2 x_4 x_7^2 x_1 x_6 - 2 x_4 x_7^2 x_5 x_2 \\ - 2 x_4 x_7^2 x_5 x_6 - 2 x_4^2 x_3 x_2 x_6 - 2 x_4^2 x_7 x_2 x_6 - 6 x_3^2 x_7 x_1 x_5 \\ - 6 x_3 x_7^2 x_1 x_5 - 2 x_3^3 x_1 x_5 - 3 x_3^2 x_7 x_1^2 - 3 x_3^2 x_7 x_5^2 \\ - 3 x_3 x_7^2 x_1^2 - 3 x_3 x_7^2 x_5^2 - 2 x_7^3 x_1 x_5 - x_4^2 x_3 x_6^2 - x_4^2 x_7 x_2^2 \\ \left. - x_4^2 x_7 x_6^2 - x_3^3 x_5^2 - x_7^3 x_1^2 - x_7^3 x_5^2 \right) \\ S_4 := (x_1, x_2, x_3, x_4, x_5 + x_8, x_6 + x_9, x_7, x_8, x_9)$$

such that

$$R_4 \circ R_3 \circ \left(R_2 \circ R_1 \circ F^{[3]} \circ S_1 \circ S_2 \right)^{[2]} \circ S_3 \circ S_4 = \\ \left(x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8 + x_4 \right. \\ \left. (x_4 x_9 + x_6 x_4 + x_7 x_8 + x_3 x_8 + x_5 x_7 + x_5 x_3 + x_7 x_1) \right. \\ \left. (x_5 x_3 + x_5 x_7 + x_7 x_1 + x_7 x_8 + 2 x_4 x_2 \right. \\ \left. + x_4 x_9 + x_3 x_8 + 2 x_3 x_1 + x_6 x_4), \right. \\ -4 x_4 x_3 x_7 x_1 x_9 - x_3^3 x_8^2 - 2 x_4 x_3^2 x_1 x_9 - 2 x_2 x_3^2 x_4 x_8 \\ - 2 x_4 x_3^2 x_5 x_9 - 2 x_4 x_3^2 x_8 x_6 - 2 x_4 x_3^2 x_8 x_9 - 6 x_3 x_7^2 x_1 x_8 \\ - 6 x_3^2 x_7 x_5 x_8 - 6 x_3 x_7^2 x_5 x_8 - 2 x_7^3 x_1 x_8 - 2 x_4^2 x_3 x_2 x_9 \\ - 2 x_4^2 x_7 x_2 x_9 - 6 x_3^2 x_7 x_1 x_8 - 2 x_4 x_7^2 x_1 x_9 - 2 x_2 x_7^2 x_4 x_8 \\ - 2 x_4 x_7^2 x_5 x_9 - 2 x_4 x_7^2 x_8 x_6 - 2 x_4 x_7^2 x_8 x_9 - 2 x_4^2 x_7 x_6 x_9 \\ - x_4^2 x_7 x_9^2 - 3 x_3^2 x_7 x_8^2 - 3 x_3 x_7^2 x_8^2 - 2 x_4^2 x_3 x_6 x_9 \\ - x_4^2 x_3 x_9^2 - 2 x_3^3 x_1 x_8 - 2 x_7^3 x_5 x_8 - x_7^3 x_8^2 - 4 x_4 x_3 x_7 x_5 x_9 \\ - 4 x_4 x_3 x_7 x_8 x_6 - 4 x_4 x_3 x_7 x_8 x_9 - 4 x_2 x_7 x_3 x_4 x_8 - 2 x_3^3 x_5 x_8 \\ + x_9 - 2 x_4 x_3^2 x_1 x_6 - 2 x_4 x_3^2 x_5 x_2 - 2 x_4 x_3^2 x_5 x_6 \\ - 4 x_4 x_3 x_7 x_1 x_2 - 4 x_4 x_3 x_7 x_1 x_6 - 4 x_4 x_3 x_7 x_5 x_2 \\ - 4 x_4 x_3 x_7 x_5 x_6 - 2 x_4 x_7^2 x_1 x_2 - 2 x_4 x_7^2 x_1 x_6 - 2 x_4 x_7^2 x_5 x_2 \\ - 2 x_4 x_7^2 x_5 x_6 - 2 x_4^2 x_3 x_2 x_6 - 2 x_4^2 x_7 x_2 x_6 - 6 x_3^2 x_7 x_1 x_5 \\ - 6 x_3 x_7^2 x_1 x_5 - 2 x_3^3 x_1 x_5 - 3 x_3^2 x_7 x_1^2 - 3 x_3^2 x_7 x_5^2 \\ - 3 x_3 x_7^2 x_1^2 - 3 x_3 x_7^2 x_5^2 - 2 x_7^3 x_1 x_5 - x_4^2 x_3 x_6^2 - x_4^2 x_7 x_2^2 \\ \left. - x_4^2 x_7 x_6^2 - x_3^3 x_5^2 - x_7^3 x_1^2 - x_7^3 x_5^2 \right)$$

The last tuple $(T'', c'') \in \mathcal{D}_2(\mathbb{C}[x_1, x_2, x_3, x_4, x_5, x_6, x_7])$ is found by looking at the last two components:

$$T'' := \left(\left(\begin{array}{cc} 1 & 0 \\ x_3 + x_7 & x_4 \end{array} \right) \right)$$

$$\begin{aligned} c'' := & \left((x_9 (x_9 + 2x_5x_7 + 2x_7x_1 + 2x_6x_4 + 2x_4x_2 + 2x_3x_1 + 2x_5x_3)), \right. \\ & \left(x_4 (x_5x_3 + x_5x_7 + x_6x_4 + x_7x_1) \right. \\ & \left. (x_5x_3 + x_5x_7 + x_7x_1 + 2x_3x_1 + x_6x_4 + 2x_4x_2), \right. \\ & -2x_4x_3^2x_1x_6 - 2x_4x_3^2x_5x_2 - 2x_4x_3^2x_5x_6 - 4x_4x_3x_7x_1x_2 \\ & - 4x_4x_3x_7x_1x_6 - 4x_4x_3x_7x_5x_2 - 4x_4x_3x_7x_5x_6 \\ & - 2x_4x_7^2x_1x_2 - 2x_4x_7^2x_1x_6 - 2x_4x_7^2x_5x_2 - 2x_4x_7^2x_5x_6 \\ & - 2x_4^2x_3x_2x_6 - 2x_4^2x_7x_2x_6 - 6x_3^2x_7x_1x_5 - 6x_3x_7^2x_1x_5 \\ & - 2x_3^3x_1x_5 - 3x_3^2x_7x_1^2 - 3x_3^2x_7x_5^2 - 3x_3x_7^2x_1^2 - 3x_3x_7^2x_5^2 \\ & - 2x_7^3x_1x_5 - x_4^2x_3x_6^2 - x_4^2x_7x_2^2 \\ & \left. \left. - x_4^2x_7x_6^2 - x_3^3x_5^2 - x_7^3x_1^2 - x_7^3x_5^2 \right) \right) \end{aligned}$$

We use this tuple to define the last couple of automorphisms:

$$\begin{aligned} R_5 := & \left(x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8 - x_4 (x_5x_3 + x_5x_7 + x_6x_4 + x_7x_1) \right. \\ & \left. (x_5x_3 + x_5x_7 + x_7x_1 + 2x_3x_1 + x_6x_4 + 2x_4x_2), \right. \\ & x_9 + 2x_4x_3^2x_1x_6 + 2x_4x_3^2x_5x_2 + 2x_4x_3^2x_5x_6 + 4x_4x_3x_7x_1x_2 \\ & + 4x_4x_3x_7x_1x_6 + 4x_4x_3x_7x_5x_2 + 4x_4x_3x_7x_5x_6 + 2x_4x_7^2x_1x_2 \\ & + 2x_4x_7^2x_1x_6 + 2x_4x_7^2x_5x_2 + 2x_4x_7^2x_5x_6 + 2x_4^2x_3x_2x_6 \\ & + 2x_4^2x_7x_2x_6 + 6x_3^2x_7x_1x_5 + 6x_3x_7^2x_1x_5 + 2x_3^3x_1x_5 \\ & + 3x_3^2x_7x_1^2 + 3x_3^2x_7x_5^2 + 3x_3x_7^2x_1^2 + 3x_3x_7^2x_5^2 + 2x_7^3x_1x_5 \\ & + x_4^2x_3x_6^2 + x_4^2x_7x_2^2 + x_4^2x_7x_6^2 + x_3^3x_5^2 + x_7^3x_1^2 + x_7^3x_5^2, \\ & \left. x_{10} \right) \end{aligned}$$

$$R_6 := (x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8 - x_4x_{10}, x_9 + (x_3 + x_7)x_{10}, x_{10})$$

$$\begin{aligned} S_5 := & \left(x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9, x_{10} + (x_3x_8 + x_7x_8 + x_4x_9) \right. \\ & \left. (x_3x_8 + x_7x_8 + x_4x_9 + 2x_5x_7 + 2x_7x_1 + 2x_6x_4 + 2x_4x_2 \right. \\ & \left. + 2x_3x_1 + 2x_5x_3) \right) \end{aligned}$$

$$S_6 := (x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8 + x_4x_{10}, x_9 - (x_3 + x_7)x_{10}, x_{10})$$

And combining all these automorphisms one gets:

$$\begin{aligned} R_6 \circ R_5 \circ \left(R_4 \circ R_3 \circ \left(R_2 \circ R_1 \circ F^{[3]} \circ S_1 \circ S_2 \right)^{[2]} \circ S_3 \circ S_4 \right)^{[1]} \circ S_5 \circ S_6 = \\ \left(x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9, x_{10} + (x_3x_8 + x_7x_8 + x_4x_9) \right. \\ \left. (x_3x_8 + x_7x_8 + x_4x_9 + 2x_5x_7 + 2x_7x_1 + 2x_6x_4 \right. \\ \left. + 2x_4x_2 + 2x_3x_1 + 2x_5x_3) \right) \end{aligned}$$

which is a triangular map and hence tame. In correspondence with corollary 5.26 we have added 6 new variables.

5.4.2 The stronger method

The quick method in the previous section is based on the $\mathcal{H}_n(A)$ -structure. The method in this section also uses the notion of $\mathcal{D}_n(A)$ and derivations. The benefit of adding this extra structure lies in the fact that we get a sharper upper bound if $n \geq 3$ for the number of extra variables needed compared to corollary 5.26: $n - 1$ instead of $\frac{n(n-1)}{2}$. Therefore we named it the stronger method.

We recall the statement of theorem 5.12: Let $n \geq 2$. Let $F = X + H$, where $H = E_n(T, c)$, for some $(T, c) \in \mathcal{D}_n(A)$. Then

$$F = \exp\left(D\left(c_n; \frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n}\right)\right) \prod_{p=0}^{n-2} \exp\left(D\left(E_{n,p}(T, c); \frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n}\right)\right).$$

From remark 5.28 below it follows that we can restrict ourselves to the automorphism $X + E_{n,p}(T, c)$ with $p = n - 2$. So if we can show that we can reduce $X + E_{n,p}(T, c)$ by adding $p + 1$ new variables we have accomplished our goal.

Remark 5.28

If $(\exp(a_i D_i), t_1, \dots, t_{m_i})$ is a tame automorphism for $i = 1, \dots, n$ and $m_i, n \in \mathbb{N}$, then

$$\prod_{i=1}^n (\exp(a_i D_i), t_1, \dots, t_{m_i}) = \left(\prod_{i=1}^n \exp(a_i D_i), t_1, \dots, t_m\right)$$

where $m = \max\{m_1, \dots, m_n\}$, is a tame automorphism.

Proposition 5.29

If $F = X + E_{n,p}(T, c)$ then $F^{[p+1]}$ is tame.

The proof is split into several parts.

By remark 4.7 and the remark just before definition 5.21 we see that we can focus on $E_{n,p}(T; c_{n-p-1})$ instead of $E_{n,p}(T, c)$. Now write c_{n-p-1} as $f = (f_1, \dots, f_{n-p-1})$ where each $f_i \in A[x_{n-p}, \dots, x_n]$. The next lemma provides another reduction without loss of generality.

Lemma 5.30

Let $(T; c), (T; d) \in \mathcal{D}_n(A)$. Then

$$(X + E_{n,p}(T; c)) \circ (X + E_{n,p}(T; d)) = X + E_{n,p}(T; c + d)$$

Proof. The key step here is to prove that

$$E_{n,p}(T; c) \circ (X + E_{n,p}(T; d)) = E_{n,p}(T; c) \tag{5.6}$$

For $c = (c_1, \dots, c_{n-p-1}, 0, \dots, 0)$ and $d = (d_1, \dots, d_{n-p-1}, 0, \dots, 0)$. Both c and d are in $A[x_{n-p}, \dots, x_n]^n$. The proof of (5.6) goes by induction on p .

- $p = 0$. By definition we have

$$E_{n,p}(T; c) = \text{Adj}(T_n) ((c_1, \dots, c_{n-1}, 0)_{|T_n X})$$

where $c_i \in A[x_n]$. The same holds for $E_{n,p}(T; d)$. Since the coefficients of T_n are scalars, this can be written in the following way as the composition of polynomial maps

$$[\text{Adj}(T_n)X] \circ (c_1, \dots, c_{n-1}, 0) \circ T_n X$$

Let $\delta = \det(T_n) \in A$. Then

$$\begin{aligned} & E_{n,0}(T; c) \circ (X + E_{n,0}(T; d)) \\ &= [\text{Adj}(T_n)X] \circ (c_1, \dots, c_{n-1}, 0) \circ T_n X \\ &\quad \circ (X + [\text{Adj}(T_n)X] \circ (d_1, \dots, d_{n-1}, 0) \circ T_n X) \\ &= [\text{Adj}(T_n)X] \circ (c_1, \dots, c_{n-1}, 0) \circ (T_n X + (\delta X) \circ (d_1, \dots, d_{n-1}, 0) \circ T_n X) \end{aligned} \tag{5.7}$$

We note that the n^{th} coordinate function of $(\delta X) \circ (d_1, \dots, d_{n-1}, 0) \circ T_n X$ is 0, and since c_1, \dots, c_{n-1} only involve x_n , the composition $(c_1, \dots, c_{n-1}, 0) \circ (T_n X + (\delta X) \circ (d_1, \dots, d_{n-1}, 0) \circ T_n X)$ is equal to $(c_1, \dots, c_{n-1}, 0) \circ T_n X$. Thus the composition of (5.7) is equal to

$$[\text{Adj}(T_n)X] \circ (c_1, \dots, c_{n-1}, 0) \circ T_n X = E_{n,0}(T; c)$$

as desired.

- $p > 0$. By the inductive definition we have

$$E_{n,p}(T; c) = \text{Adj}(T_n) ((E_{n-1,p-1}(T'; c'), 0)_{|T_n X})$$

(in the notation of [37]), which, again since T_n is a scalar, can be written as the polynomial composition:

$$[\text{Adj}(T_n)X] \circ (E_{n-1,p-1}(T'; c'), 0) \circ T_n X$$

And hence

$$\begin{aligned} & E_{n,p}(T; c) \circ (X + E_{n,p}(T; d)) \\ &= [\text{Adj}(T_n)X] \circ (E_{n-1,p-1}(T', c'), 0) \circ T_n X \\ &\quad \circ (X + [\text{Adj}(T_n)X] \circ (E_{n-1,p-1}(T'; d'), 0) \circ T_n X) \\ &= [\text{Adj}(T_n)X] \circ (E_{n-1,p-1}(T', c'), 0) \circ (T_n X + \underbrace{\delta(E_{n-1,p-1}(T'; d'), 0)}_{=(E_{n-1,p-1}(T'; \delta d'), 0)} \circ T_n X) \\ &= [\text{Adj}(T_n)X] \circ (E_{n-1,p-1}(T', c'), 0) \circ (T_n X + (E_{n-1,p-1}(T'; \delta d'), 0) \circ T_n X) \\ &= [\text{Adj}(T_n)X] \circ \underbrace{(E_{n-1,p-1}(T', c'), 0) \circ (X + (E_{n-1,p-1}(T'; \delta d'), 0))}_{=E_{n-1,p-1}(T', c')} \circ T_n X \\ &= [\text{Adj}(T_n)X] \circ (E_{n-1,p-1}(T', c'), 0) \circ T_n X \\ &= E_{n,p}(T; c) \end{aligned}$$

(where again $\delta = \det(T_n) \in A$).

It now easily follows that $(X + E_{n,p}(T; c)) \circ (X + E_{n,p}(T; d)) = X + E_{n,p}(T; c + d)$ using (5.6) and theorem 4.6. \square

In some sense this lemma says that $X + E_{n,p}(T; c)$ is additive in c . The impact is that we can split our $X + E_{n,p}(T; f)$ into

$$(X + E_{n,p}(T; (f_1, 0, \dots, 0))) \circ (X + E_{n,p}(T; (0, f_2, 0, \dots, 0))) \circ \dots \circ (X + E_{n,p}(T; (0, \dots, 0, f_{n-p-1})))$$

And for the purpose of reducing to a tame automorphism this means that we can restrict to one general $X + E_{n,p}(T; (0, \dots, 0, f_i, 0, \dots, 0))$ for some $1 \leq i \leq n - p - 1$.

The next step in the process is the observation:

Lemma 5.31

$X + E_{n,p}(T; (0, \dots, 0, f_i, 0, \dots, 0)) = \exp(hD)$ where D is a locally nilpotent derivation and $h \in A[x_1, \dots, x_n]$.

Proof. Theorem 4.6 shows

$$E_{n,p}(T, f) = \text{Adj}(T_{n-p} \Delta T_{n-p+1} \Delta \dots \Delta T_n) \begin{pmatrix} 0 \\ \vdots \\ 0 \\ f_i(x_{n-p}, \dots, x_n) \\ 0 \\ \vdots \\ 0 \end{pmatrix} |_{(T_{n-p} \Delta T_{n-p+1} \Delta \dots \Delta T_n)X}$$

It is obvious that we can split this object into two smaller parts:

$$g := \text{Adj}(T_{n-p} \Delta T_{n-p+1} \Delta \dots \Delta T_n) e_i$$

where e_i is the i -th unit vector and

$$h := f_i(x_{n-p}, \dots, x_n) |_{(T_{n-p} \Delta T_{n-p+1} \Delta \dots \Delta T_n)X}$$

Multiplying these two factors gives back the complete result. Lemma 5.32 below shows that

$$h = \begin{matrix} f_i(x_{n-p}, \dots, x_n) |_{(T_{n-p}X \circ T_{n-p+1}X \circ \dots \circ T_nX)} \\ f_i(T_{n-p}X \circ T_{n-p+1}X \circ \dots \circ T_nX) \end{matrix}$$

Now let $D = \mathcal{D}(g; \partial_1, \dots, \partial_n)$. Then hD is the same derivation as presented in corollary 3.4 in [37]. And there it is shown that this derivation is locally nilpotent and $X + E_{n,p}(T, c) = \exp(hD)$. \square

Lemma 5.32

The ‘ Δ ’ operator has the property:

$$(S_1 \Delta S_2 \Delta \cdots \Delta S_k)X = S_1 X \circ S_2 X \circ \cdots \circ S_k X \text{ for all } k \geq 2$$

Proof. The proof goes by induction on k . Note that $(S_1 \Delta S_2)X = (S_1(S_2 X) * S_2) * X = S_1(S_2 X) * S_2 * X = S_1(S_2 X) * S_2 X = S_1 X \circ S_2 X$, which proves the case $k = 2$. Now for $k > 2$ we have:

$$\begin{aligned} (S_1 \Delta S_2 \Delta \cdots \Delta S_k)X &= (S_1 \Delta (S_2 \Delta \cdots \Delta S_k))X \\ &= S_1 X \circ (S_2 \Delta \cdots \Delta S_k)X \\ &= S_1 X \circ S_2 X \circ \cdots \circ S_k X \end{aligned}$$

which proves the lemma. □

At this point we introduce a new set of matrices. We use it in the next step of the proof of proposition 5.29.

$$S_{n-r} = \begin{cases} r = 0: T_n \\ r > 0: T_{n-r}(S_{n-(r-1)} \cdots S_n X) \end{cases}$$

Note that the matrix S_{n-r} has the form

$$\begin{pmatrix} S & 0 \\ 0 & I_r \end{pmatrix}$$

for some S because T_{n-r} has this form. Furthermore this means that

$$\text{Adj}(S_{n-r}) = \begin{pmatrix} \text{Adj}(S) & 0 \\ 0 & \delta I_r \end{pmatrix}$$

where $\delta = \det(S)$.

Lemma 5.33

$T_{n-p} \Delta T_{n-p+1} \Delta \cdots \Delta T_n = S_{n-p} \cdot S_{n-p+1} \cdots S_n$, where S_n is defined as above.

Proof. The proof is with induction on p . As usual, the case $p = 0$ is clear, hence assume $p > 0$. Then

$$\begin{aligned} T_{n-p} \Delta T_{n-p+1} \Delta \cdots \Delta T_n &= T_{n-p} \Delta (T_{n-p+1} \Delta \cdots \Delta T_n) \\ &= T_{n-p} ((T_{n-p+1} \Delta \cdots \Delta T_n)X) \cdot (T_{n-p+1} \Delta \cdots \Delta T_n) \\ &= T_{n-p}(S_{n-p+1} \cdots S_n X) \cdot (S_{n-p+1} \cdots S_n) \\ &= S_{n-p} S_{n-p+1} \cdots S_n \end{aligned}$$

□

Now that we have written $X + E_{n,p}(T; f)$ as $\exp(hD)$, the next step in the proof is using Martha Smith's result of [85]. From her paper it follows that if $a \in \ker(D)$ and $\rho = (X, t + a)$ then

$$(\exp(aD), t) = \exp(tD)\rho \exp(-tD)\rho^{-1}$$

and hence we are reduced to factoring $\exp(tD)$. In order to exploit this step we have to show that $h \in \ker(D)$.

Lemma 5.34

$$D(h) = 0.$$

Proof. Let (H_1, \dots, H_n) be the coordinate functions of the map $(T_{n-p}\Delta \cdots \Delta T_n)X$. We have

$$\begin{aligned} h &= f_i(x_{n-p}, \dots, x_n)|_{(T_{n-p}\Delta \cdots \Delta T_n)X} \\ &= f_i(H_{n-p}, \dots, H_n) \end{aligned}$$

So it suffices to show that D kills H_{n-p}, \dots, H_n .

- $D(H_n) = 0$. Since $(T_{n-p}\Delta \cdots \Delta T_n)X = T_{n-p}X \circ \cdots \circ T_nX$ (lemma 5.32) and since $T_{n-p}X, \dots, T_{n-1}X$ fix x_n , it is clear that $H_n = a_{n,1}x_1 + \cdots + a_{n,n}x_n$ where $T_n = (a_{i,j}) \in M_n(A)$. Let $(b_{i,j}) = \text{Adj}(T_n) = \text{Adj}(S_n)$. Then

$$\begin{aligned} D(H_n) &= \mathcal{D}(\text{Adj}(T_{n-p}\Delta \cdots \Delta T_n)e_i; \partial_1, \dots, \partial_n)(H_n) \\ &= \mathcal{D}(\text{Adj}(S_{n-p} \cdots S_n)e_i; \partial_1, \dots, \partial_n)(H_n) \\ &= \mathcal{D}(\text{Adj}(S_n) \text{Adj}(S_{n-p} \cdots S_{n-1})e_i; \partial_1, \dots, \partial_n)(H_n) \\ &= (\partial_1 H_n, \dots, \partial_n H_n) \text{Adj}(S_n) \text{Adj}(S_{n-p} \cdots S_{n-1})e_i \\ &= (a_{n,1}, \dots, a_{n,n})(b_{i,j}) \text{Adj}(S_{n-p} \cdots S_{n-1})e_i \\ &= (0, \dots, 0, \delta) \text{Adj}(S_{n-p} \cdots S_{n-1})e_i \end{aligned}$$

where $\delta = \det(T_n)$. We have seen before that $\text{Adj}(S_{n-p} \cdots S_{n-1})$ has the form

$$\begin{pmatrix} * & \dots & * & 0 \\ \vdots & & \vdots & \vdots \\ * & \dots & * & 0 \\ 0 & \dots & 0 & * \end{pmatrix}$$

and therefore

$$\begin{aligned} D(H_n) &= (0, \dots, 0, \delta) \text{Adj}(S_{n-p} \cdots S_{n-1})e_i \\ &= (0, \dots, 0, *)e_i \\ &= 0 \end{aligned}$$

since $i \leq n - p - 1 < n$.

- $D(H_r) = 0$ for $n - p \leq r < n$. Let $G \in A[x_n]^{n-1}$ where

$$G = (G_1, \dots, G_{n-1}) = (T_{n-p} \Delta \cdots \Delta T_{n-1}) \begin{pmatrix} x_1 \\ \vdots \\ x_{n-1} \end{pmatrix}$$

Let $L = (L_1, \dots, L_n) = T_n X$. Then

$$\begin{aligned} (H_1, \dots, H_n) &= (G, x_n) \circ L \\ &= (G_1(L), \dots, G_{n-1}(L), L_n) \end{aligned}$$

So in particular $H_r = G_r(L)$.

Furthermore let $S = S_{n-p} \cdots S_n$, and $S' = S'_{n-p} \cdots S'_{n-1} = T_{n-p} \Delta \cdots \Delta T_{n-1}$ and note that

$$S = S'(L) \cdot T_n \tag{5.8}$$

As an $n \times n$ matrix, S' has the form $\begin{pmatrix} S' & 0 \\ 0 & 1 \end{pmatrix}$. (We assume the reader won't be disturbed by the slight abuse of notation in the double use of S' .) Hence $\text{Adj}(S')$ has the form

$$(s_{i,j}) = \begin{pmatrix} \text{Adj}(S') & 0 \\ 0 & d \end{pmatrix}$$

where $d = \det(S')$ and therefore $\text{Adj}(S'(L))$ has the form

$$\begin{pmatrix} \text{Adj}(S')(L) & 0 \\ 0 & d(L) \end{pmatrix}$$

Now

$$\begin{aligned} D(H_r) &= D(G_r(L)) \\ &= \mathcal{D}(\text{Adj}(T_{n-p} \Delta \cdots \Delta T_n) e_i; \partial_1, \dots, \partial_n)(G_r(L)) \\ &= \mathcal{D}(\text{Adj}(S_{n-p} \cdots S_n) e_i; \partial_1, \dots, \partial_n)(G_r(L)) \\ &= \mathcal{D}(\text{Adj}(T_n) \text{Adj}(S'(L)) e_i; \partial_1, \dots, \partial_n)(G_r(L)) \quad (\text{by (5.8)}) \\ &= (\partial_1 G_r(L), \dots, \partial_n G_r(L)) \text{Adj}(T_n) \text{Adj}(S'(L)) e_i \end{aligned}$$

As before let $T_n = (a_{i,j})$ and let $\text{Adj}(T_n) = (b_{i,j})$. The above is then equal to:

$$\begin{aligned} &\sum_{j=1}^n \sum_{u=1}^n \partial_j(G_r(L)) b_{j,u} s_{u,i}(L) \\ &= \sum_{j=1}^n \sum_{u=1}^n \sum_{v=1}^n (\partial_j L_v) (\partial_v G_r)(L) b_{j,u} s_{u,i}(L) \quad (\text{by chain rule}) \\ &= \sum_{j=1}^n \sum_{u=1}^n \sum_{v=1}^n a_{v,j} b_{j,u} (\partial_v G_r)(L) s_{u,i}(L) \quad (L_v = a_{v,1} x_1 + \cdots + a_{v,n} x_n) \end{aligned}$$

$$\begin{aligned}
&= \sum_{u=1}^n \sum_{v=1}^n \delta_{v,u} \delta(\partial_v G_r)(L) s_{u,i}(L) \quad (\delta = \det(T_n); \delta_{v,u} \text{ is Kronecker delta}) \\
&= \delta \sum_{u=1}^n (\partial_u G_r)(L) s_{u,i}(L) \\
&= \delta \sum_{u=1}^{n-1} (\partial_u G_r)(L) s_{u,i}(L) \quad (\text{because } s_{n,i} = 0, \text{ since } i \leq n - p - 1 < n) \\
&= \delta D'(G_r)(L)
\end{aligned}$$

where $D' = \mathcal{D}(\text{Adj}(S'_{n-p} \cdots S'_{n-1})e_i; \partial_1, \dots, \partial_{n-1})$. By induction on $n-r$, we know that $D'(G_r) = 0$, hence $D'(G_r)(L) = 0$, and $D(H_r) = 0$ as desired.

Hence $D(h) = 0$. □

The last step in the proof of proposition 5.29 is given by lemma 5.35. We have already added one new variable in order to get this $\exp(tD)$, hence if we can show that $\exp(tD)^{[p]}$ is tame, we have shown that $(X + E_{n,p}(T; f))^{[p+1]}$ is tame, the claim of proposition 5.29.

Lemma 5.35

The map $\exp(tD)^{[p]}$ is tame.

Proof. If we consider $A[t]$ to be the new base ring, we see that $\exp(tD) = X + E_{n,p}(T; q)$ where q is the $(n-p-1)$ -tuple $(0, \dots, 0, t, 0, \dots, 0)$. By using theorem 4.6 again we see that the middle part of $E_{n,p}(T; q)$ is given by the composition

$$\text{Adj}(T_{n-p})(0, \dots, 0, t, 0, \dots, 0, 0, \dots, 0)_{|T_{n-p}X}$$

However since t is in the base ring, the substitution has no effect. Only the product remains and we get:

$$(t \text{Adj}(T_{n-p})_{1,i}, \dots, t \text{Adj}(T_{n-p})_{n-p,i}, 0, \dots, 0)$$

Now let g be the tuple of the first $n-p$ entries. If $p = 0$ then g is an n -tuple over $A[t]$ and $X + E_{n,p}(T; q)$ is clearly tame and we have reduced the original $X + E_{n,p}(T; f)$ using one new variable. If $p > 0$ then we have $E_{n,p}(T; q) = E_{n,p-1}(T; g)$. And this expression can be factored using p new variables by induction. □

Theorem 5.36

Let $F = X + H$ with $H \in \mathcal{H}_n(A)$. Then $F^{[n-1]}$ is tame.

Proof. Proposition 5.29 shows that $(X + E_{n,p}(T; c))^{[p+1]}$ is tame. Theorem 5.12 and remark 5.28 show that it suffices to prove that $(X + E_{n,p}(T; c))^{[n-1]}$ is tame for $p \leq n-2$. Because $p+1 \leq n-1$ this is obviously the case. □

We show on our running example that this method really works:

Example 5.37

Take F as in example 5.27 (which is the same as example 4.8 with $m = 3$) and use the same $(T, c) \in \mathcal{D}_4(\mathbb{C})$:

$$F = (x_4(x_3x_1 + x_4x_2)^2, -x_3(x_3x_1 + x_4x_2)^2, x_4^3, 0)$$

The first step is splitting F into

$$F = (X + c_4) \circ (X + E_{4,0}(T, c)) \circ (X + E_{4,1}(T, c)) \circ (X + E_{4,2}(T, c))$$

In example 4.8 we have already seen that

$$\begin{aligned} X + c_4 &= X \\ X + E_{4,0}(T, c) &= X + \begin{pmatrix} 0 \\ 0 \\ x_4^3 \\ 0 \end{pmatrix} \\ X + E_{4,1}(T, c) &= X \end{aligned}$$

So the first three parts of this composition of F are already tame. Hence we only have to look at $X + E_{4,2}(T, c)$. We have seen that

$$\begin{aligned} X + E_{4,2}(T, c) &= X + E_{4,2}(T; c_1) \\ &= X + E_{4,2}(T; (x_2^2, 0, 0, 0)) \\ &= \exp(h_1 D_1) \end{aligned}$$

where $h_1 = (x_3x_1 + x_4x_2)^2$ and $D_1 = x_4\partial_1 - x_3\partial_2$. Obviously $D_1(h_1) = 0$. Now Smith tells us that

$$(\exp(h_1 D_1), x_5) = \exp(x_5 D_1) \rho_1 \exp(-x_5 D_1) \rho_1^{-1}$$

where

$$\rho_1 = (x_1, x_2, x_3, x_4, x_5 + (x_3x_1 + x_4x_2)^2)$$

and

$$\exp(x_5 D_1) = (x_1 + x_4x_5, x_2 - x_3x_5, x_3, x_4, x_5)$$

In this example we see that both ρ_1 and $\exp(x_5 D_1)$ are compositions of elementary maps and hence tame. This means that we can stop here and claim that F can be factored into tame automorphisms by adding only one variable.¹ However, for the sake of the argument, we continue with this algorithm to show that it really ends after $n - 1 = 3$ steps.

Note that

$$\begin{aligned} \exp(x_5 D_1) &= X + E_{4,2}(T; (x_5, 0, 0, 0)) \\ &= X + E_{4,1}(T; \text{Adj}(T_2)(x_5, 0, 0, 0)) \\ &= X + E_{4,1}(T; (x_4x_5, -x_3x_5, 0, 0)) \\ &= (X + E_{4,1}(T; (x_4x_5, 0, 0, 0))) \circ (X + E_{4,1}(T; (0, -x_3x_5, 0, 0))) \\ &= \exp(h_2 D_2) \circ \exp(h_3 D_3) \end{aligned}$$

¹This is a consequence of the fact that we can view the first two components of F as $(x_1, x_2) + (H_1, H_2)$ with $(H_1, H_2) \in \mathcal{H}_2(\mathbb{C}[x_3, x_4])$.

where $h_2 = x_4x_5$, $D_2 = \partial_1$, $h_3 = -x_3x_5$ and $D_3 = \partial_2$. Using Smith's result again we can write

$$\begin{aligned}(\exp(h_2D_2), x_6) &= \exp(x_6D_2)\rho_2 \exp(-x_6D_2)\rho_2^{-1} \\(\exp(h_3D_3), x_6) &= \exp(x_6D_3)\rho_3 \exp(-x_6D_3)\rho_3^{-1}\end{aligned}$$

where $\rho_2 = (x_1, x_2, x_3, x_4, x_6 + x_4x_5)$, where $\exp(x_6D_2) = (x_1 + x_6, x_2, x_3, x_4, x_6)$, where $\rho_3 = (x_1, x_2, x_3, x_4, x_6 - x_3x_5)$ and where $\exp(x_6D_3) = (x_1, x_2 + x_6, x_3, x_4, x_6)$. Now the final step gives

$$\begin{aligned}\exp(x_6D_2) &= X + E_{4,1}(T; (x_6, 0, 0, 0)) \\ &= X + E_{4,0}(T; \text{Adj}(T_3)(x_6, 0, 0, 0)) \\ &= X + E_{4,0}(T; (x_6, 0, 0, 0)) \\ &= \exp(h_4D_4) \\ \exp(x_6D_3) &= X + E_{4,1}(T; (0, x_6, 0, 0)) \\ &= X + E_{4,0}(T; \text{Adj}(T_3)(0, x_6, 0, 0)) \\ &= X + E_{4,0}(T; (0, x_6, 0, 0)) \\ &= \exp(h_5D_5)\end{aligned}$$

where $h_4 = h_5 = x_6$, $D_4 = \partial_1$ and $D_5 = \partial_2$. Then

$$\begin{aligned}(\exp(h_4D_4), x_7) &= \exp(x_7D_4)\rho_4 \exp(-x_7D_4)\rho_4^{-1} \\(\exp(h_5D_5), x_7) &= \exp(x_7D_5)\rho_5 \exp(-x_7D_5)\rho_5^{-1}\end{aligned}$$

where $\rho_4 = \rho_5 = (x_1, x_2, x_3, x_4, x_7 + x_6)$, $\exp(x_7D_4) = (x_1 + x_7, x_2, x_3, x_4, x_7)$ and $\exp(x_7D_5) = (x_1, x_2 + x_7, x_3, x_4, x_7)$.

And now we have $\exp(x_7D_4), \exp(x_7D_5) \in \mathbb{C}[x_7]$ and hence the algorithm ends. Coercing to seven-dimensional mappings in the logical way gives:

$$\begin{aligned}\rho_1 &= (x_1, x_2, x_3, x_4, x_5 + (x_3x_1 + x_4x_2)^2, x_6, x_7) \\ \rho_2 &= (x_1, x_2, x_3, x_4, x_5, x_6 + x_4x_5, x_7) \\ \rho_3 &= (x_1, x_2, x_3, x_4, x_5, x_6 - x_3x_5, x_7) \\ \rho_4 = \rho_5 &= (x_1, x_2, x_3, x_4, x_5, x_6, x_7 + x_6) \\ \exp(x_5D_1) &= (x_1 + x_4x_5, x_2 - x_3x_5, x_3, x_4, x_5, x_6, x_7) \\ \exp(x_6D_2) &= (x_1 + x_6, x_2, x_3, x_4, x_5, x_6, x_7) \\ \exp(x_6D_3) &= (x_1, x_2 + x_6, x_3, x_4, x_5, x_6, x_7) \\ \exp(x_7D_4) &= (x_1 + x_7, x_2, x_3, x_4, x_5, x_6, x_7) \\ \exp(x_7D_5) &= (x_1, x_2 + x_7, x_3, x_4, x_5, x_6, x_7)\end{aligned}$$

And with these tame automorphisms we can write down the factorisation:

$$\begin{aligned}(X + E_{4,2}(T, c), x_5, x_6, x_7) \\ = \exp(x_5D_1)\rho_1 \exp(-x_5D_1)\rho_1^{-1}\end{aligned}$$

$$\begin{aligned}
&= \exp(h_2 D_2) \exp(h_3 D_3) \rho_1 \exp(-h_3 D_3) \exp(-h_2 D_2) \rho_1^{-1} \\
&= \exp(x_6 D_2) \rho_2 \exp(-x_6 D_2) \rho_2^{-1} \exp(x_6 D_3) \rho_3 \exp(-x_6 D_3) \rho_3^{-1} \rho_1 \\
&\quad \rho_3 \exp(x_6 D_3) \rho_3^{-1} \exp(-x_6 D_3) \rho_2 \exp(x_6 D_2) \rho_2^{-1} \exp(-x_6 D_2) \rho_1^{-1} \\
&= \exp(h_4 D_4) \rho_2 \exp(-h_4 D_4) \rho_2^{-1} \exp(h_5 D_5) \rho_3 \exp(-h_5 D_5) \rho_3^{-1} \rho_1 \\
&\quad \rho_3 \exp(h_5 D_5) \rho_3^{-1} \exp(-h_5 D_5) \rho_2 \exp(h_4 D_4) \rho_2^{-1} \exp(-h_4 D_4) \rho_1^{-1} \\
&= \exp(x_7 D_4) \rho_4 \exp(-x_7 D_4) \rho_4^{-1} \rho_2 \rho_4 \exp(x_7 D_4) \rho_4^{-1} \exp(-x_7 D_4) \rho_2^{-1} \\
&\quad \exp(x_7 D_5) \rho_5 \exp(-x_7 D_5) \rho_5^{-1} \rho_3 \rho_5 \exp(x_7 D_5) \rho_5^{-1} \exp(-x_7 D_5) \rho_3^{-1} \rho_1 \\
&\quad \rho_3 \exp(x_7 D_5) \rho_5 \exp(-x_7 D_5) \rho_5^{-1} \rho_3^{-1} \rho_5 \exp(x_7 D_5) \rho_5^{-1} \exp(-x_7 D_5) \rho_2 \\
&\quad \exp(x_7 D_4) \rho_4 \exp(-x_7 D_4) \rho_4^{-1} \rho_2^{-1} \rho_4 \exp(x_7 D_4) \rho_4^{-1} \exp(-x_7 D_4) \rho_1^{-1}
\end{aligned}$$

We see that $(X + E_{4,2}(T, c))^{[3]}$ is tame and hence $F^{[3]}$ is tame.

Remark 5.38

Note that we only prove theorem 5.36 for $X + H$ with $H \in \mathcal{H}_n(A)$. If one looks at $X + H$ with $H \in \overline{\mathcal{H}_n(A)}$, we don't know whether these maps are stably tame or not. We cannot modify the proof of this theorem slightly, because it is mainly based on the $\mathcal{D}_n(A)$ -structure which is not defined for $\overline{\mathcal{H}_n(A)}$.

Chapter 6

Nilpotent matrices and polynomial maps

Introduction

The working title of this chapter has been ‘tying up the loose ends’ for a long time. Simply because this chapter deals with several topics which are not very strongly related to each other. We study polynomial maps F which belong either to the class where $JF = I_n + N$ where N is a strong nilpotent matrix or to the class where $F = X + (AX)^{*3}$ with A a D-nilpotent matrix. Furthermore we look at linearisability for certain polynomial maps. And finally we apply the pairing technique by Gorni and Zampieri to several examples to show that this is a handsome tool. This chapter is mainly based on work by van den Essen and the author, Berson, Ivanenko and Gorni, Tutaj and Zampieri.

Unless otherwise stated: let k be a field with $\text{char}(k) = 0$.

6.1 Strongly nilpotent Jacobians

As already stated before in chapter 3, understanding nilpotent matrices is crucial in studying the Jacobian Conjecture. In [8] Bass, Connell and Wright proved that it suffices to prove the Jacobian Conjecture for polynomial maps of the form $F = X + H$ where H is a cubic homogeneous polynomial map. In [90] Yagzhev proved the same result independently. Conform [8, lemma 4.1] $\det(JH) \in \mathbb{C}^*$ is equivalent with the statement JH is nilpotent.

In [73] Meisters and Olech introduced the notion of strongly nilpotent matrices. In [34] van den Essen and the author generalised this definition to:

Definition 6.1

Consider the n tuples of n variables.

$$\begin{aligned} Y_{(1)} &= (Y_{(1),1}, \dots, Y_{(1),n}) \\ &\vdots \\ Y_{(n)} &= (Y_{(n),1}, \dots, Y_{(n),n}) \end{aligned}$$

Let $M(X) \in \text{Mat}_n(k[X])$. Then $M(X)$ is nilpotent in the usual sense if

$$M(Y_{(1)})^n = 0$$

and $M(X)$ is *strong nilpotent* if

$$M(Y_{(1)}) \cdot M(Y_{(2)}) \cdots M(Y_{(n)}) = 0$$

Obviously strong nilpotence implies nilpotence. And if $M(X) = M \in \text{Mat}_n(k)$ then strong nilpotence is equivalent to nilpotence.

Remark 6.2

Note that if $M(X)$ is an upper triangular matrix with zeros on the main diagonal, $M(X)$ is always strongly nilpotent. A direct result from the fact that the matrix

$$M(Y_{(1)}) \cdot M(Y_{(2)}) \cdots M(Y_{(i)})$$

is not only an upper triangular matrix with zeros on the main diagonal, but also with zeros on the upper $i - 1$ side-diagonals. And of course if $i = n$ this means that the product is completely zero, no matter what the coefficients in the original $M(X)$ are.

Now a natural question to ask is

Question 6.3

Does nilpotence imply strong nilpotence?

In particular Meisters and Olech studied the weaker question

Question 6.4

Does nilpotence combined with symmetry imply strong nilpotence, i.e.

if $M(Y_{(1)})Y_{(2)} = M(Y_{(2)})Y_{(1)}$ and $M(Y_{(1)})^n = 0$ do we have that $M(X)$ is strong nilpotent?

Now if one takes for $M(X)$ the Jacobian matrix of a certain polynomial map $F = X + Q$ with $F(0) = 0$, the symmetry aspect implies that $Q(X)$ must be homogeneous of degree 2. Otherwise $\deg_{Y_{(1)}}(JQ(Y_{(1)})Y_{(2)}) \neq \deg_{Y_{(1)}}(JQ(Y_{(2)})Y_{(1)})$. In fact $Q(X) = \frac{1}{2}JQ(X)X$. And from this it follows that the nilpotence property is equivalent to $\det(JF(X)) = 1$. In the remainder of this section $Q(X)$ will be quadratic homogeneous. In their paper Meisters and Olech proved

Theorem 6.5

If $F = X + Q$ and

- 1. $n \leq 4$ then question 6.4 has an affirmative answer.*
- 2. $JQ(X)^2 = 0$ then question 6.4 has an affirmative answer for any dimension n .*
- 3. $n \geq 5$ and $JQ(X)^2 \neq 0$ then question 6.4 has a negative answer.*

For the proof see [73]. We only copy the counterexample in dimension five, found by Vasyunin, which leads to the third statement in theorem 6.5.

Example 6.6

Let $F = X + Q$ be the polynomial map $\mathbb{C}^5 \rightarrow \mathbb{C}^5$ where

$$Q = \begin{pmatrix} \frac{1}{2} \alpha x_3^2 \\ x_3 x_1 + \frac{1}{2} \beta x_3^2 - \alpha x_5 x_3 \\ -x_4 \alpha x_5 + x_4 x_1 + \frac{1}{2} x_2^2 - \beta x_5 x_2 + \frac{1}{2} \beta^2 x_5^2 \\ x_3 \beta x_5 + x_5 x_1 - x_3 x_2 - \alpha x_5^2 \\ \frac{1}{2} x_3^2 \end{pmatrix}$$

Then JQ is given by

$$\begin{pmatrix} 0 & 0 & \alpha x_3 & 0 & 0 \\ x_3 & 0 & x_1 + \beta x_3 - \alpha x_5 & 0 & -\alpha x_3 \\ x_4 & x_2 - \beta x_5 & 0 & x_1 - \alpha x_5 & -\beta x_2 - \alpha x_4 + \beta^2 x_5 \\ x_5 & -x_3 & -x_2 + \beta x_5 & 0 & x_1 + \beta x_3 - 2\alpha x_5 \\ 0 & 0 & x_3 & 0 & 0 \end{pmatrix}$$

One easily verifies that $JQ^5 = 0$ and $JQ^4 \neq 0$.¹ Hence JQ is nilpotent. Because Q is quadratic homogeneous, the symmetry part automatically holds. However JQ is not strongly nilpotent. For instance take $Y_{(1)} = e_1$, $Y_{(2)} = e_2$, $Y_{(3)} = e_1$, $Y_{(4)} = e_2$ and $Y_{(5)} = e_4$. Then

$$JQ(Y_{(1)})JQ(Y_{(2)})JQ(Y_{(3)})JQ(Y_{(4)})JQ(Y_{(5)}) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & -\alpha \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

And this matrix should have been the 0-matrix for JQ to be strongly nilpotent. Note that this matrix is nilpotent on its turn. This is not the general case. For instance if we take $Y_{(5)} = e_5$ instead of $Y_{(5)} = e_4$ we get the matrix

$$JQ(Y_{(1)})JQ(Y_{(2)})JQ(Y_{(3)})JQ(Y_{(4)})JQ(Y_{(5)}) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\alpha & 0 & 0 \\ 0 & -\beta & 0 & -\alpha & \beta^2 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

which is only nilpotent if either $\alpha = 0$ or $\beta = 0$.

Having introduced the notion of strongly nilpotent matrices and knowing that for the Jacobian Conjecture it is crucial to understand maps $F = X + H$ with JH nilpotent, it seems natural to consider the next question.

¹In [73] there are also examples with $JQ^4 = 0$ but still JQ is not strongly nilpotent.

Question 6.7

Is the Jacobian Conjecture true for maps of the form $F = X + H$ with JH is strongly nilpotent?

In [34] van den Essen and the author provide an answer to this question. It turns out that this question has an affirmative answer. In fact the answer is even stronger. It is shown that JH is strongly nilpotent if and only if JH is linearly triangularisable if and only if the map $F = X + H$ is linearly triangularisable.

Before we can prove this statement, we have to specify some lemmas. First recall the definition of triangular and linearly triangularisable maps in definition 1.9. Now one easily verifies the following lemma:

Lemma 6.8

Let $F = X + H$ be a polynomial map. Then F is in upper triangular form if and only if JH is upper triangular with zeros on the main diagonal.

The following two lemmas are of a more technical nature:

Lemma 6.9

Let $JH = \sum_{|\alpha| \leq d} A_\alpha X^\alpha$, where $d = \max_i (\deg(H_i)) - 1$ and $A_\alpha \in \text{Mat}_n(k)$ for all α . Then JH is strongly nilpotent if and only if $A_{\alpha_{(1)}} \cdots A_{\alpha_{(n)}} = 0$, for all multi-indices $\alpha_{(i)}$ with $|\alpha_{(i)}| \leq d$.

Proof. By definition 6.1 we obtain

$$\left(\sum_{|\alpha_{(1)}| \leq d} A_{\alpha_{(1)}} Y_{(1)}^{\alpha_{(1)}} \right) \cdots \left(\sum_{|\alpha_{(n)}| \leq d} A_{\alpha_{(n)}} Y_{(n)}^{\alpha_{(n)}} \right) = 0.$$

Now if we expand this product and look at the coefficients of $Y_{(1)}^{\alpha_{(1)}} \cdots Y_{(n)}^{\alpha_{(n)}}$ we see that the claim holds. \square

Lemma 6.10

Let V be a finite dimensional k -vector-space and ℓ_1, \dots, ℓ_p k -linear maps from V to V . Let $r \in \mathbb{N}$, $r \geq 1$. If $\ell_{i_1} \circ \cdots \circ \ell_{i_r} = 0$ for each r -tuple $\ell_{i_1}, \dots, \ell_{i_r}$ with $1 \leq i_1, \dots, i_r \leq p$, then there exists a basis (v) of V such that $\text{Mat}(\ell_i, (v)) = D_i$ where D_i is an upper triangular matrix with zeros on the main diagonal.

Proof. Let $d := \dim(V)$. We use induction on d . First let $d = 1$. Then the hypothesis implies that $\ell_i^r = 0$ for each i . Hence $\ell_i = 0$ for each i and the statement is true. Now let $d > 1$ and assume that the assertion is proved for all $d - 1$ dimensional vector-spaces. Now we (also) use induction on r . If $r = 1$ then each $\ell_i = 0$. So let $r \geq 2$. Then for each $(r - 1)$ -tuple $\ell_{i_2}, \dots, \ell_{i_r}$ with $1 \leq i_2, \dots, i_r \leq p$ we have

$$\begin{aligned} \ell_1 \ell_{i_2} \cdots \ell_{i_r} &= 0 \\ &\vdots \\ \ell_p \ell_{i_2} \cdots \ell_{i_r} &= 0 \end{aligned} \tag{6.1}$$

If $\ell_{i_2} \cdots \ell_{i_r} = 0$ for each such $(r-1)$ -tuple we are done by the induction hypothesis on r . So we may assume that for some $(r-1)$ -tuple $\ell_{i_2}, \dots, \ell_{i_r}$ the map $\ell_{i_2} \cdots \ell_{i_r} \neq 0$. So there exists $v \neq 0, v \in V$ with $\ell_{i_2} \cdots \ell_{i_r} v \neq 0$. Let $v_1 := \ell_{i_2} \cdots \ell_{i_r} v$. From (6.1) we deduce that $\ell_i v_1 = 0$ for all i . Then consider $\bar{V} := V/kv_1$. Since $\ell_i v_1 = 0$ for all i we get induced k -linear maps $\bar{\ell}_i : \bar{V} \rightarrow \bar{V}$. Since $\dim(\bar{V}) = d-1$ the induction hypothesis implies that there exist v_2, \dots, v_r in V such that $(\bar{v}_2, \dots, \bar{v}_r)$ is a k -basis of \bar{V} and $\text{Mat}(\bar{\ell}_i, (\bar{v}_2, \dots, \bar{v}_r))$ is on upper triangular form. Then $(v) = (v_1, v_2, \dots, v_r)$ is as desired. \square

Corollary 6.11

Let $A_1, \dots, A_p \in \text{Mat}_n(k)$. Let $r \in \mathbb{N}, r \geq 1$. If $A_{i_1} \cdots A_{i_r} = 0$ for each r -tuple A_{i_1}, \dots, A_{i_r} with $1 \leq i_1, \dots, i_r \leq p$, then there exists $T \in \text{Gl}_n(k)$ such that $T^{-1}A_i T = D_i$, where each D_i is an upper triangular matrix with zeros on the main diagonal.

Now we have enough tools to present and prove the announced theorem.

Theorem 6.12

Let $H = (H_1, \dots, H_n) : k^n \rightarrow k^n$ be a polynomial map. Then there is equivalence between

1. JH is strongly nilpotent.
2. There exists $T \in \text{Gl}_n(k)$ such that $J(T^{-1}HT)$ is upper triangular with zeros on the main diagonal.
3. $F := X + H$ is linearly triangularisable.

Proof. Assume 2 holds. Then by lemma 6.8 also 3 holds. Now assume 3 holds. If $F = X + H$ is linearly triangularisable, then by lemma 6.8 $J(T^{-1}HT)$ is an upper triangular matrix with zeros on the main diagonal. As noted in remark 6.2 this implies that $J(T^{-1}HT)$ is strongly nilpotent. Finally observe that $J(T^{-1}HT) = T^{-1}JH(TX)T$. So the strong nilpotence of $J(T^{-1}HT)$ implies that $JH(TY_{(1)}) \dots JH(TY_{(n)}) = 0$, which implies on its turn that JH is strongly nilpotent.

Finally we prove that statement 1 implies statement 2. So let JH be strongly nilpotent. Now if we write $JH = \sum_{|\alpha| \leq d} A_\alpha X^\alpha$, then by lemma 6.9 $A_{\alpha_{(1)}} \cdots A_{\alpha_{(n)}} = 0$ for all n -tuples with $|\alpha_{(i)}| \leq d$. So by corollary 6.11 there exists $T \in \text{Gl}_n(k)$ such that $T^{-1}A_\alpha T = D_\alpha$ for all α with $|\alpha| \leq d$, where D_α is an upper triangular matrix with zeros on the main diagonal. Consequently this also holds for $T^{-1}JH(X)T (= \sum T^{-1}A_\alpha TX^\alpha)$ and hence also for $J(T^{-1}HT) = T^{-1}JH(TX)T$, which is obtained by replacing X by TX in $T^{-1}JH(X)T$. \square

A direct consequence of this theorem is given by:

Corollary 6.13

If $F = X + H$ with JH strongly nilpotent, then F is invertible and hence the Jacobian Conjecture holds for these maps.

6.2 Linearisations

The main theorem of the previous section turned out to be a good tool in research on linearisations. In our paper [34] we used it to examine a conjecture by Meisters. Later on Ivanenko used it to give a generalisation of our results in his paper [55].

Meisters conjecture was a result of his work with Deng and Zampieri. In [19] they studied dilations of polynomial maps with $\det(JF) \in \mathbb{C}^*$. They were able to prove that for large enough $s \in \mathbb{C}$ the map sF is locally linearisable to $sJF(0)X$ by means of an analytic map φ_s , the so-called Schröder map, which inverse is an entire function and satisfies some nice properties. Their original aim was to show that φ_s is entire analytic, which would imply that sF and hence F is injective, which in turn would imply the Jacobian Conjecture.

Although they were not able to prove the ‘entireness’ of φ_s , calculations of many examples of polynomial maps of the form $X + H$ with H cubic homogeneous showed that in all these cases the Schröder map was even much better than expected: each time it turned out to be a polynomial automorphism! (see [69]). These observations lead Meisters to the following conjecture:

Conjecture 6.14 (Linearisation Conjecture)

Let $F = X + H$ be a cubic homogeneous polynomial map with JH nilpotent. Then for almost all $s \in \mathbb{C}$ (except a finite number of roots of unity) there exists a polynomial automorphism φ_s such that $\varphi_s^{-1}sF\varphi_s = sX$.

In [28] it was shown by van den Essen that this conjecture is false if $n \geq 4$ and true if $n \leq 3$. Unfortunately there are some typing errors in this publication. Therefore we present the correct proofs here again.

Example 6.15

Define $d(x) := x_3x_1 + x_4x_2 \in \mathbb{C}[x]$. Let F be the polynomial automorphism

$$F = (x_1 + x_4d(x), x_2 - x_3d(x), x_3 + x_4^3, x_4, \dots, x_n)$$

(Compare this example with example 2.17.)

Theorem 6.16

The map F is a counterexample to conjecture 6.14.

Proof. Let $(sF)^m$ denote the iteration $sF \circ sF \circ \dots \circ sF$ and let $(sF)_i^m$ denote the i -th component of this map. Then for each $m \geq 1$ there exist positive integers p and q and polynomials $a(x_4)$ and $b(x_4)$ in $\mathbb{C}[x_4]$ such that:

1. $(sF)_1^m = a(x_4)d(x) + b(x_4)x_1$
2. $a(x_4) = s^p(x_4^4)^{m-1} \cdot x_4 + \text{lot}(x_4)$ where $\text{lot}(x_4)$ is a polynomial with lower order terms in x_4 .
3. $b(x_4) = s^q(x_4^4)^{m-1} + \text{lot}(x_4)$.

For $m = 1$ these statements are obviously true since $(sF)_1 = sx_1 + sx_4d(x)$. If $m > 1$ we use induction. Note that

$$\begin{aligned}
& (sF)_1^{m+1} \\
&= (sF)_1^m \circ sF \\
&= (a(x_4)d(x) + b(x_4)x_1) \circ (sF) \\
&= a(sx_4)d(sF) + b(sx_4)(sx_1 + sx_4d(x)) \\
&= a(sx_4)(s^2(1 + x_4^4)d(x) + x_4^3x_1) + b(sx_4)(sx_1 + sx_4d(x)) \\
&= (a(sx_4)s^2(1 + x_4^4) + b(sx_4))d(x) + (a(sx_4)x_4^3x_1 + b(sx_4)sx_1) \\
&= (a(sx_4)s^2(1 + x_4^4) + b(sx_4))d(x) + (a(sx_4)x_4^3 + b(sx_4)s)x_1 \\
&= ((s^p((sx_4)^4)^{m-1}sx_4 + \text{lot}(x_4))s^2(1 + x_4^4) + (s^q((sx_4)^4)^{m-1} + \text{lot}(x_4)))d(x) \\
&\quad + ((s^p((sx_4)^4)^{m-1}sx_4 + \text{lot}(x_4))s^2x_4^3 + (s^q((sx_4)^4)^{m-1} + \text{lot}(x_4))s)x_1 \\
&= (s^p((sx_4)^4)^{m-1}sx_4s^2x_4^4 + \text{lot}(x_4))d(x) \\
&\quad + (s^p((sx_4)^4)^{m-1}sx_4s^2x_4^3 + \text{lot}(x_4))x_1 \\
&= (s^{p+4(m-1)+3}(x_4^4)^m x_4 + \text{lot}(x_4))d(x) \\
&\quad + (s^{p+4(m-1)+3}(x_4^4)^m + \text{lot}(x_4))x_1
\end{aligned}$$

Hence the statements also hold for $(sF)^{m+1}$.

Now suppose that Meisters' conjecture is true. Then for almost all $s \in \mathbb{C}^*$ sF is linearisable. Pick such a good $s \in \mathbb{C}^*$. Now there exists $\varphi \in \text{Aut}(\mathbb{C}[X])$ with $\varphi^{-1} \circ sF \circ \varphi = sX$, hence in particular $\varphi^{-1} \circ (sF)^m \circ \varphi = s^m X$ for all $m \geq 1$. Composing with φ on the left and φ^{-1} on the right we get $(sF)^m = \varphi(s^m \varphi^{-1})$. Looking at the x_4 -degrees gives the contradiction: both $\deg_{x_4}(\varphi) \leq N$ and $\deg_{x_4}(\varphi^{-1}) \leq N$ for some $N \in \mathbb{N}$. So $\deg_{x_4}(\varphi(s^m \varphi^{-1})) \leq N^2$ for all $m \geq 1$. However the three statements above show that

$$\begin{aligned}
\deg_{x_4}((sF)_1^m) &= \deg_{x_4}(a(x_4)) + \deg_{x_4}(d(x)) \\
&= (4(m-1) + 1) + 1 \\
&= 4m - 2
\end{aligned}$$

And this degree is not bounded for $m \geq 1$. Hence a contradiction and this F cannot satisfy conjecture 6.14. \square

Theorem 6.17

Conjecture 6.14 is true for $n \leq 3$.

Proof. Let $n \leq 3$ and $F = X + H$ where H is cubic homogeneous and $\det(JF) = 1$. Wright's main result in [89] shows that there exists $T \in \text{Gl}_n(\mathbb{C})$ such that $T^{-1}FT = \tilde{F}$ where \tilde{F} is in triangular form. Since $T \in \text{Gl}_n(\mathbb{C})$ it suffices to prove that $s\tilde{F}$ is linearisable for almost all $s \in \mathbb{C}^*$. Now this means that we may assume that F is already on triangular form. The rest of this proof goes straightforward by presenting the actual Schröder maps.

- $n = 1$. Now $F = (x_1) = X$ and take $\varphi_s = (x_1)$.

• $n = 2$. Now $F = (x_1 + ax_2^3, x_2)$, $a \in \mathbb{C}$. Take $\varphi_s = (x_1 + \frac{a}{1-s^2}x_2^3, x_2)$. Then $\varphi_s(sF)\varphi^{-1} = sX$.

• $n = 3$. Now

$$F = \begin{pmatrix} x_1 + ax_2^3 + bx_2^2x_3 + cx_2x_3^2 + dx_3^3 \\ x_2 + ex_3^3 \\ x_3 \end{pmatrix}$$

Let

$$\begin{aligned} t(s) &= \frac{ae^3s^2(1+s^2)(1-s^2+3s^4-s^6+s^8)}{(-1+s^2)(s^4-1)(s^6-1)(s^8-1)} \\ u(s) &= \frac{es^2}{(-1+s^2)(s^4-1)} \\ v(s) &= -(-1+s^2)^{-1} \\ w(s) &= -\frac{s^2(s^4+1)}{(-1+s^2)(s^4-1)(s^6-1)} \\ \varphi_s &= \begin{pmatrix} x_1 + t(s)x_3^9 + u(s)x_3^3(3ax_2^2 + 2bx_2x_3 + cx_3^2) \\ \quad + v(s)(ax_2^3 + bx_2^2x_3 + cx_2x_3^2 + dx_3^3) \\ \quad + w(s)(3ae^2x_2x_3^6 + be^2x_3^7) \\ x_2 + \frac{ex_3^3}{1-s^2} \\ x_3 \end{pmatrix} \end{aligned}$$

By computer one easily verifies that $\varphi_s(sF)\varphi^{-1} = sX$.

So conjecture 6.14 holds for $n \leq 3$. □

The last φ_s was originally computed by Gary Meisters. Only due to the aforementioned typing errors in [28] the author had to do the computation again.

Now that we have recalled the status of conjecture 6.14 we can get back to the notion of strongly nilpotent matrices. It turns out that if we replace *JH is nilpotent* by *JH is strongly nilpotent* the conjecture becomes true for all $n \geq 1$. In fact we don't even need the assumption that this H is cubic homogeneous. We'll prove the following theorem:

Theorem 6.18

Let k be a field. Let $k(s)$ be the field of rational functions in one variable. And let $F : k^n \rightarrow k^n$ be a polynomial map of the form $F = X + H$ with $F(0) = 0$ and JH strongly nilpotent. Then there exists a polynomial automorphism $\varphi_s \in \text{Aut}_{k(s)}(k(s)[X])$ such that

$$\varphi_s^{-1}sF\varphi_s = sJF(0)X.$$

and φ_s is linearly triangularisable over $k(s)$.

Furthermore, the zeros of the denominators of the coefficients of the X -monomials appearing in φ_s are roots of unity.

Before we can prove this result we need another definition and some lemmas.

Definition 6.19

We say that $x_1^{i_1} \cdots x_n^{i_n} > x_1^{i'_1} \cdots x_n^{i'_n}$ if and only if $\sum_{j=1}^n i_j > \sum_{j=1}^n i'_j$ or if $\sum_{j=1}^n i_j = \sum_{j=1}^n i'_j$ and there exists some $l \in \{1, 2, \dots, n\}$ such that $i_j = i'_j$ for all $j < l$ and $i_l > i'_l$.

Furthermore we say that the *rank* of the monomial $M := x_1^{i_1} \cdots x_n^{i_n}$ is the index of this monomial in the ascending ordered list of all monomials M' in x_1, \dots, x_n with $\deg(M') \leq \deg(M)$ (total degree).

Example 6.20

The rank of $x_1 x_2 x_3$ is 15, since the ascending ordered list of all monomials in x_1, x_2 and x_3 of total degree at most three is:

$$\begin{aligned} & x_3, x_2, x_1, \\ & x_3^2, x_2 x_3, x_2^2, x_1 x_3, x_1 x_2, x_1^2, \\ & x_3^3, x_2 x_3^2, x_2^2 x_3, x_2^3, x_1 x_3^2, x_1 x_2 x_3, x_1 x_2^2, x_1^2 x_3, x_1^2 x_2, x_1^3 \end{aligned}$$

Lemma 6.21

For each $2 \leq j \leq n - 1$ let $\ell_j(x_{j+1}, \dots, x_n)$ be a linear form in x_{j+1}, \dots, x_n and let $\mu \in k$. Then the leading monomial with respect to the order of definition 6.19 in the expansion of

$$\mu \prod_{j=2}^n (s x_j + s \ell_j(x_{j+1}, \dots, x_n))^{i_j} \quad (6.2)$$

is

$$\mu s^{i_2 + \dots + i_n} x_2^{i_2} \cdots x_n^{i_n}.$$

Proof. It is obvious that the monomial $\mu s^{i_2 + \dots + i_n} x_2^{i_2} \cdots x_n^{i_n}$ appears in the expansion of (6.2). Now we have to show that this is really the leading monomial. Note that all monomials in the expansion have the same (total) degree: $i_2 + \dots + i_n$. For each $j = 2, \dots, n$ we get a contribution of $(s x_j + s \ell_j(x_{j+1}, \dots, x_n))^{i_j}$ that is of the form

$$\sum_{k=0}^{i_j} \binom{i_j}{k} x_j^k (\ell_j(x_{j+1}, \dots, x_n))^{i_j - k}$$

and since ℓ_j is a linear term that does not contain x_j it is clear that we get the highest order monomial if we take $k = i_j$. So if we start with $j = 2$, we see that the highest x_2 power is i_2 . And if we apply this result to $j = 3$ we see that the leading power product must begin with $x_2^{i_2} x_3^{i_3}$. If we do this for all j we see that it follows that the leading monomial is $\mu s^{i_2 + \dots + i_n} x_2^{i_2} \cdots x_n^{i_n}$. \square

Lemma 6.22

Let F be a polynomial map of the form:

$$F = \begin{pmatrix} x_1 + a(x_2, \dots, x_n) + \ell_1(x_2, \dots, x_n) \\ x_2 + \ell_2(x_3, \dots, x_n) \\ \vdots \\ x_{n-1} + \ell_{n-1}(x_n) \\ x_n \end{pmatrix}$$

where $a(x_2, \dots, x_n)$ is a polynomial with leading monomial (with respect to the order of definition 6.19) $\lambda x_2^{i_2} \cdots x_n^{i_n}$ and $i_2 + \cdots + i_n \geq 2$. Furthermore $\ell_i(x_{i+1}, \dots, x_n)$ are some linear forms. Then there exists a polynomial map φ on triangular form such that

$$\varphi^{-1} s F \varphi = s \begin{pmatrix} x_1 + \tilde{a}(x_2, \dots, x_n) + \ell_1(x_2, \dots, x_n) \\ x_2 + \ell_2(x_3, \dots, x_n) \\ \vdots \\ x_{n-1} + \ell_{n-1}(x_n) \\ x_n \end{pmatrix} \quad (6.3)$$

where the leading monomial of $\tilde{a}(x_2, \dots, x_n)$, say $\tilde{\lambda} x_2^{j_2} \cdots x_n^{j_n}$, is of strict lower order than the leading monomial of $a(x_2, \dots, x_n)$, i.e.:

$$x_2^{j_2} \cdots x_n^{j_n} < x_2^{i_2} \cdots x_n^{i_n}.$$

Proof. Let

$$\varphi = \begin{pmatrix} x_1 + \mu x_2^{i_2} \cdots x_n^{i_n} \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$$

for some $\mu \in k$. Obviously φ is on triangular form. Proving that the equation (6.3) is valid is equivalent with showing that

$$s F \varphi = \varphi \left(s \begin{pmatrix} x_1 + \tilde{a}(x_2, \dots, x_n) + \ell_1(x_2, \dots, x_n) \\ x_2 + \ell_2(x_3, \dots, x_n) \\ \vdots \\ x_{n-1} + \ell_{n-1}(x_n) \\ x_n \end{pmatrix} \right) \quad (6.4)$$

is valid. We do this by looking at the n components. For $i \geq 2$ it is easy to see that the i -th component of the left hand side of (6.4) equals that of the right hand side of (6.4). Hence our only concern is the first component. Put $\hat{a}(x_2, \dots, x_n) := a(x_2, \dots, x_n) - \lambda x_2^{i_2} \cdots x_n^{i_n}$. On the left hand side we have:

$$s F \varphi|_1 = s x_1 + s \mu x_2^{i_2} \cdots x_n^{i_n} + s \lambda x_2^{i_2} \cdots x_n^{i_n} + s \hat{a}(x_2, \dots, x_n) + s \ell_1(x_2, \dots, x_n) \quad (6.5)$$

and on the right hand side:

$$\begin{aligned} & \varphi \left(s \begin{pmatrix} x_1 + \tilde{a}(x_2, \dots, x_n) + \ell_1(x_2, \dots, x_n) \\ x_2 + \ell_2(x_3, \dots, x_n) \\ \vdots \\ x_{n-1} + \ell_{n-1}(x_n) \\ x_n \end{pmatrix} \right)|_1 \quad (6.6) \\ &= s x_1 + s \tilde{a}(x_2, \dots, x_n) + s \ell_1(x_2, \dots, x_n) + \mu \prod_{j=2}^n (s x_j + s \ell_j(x_{j+1}, \dots, x_n))^{i_j} \end{aligned}$$

By subtracting equation (6.6) from equation (6.5) under the assumption that equation (6.4) holds, we get:

$$s(\mu + \lambda)x_2^{i_2} \cdots x_n^{i_n} + s\widehat{a}(x_2, \dots, x_n) = \mu \prod_{j=2}^n (sx_j + s\ell_j(x_{j+1}, \dots, x_n))^{i_j} \quad (6.7)$$

where $\widehat{a} = \widehat{a} - \widetilde{a}$. Now we have to derive a relation for μ to achieve that equation (6.4) indeed holds. We can do this by restricting equation (6.7) to the coefficients of $x_2^{i_2} \cdots x_n^{i_n}$. With lemma 6.21 we see that the restriction of the right hand side of (6.7) to $x_2^{i_2} \cdots x_n^{i_n}$ gives $\mu s^{i_2 + \dots + i_n}$, so we get:

$$s\mu + s\lambda = s^{i_2 + \dots + i_n} \mu$$

and from this equation we can compute μ :

$$\mu = \frac{\lambda}{s^{i_2 + \dots + i_n - 1} - 1}$$

Note that we have assumed that $i_2 + \dots + i_n \geq 2$ so $s^{i_2 + \dots + i_n - 1} - 1 \neq 0$, hence μ is well defined. \square

Using these lemmas we get:

Proof of theorem 6.18. Theorem 6.12 tells that we may assume without loss of generality that $F = (F_1, \dots, F_n)$ is on triangular form. We use induction on n . If $n = 1$ F degenerates to the identity map x_1 and the theorem follows immediately. Now if $n = 2$ we can write

$$F = \begin{pmatrix} x_1 + a(x_2) + \ell_1(x_2) \\ x_2 \end{pmatrix}$$

where $a = \sum_{i=2}^m a_i x_2^i$ and $\ell_1 = ax_2$, the linear part. In particular we have that the leading monomial of a is $a_m x_2^m$. So with lemma 6.22 we know that there exists a map φ_m on triangular form such that

$$\varphi_m^{-1} sF \varphi_m = \begin{pmatrix} sx_1 + \widetilde{a}(x_2) + s\ell_1(x_2) \\ sx_2 \end{pmatrix}$$

where $\deg(\widetilde{a}) < m$. By applying the same lemma m times (if necessary we can use φ_j is the identity) we find a sequence $\varphi_1, \dots, \varphi_m$ such that

$$\varphi_1^{-1} \cdots \varphi_m^{-1} sF \varphi_m \cdots \varphi_1 = s \begin{pmatrix} x_1 + \ell_1(x_2) \\ x_2 \end{pmatrix}$$

So $\varphi_s := \varphi_m \circ \dots \circ \varphi_1$ is as desired. Now let $n > 2$ and consider the map $F = (F_1, F_2, \dots, F_n)$. Put $\widetilde{F} := (F_2, \dots, F_n)$ and $\widetilde{X} := (x_2, \dots, x_n)$. Then by the induction hypothesis we know that there exists an invertible polynomial map $\widetilde{\varphi}_s$ such that

$$\widetilde{\varphi}_s^{-1} s\widetilde{F} \widetilde{\varphi}_s = sJ_{\widetilde{X}} \widetilde{F}(0).$$

So with $\chi = (x_1, \tilde{\varphi}_s)$ and with the notation

$$F = (x_1 + a(x_2, \dots, x_n) + \ell_1(x_2, \dots, x_n), \tilde{F})$$

we get

$$\chi^{-1} s F \chi = s \begin{pmatrix} x_1 + \tilde{a}(x_2, \dots, x_n) + \ell_1(x_2, \dots, x_n) \\ x_2 + \ell_2(x_3, \dots, x_n) \\ \vdots \\ x_{n-1} + \ell_{n-1}(x_n) \\ x_n \end{pmatrix}$$

Now we only have to make the first component linear. Let r be the rank of the leading monomial in $\tilde{a}(x_2, \dots, x_n)$. Lemma 6.22 implies that there exists a φ_r such that

$$\varphi_r^{-1} \chi^{-1} s F \chi \varphi_r = s \begin{pmatrix} x_1 + \tilde{a}_r(x_2, \dots, x_n) + \ell_1(x_2, \dots, x_n) \\ x_2 + \ell_2(x_3, \dots, x_n) \\ \vdots \\ x_{n-1} + \ell_{n-1}(x_n) \\ x_n \end{pmatrix}$$

where the rank of the leading monomial of $\tilde{a}_r(x_2, \dots, x_n) < r$. So after r applications of lemma 6.22 we have obtained a sequence $\varphi_1, \dots, \varphi_r$ such that

$$\varphi_1^{-1} \cdots \varphi_r^{-1} \chi s F \chi \varphi_r \cdots \varphi_1 = s \begin{pmatrix} x_1 + \ell_1(x_2, \dots, x_n) \\ x_2 + \ell_2(x_3, \dots, x_n) \\ \vdots \\ x_{n-1} + \ell_{n-1}(x_n) \\ x_n \end{pmatrix}$$

which proves the theorem. □

Theorem 6.18 concerns multiplication by a scalar s . Ivankenko made a generalisation of this theorem by replacing this scalar s by a vector s_1, \dots, s_n , during his Master Class program at the MRI in the Netherlands (cf. [55]). His main theorem is:

Theorem 6.23

Let k be a field. Let $F : k^n \rightarrow k^n$ be a triangular map. Then for almost all $A \in \text{Gl}_n(k)$ such that A is a lower triangular matrix, there exists a triangular automorphism φ such that $\varphi^{-1} A F \varphi \in \mathfrak{A}_n(k)$. More precisely, if

$$A = \begin{pmatrix} s_1 & 0 & \dots & 0 \\ a_{21} & s_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \dots & s_n \end{pmatrix} \tag{6.8}$$

then for all matrices such that $s_1^{i_1} \cdots s_n^{i_n} \neq 1$ for some finite set of vectors $(i_1, \dots, i_n) \in \mathbb{Z}^n$ depending on F there exists $\varphi \in \mathfrak{J}_n(k)$ of the form

$$\varphi = \begin{pmatrix} x_1 + a_1 \\ x_2 + a_2(x_1) \\ \vdots \\ x_n + a_n(x_1, \dots, x_{n-1}) \end{pmatrix}$$

such that

$$\varphi^{-1} A F \varphi = \begin{pmatrix} s_1 x_1 \\ s_2 x_2 \\ \vdots \\ s_{n-1} x_{n-1} \\ s_n x_n \end{pmatrix} + \begin{pmatrix} \ell_1 \\ \ell_2(x_1) \\ \vdots \\ \ell_{n-1}(x_1, \dots, x_{n-2}) \\ \ell_n(x_1, \dots, x_{n-1}) \end{pmatrix}$$

where $\ell_i(x_1, \dots, x_{i-1})$ are linear in all variables x_1, \dots, x_{i-1} together.

The proof of this theorem goes in the same way as the proof of theorem 6.18. It is written down in [55]. In this paper this theorem is used to prove the next theorem concerning maps of finite order.

Theorem 6.24

Let k be a field and $F : k \rightarrow k$ a triangular polynomial map of the form

$$F = \begin{pmatrix} s_1 x_1 + a_1 \\ s_2 x_2 + a_2(x_1) \\ \vdots \\ s_n x_n + a_n(x_1, \dots, x_{n-1}) \end{pmatrix}$$

Now if $F^m = I_n$ for some $m \in \mathbb{N}$ and $\text{char}(k) \nmid m$ then there exists a triangular automorphism $\varphi \in \text{Aut}_{k(s)}(k(s)[X])$ such that $\varphi^{-1} F \varphi \in \mathfrak{A}_n(k)$.

Note that $\text{char}(k)$ need not be zero for this theorem as long as $\text{char}(k) \nmid m$.

In 1997 a student at the University of Nijmegen, Joost Berson, has done an assignment concerning polynomial maps and linearisability. He constructs polynomial maps $F \in \text{Aut}_{\mathbb{C}}(\mathbb{C}[X])$ such that they have finite order. In particular they all have order 2. He uses theorem 6.24 by Ivanenko. Because Berson's work ([10]) is not widely available, his results are repeated here.

Let $F_X = x_1 T + \cdots + x_d T^d \in \mathbb{C}[x_1, \dots, x_d][T]$. Consider $F_X(F_X(T)) - T$ as a polynomial in T and let \mathfrak{i} be the ideal generated by its coefficients. Let

$$R := \mathbb{C}[x_1, \dots, x_d] / \mathfrak{i}$$

and $c_i := \overline{x_i}$ for all i . His main result is the following theorem.

Theorem 6.25

Let $F := \overline{F_X}$. Then $F^2 = I_n$ and $n := \dim_{\mathbb{C}}(R)$ is finite. Furthermore define $\varphi : \mathbb{C}^n \rightarrow R$ by $\varphi(z_1, \dots, z_n) = d_1 z_1 + \cdots + d_n z_n$ where (d_1, \dots, d_n) is a \mathbb{C} -basis of R . Then $\varphi^{-1} F \varphi$ is linearisable.

We need some lemmas before we can prove this theorem.

Lemma 6.26

Let $\mathfrak{n} = \mathfrak{r}(0) = \{x \in R : x \text{ nilpotent}\}$. Then $c_1^2 = 1$ and $c_i \in \mathfrak{n}$ if $i \geq 2$.

Proof. $F(F(T)) = T$ hence

$$c_1(c_1T + \cdots + c_dT^d) + c_2(c_1T + \cdots + c_dT^d)^2 + \cdots + c_d(c_1T + \cdots + c_dT^d)^d = T \quad (6.9)$$

Looking at the coefficients of T in both sides of (6.9) immediately gives $c_1^2 = 1$. Now define $\overline{R} := R/\mathfrak{n}$. Then $\overline{F}(\overline{F}(T)) = T$ with $\overline{F} = \overline{c}_1T + \cdots + \overline{c}_dT^d$. In \overline{R} we have: $\mathfrak{r}(0) = (0)$. Hence we are done if $\overline{c}_i \in (0) = \mathfrak{r}(0)$ in \overline{R} for all i since this implies $c_i \in \mathfrak{n}$ for all i . Now let $\mathfrak{p} \in \text{Spec}(\overline{R})$. Assume $\overline{c}_d \neq 0$ and $d \geq 2$. Then $\overline{R}/\mathfrak{p}$ is a domain and $F_{\mathfrak{p}}(F_{\mathfrak{p}}(T)) = T$ with $F_{\mathfrak{p}} = (\overline{c}_1 + \mathfrak{p})T + \cdots + (\overline{c}_d + \mathfrak{p})T^d$. Comparing the coefficients of T^{d^2} one sees that $(\overline{c}_d + \mathfrak{p})^{d+1} = 0$ and because $\overline{R}/\mathfrak{p}$ is a domain, this implies $\overline{c}_d + \mathfrak{p} = 0$ or equivalently $\overline{c}_d \in \mathfrak{p}$. Because this holds for any $\mathfrak{p} \in \text{Spec}(\overline{R})$ we have that $\overline{c}_d \in \bigcap_{\mathfrak{p} \in \text{Spec}(\overline{R})} \mathfrak{p} = \mathfrak{r}(0) = (0)$. However this is in contradiction with the assumption that $\overline{c}_d \neq 0$ and $d \geq 2$. So if $d \geq 2$ then $\overline{c}_d = 0$ and by induction we can prove that $\overline{c}_i = 0$ for all $i \geq 2$ hence $c_i \in \mathfrak{n}$. Note that if $d = 1$ then also $c_i \in \mathfrak{n}$ for $i \geq 2$. \square

Lemma 6.26 leads to the conclusion that R is a \mathbb{C} -vector-space of finite dimension:

$$R = \sum_{\alpha_{j_1 \dots j_d}} \alpha_{j_1 \dots j_d} c_1^{j_1} \cdots c_d^{j_d} \quad (6.10)$$

where $j_1 \in \{0, 1\}$ and $j_2, \dots, j_d \in \{0, \dots, \rho - 1\}$ where $\rho \in \mathbb{N}$ such that $\mathfrak{n}^{\rho-1} \neq (0)$ but $\mathfrak{n}^{\rho} = (0)$.

The φ defined in theorem 6.25 depends on the basis (d_1, \dots, d_n) . Lemma 6.26 shows that such a basis really exists. Now we have to show that the choice of (d_1, \dots, d_n) has no influence on the claim of the theorem.

Lemma 6.27

Let (d_1, \dots, d_n) be a \mathbb{C} -basis of R such that $\varphi^{-1}F\varphi$ is linearisable. Let $(\tilde{d}_1, \dots, \tilde{d}_n)$ be another basis of R . Then also $\tilde{\varphi}^{-1}F\tilde{\varphi}$ is linearisable where $\tilde{\varphi}$ corresponds with the basis $(\tilde{d}_1, \dots, \tilde{d}_n)$.

Proof. Let $\psi \in \text{Aut}_{\mathbb{C}}(\mathbb{C}[X])$ such that $\psi^{-1}(\varphi^{-1}F\varphi)\psi$ is linear. Naturally there exists $A \in \text{Mat}_n(\mathbb{C})$ such that $AD = \tilde{D}$ where D is the matrix with rows d_1, \dots, d_n and \tilde{D} likewise. Now for $z \in \mathbb{C}^n$:

$$\begin{aligned} \tilde{\varphi}(Z) &= (\tilde{D})^t z \\ &= (AD)^t z \\ &= D^t (A^t z) \\ &= \varphi(A^t z) \end{aligned}$$

Hence $\tilde{\varphi} = \varphi \circ A^t$. This implies $(\tilde{\varphi})^{-1}F\tilde{\varphi} = (A^t)^{-1}\varphi^{-1}F\varphi A^t$. Now take $\tilde{\psi} = (A^t)\varphi^{-1}$. Then

$$\begin{aligned} (\tilde{\psi})^{-1} \left(\tilde{\varphi}^{-1}F\tilde{\varphi} \right) \tilde{\psi} &= (\psi^{-1}A^t) \left((A^t)^{-1}\varphi^{-1}F\varphi A^t \right) (A^t)^{-1}\psi \\ &= \psi^{-1}(\varphi^{-1}F\varphi)\psi \in \mathfrak{A}_n(\mathbb{C}) \end{aligned}$$

which completes the proof. \square

So now we only have to prove the existence of a basis (d_1, \dots, d_n) such that $\varphi^{-1}F\varphi$ is linearisable. We give a construction. Let ρ be as above. Note that if $\rho = 1$ then $\mathfrak{n} = (0)$ and hence $c_i = 0$ for $i \geq 2$ (lemma 6.26) and hence F and surely $\varphi^{-1}F\varphi$ are linear. So we may assume $\rho \geq 2$. Now take an arbitrary basis $(d_n, d_{n-1}, \dots, d_{q_1})$ of $\mathfrak{n}^{\rho-1}$ (for certain q_1). Extend this basis with elements of $\mathfrak{n}^{\rho-2}$ to a basis $(d_n, d_{n-1}, \dots, d_{q_2})$ of $\mathfrak{n}^{\rho-2}$. Continuing likewise one gets a basis $(d_n, d_{n-1}, \dots, d_q)$ of \mathfrak{n} after a finite number of steps (for certain $2 \leq q \leq n$). We are left to extend this basis to a basis of R . Therefore we use the next lemma.

Lemma 6.28

Let $B := (1 + c_1, 1 - c_1, d_q, d_{q+1}, \dots, d_n)$. Then B is a basis of R . Hence $q = 3$.

Proof. First we show that $B_0 := (1, c_1, d_q, d_{q+1}, \dots, d_n)$ is a basis of R .

- Lemma 6.26 and formula (6.10) show that for each $r \in R$ there exist $a, b \in \mathbb{C}$ and $N \in \mathfrak{n}$ such that $r = a + bc_1 + N$. Hence B_0 is complete.
- Assume $a + bc_1 + N = 0$ for certain $a, b \in \mathbb{C}$ and $N \in \mathfrak{n}$. Hence $a + bc_1 = -N \in \mathfrak{n}$. Hence there exists $m \in \mathbb{N}$ such that $(a + bc_1)^m = 0$. This means that $(a + bx_1)^m \in \mathfrak{i}$, the ideal generated by the coefficients of the one-variable polynomial in T , $F_X(F_X(T)) - T = x_1(x_1T + \dots + x_dT^d) + x_2(x_1T + \dots + x_dT^d)^2 + \dots + x_d(x_1T + \dots + x_dT^d)^d - T$. Now if we take $x_2 = x_3 = \dots = x_d = 0$ and substitute in the map we get $F_X(F_X(T)) - T = (x_1^2 - 1)T$. Now if we substitute also in the relation $(a + bx_1)^m \in \mathfrak{i}$ we get: $(a + bx_1)^m \in \mathbb{C}[x_1](x_1^2 - 1)$. Using both the substitutions $x_1 = 1$ and $x_1 = -1$ one gets $(a + b)^m = 0$ and $(a - b)^m = 0$ (in \mathbb{C}), hence $a + b = a - b = 0$ or $a = b = 0$. Now write $N = \lambda_q d_q + \dots + \lambda_n d_n$. Because $a + bc_1 + N = 0$ and $a = b = 0$ we must have $N = 0$. However (d_q, \dots, d_n) is a basis of \mathfrak{n} . Therefore $\lambda_q = \lambda_{q+1} = \dots = \lambda_n = 0$. And this means that the elements of B_0 are independent.

The conclusion is that B_0 is a basis of R . Now because for all $a, b \in \mathbb{C}$ $a(1 + c_1) + b(1 - c_1) = (a + b) + (a - b)c_1$ we have that also B is a basis of R . \square

Now put $d_1 = 1 + c_1$ and $d_2 = 1 - c_1$. Then (d_1, d_2, \dots, d_n) is a \mathbb{C} -basis of R . Let $\varphi : \mathbb{C}^n \rightarrow R$ be as before. Then $\varphi^{-1}F\varphi$ is linearisable. We need two more lemmas before we can prove this.

Lemma 6.29

$c_1 c_i = -c_i$ for all $2 \leq i \leq d$.

Proof. By induction on i .

- If $i = 2$ we look at the coefficient of T^2 in $F(F(T)) = c_1(c_1T + \cdots + c_dT^d) + c_2(c_1T + \cdots + c_dT^d)^2 + \cdots + c_d(c_1T + \cdots + c_dT^d)^d$. Because $F(F(T)) = T$ this coefficient must be 0. Hence $c_1c_2 + c_2c_1^2 = 0$. However we have seen that $c_1^2 = 1$ so $c_1c_2 + c_2 = 0$ or equivalently $c_1c_2 = -c_2$.
- Let $i > 2$. Assume $c_1c_k = -c_k$ for all $2 \leq k < i$. Let $\mathfrak{q} = (c_2, c_3, \dots, c_{i-1}) \subseteq R$. Then for each $q \in \mathfrak{q}$: $c_1q = -q$. Now consider

$$\begin{aligned} \overline{F}(\overline{F}(T)) &= \overline{c_1}(\overline{c_1}T + \overline{c_i}T^i + \cdots + \overline{c_d}T^d) + \\ &\quad \overline{c_i}(\overline{c_1}T + \overline{c_i}T^i + \cdots + \overline{c_d}T^d)^i + \cdots + \\ &\quad \overline{c_d}(\overline{c_1}T + \overline{c_i}T^i + \cdots + \overline{c_d}T^d)^d \in R/\mathfrak{q}[T] \end{aligned}$$

Since $\overline{F}(\overline{F}(T)) = T$ we get that the coefficient of T^i must be 0. Hence $\overline{c_1c_i} + \overline{c_i c_1^i} = 0$, hence $c_1c_i + c_i c_1^i \in \mathfrak{q}$. Now this means $c_1(c_1c_i + c_i c_1^i) = -(c_1c_i + c_i c_1^i)$. We know that $c_1^2 = 1$. Hence if i is odd we get: $c_1(c_1c_i + c_i c_1) = -(c_1c_i + c_i c_1)$ or $c_1 \cdot 2c_1c_i = -2c_1c_i$ hence $c_i = -c_1c_i$. Otherwise if i is even we get $c_1(c_1c_i + c_i) = -(c_1c_i + c_i)$ or $c_i + c_1c_i = -(c_1c_i + c_i)$ hence $c_1c_i + c_i = 0$. Both cases lead to the conclusion that $c_1c_i = -c_i$.

And hence the lemma holds. □

Lemma 6.30

The ideal $\mathfrak{n} = (c_2, \dots, c_d)$.

Proof. The inclusion $(c_2, \dots, c_d) \subseteq \mathfrak{n}$ follows from 6.26. The other inclusion goes like this: Let $q \in \mathfrak{n}$. Formula (6.10) shows that there exists $\lambda_0, \lambda_1, \dots, \lambda_d \in R$ (not necessarily unique) such that $\lambda_0 + \lambda_1c_1 + \lambda_2c_2 + \cdots + \lambda_dc_d = q$. One may assume that $\lambda_0, \lambda_1 \in \mathbb{C}$. Now some rewriting gives

$$\frac{\lambda_0 + \lambda_1}{2}d_1 + \frac{\lambda_0 - \lambda_1}{2}d_2 + \lambda_2c_2 + \cdots + \lambda_dc_d = q$$

However $\lambda_2c_2 + \cdots + \lambda_dc_d \in \mathfrak{n}$ and q has a unique expansion with respect to the basis (d_1, \dots, d_n) . Hence $\frac{\lambda_0 + \lambda_1}{2} = \frac{\lambda_0 - \lambda_1}{2} = 0$, or $\lambda_0 = \lambda_1 = 0$. Hence $q \in (c_2, \dots, c_d)$. □

So finally we can prove theorem 6.25.

Proof of theorem 6.25. For all $z_1, \dots, z_n \in \mathbb{C}^n$ we have: $F\varphi(z_1, \dots, z_n) = c_1(z_1d_1 + \cdots + z_nd_n) + c_2(z_1d_1 + \cdots + z_nd_n)^2 + \cdots + c_d(z_1d_1 + \cdots + z_nd_n)^d$. Because $d_3, \dots, d_n \in \mathfrak{n}$ we have by lemma 6.29 that $c_1d_i = -d_i$ for $i \geq 3$. Furthermore $c_1d_1 = c_1(1 + c_1) = c_1 + c_1^2 = c_1 + 1 = d_1$ and also $c_1d_2 = c_1(1 - c_1) = c_1 - c_1^2 = c_1 - 1 = -d_2$. This implies $c_1(z_1d_1 + \cdots + z_nd_n) = z_1d_1 - \cdots - z_nd_n$. Now put $c_2(z_1d_1 + \cdots + z_nd_n)^2 + \cdots + c_d(z_1d_1 + \cdots + z_nd_n)^d = \lambda_1d_1 + \lambda_2d_2 + \cdots + \lambda_nd_n$ for $\lambda_1, \dots, \lambda_n \in \mathbb{C}$. Because $\lambda_1d_1 + \cdots + \lambda_nd_n \in (c_2, \dots, c_d) = \mathfrak{n}$ (cf. lemma 6.30) and $\mathfrak{n} = \mathbb{C}d_3 \oplus \cdots \oplus \mathbb{C}d_n$ it follows that both $\lambda_1 = 0$ and $\lambda_2 = 0$. Now let $k, m \in \{3, \dots, n\}$ such that $m \geq k$. Expanding $c_2(z_1d_1 + \cdots + z_nd_n)^2 + \cdots + c_d(z_1d_1 + \cdots + z_nd_n)^d$ gives that an arbitrary

monomial in this expansion looks like $\alpha c_l \cdot z_{i_1} d_{i_1} \cdots z_{i_l} d_{i_l}$ with $\alpha \in \mathbb{N}^*$, $l \in \{2, \dots, d\}$ and $i_1, \dots, i_l \in \{1, \dots, n\}$. Assume there exists $j \in \{1, \dots, l\}$ such that $i_j = m$. Then $d_{i_j} = d_m$. Now let $p \in \mathbb{N}^*$ such that $d_k \in \mathfrak{n}^p$ but $d_k \notin \mathfrak{n}^{p+1}$. Then certainly $d_m \in \mathfrak{n}^p$. (See our construction of (d_1, \dots, d_n) .) Because $c_l \in \mathfrak{n}$ $\alpha c_l \cdot z_{i_1} d_{i_1} \cdots z_{i_l} d_{i_l} \in \mathfrak{n}^{p+1}$. Because $d_k \notin \mathfrak{n}^{p+1}$ it follows that $\alpha c_l \cdot z_{i_1} d_{i_1} \cdots z_{i_l} d_{i_l} \in \mathbb{C}d_{k+1} \oplus \cdots \oplus \mathbb{C}d_n$. The conclusion is that for each $k \in \{3, \dots, n\}$ $\lambda_k \in \mathbb{C}[z_1, \dots, z_{k-1}]$. Hence

$$\begin{aligned} F\varphi(z_1, \dots, z_n) &= (z_1 d_1 - z_2 d_2 - \cdots - z_n d_n) + (\lambda_3 d_3 + \cdots + \lambda_n d_n) \\ &= z_1 d_1 - z_2 d_2 + (\lambda_3 - z_3) d_3 + \cdots + (\lambda_n - z_n) d_n \end{aligned}$$

And hence

$$\varphi^{-1}F\varphi(z_1, \dots, z_n) = (z_1, -z_2, \lambda_3 - z_3, \dots, \lambda_n - z_n)$$

where $\lambda_k \in \mathbb{C}[z_1, \dots, z_{k-1}]$ for each $k \in \{3, \dots, n\}$. Applying Ivanenko's theorem 6.24 now gives that $\varphi^{-1}F\varphi$ is linearisable. \square

6.3 D-nilpotent automorphisms

In this section we present a class of automorphisms defined by Gorni, Tutaj and Zampieri. It is based on their paper [45].

In [20], Drużkowski proved that it is sufficient to prove the Jacobian Conjecture for cubic-linear mappings. These mappings are described by an $n \times n$ matrix A : the cubic-linear map associated to A is defined as $F(X) = X - (AX)^{*3}$. See definition 1.47 for the definition of $(AX)^{*3}$. Furthermore if $\det(JF) = 1$ then F is called a Drużkowski map and the corresponding matrix A a Drużkowski matrix. (See also definition 8.1.) Unfortunately the classification of these Drużkowski matrices is far from trivial. Up to dimension four work has been done by Meisters. See for instance [69]. Fall 1993 the author examined the four-dimensional case and came to the same conclusion as Meisters. In 1996 he worked on the five-dimensional case and managed to get a complete classification. However this took a lot of work. See chapter 8. From this and from the important paper [22] by Drużkowski, it follows that the Jacobian Conjecture holds for Drużkowski maps if $n \leq 7$. The effort of Gorni, Tutaj and Zampieri on this topic is that they present a method to generate equations for the Jacobian condition on cubic-linear maps, only in the coefficients of the corresponding matrix A . While working out details for this method they found that there exists a natural subclass of Drużkowski matrices: the D-nilpotent matrices.

Definition 6.31

A matrix $A \in \text{Mat}_n(\mathbb{C})$ is D-nilpotent if $(DA)^n = 0$ for all diagonal matrices $D \in \text{Mat}_n(\mathbb{C})$.

The 'natural' point in this definition lies in the fact that if one tries to reformulate the fact that $JF - I_n$ is a nilpotent matrix, which must be the case for A to be a Drużkowski matrix, one gets for all $x \in \mathbb{C}^n$ the equation

$$\left(\text{diag} \left((Ax)^{*2} \right) A \right)^n = 0 \tag{6.11}$$

And obviously $\text{diag}((AX)^{*2})$ is a diagonal matrix and hence definition 6.31 is a natural generalisation of equation (6.11). Evidently all D-nilpotent matrices will be Drużkowski matrices. The converse is not true. Let A be the matrix

$$\begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

Then

$$\text{diag}((AX)^{*2})A = \begin{pmatrix} 0 & x_2^2 & 0 \\ -(-x_1 + x_3)^2 & 0 & (-x_1 + x_3)^2 \\ 0 & x_2^2 & 0 \end{pmatrix}$$

and clearly $(\text{diag}((AX)^{*2})A)^3 = 0$ which implies that A is a Drużkowski matrix.

However, if $D = \text{diag}(-1, 1, 1)$ then $DA = \begin{pmatrix} 0 & -1 & 0 \\ -1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$ and hence $(DA)^3 =$

$$\begin{pmatrix} 0 & -2 & 0 \\ -2 & 0 & 2 \\ 0 & 2 & 0 \end{pmatrix},$$

which proves that the class of Drużkowski matrices is strictly larger than the class of D-nilpotent matrices, because A is not D-nilpotent. In fact even in dimension two we have that the class of Drużkowski matrices is strictly larger than the class of D-nilpotent matrices. If one takes $A = \begin{pmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{pmatrix}$. Solving the system of equations one gets from $(\text{diag}((AX)^{*2})A)^2 = 0$ gives two solutions (besides permutations):

$$A = \begin{pmatrix} 0 & a_{1,2} \\ 0 & 0 \end{pmatrix} \quad \text{or} \quad A = \begin{pmatrix} a_{1,1} & -\frac{a_{1,1}^4}{a_{2,1}^3} \\ a_{2,1} & -\frac{a_{1,1}^3}{a_{2,1}^2} \end{pmatrix}$$

The first solution is obviously D-nilpotent. Now let $D = \text{diag}(d_1, d_2)$. If we consider the second solution we get that

$$(DA)^2 = \begin{pmatrix} d_1^2 a_{1,1}^2 - \frac{d_1 a_{1,1}^4 d_2}{a_{2,1}^2} & -\frac{d_1^2 a_{1,1}^5}{a_{2,1}^3} + \frac{d_1 a_{1,1}^7 d_2}{a_{2,1}^5} \\ d_2 a_{2,1} d_1 a_{1,1} - \frac{d_2^2 a_{1,1}^3}{a_{2,1}} & -\frac{d_1 a_{1,1}^4 d_2}{a_{2,1}^2} + \frac{d_2^2 a_{1,1}^6}{a_{2,1}^4} \end{pmatrix}$$

If A is D-nilpotent, this matrix must be 0 for all choices of d_1 and d_2 . However if one solves the corresponding equations, one gets that this matrix is 0 only if $d_1 = \frac{a_{1,1}^2}{a_{2,1}} d_2$.

So for instance if $a_{1,1} = a_{2,1} = 1$ we get that $A = \begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix}$. Taking $D = \text{diag}(1, 2)$ gives $DA = \begin{pmatrix} 1 & -1 \\ 2 & -2 \end{pmatrix}$ and hence $(DA)^2 = \begin{pmatrix} -1 & 1 \\ -2 & 2 \end{pmatrix}$. So this $A \in \text{Mat}_2(\mathbb{C})$ is a Drużkowski matrix but not a D-nilpotent matrix.

In the two-dimensional example it turns out that the D-nilpotent matrix is an upper triangular matrix with zero diagonal. Evidently one can transform this matrix into a

lower triangular matrix by a permutation. Question is whether this permutation destroys the D-nilpotence of the matrix. In the two dimensional case one easily verifies that this property remains valid. In fact it can be shown in any dimension:

Lemma 6.32

Let A be a D-nilpotent matrix and P a permutation matrix of the correct dimensions. Then $P^{-1}AP$ is also D-nilpotent.

Proof. Note that PDP^{-1} is again a diagonal matrix. Hence

$$\begin{aligned} P(DP^{-1}AP)^n P^{-1} &= P(DP^{-1}AP)(DP^{-1}AP) \cdots (DP^{-1}AP)P^{-1} \\ &= (PDP^{-1}A)(PDP^{-1}A) \cdots (PDP^{-1}A)PP^{-1} \\ &= ((PDP^{-1})A)^n \\ &= 0 \end{aligned}$$

because PDP^{-1} is a diagonal matrix and A is D-nilpotent. Now obviously

$$P^{-1}P(DP^{-1}AP)^n P^{-1}P = 0$$

and hence $(DP^{-1}AP)^n = 0$ for all diagonal D and hence $P^{-1}AP$ is D-nilpotent. □

This lemma is the basis for the characterisation theorem of D-nilpotent matrices. Now let $S = (i_1, \dots, i_s)$ be an s -tuple of (distinct) elements of $\{1, 2, \dots, n\}$. Let $A \in \text{Mat}_n(\mathbb{C})$. Then $A_S \in \text{Mat}_s(\mathbb{C})$, where A_S is defined by erasing the rows and columns of A which do not appear in the tuple (i_1, \dots, i_s) . Furthermore the product

$$\pi_S(A) = A_{i_1, i_2} A_{i_2, i_3} \cdots A_{i_s, i_1}$$

is defined for all tuples S . These definitions lead to the following lemma:

Lemma 6.33

If A is a D-nilpotent matrix, $1 \leq s \leq n$ and $S = (i_1, \dots, i_s)$ is an s -tuple of $\{1, 2, \dots, n\}$, then A_S is also D-nilpotent.

Proof. By lemma 6.32 above we may assume that $S = (1, 2, \dots, s)$. Hence we have the following block structure in A :

$$A = \begin{pmatrix} A_S & U \\ V & W \end{pmatrix}$$

for certain $U \in \text{Mat}_{s, n-s}(\mathbb{C})$, $V \in \text{Mat}_{n-s, s}(\mathbb{C})$ and $W \in \text{Mat}_{n-s}(\mathbb{C})$. Now if we put $D = \text{diag}(d_1, \dots, d_s, 0, \dots, 0) \in \text{Mat}_n(\mathbb{C})$. Then

$$DA = \begin{pmatrix} D_S A_S & D_S U \\ 0 & 0 \end{pmatrix}$$

However $(DA)^n = 0$. And this implies that also $(D_S A_S)^n = 0$ which proves that A_S is D-nilpotent. □

With these two lemmas in mind we can recall the main theorem of the paper [45] by Gorni, Tutaj and Zampieri: a characterisation of D-nilpotent matrices.

Theorem 6.34

For any $A \in \text{Mat}_n(\mathbb{C})$ the following statements are equivalent:

1. A is D-nilpotent
2. The product $\pi_S(A) = 0$ for all $S = (i_1, \dots, i_s)$ for $i_1, \dots, i_s \in \{1, 2, \dots, n\}$.
3. There exists a permutation matrix $P \in \text{Mat}_n(\mathbb{C})$ such that $P^{-1}AP$ is upper triangular with zeros on the diagonal.

Because the proof is rather technical, we will not recall it here but refer to the paper [45] for it.

An immediate consequence of the last theorem is

Corollary 6.35

If $F = X - (AX)^{*3}$ and A is D-nilpotent, then the Jacobian Conjecture holds for F , i.e. F is invertible.

Note that $\det(JF) = 1$ automatically if A is D-nilpotent.

The next thing we describe is a comparison of D-nilpotent matrices with strongly nilpotent matrices. As we have seen in remark 6.2 any matrix on upper triangular form with zeros on the diagonal is strong nilpotent. This leads to:

Lemma 6.36

If A is D-nilpotent then A is strong nilpotent.

Proof. Because $A \in \text{Mat}_n(\mathbb{C})$ it suffices to show that A is nilpotent. Theorem 6.34 implies that there exists permutation P such that $P^{-1}AP$ is on triangular form with zeros on the diagonal. Hence $P^{-1}AP$ is nilpotent. Hence

$$\begin{aligned}
 0 &= P \cdot 0 \cdot P^{-1} \\
 &= P(P^{-1}AP) \cdot (P^{-1}AP) \cdots (P^{-1}AP)P^{-1} \\
 &= PP^{-1}APP^{-1}AP \cdots P^{-1}APP^{-1} \\
 &= A \cdot A \cdots A
 \end{aligned}$$

which implies that A is indeed a nilpotent and hence a strongly nilpotent matrix. \square

It doesn't work the other way around: take

$$A = \begin{pmatrix} -6 & -8 \\ \frac{9}{2} & 6 \end{pmatrix}$$

Then one easily verifies that A is strong nilpotent. However according to theorem 6.34 $\pi_S(A) = 0$ for all S if A is D-nilpotent. All possible tuples S are given by (1), (2) and

(1, 2). Computation of $\pi_S(A)$ in these three cases gives:

$$\begin{aligned}\pi_{(1)} &= A_{1,1} = -6 \\ \pi_{(2)} &= A_{2,2} = 6 \\ \pi_{(1,2)} &= A_{1,2}A_{2,1} = -8 \cdot \frac{9}{2} = -36\end{aligned}$$

So instead of finding that $\pi_S(A) = 0$ for all possible tuples S , we find that $\pi_S(A) \neq 0$ for all these S . And hence A cannot be D-nilpotent. The construction of this counterexample is pretty trivial. It is based upon theorem 6.12 and on theorem 6.34. By the first of these theorems we know that any strongly nilpotent matrix, can be transformed to an upper triangular matrix with zeros on the diagonal by a *linear map*. The second theorem states that each D-nilpotent matrix can be transformed to an upper triangular matrix with zeros on the diagonal by a *permutation*. So one only has to start with such an upper triangular matrix U and compute $T^{-1}UT$ for some linear T which is not a permutation. Here we used the matrices

$$U = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad T = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$$

to get $A = T^{-1}UT$.

We conclude this section by stating that the class of D-nilpotent matrices has nice features, as one can see by theorem 6.34. Unfortunately the class of D-nilpotent matrices is not very large. We have seen that it is a subclass of the class of Drużkowski matrices and of the class of strongly nilpotent matrices. However there is also one point which votes in favour of the introduction of D-nilpotent matrices. The second item of theorem 6.34 provides a criterion to decide whether a matrix can be brought into upper triangular form with zeros on the main diagonal by a permutation: one only has to compute $\pi_S(A)$ for all S , which can be done by a computer fairly easy.

Example 6.37

Let $F : \mathbb{C}^6 \rightarrow \mathbb{C}^6$ be the map in figure 6.1 on the next page. A computer check reveals that this map F is in fact the cubic-linear map $F = X + (AX)^{*3}$ where

$$A = \begin{pmatrix} 0 & 0 & 0 & 38 & 0 & 0 \\ 88 & 0 & 0 & 1 & 0 & 11 \\ 4 & -73 & 0 & -59 & -43 & 25 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 40 & 25 & 0 & 61 & 0 & 9 \\ 4 & 0 & 0 & -11 & 0 & 0 \end{pmatrix}$$

Theorem 6.34 states that this matrix A and hence the map F is triangularisable by a permutation if $\pi_S(A) = 0$ for all possible tuples. The list of all tuples -ordered by length- is given by

- (1), (2), (3), (4), (5), (6),
 (1, 2), (1, 3), (1, 4), (1, 5), (1, 6), (2, 3), (2, 4), (2, 5), (2, 6), (3, 4), (3, 5),

$$F := \left(\begin{array}{l}
x_1 + 54872 x_4^3 \\
x_2 + 681472 x_1^3 + 23232 x_1^2 x_4 + 255552 x_1^2 x_6 + 264 x_1 x_4^2 \\
\quad + 5808 x_1 x_4 x_6 + 31944 x_1 x_6^2 + x_4^3 + 33 x_4^2 x_6 + 363 x_4 x_6^2 \\
\quad + 1331 x_6^3 \\
x_3 - 35400 x_1 x_4 x_6 + 60888 x_1 x_4 x_5 - 43800 x_1 x_2 x_6 + 380550 x_4 x_5 x_6 \\
\quad - 1111206 x_2 x_4 x_5 + 470850 x_2 x_5 x_6 + 646050 x_2 x_4 x_6 - 389017 x_2^3 \\
\quad + 41772 x_1 x_4^2 + 7500 x_1 x_6^2 - 110625 x_4 x_6^2 + 1200 x_1^2 x_6 + 64 x_1^3 \\
\quad - 2064 x_1^2 x_5 + 63948 x_1 x_2^2 + 103368 x_1 x_2 x_4 + 75336 x_1 x_2 x_5 \\
\quad - 205379 x_4^3 - 943233 x_2^2 x_4 - 404931 x_2 x_5^2 - 687441 x_2^2 x_5 \\
\quad - 762339 x_2 x_4^2 - 136875 x_2 x_6^2 + 22188 x_1 x_5^2 - 2832 x_1^2 x_4 \\
\quad + 138675 x_5^2 x_6 + 399675 x_2^2 x_6 - 449049 x_4^2 x_5 - 80625 x_5 x_6^2 \\
\quad - 79507 x_5^3 - 327273 x_4 x_5^2 + 261075 x_4^2 x_6 + 15625 x_6^3 \\
\quad - 3504 x_1^2 x_2 - 25800 x_1 x_5 x_6 \\
x_4 \\
x_5 + 131760 x_1 x_4 x_6 + 54000 x_1 x_2 x_6 + 82350 x_2 x_4 x_6 + 15625 x_2^3 \\
\quad + 446520 x_1 x_4^2 + 9720 x_1 x_6^2 + 14823 x_4 x_6^2 + 43200 x_1^2 x_6 \\
\quad + 64000 x_1^3 + 75000 x_1 x_2^2 + 366000 x_1 x_2 x_4 + 226981 x_4^3 \\
\quad + 114375 x_2^2 x_4 + 279075 x_2 x_4^2 + 6075 x_2 x_6^2 + 292800 x_1^2 x_4 \\
\quad + 16875 x_2^2 x_6 + 100467 x_4^2 x_6 + 729 x_6^3 + 120000 x_1^2 x_2 \\
x_6 + 64 x_1^3 - 528 x_1^2 x_4 + 1452 x_1 x_4^2 - 1331 x_4^3
\end{array} \right)$$

Figure 6.1: The map of example 6.37

$(3, 6), (4, 5), (4, 6), (5, 6),$
 $(1, 2, 3), (1, 2, 4), (1, 2, 5), (1, 2, 6), (1, 3, 4), (1, 3, 5), (1, 3, 6), (1, 4, 5),$
 $(1, 4, 6), (1, 5, 6), (2, 3, 4), (2, 3, 5), (2, 3, 6), (2, 4, 5), (2, 4, 6), (2, 5, 6),$
 $(3, 4, 5), (3, 4, 6), (3, 5, 6), (4, 5, 6),$
 $(1, 2, 3, 4), (1, 2, 3, 5), (1, 2, 3, 6), (1, 2, 4, 5), (1, 2, 4, 6), (1, 2, 5, 6),$
 $(1, 3, 4, 5), (1, 3, 4, 6), (1, 3, 5, 6), (1, 4, 5, 6), (2, 3, 4, 5), (2, 3, 4, 6),$
 $(2, 3, 5, 6), (2, 4, 5, 6), (3, 4, 5, 6),$
 $(1, 2, 3, 4, 5), (1, 2, 3, 4, 6), (1, 2, 3, 5, 6), (1, 2, 4, 5, 6), (1, 3, 4, 5, 6),$
 $(2, 3, 4, 5, 6),$
 $(1, 2, 3, 4, 5, 6)$

A pretty easy check on a computer shows that indeed for all these 63 tuples S , $\pi_S(A) = 0$. And hence F is triangularisable by a permutation. In this case take $P = (x_3, x_5, x_2, x_6, x_1, x_4)$. All computations in this example were handled by Maple in a few seconds.

6.4 Pairing cubic homogeneous and cubic-linear maps

In this section we look at an algorithm also due to the Italians Gorni and Zampieri. In their paper [46] they define a relation between cubic homogeneous maps and cubic-linear maps. From Drużkowski it was already known that given a cubic homogeneous map one can build a higher dimensional cubic-linear map such that one of them is invertible if and only if the other one is invertible. In this paper Gorni and Zampieri formalise this by introduction of the so-called pairing mechanism.

Definition 6.38

Let $f : k^n \rightarrow k^n$ be a cubic homogeneous map and let $F = X + (AX)^{*3} : k^N \rightarrow k^N$ be a cubic-linear map, such that $N > n$. The maps f and F are *paired through the matrices B and C* if $\ker(A) = \ker(B)$ and the diagrams in figure 6.2 commute. or in other words, if $BC = I_n$ and $f(x) = BF(Cx)$ for all $x \in k^n$.



Figure 6.2: Commutative pairing diagrams

The conditions on B and C imply that $B \in \text{Mat}_{n,N}(k)$ and $C \in \text{Mat}_{N,n}(k)$. Now the main theorem in [46] is given by:

Theorem 6.39

Each cubic homogeneous map can be paired to a cubic-linear map and vice versa. Moreover, if f and F are paired through B and C , each of the following properties for one of the two mappings implies the same property for the other, for a given $\lambda \in k^$, $|\lambda| \neq 1$:*

1. *the map is injective,*
2. *the map is surjective,*
3. *the map is invertible with polynomial inverse,*
4. *the map has constant Jacobian determinant,*

In fact the original theorem also stated some facts concerning (pre-)conjugations, but because we don't use them here we have left them out. We can express some of the items in theorem 6.39 in formulas. Let $x, y \in k^n$ and $X, Y \in k^N$. Then:

$$f(BX) = CF(X)$$

$$\begin{aligned}
\det(Jf(x)) &= \det(JF(Cx)) \\
\det(JF(X)) &= \det(Jf(BX)) \\
f^{-1}(y) &= BF^{-1}(Cy) \\
F^{-1}(Y) &= Y + (ACf^{-1}(BY))^*3
\end{aligned}$$

Some of these formulas were already implicitly stated in the papers [21] and [27].

This pairing mechanism is very useful. Sometimes it is easier to find cubic-linear examples with certain properties, sometimes it is easier to find cubic homogeneous examples. By this mechanism finding the easiest form is often good enough to construct an example of the other kind. For instance, in [21] Drużkowski presents a cubic-linear example which does not fulfill the so-called Yagzhev's condition:

$$\det(JF(x)) = 1 \quad \forall x \in k^n, \quad \det(JF(x) + JF(y)) \neq 0 \quad \forall x, y \in k^n \quad (6.12)$$

Example 6.40

Let $F : k^{15} \rightarrow k^{15}$ be the map

$$\begin{pmatrix}
x_1 \\
x_2 \\
x_3 + (-2x_4 - x_5 + x_6 + x_7 + x_8 - x_{11} - x_{14})^3 \\
x_4 + \left(-x_3 - x_5 + \frac{1}{2}x_7 + \frac{1}{2}x_{10} - \frac{1}{2}x_{12} - \frac{1}{2}x_{13}\right)^3 \\
x_5 + (x_3 - 2x_4 + x_8 - x_9 - x_{10} - x_{11} + x_{15})^3 \\
x_6 + (x_1 + x_3 - 2x_4 + x_8 - x_9 - x_{10} - x_{11} + x_{15})^3 \\
x_7 + (x_2 + x_3 - 2x_4 + x_8 - x_9 - x_{10} - x_{11} + x_{15})^3 \\
x_8 + \left(x_1 - x_3 - x_5 + \frac{1}{2}x_7 + \frac{1}{2}x_{10} - \frac{1}{2}x_{12} - \frac{1}{2}x_{13}\right)^3 \\
x_9 + (x_1 - 2x_4 - x_5 + x_6 + x_7 + x_8 - x_{11} - x_{14})^3 \\
x_{10} + (x_2 - 2x_4 - x_5 + x_6 + x_7 + x_8 - x_{11} - x_{14})^3 \\
x_{11} + \left(x_1 + x_3 + x_5 - \frac{1}{2}x_7 - \frac{1}{2}x_{10} + \frac{1}{2}x_{12} + \frac{1}{2}x_{13}\right)^3 \\
x_{12} + (x_2 - x_3 + 2x_4 - x_8 + x_9 + x_{10} + x_{11} - x_{15})^3 \\
x_{13} + (x_2 + 2x_4 + x_5 - x_6 - x_7 - x_8 + x_{11} + x_{14})^3 \\
x_{14} + (x_1 + x_2 + x_3 - 2x_4 + x_8 - x_9 - x_{10} - x_{11} + x_{15})^3 \\
x_{15} + (x_1 + x_2 - 2x_4 - x_5 + x_6 + x_7 + x_8 - x_{11} - x_{14})^3
\end{pmatrix}$$

Then Yagzhev's condition (6.12) is not fulfilled is the claim by Drużkowski. However his proof is 'one can check'. First note that $\det(JF(x)) = 1$ for all $x \in k^{15}$. The brute force method to find $x, y \in k^n$ such that Yagzhev's condition is not fulfilled says: compute $JF(x)$, compute $JF(y)$, compute $JF(x) + JF(y)$, compute $\det(JF(x) + JF(y))$ and find solutions of $\det(JF(x) + JF(y)) = 0$. However Maple has problems computing $\det(JF(x) + JF(y))$ and hence the method presented doesn't work. Now applying Gorni and Zampieri's method to F we find $f : k^5 \rightarrow k^5$ which is

paired to F through B and C . Here f is given by

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 - 3x_2^2x_3 + 12x_2^2x_4 - 3x_2^2x_5 \\ x_4 + 3x_1^2x_3 - \frac{3}{2}x_2^2x_3 + 6x_2^2x_4 - \frac{3}{2}x_2^2x_5 \\ \quad + 6x_1x_2x_4 - \frac{3}{2}x_1x_2^2 - \frac{3}{2}x_1^2x_2 - 3x_1x_2x_5 \\ x_5 - 3x_2^2x_3 + 12x_2^2x_4 - 3x_2^2x_5 + 24x_1x_2x_4 - 6x_1x_2^2 \\ \quad - 6x_1^2x_2 - 6x_1x_2x_3 - 6x_1x_2x_5 \end{pmatrix}$$

and B by

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & -\frac{1}{2} & 0 & 0 & -\frac{1}{2} & 0 & \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & 1 & \frac{1}{2} & 0 & -\frac{1}{4} & -\frac{1}{2} & \frac{1}{2} & \frac{1}{4} & \frac{1}{2} & \frac{1}{4} & \frac{1}{4} & 0 & -\frac{1}{2} \\ 0 & 0 & 0 & 0 & 0 & 1 & \frac{1}{2} & 0 & 1 & \frac{1}{2} & 0 & \frac{1}{2} & \frac{1}{2} & -1 & -1 \end{pmatrix}$$

and C^T by

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -2 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Computation of $Jf(x)$ and $Jf(y)$ leads to

$$\det(Jf(x) + Jf(y)) = 32 - 288x_1^2y_2^2 + 576x_1x_2y_1y_2 - 288y_1^2x_2^2$$

Obviously $\{x_1 = \frac{1}{3y_2}, y_1 = 0\}$ gives $\det(Jf(x) + Jf(y)) = 0$. Now put $a := (\frac{1}{3}, -1, 0, 0, 0)$ and $b := (0, 1, 0, 0, 0)$. This gives $\det(Jf(a) + Jf(b)) = 0$. And because $\det(Jf(x)) = 1$ for all x this is a five-dimensional cubic homogeneous example which does not fulfill Yagzhev's condition.² And if we put

$$\begin{aligned} x &= Ca \\ &= (\frac{1}{3}, -1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0) \\ y &= Cb \\ &= (0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0) \end{aligned}$$

then we get $\det(JF(x) + JF(y)) = 0$ and hence F is indeed a cubic-linear map which doesn't obey Yagzhev's condition.³

²Note that Rusek already gave such an example in [82].

³By 'guessing' $x_i = 0$ and $y_i = 0$ for $i = 3, \dots, 15$, the brute method works also and one doesn't need the pairing mechanism to find appropriate x and y .

to the cubic linear map f which is defined as

$$\left(\begin{array}{l} x_1 + \left(\frac{1}{6}x_4 + \frac{1}{6}x_5 - \frac{1}{3}x_6 - \frac{1}{6}x_7 - \frac{1}{6}x_8 + \frac{1}{3}x_9 + x_{17} \right)^3 \\ x_2 + \left(\frac{1}{6}x_4 + \frac{1}{6}x_5 - \frac{1}{3}x_6 - \frac{1}{6}x_7 - \frac{1}{6}x_8 + \frac{1}{3}x_9 - x_{17} \right)^3 \\ x_3 + \left(\frac{1}{6}x_4 + \frac{1}{6}x_5 - \frac{1}{3}x_6 - \frac{1}{6}x_7 - \frac{1}{6}x_8 + \frac{1}{3}x_9 \right)^3 \\ x_4 + \left(\frac{1}{6}x_1 + \frac{1}{6}x_2 - \frac{1}{3}x_3 + x_{17} \right)^3 \\ x_5 + \left(-\frac{1}{6}x_1 - \frac{1}{6}x_2 + \frac{1}{3}x_3 + x_{17} \right)^3 \\ x_6 + x_{17}^3 \\ x_7 + \left(-\frac{1}{6}x_{10} - \frac{1}{6}x_{11} + \frac{1}{3}x_{12} + \frac{1}{12}x_{13} + \frac{1}{12}x_{14} - \frac{1}{12}x_{15} - \frac{1}{12}x_{16} + x_{17} \right)^3 \\ x_8 + \left(-\frac{1}{6}x_{10} - \frac{1}{6}x_{11} + \frac{1}{3}x_{12} + \frac{1}{12}x_{13} + \frac{1}{12}x_{14} - \frac{1}{12}x_{15} - \frac{1}{12}x_{16} - x_{17} \right)^3 \\ x_9 + \left(-\frac{1}{6}x_{10} - \frac{1}{6}x_{11} + \frac{1}{3}x_{12} + \frac{1}{12}x_{13} + \frac{1}{12}x_{14} - \frac{1}{12}x_{15} - \frac{1}{12}x_{16} \right)^3 \\ x_{10} + \left(\frac{1}{6}x_1 - \frac{1}{6}x_2 - \frac{1}{3}x_6 + x_{17} \right)^3 \\ x_{11} + \left(\frac{1}{6}x_1 - \frac{1}{6}x_2 - \frac{1}{3}x_6 - x_{17} \right)^3 \\ x_{12} + \left(\frac{1}{6}x_1 - \frac{1}{6}x_2 - \frac{1}{3}x_6 \right)^3 \\ x_{13} + \left(\frac{1}{6}x_1 + \frac{1}{6}x_2 - \frac{1}{3}x_3 + \frac{1}{6}x_4 + \frac{1}{6}x_5 - \frac{1}{3}x_6 - \frac{1}{6}x_7 - \frac{1}{6}x_8 + \frac{1}{3}x_9 + x_{17} \right)^3 \\ x_{14} + \left(-\frac{1}{6}x_1 - \frac{1}{6}x_2 + \frac{1}{3}x_3 - \frac{1}{6}x_4 - \frac{1}{6}x_5 + \frac{1}{3}x_6 + \frac{1}{6}x_7 + \frac{1}{6}x_8 - \frac{1}{3}x_9 + x_{17} \right)^3 \\ x_{15} + \left(-\frac{1}{6}x_1 - \frac{1}{6}x_2 + \frac{1}{3}x_3 + \frac{1}{6}x_4 + \frac{1}{6}x_5 - \frac{1}{3}x_6 - \frac{1}{6}x_7 - \frac{1}{6}x_8 + \frac{1}{3}x_9 + x_{17} \right)^3 \\ x_{16} + \left(\frac{1}{6}x_1 + \frac{1}{6}x_2 - \frac{1}{3}x_3 - \frac{1}{6}x_4 - \frac{1}{6}x_5 + \frac{1}{3}x_6 + \frac{1}{6}x_7 + \frac{1}{6}x_8 - \frac{1}{3}x_9 + x_{17} \right)^3 \\ x_{17} \end{array} \right)$$

Hence f is a counterexample to the cubic-linear linearisation conjecture.

Also using this method the author tried to give a cubic-linear counterexample to the Real Jacobian Conjecture.

Example 6.43

Let F be the two-dimensional counterexample to the Real Jacobian Conjecture found by Pinchuk. See example 1.24. Note that $\deg(F) = 25$. So before we can use Gorni and Zampieri's pairing mechanism, we must reduce this map to a cubic homogeneous form. However using the Maple implementation of the reduction to cubic homogeneous maps by Bass, Connell and Wright ([8]), this results in a cubic homogeneous map $F' : k^{715} \rightarrow k^{715}$. And this is even before the pairing algorithm starts. Therefore we improved the performance of the reduction algorithm by letting it work on polynomials instead of on monomials. Using this result we found that there exists $L_1, \dots, L_{133}, R_1, \dots, R_{133}$ such that

$$L_{133} \circ \dots \circ L_1 \circ F^{[106]} \circ R_1 \circ \dots \circ R_{133} = F'$$

where $\deg(F') = 3$ and $F : k^{108} \rightarrow k^{108}$. Using the standard method this means that we can transform this map into a $F'' : k^{217} \rightarrow k^{217}$ where F'' is cubic homogeneous. After

a suitable permutation P we can arrange that the last 42 components of $g := P \circ F'' \circ P$ are x_{176}, \dots, x_{217} . Because this single map takes over 60 pages to print we shall not print the map here. So now we can try to find a cubic linear map G which is paired to g . Because the algorithm crashed because of memory problems, we can only present the first stages. Gorni and Zampieri's algorithm is divided into several parts:

1. First write all monomials as a sum of cubic powers of linear forms. Put the linear forms in a matrix D_0 and the coefficients in B_0 . If one uses the same order for both matrices one gets the identity:

$$f = X - B_0(D_0Z)^{*3}$$

Here $B_0 \in \text{Mat}_{217,1977}(k)$ and $D_0 \in \text{Mat}_{1977,217}(k)$.

2. Next step is to add columns to B_0 and rows to D_0 in order to get full rank for B_0 and the right dimension for D_0 . One wants to add as less columns as possible. However adding only one column and checking whether the rank had increased took about two days computation time (on an eight-processor machine with 170MHz Ultra Sparc processors), which is obviously too long. Therefore we simply add the I_{217} matrix. And a zero-matrix of the same dimensions to D_0 .
3. Now we have to make D_0 of full rank and add zeros to B_0 . Again we add I_{217} . This results in a $B \in \text{Mat}_{217,2411}(k)$ and $D \in \text{Mat}_{2411,217}(k)$ such that

$$g = X - B(DX)^{*3}$$

4. Next step is computation of C , a right inverse of B . By solving $BC_i = e_i$ for $i = 1, \dots, 217$ Maple was able to find this C .
5. Next we compute $M := \ker(B)$. This was done straightforward.
6. This is probably the most difficult part: compute $(C|M)^{-1}$. Unfortunately after two weeks, not even the first column of this matrix was found. Therefore we abandoned the computation.

So the only thing we have really found is that there exists cubic-linear $G : k^{2411} \rightarrow k^{2411}$ which is a counterexample to the Real Jacobian Conjecture. Or in other words: though the existence of such a cubic-linear counterexample was already known, now we also have a bound for the dimension of this counterexample. This bound is a worse scenario bound. The actual bound can be computed using:

$$N = 1977 + (217 - \text{rank}(B_0)) + (217 - \text{rank}(D_0))$$

Maple was able to compute these ranks: $\text{rank}(B_0) = 164$ and $\text{rank}(D_0) = 214$. Therefore we can conclude that there exists a cubic-linear counterexample to the Real Jacobian Conjecture in dimension 2033. See [53] for more details about the reduction process.

Note that this bound of 2033 indicates that corollary 1.51 where we claim that the Jacobian Conjecture holds for all cubic-linear Keller maps up to dimension seven doesn't mean very much. Compared to 2033, a positive result up to 7 is not a strong indication that the Jacobian Conjecture holds in general!

Chapter 7

Quadratic homogeneous maps in dimension five

Introduction

In this chapter we present a classification of quadratic homogeneous maps in dimension five with nilpotent Jacobian under the assumption that the Dependence Problem is true in some specific cases. We show that by linear transformations all these maps can be reduced to a triangular map or to one specific map which is not triangularisable by linear mappings. This chapter is divided into three parts. First we describe how we can reduce any quadratic homogeneous map in dimension five to a specific form. Second we present the process of classifying all these maps with nilpotent Jacobian. And third, we show that these maps can be transformed to triangular maps or to a specific non-triangularisable map.

7.1 Reduction to computable cases

If one considers a generic quadratic homogeneous map Q in dimension five, one gets fifteen variables in each component, and hence 75 variables in total. Computation of JQ^5 and using the assumption that JQ is nilpotent hence all elements of JQ^5 must be equal to zero, gives a system of 3150 equations. Solving this set of equations turned out to be pretty hopeless. Therefore we tried to reduce this number of variables as much as possible before actually starting the computations. We begin with describing the assumption concerning the Dependence Problem.

Problem 7.1 (DPLQ)

Let $F = X + H_{(1)} + H_{(2)}$ where $H_{(1)}$ is homogeneous of degree 1 and $H_{(2)}$ is homogeneous of degree 2. Assume $J(H_{(1)} + H_{(2)})$ is nilpotent. Then the rows of $J(H_{(1)} + H_{(2)})$ are linearly dependent over k , i.e. there exists $T \in \text{Gl}_n(k)$ such that $T^{-1}FT = X + H'_{(1)} + H'_{(2)}$ where $H'_{(1)}$ is homogeneous of degree 1 and $H'_{(2)}$ is homogeneous of degree 2 with $H'_{(1)5} = H'_{(2)5} = 0$.

The rest of this chapter is based on the assumption that DPLQ is true. Now let $F = X + Q$ where Q is quadratic homogeneous, $F(0) = 0$ and JQ is nilpotent.

Lemma 7.2

Let $F = X + Q$ as above. There exists $T \in \text{Gl}_5(k)$ such that $(T^{-1}FT)_5 = x_5$.

Proof. If $F = X + Q$ with Q quadratic homogeneous and JQ is nilpotent then F is as in the DPLQ and hence the lemma holds. \square

The result of this theorem is that we may assume that the last row of Q is equal to zero. This decreases the number of variables by 15.

Lemma 7.3

If $F = X + Q$ as above then we may assume that the coefficients of the monomial x_5^2 in all 5 components are 0.

Proof. Write

$$F = X + \begin{pmatrix} H_1(x_1, x_2, x_3, x_4) \\ H_2(x_1, x_2, x_3, x_4) \\ H_3(x_1, x_2, x_3, x_4) \\ H_4(x_1, x_2, x_3, x_4) \\ 0 \end{pmatrix} + \begin{pmatrix} x_5 \ell_1(x_1, x_2, x_3, x_4) \\ x_5 \ell_2(x_1, x_2, x_3, x_4) \\ x_5 \ell_3(x_1, x_2, x_3, x_4) \\ x_5 \ell_4(x_1, x_2, x_3, x_4) \\ 0 \end{pmatrix} + \begin{pmatrix} \lambda_1 x_5^2 \\ \lambda_2 x_5^2 \\ \lambda_3 x_5^2 \\ \lambda_4 x_5^2 \\ 0 \end{pmatrix}$$

where $H_i(x_1, x_2, x_3, x_4)$ is quadratic homogeneous and $\ell_i(x_1, x_2, x_3, x_4)$ is linear. Let E be the elementary map

$$E = X - \begin{pmatrix} \lambda_1 x_5^2 \\ \lambda_2 x_5^2 \\ \lambda_3 x_5^2 \\ \lambda_4 x_5^2 \\ 0 \end{pmatrix}$$

Then

$$E \circ F = X + \begin{pmatrix} H_1(x_1, x_2, x_3, x_4) \\ H_2(x_1, x_2, x_3, x_4) \\ H_3(x_1, x_2, x_3, x_4) \\ H_4(x_1, x_2, x_3, x_4) \\ 0 \end{pmatrix} + \begin{pmatrix} x_5 \ell_1(x_1, x_2, x_3, x_4) \\ x_5 \ell_2(x_1, x_2, x_3, x_4) \\ x_5 \ell_3(x_1, x_2, x_3, x_4) \\ x_5 \ell_4(x_1, x_2, x_3, x_4) \\ 0 \end{pmatrix}$$

Note that $\det(JE) = 1$. Let $N = J(E \circ F - X)$. Because JQ and N are homogeneous we know by [8, lemma 4.1] that JQ is nilpotent if and only if $\det(JF) = 1$. And $\det(JF) = 1$ if and only if $\det(JEF) = 1$. And finally $\det(JEF) = 1$ if and only if N is nilpotent. Hence JQ is nilpotent if and only if N is nilpotent. \square

So we may assume that $\lambda_i = 0$ for $i = 1, \dots, 4$. And in particular we have that $Q(0) = 0$ if we view $Q \in k(x_5)[x_1, \dots, x_4]$.

Lemma 7.4

There exists $T \in \text{Gl}_5(k)$ such that $T^{-1}FT$ is of the form

$$X + \begin{pmatrix} H_1(x_2, x_3, x_4) \\ H_2(x_3, x_4) \\ H_3(x_4) \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} x_5 \ell_1(x_1, x_2, x_3, x_4) \\ x_5 \ell_2(x_1, x_2, x_3, x_4) \\ x_5 \ell_3(x_1, x_2, x_3, x_4) \\ x_5 \ell_4(x_1, x_2, x_3, x_4) \\ 0 \end{pmatrix}$$

Proof. JQ must be nilpotent for all values of x_i . Hence in particular JQ must be nilpotent for $x_5 = 0$. However this means:

$$JQ|_{x_5=0} = \left(\begin{array}{c|ccc} J_{x_1, x_2, x_3, x_4} \begin{pmatrix} H_1(x_1, x_2, x_3, x_4) \\ H_2(x_1, x_2, x_3, x_4) \\ H_3(x_1, x_2, x_3, x_4) \\ H_4(x_1, x_2, x_3, x_4) \end{pmatrix} & 0 \\ \hline 0 & 0 & 0 & 0 \end{array} \right)$$

And this matrix is nilpotent if and only if $J_{x_1, x_2, x_3, x_4}(H_1, H_2, H_3, H_4)$ is nilpotent. However because $\deg_{x_1, x_2, x_3, x_4}(H_i) = 2$ we know that $J_{x_1, x_2, x_3, x_4}(H_1, H_2, H_3, H_4)$ is in fact strong nilpotent by Meisters and Olech ([73]). Hence by theorem 6.12 we know that there exists $T \in \text{Gl}_4(k)$ such that $T^{-1}J_{x_1, x_2, x_3, x_4}(H_1, H_2, H_3, H_4)T$ is on triangular form. If we expand this T to a five-dimensional map by adding a one-dimensional identity, we get a T as in the lemma. \square

Lemma 7.5

If $F = X + Q$ as above then we may assume that $\ell_4(x_1, x_2, x_3, x_4) = 0$.

Proof. Note that JQ is given by

$$\left(\begin{array}{c|ccc} J_{x_1, \dots, x_4} \begin{pmatrix} H_1(x_2, x_3, x_4) + x_5 \ell_1 \\ H_2(x_3, x_4) + x_5 \ell_2 \\ H_3(x_4) + x_5 \ell_3 \\ x_5 \ell_4 \end{pmatrix} & * \\ \hline 0 & & 0 \end{array} \right)$$

Obviously JQ is nilpotent implies that $J_{x_1, \dots, x_4} \begin{pmatrix} H_1(x_2, x_3, x_4) + x_5 \ell_1 \\ H_2(x_3, x_4) + x_5 \ell_2 \\ H_3(x_4) + x_5 \ell_3 \\ x_5 \ell_4 \end{pmatrix}$ is nilpotent.

Because of this nilpotence and $\deg_{x_1, \dots, x_4}(H_i + x_5 \ell_i) \leq 2$ and $Q(0) = 0$ we can use the DPLQ and we know that there exist $a_1, \dots, a_4 \in k(x_5)$, not all zero, such that

$$\sum_{i=1}^4 a_i(x_5) (H_i(x_{i+1}, \dots, x_4) + x_5 \ell_i(x_1, x_2, x_3, x_4)) = 0 \tag{7.1}$$

Clearing denominators we find that there exist $a_i(x_5) \in k[x_5]$, not all zero, such that (7.1) holds. Furthermore we can also arrange that $\text{gcd}(a_1(x_5), \dots, a_4(x_5)) = 1$. This

implies that there exists $i \in \{1, \dots, 4\}$ such that $x_5 \nmid a_i(x_5)$. Consequently there exists $i \in \{1, \dots, 4\}$ such that $a_i(0) \neq 0$. Now write $a_i(x_5) = a_i(0) + \text{hot}(x_5)$ where $\text{hot}(x_5)$ stands for higher order terms with respect to x_5 . By (7.1) we know that

$$\begin{aligned} 0 &= \sum_{i=1}^4 (a_i(0) + \text{hot}(x_5)) (H_i(x_{i+1}, \dots, x_4) + x_5 \ell_i(x_1, x_2, x_3, x_4)) \\ &= \sum_{i=1}^4 a_i(0) (H_i(x_{i+1}, \dots, x_4) + x_5 \ell_i(x_1, x_2, x_3, x_4)) \\ &\quad + \sum_{i=1}^4 \text{hot}(x_5) (H_i(x_{i+1}, \dots, x_4) + x_5 \ell_i(x_1, x_2, x_3, x_4)) \end{aligned}$$

Note that

$$\deg_{x_5} (a_i(0) (H_i(x_{i+1}, \dots, x_4) + x_5 \ell_i(x_1, x_2, x_3, x_4))) = 1$$

and that

$$\deg_{x_5} (\text{hot}(x_5) (H_i(x_{i+1}, \dots, x_4) + x_5 \ell_i(x_1, x_2, x_3, x_4))) > 1$$

Hence we must have

$$\sum_{i=1}^4 a_i(0) (H_i(x_{i+1}, \dots, x_4) + x_5 \ell_i(x_1, x_2, x_3, x_4)) = 0 \quad (7.2)$$

And because there exists at least one i with $a_i(0) \neq 0$, we see that there exists a non-trivial dependency relation between $H_i + x_5 \ell_i$. Now that we know that there exists such a relation, we must make sure that this relation and the corresponding T does not interfere with the triangular form we have already. Obviously adding rows with a higher index to a row with lower index is a safe method: the result remains on triangular form.

- Assume $a_1(0) \neq 0$. Then there exists T such that

$$T^{-1}FT = X + \begin{pmatrix} 0 \\ H_2(x_3, x_4) \\ H_3(x_4) \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ x_5 \ell_2(x_1, x_2, x_3, x_4) \\ x_5 \ell_3(x_1, x_2, x_3, x_4) \\ x_5 \ell_4(x_1, x_2, x_3, x_4) \\ 0 \end{pmatrix}$$

And hence

$$P_{(4321)}^{-1} T^{-1} F T P_{(4321)} = X + \begin{pmatrix} H_2(x_2, x_3) \\ H_3(x_3) \\ 0 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} x_5 \ell_2(x_1, x_2, x_3, x_4) \\ x_5 \ell_3(x_1, x_2, x_3, x_4) \\ x_5 \ell_4(x_1, x_2, x_3, x_4) \\ 0 \\ 0 \end{pmatrix}$$

where $P_{(4321)}$ is the appropriate permutation.

- Assume $a_1(0) = 0$ and $a_2(0) \neq 0$. Now there exists T such that

$$T^{-1}FT = X + \begin{pmatrix} H_1(x_2, x_3, x_4) \\ 0 \\ H_3(x_4) \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} x_5 \ell_1(x_1, x_2, x_3, x_4) \\ 0 \\ x_5 \ell_3(x_1, x_2, x_3, x_4) \\ x_5 \ell_4(x_1, x_2, x_3, x_4) \\ 0 \end{pmatrix}$$

And by conjugation with $P_{(432)}$ we get

$$P_{(432)}^{-1}T^{-1}FTP_{(432)} = X + \begin{pmatrix} H_1(x_2, x_3, x_4) \\ H_3(x_3) \\ 0 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} x_5 \ell_1(x_1, x_2, x_3, x_4) \\ x_5 \ell_3(x_1, x_2, x_3, x_4) \\ x_5 \ell_4(x_1, x_2, x_3, x_4) \\ 0 \\ 0 \end{pmatrix}$$

- $a_1(0) = 0$, $a_2(0) = 0$ and $a_3(0) \neq 0$. Then there exists T such that

$$P_{(43)}^{-1}T^{-1}FTP_{(43)} = X + \begin{pmatrix} H_1(x_2, x_3, x_4) \\ H_2(x_3, x_4) \\ 0 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} x_5 \ell_1(x_1, x_2, x_3, x_4) \\ x_5 \ell_2(x_1, x_2, x_3, x_4) \\ x_5 \ell_4(x_1, x_2, x_3, x_4) \\ 0 \\ 0 \end{pmatrix}$$

- $a_1(0) = 0$, $a_2(0) = 0$, $a_3(0) = 0$ and $a_4(0) \neq 0$. Then automatically $\ell_4(x_1, x_2, x_3, x_4) = 0$ and we have

$$F = X + \begin{pmatrix} H_1(x_2, x_3, x_4) \\ H_2(x_3, x_4) \\ H_3(x_4) \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} x_5 \ell_1(x_1, x_2, x_3, x_4) \\ x_5 \ell_2(x_1, x_2, x_3, x_4) \\ x_5 \ell_3(x_1, x_2, x_3, x_4) \\ 0 \\ 0 \end{pmatrix}$$

Note that the result of the first three cases are special instances of the last case. Hence we get that there exists invertible $T : k^5 \rightarrow k^5$ such that

$$T^{-1}FT = X + \begin{pmatrix} H_1(x_2, x_3, x_4) \\ H_2(x_3, x_4) \\ H_3(x_4) \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} x_5 \ell_1(x_1, x_2, x_3, x_4) \\ x_5 \ell_2(x_1, x_2, x_3, x_4) \\ x_5 \ell_3(x_1, x_2, x_3, x_4) \\ 0 \\ 0 \end{pmatrix}$$

This completes the proof. \square

So at this point we have achieved that Q has the last two rows equal to zero. Now we can repeat the trick of lemma 7.3.

Lemma 7.6

If $F = X + Q$ as above then we may assume that the coefficients of the monomial x_4^2 and the monomial x_4x_5 in all 5 components are 0.

Proof. Write $F = X + Q$ where

$$Q = \begin{pmatrix} H_1(x_2, x_3) \\ H_2(x_3) \\ 0 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} x_4\ell_1(x_2, x_3) \\ x_4\ell_2(x_3) \\ 0 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} x_5\ell'_1(x_1, x_2, x_3) \\ x_5\ell'_2(x_1, x_2, x_3) \\ x_5\ell'_3(x_1, x_2, x_3) \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} \sigma_1x_4^2 \\ \sigma_2x_4^2 \\ \sigma_3x_4^2 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} \rho_1x_4x_5 \\ \rho_1x_4x_5 \\ \rho_1x_4x_5 \\ 0 \\ 0 \end{pmatrix}$$

Now define the elementary maps

$$E_1 = X - \begin{pmatrix} \sigma_1x_4^2 \\ \sigma_2x_4^2 \\ \sigma_3x_4^2 \\ 0 \\ 0 \end{pmatrix}, \quad E_2 = X - \begin{pmatrix} \rho_1x_4x_5 \\ \rho_1x_4x_5 \\ \rho_1x_4x_5 \\ 0 \\ 0 \end{pmatrix}$$

Composition on the left side gives

$$E_1 \circ E_2 \circ F = X + \begin{pmatrix} H_1(x_2, x_3) \\ H_2(x_3) \\ 0 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} x_4\ell_1(x_2, x_3) \\ x_4\ell_2(x_3) \\ 0 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} x_5\ell'_1(x_1, x_2, x_3) \\ x_5\ell'_2(x_1, x_2, x_3) \\ x_5\ell'_3(x_1, x_2, x_3) \\ 0 \\ 0 \end{pmatrix}$$

And by the same argument as in lemma 7.3 we get that JQ is nilpotent if and only if $J(E_1 \circ E_2 \circ F - X)$ is nilpotent. \square

Now let $\ell'_i(x_1, x_2, x_3) = d_ix_1 + e_ix_2 + f_ix_3$ for $i = 1, 2, 3$. Substituting $x_1 = x_2 = x_3 = x_4 = 0$ and $x_5 = 1$ in JQ gives

$$JQ|_{x_1=0, x_2=0, x_3=0, x_4=0, x_5=1} = \begin{pmatrix} d_1 & e_1 & f_1 & 0 & 0 \\ d_2 & e_2 & f_2 & 0 & 0 \\ d_3 & e_3 & f_3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

This matrix must be nilpotent also. Now the block structure implies that

$$\begin{pmatrix} d_1 & e_1 & f_1 \\ d_2 & e_2 & f_2 \\ d_3 & e_3 & f_3 \end{pmatrix}$$

is nilpotent. Hence there must exist $\mu_1, \mu_2, \mu_3 \in k$ such that

$$\mu_1 \ell'_1 + \mu_2 \ell'_2 + \mu_3 \ell'_3 = 0$$

where $\{\mu_1, \mu_2, \mu_3\} \neq \{0\}$. Here we must consider three cases:

1. $\mu_1 \neq 0$. Hence $\ell'_1 + \frac{\mu_2}{\mu_3}x_2 + \frac{\mu_3}{\mu_2}x_3 = 0$. Now put

$$T = (x_1 - \frac{\mu_2}{\mu_3}x_2 - \frac{\mu_3}{\mu_2}x_3, x_2, x_3, x_4, x_5)$$

and we get that $T^{-1}FT$ is of the form

$$X + \begin{pmatrix} H_1(x_2, x_3) \\ H_2(x_3) \\ 0 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} x_4 \ell'_1(x_2, x_3) \\ x_4 \ell'_2(x_3) \\ 0 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ x_5 \ell'_2(x_1, x_2, x_3) \\ x_5 \ell'_3(x_1, x_2, x_3) \\ 0 \\ 0 \end{pmatrix}$$

Like before this T does not interfere with the nice forms of the triangular part and the $x_4 \ell$ -part.

Note that

$$JQ|_{x_1=0, x_2=0, x_3=0, x_4=0, x_5=1} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ d_2 & e_2 & f_2 & 0 & 0 \\ d_3 & e_3 & f_3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

and hence the matrix $\begin{pmatrix} 0 & 0 & 0 \\ d_2 & e_2 & f_2 \\ d_3 & e_3 & f_3 \end{pmatrix}$ is nilpotent. From the trace it immediately follows that $f_3 = -e_2$. If we put

$$\begin{aligned} H_1(x_2, x_3) &= \mu x_2 x_3 + \nu x_2^2 + \tau_1 x_3^2 \\ H_2(x_3) &= \tau_2 x_3^2 \\ \ell_1(x_2, x_3) &= b_1 x_2 + c_1 x_3 \\ \ell_2(x_3) &= c_2 x_3 \end{aligned}$$

then we see that we have reduced our original map to $F = X + Q$ where Q is given by:

$$\begin{pmatrix} \mu x_2 x_3 + \nu x_2^2 + \tau_1 x_3^2 + x_4(b_1 x_2 + c_1 x_3) & & & & \\ \tau_2 x_3^2 + x_4(c_2 x_3) & & & & x_5(d_2 x_1 + e_2 x_2 + f_2 x_3) \\ & & & & x_5(d_3 x_1 + e_3 x_2 - e_2 x_3) \\ & & & 0 & \\ & & & 0 & \end{pmatrix} \quad (7.3)$$

2. $\mu_1 = 0$ and $\mu_2 \neq 0$. Now take

$$T = (x_1, x_2 - \frac{\mu_3}{\mu_2}x_3, x_3, x_4, x_5)$$

and we have that $T^{-1}FT$ is of the form

$$X + \begin{pmatrix} H_1(x_2, x_3) \\ H_2(x_3) \\ 0 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} x_4 \ell_1(x_2, x_3) \\ x_4 \ell_2(x_3) \\ 0 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} x_5 \ell'_1(x_1, x_2, x_3) \\ 0 \\ x_5 \ell'_3(x_1, x_2, x_3) \\ 0 \\ 0 \end{pmatrix}$$

We put H_1, H_2, ℓ_i and ℓ'_i as above. And by computing $JQ|_{x_1=0, x_2=0, x_3=0, x_4=0, x_5=1}$ we can eliminate one variable using the trace. This time we have $f_3 = -d_1$. The resulting Q equals

$$\begin{pmatrix} \mu x_2 x_3 + \nu x_2^2 + \tau_1 x_3^2 + x_4(b_1 x_2 + c_1 x_3) + x_5(d_1 x_1 + e_1 x_2 + f_1 x_3) \\ \tau_2 x_3^2 + x_4(c_2 x_3) & & x_5(d_3 x_1 + e_3 x_2 - d_1 x_3) \\ & 0 & \\ & 0 & \end{pmatrix} \quad (7.4)$$

3. $\mu_1 = 0$ and $\mu_2 = 0$. Hence $\mu_3 \neq 0$ and $\ell'_3 = 0$. Hence F is of the form

$$X + \begin{pmatrix} H_1(x_2, x_3) \\ H_2(x_3) \\ 0 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} x_4 \ell_1(x_2, x_3) \\ x_4 \ell_2(x_3) \\ 0 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} x_5 \ell'_1(x_1, x_2, x_3) \\ x_5 \ell'_2(x_1, x_2, x_3) \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

Note that we have three rows equal to zero. So if we use the same values for H_1, H_2, ℓ_i and ℓ'_i as above, we can arrange that the coefficients of the monomials $x_3^2, x_3 x_4$ and $x_3 x_5$ can be set to zero using the same trick as in lemma 7.3 with an elementary map. Furthermore the trace of $JQ|_{x_1=0, x_2=0, x_3=0, x_4=0, x_5=1}$ gives $d_1 = -e_2$. So this case can be reduced to the map $F = X + Q$ where Q is presented by:

$$\begin{pmatrix} \mu x_2 x_3 + \nu x_2^2 + x_4(b_1 x_2) + x_5(-e_2 x_1 + e_1 x_2) \\ & & x_5(d_2 x_1 + e_2 x_2) \\ & 0 & \\ & 0 & \\ & 0 & \end{pmatrix} \quad (7.5)$$

In section 7.2 we start the computations with the Q 's from (7.3), (7.4) and (7.5).

7.2 Finding $F = X + Q$ with JQ nilpotent

7.2.1 Start with map (7.3)

If we start with the map from (7.3) we find 12 solutions. We find those solutions by starting with the complete system, extracting simple equations, solve them and substitute the results in the original system. We repeat this until we have solved the system completely. Unfortunately we have to make choices solving those simple equations. Therefore we get a tree-like graph starting with the complete set of equations at the top and ending with all solutions in the leaves. We will refer to these graphs as solution graphs.¹

As mentioned before this case gives 12 solutions:

Lemma 7.7

If Q is as in (7.3) then we have the following possibilities for Q :

$$1. \begin{pmatrix} \mu x_2 x_3 + \nu x_2^2 + \tau_1 x_3^2 + (b_1 x_2 + c_1 x_3) x_4 \\ \tau_2 x_3^2 + c_2 x_3 x_4 + f_2 x_3 x_5 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$2. \begin{pmatrix} \tau_1 x_3^2 + c_1 x_3 x_4 \\ \tau_2 x_3^2 + c_2 x_3 x_4 + (d_2 x_1 + f_2 x_3) x_5 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$3. \begin{pmatrix} 0 \\ \tau_2 x_3^2 + c_2 x_3 x_4 + d_2 x_1 x_5 \\ d_3 x_1 x_5 \\ 0 \\ 0 \end{pmatrix}$$

$$4. \begin{pmatrix} (b_1 x_2 - \frac{b_1 d_2 x_3}{d_3}) x_4 \\ d_2 x_1 x_5 \\ d_3 x_1 x_5 \\ 0 \\ 0 \end{pmatrix}$$

¹Originally these graphs were called solution trees. However Kristi Lampe, Washington University St. Louis, pointed out that trees usually don't grow down wards.

$$5. \begin{pmatrix} -2 \frac{\nu d_2 x_2 x_3}{d_3} + \nu x_2^2 + \frac{d_2^2 \nu x_3^2}{d_3^2} + (b_1 x_2 - \frac{b_1 d_2 x_3}{d_3}) x_4 \\ d_2 x_1 x_5 \\ d_3 x_1 x_5 \\ 0 \\ 0 \end{pmatrix}$$

$$6. \begin{pmatrix} 0 \\ \tau_2 x_3^2 + c_2 x_3 x_4 + (d_2 x_1 + f_2 x_3) x_5 \\ d_3 x_1 x_5 \\ 0 \\ 0 \end{pmatrix}$$

$$7. \begin{pmatrix} \mu x_2 x_3 + \nu x_2^2 + \tau_1 x_3^2 + (b_1 x_2 + c_1 x_3) x_4 \\ (e_2 x_2 - \frac{e_2^2 x_3}{e_3}) x_5 \\ (e_3 x_2 - e_2 x_3) x_5 \\ 0 \\ 0 \end{pmatrix}$$

$$8. \begin{pmatrix} (b_1 x_2 - \frac{b_1 e_2 x_3}{e_3}) x_4 \\ -\frac{b_1 d_2 x_3 x_4}{e_3} + (d_2 x_1 + e_2 x_2 - \frac{e_2^2 x_3}{e_3}) x_5 \\ (e_3 x_2 - e_2 x_3) x_5 \\ 0 \\ 0 \end{pmatrix}$$

$$9. \begin{pmatrix} -2 \frac{\nu e_2 x_2 x_3}{e_3} + \nu x_2^2 + \frac{e_2^2 \nu x_3^2}{e_3^2} + (b_1 x_2 - \frac{b_1 e_2 x_3}{e_3}) x_4 \\ (\frac{e_2 d_3 x_1}{e_3} + e_2 x_2 - \frac{e_2^2 x_3}{e_3}) x_5 \\ (d_3 x_1 + e_3 x_2 - e_2 x_3) x_5 \\ 0 \\ 0 \end{pmatrix}$$

$$10. \begin{pmatrix} 0 \\ (d_2 x_1 + e_2 x_2 - \frac{e_2^2 x_3}{e_3}) x_5 \\ (d_3 x_1 + e_3 x_2 - e_2 x_3) x_5 \\ 0 \\ 0 \end{pmatrix}$$

$$11. \begin{pmatrix} -\frac{\tau_2 e_3 x_3^2}{d_3} \\ \tau_2 x_3^2 + \left(d_2 x_1 + \frac{e_3 d_2 x_2}{d_3} - \frac{e_3 d_2^2 x_3}{d_3^2} \right) x_5 \\ (d_3 x_1 + e_3 x_2 - \frac{e_3 d_2 x_3}{d_3}) x_5 \\ 0 \\ 0 \end{pmatrix}$$

$$12. \begin{pmatrix} -\frac{\tau_2 e_3 x_3^2}{d_3} - \frac{c_2 e_3 x_3 x_4}{d_3} \\ \tau_2 x_3^2 + c_2 x_3 x_4 + \left(d_2 x_1 + \frac{e_3 d_2 x_2}{d_3} - \frac{e_3 d_2^2 x_3}{d_3^2} \right) x_5 \\ (d_3 x_1 + e_3 x_2 - \frac{e_3 d_2 x_3}{d_3}) x_5 \\ 0 \\ 0 \end{pmatrix}$$

For the solution graph corresponding with these solutions see figure 7.1. The boxed numbers coincide with the numbers in the lemma above.

7.2.2 Start with map (7.4)

Lemma 7.8

If Q is as in (7.4) then we have the following possibilities for Q :

$$13. \begin{pmatrix} \mu x_2 x_3 + \nu x_2^2 + \tau_1 x_3^2 + (b_1 x_2 + c_1 x_3) x_4 + (e_1 x_2 + f_1 x_3) x_5 \\ \tau_2 x_3^2 + c_2 x_3 x_4 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$14. \begin{pmatrix} \mu x_2 x_3 + \nu x_2^2 + \tau_1 x_3^2 + (b_1 x_2 + c_1 x_3) x_4 + (e_1 x_2 + f_1 x_3) x_5 \\ 0 \\ e_3 x_2 x_5 \\ 0 \\ 0 \end{pmatrix}$$

$$15. \begin{pmatrix} \nu x_2^2 + b_1 x_2 x_4 + \left(d_1 x_1 + e_1 x_2 - \frac{d_1^2 x_3}{d_3} \right) x_5 \\ 0 \\ (d_3 x_1 + e_3 x_2 - d_1 x_3) x_5 \\ 0 \\ 0 \end{pmatrix}$$

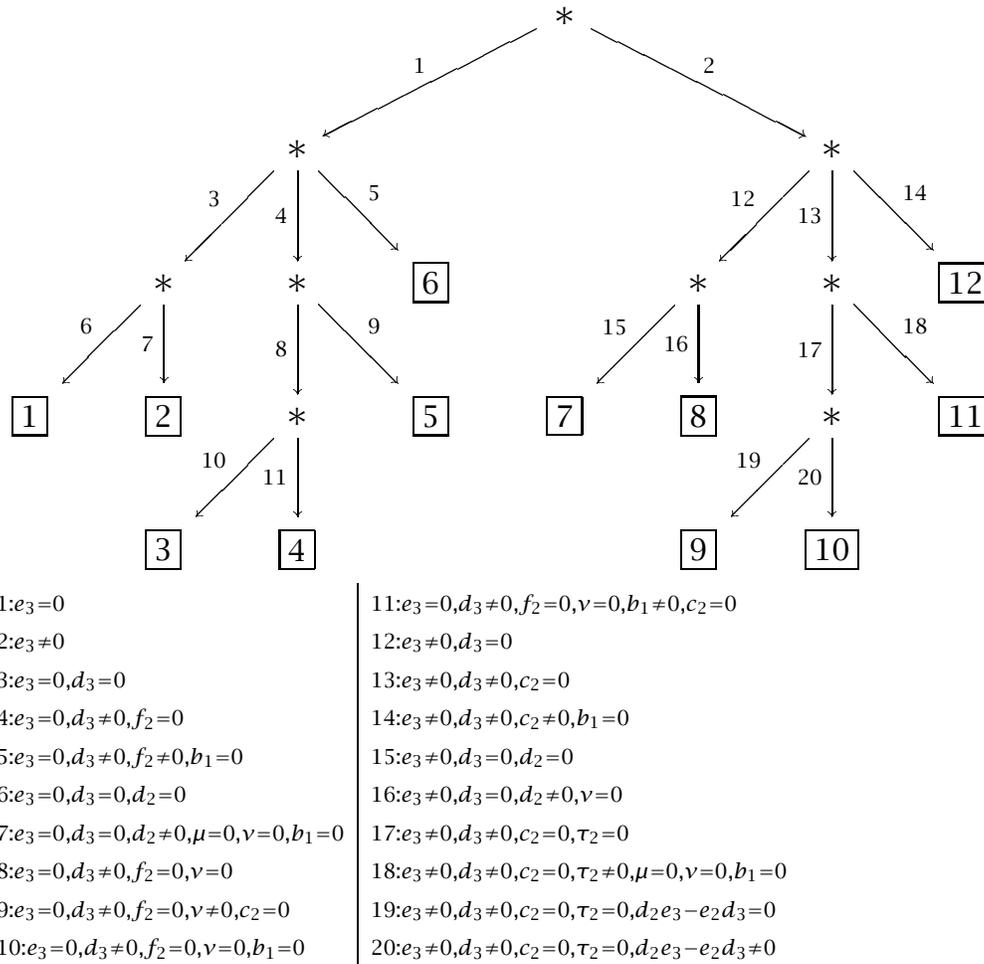


Figure 7.1: Solution graph for case (7.3)

$$16. \begin{pmatrix} -\frac{\tau_2 e_3 x_3^2}{d_3} + (d_1 x_1 + \frac{d_1 e_3 x_2}{d_3} - \frac{d_1^2 x_3}{d_3}) x_5 \\ \tau_2 x_3^2 \\ (d_3 x_1 + e_3 x_2 - d_1 x_3) x_5 \\ 0 \\ 0 \end{pmatrix}$$

$$17. \begin{pmatrix} -\frac{\tau_2 e_3 x_3^2}{d_3} - \frac{c_2 e_3 x_3 x_4}{d_3} + (d_1 x_1 + \frac{d_1 e_3 x_2}{d_3} - \frac{d_1^2 x_3}{d_3}) x_5 \\ \tau_2 x_3^2 + c_2 x_3 x_4 \\ (d_3 x_1 + e_3 x_2 - d_1 x_3) x_5 \\ 0 \\ 0 \end{pmatrix}$$

The corresponding solution graph for this lemma is presented in figure 7.2.

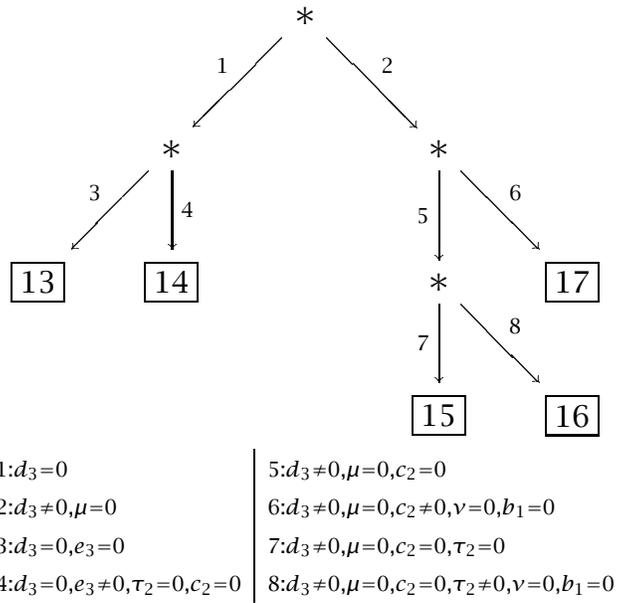


Figure 7.2: Solution graph for case (7.4)

7.2.3 Start with map (7.5)

Lemma 7.9

If Q is as in (7.5) then we have the following possibilities for Q :

$$18. \begin{pmatrix} \mu x_2 x_3 + v x_2^2 + b_1 x_2 x_4 + e_1 x_2 x_5 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$19. \begin{pmatrix} (-e_2 x_1 - \frac{e_2^2 x_2}{d_2}) x_5 \\ (d_2 x_1 + e_2 x_2) x_5 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

The solution graph for this case is very simple; it is shown in figure 7.3.

7.3 Linear triangularisable or not?

In the previous section we have given 19 standard forms on which Q can appear if we demand that JQ is nilpotent. In section 7.1 we have shown how we could reduce the

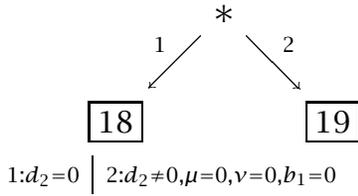


Figure 7.3: Solution graph for case (7.5)

original Q to the five values we actually started the computations with. These reductions were made using affine and elementary maps. In order to present a theorem of the form ‘for every $F = X + Q$ with $\det(JF) = 1$ in dimension five, there exists linear T such that $T^{-1}FT$ is of the form’, we must add the parts of the original Q we have taken out by elementary maps. Note that this makes no difference for the nilpotence of JQ . Therefore define:

$$E_1 := \begin{pmatrix} \alpha_1 x_4^2 + \beta_1 x_4 x_5 + \gamma_1 x_5^2 \\ \alpha_2 x_4^2 + \beta_2 x_4 x_5 + \gamma_2 x_5^2 \\ \alpha_3 x_4^2 + \beta_3 x_4 x_5 + \gamma_3 x_5^2 \\ \gamma_1 x_5^2 \\ 0 \end{pmatrix}, \quad E_2 := \begin{pmatrix} \tau_1 x_3^2 + c_1 x_3 x_4 + f_1 x_3 x_5 \\ \tau_2 x_3^2 + c_2 x_3 x_4 + f_2 x_3 x_5 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

Now we can present the classification theorem.

Theorem 7.10

Let $F = X + Q : k^5 \rightarrow k^5$ with Q quadratic homogeneous, $F(0) = 0$ and $\det(JF) = 1$. Then under the assumption that DPLQ is true we have that there exists $T \in \mathfrak{A}_5$ such that $T^{-1}FT$ is of the form

- $X + Q_i + E_1$ for $i = 1, \dots, 17$ or
- $X + Q_i + E_1 + E_2$ for $i = 18, 19$.

where Q_i are the corresponding Q 's with label i in the lemmas 7.7, 7.8 and 7.9.

Proof. See the solution graphs in section 7.2 or the Maple command listings in appendix A. □

We use theorem 7.10 to get to the main theorem of this chapter:

Theorem 7.11

Let $F = X + Q : k^5 \rightarrow k^5$ with Q quadratic homogeneous, $F(0) = 0$ and $\det(JF) = 1$. Then there exists $T \in \mathfrak{A}_5$ such that $T^{-1}FT$ is

- on triangular form or

- of the form $X + N + E_1$ where

$$N = \begin{pmatrix} b_1 x_2 x_4 \\ -\frac{b_1 d_2}{e_3} x_3 x_4 + d_2 x_1 x_5 \\ e_3 x_2 x_5 \\ 0 \\ 0 \end{pmatrix} \quad (7.6)$$

which is not linear triangularisable.

Proof. The proof is based on theorem 6.5. By theorem 7.10 we know that we may assume that F is of the form $X + Q_i + E_1$ (or $X + Q_i + E_1 + E_2$). So we only have to show that we can reduce the 19 cases of theorem 7.10 to either a triangular map or to the map $X + N + E_1$. We divide the proof according to the labels in section 7.2.

1. If $i \in \{1, \dots, 17\} \setminus \{8\}$, then $J(Q_i + E_1)$ is a strongly nilpotent matrix. And hence $X + Q_i + E_1$ is linearly triangularisable.
2. If $i \in \{18, 19\}$, then $J(Q_i + E_1 + E_2)$ is a strongly nilpotent matrix. And hence $X + Q_i + E_1 + E_2$ is linearly triangularisable.
3. If $i = 8$ then take

$$T = \begin{pmatrix} x_1 \\ x_2 + \frac{e_2}{e_3} x_3 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix}$$

Computation now shows that $T^{-1} \circ (X + Q_8 + E_1) \circ T = X + N + E'_1$, where E'_1 has the same form as E_1 but with (possibly) different coefficients.

Now we are only left to prove that JN is not strong nilpotent. Computation of

$$M = JN(Y_{(1)})JN(Y_{(2)})JN(Y_{(3)})JN(Y_{(4)})JN(Y_{(5)})$$

gives a matrix which has

$$M_{1,5} = b_1^3 d_2^2 Y_{(1)4} (-Y_{(2)5} Y_{(3)4} + Y_{(2)4} Y_{(3)5}) Y_{(4)4} Y_{(5)2}$$

Hence JN clearly is not strong nilpotent. □

Because the proof of the triangularisable cases is only a statement instead of a real proof, we write down the actual T_i 's used to bring $X + Q_i + E_1$ (or $X + Q_i + E_1 + E_2$) on triangular form in section A.1.2 in appendix A.

7.4 Note

In recent computations the author found a counterexample to problem 7.1 in dimension four.

Example 7.12

Let Q be the map

$$\left(\begin{array}{l} g_3 x_1 - f_3 x_2 \\ \frac{1}{2} \frac{(12g_3^2 + 2f_3 g_2)}{f_3} x_1 - 3g_3 x_2 + f_3 x_3 \\ g_1 x_1 + g_2 x_2 + g_3 x_3 + \frac{1}{2} \frac{f_3}{d_5} x_4 \\ 2 \frac{d_5 g_3^4}{f_3^3} x_1 + \frac{(-8d_5 f_3 g_3^3 + 2d_5 f_3^3 g_1)}{f_3^3} x_2 + g_3 x_4 \end{array} \begin{array}{l} + \frac{1}{2} x_1^2 \\ \\ + x_1 x_2 \\ + d_5 x_2^2 \end{array} \right)$$

where $d_5, f_3, g_1, g_2, g_3 \in K$ for an algebraically closed field K . Then JQ is nilpotent. However for all $a_i \in K$ $a_1 Q_1 + a_2 Q_2 + a_3 Q_3 + a_4 Q_4 \neq 0$ unless $a_1 = a_2 = a_3 = a_4 = 0$. In fact if $g_1 = g_2 = g_3 = 0$ this is still a counterexample to problem 7.1.

The consequence of this example is that lemma 7.5 must be extended. The claim should be that either $\ell_4(x_1, x_2, x_3, x_4) = 0$ or Q is of the form as in example 7.12, where $K = k(x_5)$. Actually, since $\deg(X + Q) = 2$ the variables are bounded by their x_5 -degree.

Note that JQ is nilpotent but not strongly nilpotent. Hence this map is not linearly triangularisable.

Chapter 8

Cubic similarity

Introduction

Like chapter 7 also this chapter is highly based on computations. In this chapter we look again at polynomial maps $F : k^5 \rightarrow k^5$, where k is an algebraically closed field with $\text{char}(k) = 0$. This time we look at cubic homogeneous instead of quadratic homogeneous maps. Since we could classify the quadratic homogeneous maps only under the assumption that DPLQ is true, it seems impossible to consider all cubic homogeneous maps. Therefore we restrict to the cubic-linear mappings $F = X + (AX)^{*3}$ with $\det(JF) = 1$. Like in section 6.3 these maps are called Drużkowski maps. We will give a complete classification of these five dimensional Drużkowski maps. And at the end of this chapter we use this classification to find a complete list of representatives of Meisters' cubic similarity relation in dimension five. This chapter is based on [52].

8.1 Drużkowski's reduction

In [51] the author found a classification of the Drużkowski maps in dimension four and used this to prove that Meisters' list of generators in dimension four was complete. The research for [51] was done at the end of 1993. Unfortunately it was only in 1996 that we realized that we could use the reduction scheme presented by Drużkowski in [22] in conjunction with a theorem of [51] to make a very strong reduction.

Definition 8.1

Let A be a linear matrix over k . Then the map $F = X + (AX)^{*3}$ is called *cubic-linear*. If in addition $\det(JF) = 1$ then the map is a *Drużkowski map*.

By [8] we know that classifying the Drużkowski maps is equivalent to classifying the cubic linear mappings for which $J((AX)^{*3})$ is nilpotent. So from now on we will be working with this nilpotence property.

Lemma 8.2

Let $F = X + (AX)^{*3}$ with $A \in \text{Mat}_5(k)$ and $J((AX)^{*3})$ is nilpotent. Then there exists linear invertible T such that $T^{-1}FT = X + (BX)^{*3}$ where the last row of B is a null row.

Proof. It is well known that the hypothesis on A implies that $r := \text{rank}(A) \leq 4$. Let $AX = \begin{pmatrix} \ell_i(X) \end{pmatrix}$ for $i = 1, \dots, 5$. Therefore there exists $T \in \text{Gl}_5(k)$ such that AT is on column Echelon form. Now define $G = T^{-1}FT$. Then

$$G = X + T^{-1}(ATX)^{*3} = \begin{pmatrix} x_1 + h_1(x_1, \dots, x_r) \\ x_2 + h_2(x_1, \dots, x_r) \\ \vdots \\ x_5 + h_5(x_1, \dots, x_r) \end{pmatrix}$$

where h_i is homogeneous of degree three. Since $r \leq 4$, $h_i(x_1, \dots, x_r)$ does not contain x_5 . It follows that $J_{x_1, \dots, x_4}(h_1, \dots, h_4)$ is nilpotent. But then by [51, Corollary 2.8] we have that

$$\dim_k[h_1(X), h_2(X), h_3(X), h_4(X)] < 4$$

but then also

$$\dim_k[\ell_1^{*3}(TX), \ell_2^{*3}(TX), \ell_3^{*3}(TX), \ell_4^{*3}(TX), \ell_5^{*3}(TX)] < 5.$$

And substituting $X := T^{-1}X$ gives

$$\dim_k[\ell_1^{*3}(X), \ell_2^{*3}(X), \ell_3^{*3}(X), \ell_4^{*3}(X), \ell_5^{*3}(X)] < 5$$

Hence there exists $S \in \text{Gl}_5(k)$ such that $S^{-1}FS$ is a Drużkowski map and has the last row equal to zero. \square

We now present an improvement of [22, Theorem 2.1] for the case $n = 5$.

Theorem 8.3

*If a polynomial map $F = X + (AX)^{*3} : k^5 \rightarrow k^5$ has $\det(JF) = 1$ and $\text{rank}(A) < 3$ or $\text{corank}(A) < 3$, then there exists an invertible linear map L such that $L \circ F \circ L^{-1} = X + (BX)^{*3}$, where B is upper triangular with zeros on the diagonal.*

Proof. Though the original theorem in [22] only claims that F is a tame automorphism, we can almost copy the proof in that paper. Simply because in three of the four cases it is already shown that LFL^{-1} has the desired form (and hence F tame).

- $\text{rank}(A) = 1$. The proof is exactly the same as in [22].
- $\text{corank}(A) = 1$. From lemma 8.2 it follows that we are always in case (i) of Drużkowski's paper.
- $\text{corank}(A) = 2$. Again lemma 8.2 shows that we are always in case (iii) of Drużkowski's paper.
- $\text{rank}(A) = 2$. This is the only part where Drużkowski doesn't show that F can be transformed to the desired form. To prove this case we use the lemmas 8.4, 8.5 and 8.7 below.

\square

Lemma 8.4

Assume $\text{rank}(A) = 2$. By lemma 8.2 we have that the last row is equal to zero. Now if we write

$$A = \begin{pmatrix} a_1 & a_2 & a_3 & a_4 & a_5 \\ b_1 & b_2 & b_3 & b_4 & b_5 \\ c_1 & c_2 & c_3 & c_4 & c_5 \\ d_1 & d_2 & d_3 & d_4 & d_5 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} & & & & a_5 \\ & & & & b_5 \\ & A' & & & c_5 \\ & & & & d_5 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

and we consider the Drużkowski form $X' + (A'X')^{*3}$ (where $X' = (x_1, \dots, x_4)$) we may assume that

$$\begin{aligned} \bullet A' &= \begin{pmatrix} a_1 & a_2 & a_3 & a_4 \\ \lambda_2 a_1 & \lambda_2 a_2 & \lambda_2 a_3 & \lambda_2 a_4 \\ \lambda_3 a_1 & \lambda_3 a_2 & \lambda_3 a_3 & \lambda_3 a_4 \\ \lambda_4 a_1 & \lambda_4 a_2 & \lambda_4 a_3 & \lambda_4 a_4 \end{pmatrix} \text{ or} \\ \bullet A' &= \begin{pmatrix} a_1 & a_2 & a_3 & a_4 \\ b_1 & b_2 & b_3 & b_4 \\ \lambda a_1 + \mu b_1 & \lambda a_2 + \mu b_2 & \lambda a_3 + \mu b_3 & \lambda a_4 + \mu b_4 \\ 0 & 0 & 0 & 0 \end{pmatrix} \text{ and } d_5 = 0. \end{aligned}$$

Proof. This lemma is again based on Drużkowski's paper. Naturally if $\text{rank}(A) = 2$ we have $\text{rank}(A') = 1$ or $\text{rank}(A') = 2$. The first case obviously coincides with the first matrix. Now if $\text{rank}(A') = 2$ we know that we can transform the matrix such that the first two rows are independent. But since we deal with a 4×4 matrix, $\text{rank}(A') = 2$ means also $\text{corank}(A') = 2$ and here we can use Drużkowski's proof, since $J((A'X')^{*3})$ is nilpotent if $J((AX)^{*3})$ is nilpotent, where he states that at least one of the rows of this 4×4 matrix is parallel to another row. Say

$$(d_1, \dots, d_4) = \lambda(a_1, \dots, a_4) \quad (8.1)$$

Furthermore $(d_1, \dots, d_5) = \mu_1(a_1, \dots, a_5) + \mu_2(b_1, \dots, b_5)$ (since $\text{rank}(A) = 2$ and $(a_1, \dots, a_5), (b_1, \dots, b_5)$ are independent). So in particular

$$(d_1, \dots, d_4) = \mu_1(a_1, \dots, a_4) + \mu_2(b_1, \dots, b_4) \quad (8.2)$$

Since (a_1, \dots, a_4) and (b_1, \dots, b_4) are independent it follows from (8.1) and (8.2) that $\mu_1 = \lambda$ and $\mu_2 = 0$. So $(d_1, \dots, d_5) = \lambda(a_1, \dots, a_5)$. Then making a change of coordinates, we may assume that $d_1 = \dots = d_5 = 0$, which proves the lemma. \square

Lemma 8.5

Let A and A' be as in lemma 8.4. Assume

$$A' = \begin{pmatrix} a_1 & a_2 & a_3 & a_4 \\ \lambda_2 a_1 & \lambda_2 a_2 & \lambda_2 a_3 & \lambda_2 a_4 \\ \lambda_3 a_1 & \lambda_3 a_2 & \lambda_3 a_3 & \lambda_3 a_4 \\ \lambda_4 a_1 & \lambda_4 a_2 & \lambda_4 a_3 & \lambda_4 a_4 \end{pmatrix}$$

Then there exists a linear invertible map $T \in k[X]$ and $B \in \text{Gl}_5(k)$ such that $T^{-1} \circ (X + (AX)^{*3}) \circ T = X + (BX)^{*3}$ with B is upper triangular with null diagonal.

Proof. Note that if either λ_2, λ_3 or λ_4 equals zero, we are in a special case of lemma 8.7. Hence we may assume $\lambda_2\lambda_3\lambda_4 \neq 0$. If we now look at A itself, we see that we are done if A is triangularisable or if we have that two rows of A are parallel to each other. After these observations we now start by showing we may assume $a_5 = 0$. Take $T = (x_1 - \frac{a_5}{a_1}x_5, x_2, x_3, x_4, x_5)$. (Of course we may assume $a_1 \neq 0$.) Then $T^{-1}FT$ is on Drużkowski form with the matrix:

$$\begin{pmatrix} a_1 & a_2 & a_3 & a_4 & 0 \\ \lambda_2 a_1 & \lambda_2 a_2 & \lambda_2 a_3 & \lambda_2 a_4 & -\lambda_2 a_5 + b_5 \\ \lambda_3 a_1 & \lambda_3 a_2 & \lambda_3 a_3 & \lambda_3 a_4 & -\lambda_3 a_5 + c_5 \\ \lambda_4 a_1 & \lambda_4 a_2 & \lambda_4 a_3 & \lambda_4 a_4 & -\lambda_4 a_5 + d_5 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

and by putting $b'_5 = -\lambda_2 a_5 + b_5$, $c'_5 = -\lambda_3 a_5 + c_5$ and $d'_5 = -\lambda_4 a_5 + d_5$ we get the same structure as our original A , only with $a_5 = 0$.

Now put $Y_1 = a_1 x_1 + a_2 x_2 + a_3 x_3 + a_4 x_4$. Using the nilpotence of the corresponding Jacobian matrix we obtain the following polynomial in Y_1 and x_5 by looking at the 1×1 principal minors.

$$M_1 := a_1 Y_1^2 + \lambda_2 a_2 (\lambda_2 Y_1 + b_5 x_5)^2 + \lambda_3 a_3 (\lambda_3 Y_1 + c_5 x_5)^2 + \lambda_4 a_4 (\lambda_4 Y_1 + d_5 x_5)^2$$

Collecting the coefficients of the monomials x_5^2 , Y_1^2 and $x_5 Y_1$ in M_1 we get three equations:

$$\begin{cases} 0 = \lambda_3 a_3 c_5^2 + \lambda_2 a_2 b_5^2 + \lambda_4 a_4 d_5^2 \\ 0 = \lambda_3^3 a_3 + \lambda_2^3 a_2 + a_1 + \lambda_4^3 a_4 \\ 0 = 2 \lambda_2^2 a_2 b_5 + 2 \lambda_4^2 a_4 d_5 + 2 \lambda_3^2 a_3 c_5 \end{cases}$$

By making some assumptions during the process, this system can be solved completely. It gives eleven solutions. The solution graph in figure 8.1 shows the construction of these solutions. If we substitute the solutions [\[1\]](#) through [\[10\]](#) and put $a_5 = 0$, we get ten matrices. Each of these matrices has two rows parallel to each other and hence we are done for these ten cases. Substitution of solution [\[11\]](#) gives:¹

$$B := \begin{pmatrix} -\frac{\lambda_3 a_3 \lambda_4 a_4 \%2^2}{\%1} & -\frac{\%3^2}{\lambda_2^3 \%1} & a_3 & a_4 & 0 \\ -\frac{\lambda_2 \lambda_3 a_3 \lambda_4 a_4 \%2^2}{\%1} & -\frac{\%3^2}{\lambda_2^2 \%1} & \lambda_2 a_3 & \lambda_2 a_4 & \frac{\%1 \lambda_2}{\%3} \\ -\frac{\lambda_3^2 a_3 \lambda_4 a_4 \%2^2}{\%1} & -\frac{\lambda_3 \%3^2}{\lambda_2^3 \%1} & \lambda_3 a_3 & \lambda_3 a_4 & c_5 \\ -\frac{\lambda_4^2 \lambda_3 a_3 a_4 \%2^2}{\%1} & -\frac{\lambda_4 \%3^2}{\lambda_2^3 \%1} & \lambda_4 a_3 & \lambda_4 a_4 & d_5 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

¹This output is generated by Maple. The '%1' means: substitute the expression below the matrix at this place.

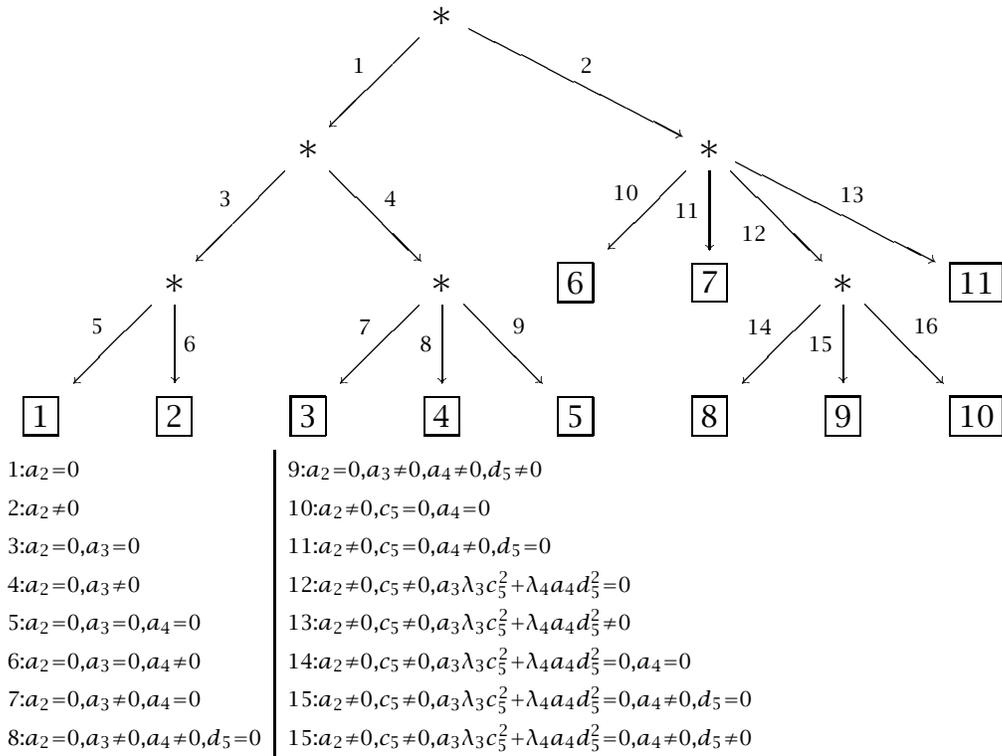


Figure 8.1: Solution graph for lemma 8.5.

$$\begin{aligned} \%1 &:= \lambda_3 a_3 c_5^2 + \lambda_4 a_4 d_5^2 \\ \%2 &:= -\lambda_4 c_5 + \lambda_3 d_5 \\ \%3 &:= \lambda_4^2 a_4 d_5 + \lambda_3^2 a_3 c_5 \end{aligned}$$

However if we define

$$\begin{aligned} T := & \left(x_1, x_2, x_3, \left(-2 \lambda_3^2 a_3 \lambda_4^2 a_4 x_1 \lambda_2^3 c_5 d_5 + \lambda_3 a_3 \lambda_4^3 a_4 x_1 \lambda_2^3 c_5^2 \right. \right. \\ & + \lambda_3^3 a_3 \lambda_4 a_4 x_1 \lambda_2^3 d_5^2 + x_2 \lambda_3^4 a_3^2 c_5^2 + x_2 \lambda_4^4 a_4^2 d_5^2 \\ & + 2 x_2 \lambda_4^2 a_4 d_5 \lambda_3^2 a_3 c_5 - a_3 x_3 \lambda_2^3 \lambda_4 a_4 d_5^2 - a_3^2 x_3 \lambda_2^3 \lambda_3 c_5^2 \\ & \left. \left. - a_4^2 x_4 \lambda_2^3 \lambda_4 d_5^2 - a_4 x_4 \lambda_2^3 \lambda_3 a_3 c_5^2 \right) / \left(a_4 \lambda_2^3 \left(\lambda_4 a_4 d_5^2 + \lambda_3 a_3 c_5^2 \right) \right), x_5 \right) \end{aligned}$$

we get $T^{-1} \circ (X + (BX)^{*3}) \circ T = X + (CX)^{*3}$ where C is given by

$$\begin{pmatrix} 0 & 0 & 0 & -a_4 & 0 \\ 0 & 0 & 0 & -\lambda_2 a_4 & \frac{(\lambda_4 a_4 d_5^2 + \lambda_3 a_3 c_5^2) \lambda_2}{\lambda_4^2 a_4 d_5 + \lambda_3^2 a_3 c_5} \\ 0 & 0 & 0 & -\lambda_3 a_4 & \frac{c_5}{(-\lambda_4 c_5 + d_5 \lambda_3)^{2/3} c_5^{1/3} a_3^{1/3} d_5^{1/3}} \\ 0 & 0 & 0 & 0 & -\frac{(\lambda_4^2 a_4 d_5 + \lambda_3^2 a_3 c_5)^{1/3}}{(\lambda_4^2 a_4 d_5 + \lambda_3^2 a_3 c_5)^{1/3}} \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

and in particular we see that C is on triangular form. Before we can finish the proof we must add a minor remark: this solution [11] excludes the branches with labels 10, 11 and 12. Therefore the factor $\lambda_4 a_4 d_5^2 + \lambda_3 a_3 c_5^2 \neq 0$. So the only way that this T might be undefined is if $a_4 = 0$. However, substitution of $a_4 = 0$ into B automatically gives a matrix where the second and third row are parallel to each other. So the case $a_4 = 0$ is also not a problem. Hence the proof is finally completed. \square

Remark 8.6

At first glance the given T may seem to appear out of nowhere, but in fact it doesn't. It is of the quite natural form:

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ -x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \frac{B_{1,1}}{B_{1,4}}x_1 + \frac{B_{1,2}}{B_{1,4}}x_2 + \frac{B_{1,3}}{B_{1,4}}x_3 + \frac{B_{1,5}}{B_{1,4}}x_5 \\ 0 \end{pmatrix}$$

Lemma 8.7

Let A and A' be as in lemma 8.4. Assume

$$A' = \begin{pmatrix} a_1 & a_2 & a_3 & a_4 \\ b_1 & b_2 & b_3 & b_4 \\ \lambda a_1 + \mu b_1 & \lambda a_2 + \mu b_2 & \lambda a_3 + \mu b_3 & \lambda a_4 + \mu b_4 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Then there exists a linear invertible map $T \in k[X]$ and $B \in \text{Gl}_5(k)$ such that $T^{-1} \circ (X + (AX)^{*3}) \circ T = X + (BX)^{*3}$ where B is upper triangular with zeros on the diagonal.

Proof. Obviously, since the first two rows of A' are independent, also the first two rows of A are independent. From this it follows that $c_5 = \lambda a_5 + \mu b_5$. Furthermore if $d_5 \neq 0$ we have that the fourth row is parallel to either the first or the second row. It cannot be a non-trivial combination of these rows since a_1, a_2, a_3, a_4 and b_1, b_2, b_3, b_4 are independent. So if $d_5 \neq 0$ we can conjugate with a suitable transformation and get the complete fourth row equal to zero. Hence we may assume $d_5 = 0$ to begin with. We divide the rest of this proof into three cases:

- Assume both $\lambda \neq 0$ and $\mu \neq 0$. Put $Y_1 := a_1 x_1 + a_2 x_2 + a_3 x_3 + a_4 x_4$ and $Y_2 := b_1 x_1 + b_2 x_2 + b_3 x_3 + b_4 x_4$. Now if we look at the principal minors, we get the following polynomials in Y_1, Y_2 and x_5 . Here M_1 stands for the polynomial we get by looking at the 1×1 principal minors and M_2 for the 2×2 principal minors.

$$M_1 := a_1 (Y_1 + a_5 x_5)^2 + b_2 (Y_2 + b_5 x_5)^2 + (\lambda a_3 + \mu b_3) (\lambda Y_1 + \mu Y_2 + (a_5 \lambda + b_5 \mu) x_5)^2 \quad (8.3)$$

$$M_2 := (a_1 b_2 - a_2 b_1) (Y_1 + a_5 x_5)^2 (Y_2 + b_5 x_5)^2 + (a_1 \mu b_3 - a_3 \mu b_1) (Y_1 + a_5 x_5)^2 (\lambda Y_1 + \mu Y_2 + (a_5 \lambda + b_5 \mu) x_5)^2 + (b_2 \lambda a_3 - b_3 \lambda a_2) (Y_2 + b_5 x_5)^2 (\lambda Y_1 + \mu Y_2 + (a_5 \lambda + b_5 \mu) x_5)^2 \quad (8.4)$$

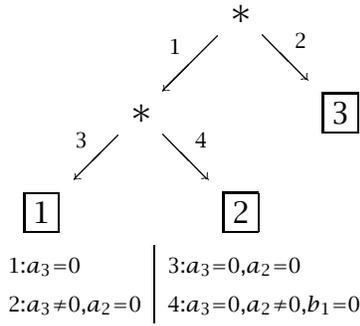


Figure 8.2: Solution graph for lemma 8.7, case $\lambda \neq 0$ and $\mu \neq 0$.

Collecting the coefficients of the monomials in Y_1, Y_2 and x_5 gives us a set of 21 equations. Solving this set gives only three solutions. See figure 8.2.

Substituting these three solutions in the matrix A gives:

$$\begin{aligned}
 \boxed{1} & : \begin{pmatrix} 0 & 0 & 0 & a_4 & a_5 \\ b_1 & 0 & 0 & b_4 & b_5 \\ \mu b_1 & 0 & 0 & \lambda a_4 + \mu b_4 & a_5 \lambda + b_5 \mu \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \\
 \boxed{2} & : \begin{pmatrix} 0 & a_2 & 0 & a_4 & a_5 \\ 0 & 0 & 0 & b_4 & b_5 \\ 0 & \lambda a_2 & 0 & \lambda a_4 + \mu b_4 & a_5 \lambda + b_5 \mu \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \\
 \boxed{3} & : \begin{pmatrix} 0 & 0 & a_3 & a_4 & a_5 \\ 0 & 0 & -\frac{\lambda a_3}{\mu} & b_4 & b_5 \\ 0 & 0 & 0 & \lambda a_4 + \mu b_4 & a_5 \lambda + b_5 \mu \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}
 \end{aligned}$$

The third matrix is already on triangular form. Using the permutation $P_{(13)}$ brings the first matrix on triangular form. The permutation $P_{(23)}$ does the same for the third matrix. So all three cases are linearly triangularisable.

- $\lambda = 0$ and $\mu \neq 0$ or $\lambda \neq 0$ and $\mu \neq 0$. Without loss of generality we may assume $\lambda = 0$. In the same way as before we find the polynomials:

$$M_1 := a_1 (Y_1 + a_5 x_5)^2 + b_2 (Y_2 + b_5 x_5)^2 + \mu b_3 (\mu Y_2 + b_5 \mu x_5)^2 \tag{8.5}$$

$$M_2 := (a_1 b_2 - a_2 b_1) (Y_1 + a_5 x_5)^2 (Y_2 + b_5 x_5)^2 + (a_1 \mu b_3 - a_3 \mu b_1) (Y_1 + a_5 x_5)^2 (\mu Y_2 + b_5 \mu x_5)^2 \tag{8.6}$$

The resulting system of equations is even simpler than the previous one. We get only two solutions. See figure 8.3.

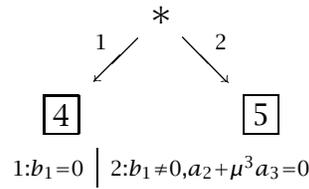


Figure 8.3: Solution graph for lemma 8.7, case $\lambda \neq 0$ and $\mu = 0$.

Substituting these three solutions in the matrix A gives:

$$\boxed{4} : \begin{pmatrix} 0 & a_2 & a_3 & a_4 & a_5 \\ 0 & -\mu^3 b_3 & b_3 & b_4 & b_5 \\ 0 & -\mu^4 b_3 & \mu b_3 & \mu b_4 & b_5 \mu \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\boxed{5} : \begin{pmatrix} 0 & -\mu^3 a_3 & a_3 & a_4 & a_5 \\ b_1 & -\mu^3 b_3 & b_3 & b_4 & b_5 \\ \mu b_1 & -\mu^4 b_3 & \mu b_3 & \mu b_4 & b_5 \mu \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

And here we see that in both cases the third row is μ times the second row and due to this parallelism we can transform this case to the case where $\lambda = 0$ and $\mu = 0$, which we will describe next:

- $\lambda = 0$ and $\mu = 0$.

$$M_1 := a_1 (Y_1 + a_5 x_5)^2 + b_2 (Y_2 + b_5 x_5)^2 \quad (8.7)$$

$$M_2 := (a_1 b_2 - a_2 b_1) (Y_1 + a_5 x_5)^2 (Y_2 + b_5 x_5)^2 \quad (8.8)$$

Again only two solutions. See figure 8.4. And these solutions result in the ma-

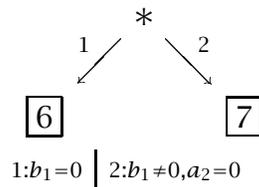


Figure 8.4: Solution graph for lemma 8.7, case $\lambda = 0$ and $\mu = 0$.

trices:

$$\boxed{6} : \begin{pmatrix} 0 & a_2 & a_3 & a_4 & a_5 \\ 0 & 0 & b_3 & b_4 & b_5 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\boxed{7} : \begin{pmatrix} 0 & 0 & a_3 & a_4 & a_5 \\ b_1 & 0 & b_3 & b_4 & b_5 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

And indeed these are the matrices we get by substitution of $\mu = 0$ into $\boxed{4}$ and $\boxed{5}$ given by cca and ccb where $\mu = 0$. Note that $\boxed{6}$ is already on triangular form and that $P_{(12)}$ brings $\boxed{7}$ onto triangular form.

So all seven solutions together show that the case of lemma 8.7 is always linearly triangularisable. \square

Hence theorem 8.3 is true. And because we are looking at dimension five, we automatically have that either $\text{rank}(A) < 3$ or $\text{corank}(A) < 3$ so we conclude this section with the corollary:

Corollary 8.8

Let $F = X + (AX)^{*3} : k^5 \rightarrow k^5$ such that $\det(JF) = 1$. Then there exists an invertible linear map L such that $L \circ F \circ L^{-1} = X + (BX)^{*3}$, where B is upper triangular with zeros on the diagonal.

8.2 Cubic similarity

With the reduction of corollary 8.8 we can give a complete classification of all cubic-linear automorphisms in dimension five. We do this in section 8.3. Before we recall in this section some theory regarding Meisters's cubic similarity relation.

Definition 8.9

Let $F = X + (AX)^{*3}$ and $G = X + (BX)^{*3}$ be two Drużkowski maps. Then the matrices $A, B \in \text{Mat}_n(k)$ are called *cubic similar* ($A \overset{3}{\sim} B$) if there exists a linear invertible polynomial map T with $T^{-1}FT = G$.

The idea behind this definition is that it is rather special that if T is a linear invertible map and F is a Drużkowski map one has that $T^{-1}FT$ is again a Drużkowski map.

Definition 8.9 is in terms of maps. For computational use however it is often preferable to work in terms of matrices.

Lemma 8.10

Let $F = X + (AX)^{*3}$ and $G = X + (BX)^{*3}$ be two Drużkowski maps. Then $A \stackrel{3}{\sim} B$ if and only if there exists $T \in \text{Gl}_n(k)$ with $(ATX)^{*3} = T(BX)^{*3}$.

Proof. The following statements can be read from top to bottom or the other way round. In either case each statement is equivalent to the next one in the sequence.

- $A \stackrel{3}{\sim} B$.
- There exists an invertible map T with $T^{-1}FT = G$.
- There exists an invertible map T with $T^{-1}(TX + (ATX)^{*3}) = X + (BX)^{*3}$.
- There exists an invertible map T with $X + T^{-1}(ATX)^{*3} = X + (BX)^{*3}$.
- There exists an invertible map T with $T^{-1}(ATX)^{*3} = (BX)^{*3}$.
- There exists an invertible matrix T with $T^{-1}(ATX)^{*3} = (BX)^{*3}$.
- There exists an invertible matrix T with $(ATX)^{*3} = T(BX)^{*3}$.

This proves the lemma. □

In [69] Meisters presents a list of seventeen mutually inequivalent representatives with respect to the cubic similarity relation in dimension five. The names of these matrices are based on the following notions:

- A J indicates that the matrix is on Jordan normal form.
- An N indicates that it is a nilpotent matrix which is not on Jordan normal form, but doesn't need extra parameters in it.
- A P indicates that it is a nilpotent matrix which contains parameters in it which cannot be reduced to some fixed $a \in k$.
- The first number is the rank of the matrix. In [68] it is shown that this is an invariant of the cubic similarity relation.
- The second number is the nilpotence index of $J((AX)^{*3})$ where A is the matrix in the list. This is also an invariant of the relation.
- The non-capitals at the end are used as an index.
- If a P matrix contains more than one parameter, the number of these parameters is appended to the name.

The representatives are:

$$\begin{array}{ccc}
\begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \\
J(1,2) & J(2,2) & J(2,3) \\
\\
\begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \\
N(2,3a) & J(3,3) & J(3,4) \\
\\
\begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \\
N(3,3a) & N(3,4a) & N(3,4b) \\
\\
\begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \\
N(3,4c) & J(4,5) & N(4,5a) \\
\\
\begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \\
N(4,5b) & N(4,5c) & N(4,5d) \\
\\
\begin{pmatrix} 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & a & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 1 & 1 & b & 0 \\ 0 & 0 & 1 & a & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \\
P(4,5c) & P(4,5c2)
\end{array}$$

Remark 8.11

Note the following points:

- $P(4, 5c)$ is not called $P(4, 5a)$, which should be natural if one uses the non-capital *only* as an index as is done for the N -matrices. However in this case the c is used because $P(4, 5c)|_{a=1} = N(4, 5c)$, where $P(4, 5c)|_{a=1}$ means substitute $a = 1$ in $P(4, 5c)$.
- Note also that $P(4, 5c)|_{a=0} = N(4, 5a)$. Hence we add the restriction that $a \notin \{0, 1\}$ for $P(4, 5c)$.
- $P(4, 5c)|_{a=a_1} \not\stackrel{3}{=} P(4, 5c)|_{a=a_2}$ if $a_1 \neq a_2$.

- $P(4, 5c2)|_{b=0} = P(4, 5c)$, hence we add the restriction $b \neq 0$ for $P(4, 5c2)$. Note that there are no restrictions on the a in $P(4, 5c2)$.

Meisters already stated that these matrices were not a complete set of representatives. But due to lack of time he hasn't found more matrices.

8.3 Classification in dimension five

As we have seen in corollary 8.8, we may assume that the Drużkowski map is on triangular form. So the most general Drużkowski map in dimension five is $F = X + (AX)^{*3}$ where A is the matrix:

$$A := \begin{pmatrix} 0 & a_2 & a_3 & a_4 & a_5 \\ 0 & 0 & b_3 & b_4 & b_5 \\ 0 & 0 & 0 & c_4 & c_5 \\ 0 & 0 & 0 & 0 & d_5 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Obviously the Jacobian matrix $J((AX)^{*3})$ is always nilpotent, independent of the choices of the ten parameters. In fact it is even strongly nilpotent. However since our goal is a classification with respect to cubic similarity and the nilpotence index of this Jacobian matrix is an invariant of this relation (see [68]), we divide the general case into the five possible values for the nilpotence index.² For each of these values n we compute the matrix $(J((AX)^{*3}))^n$ and assume it is equal to the null matrix. This gives each time a set of equations which turns out to be easy to solve. Before we give the results of this process, we remark that $J((AX)^{*3})$ has nilpotence index one if and only if A equals the null matrix itself. So we only consider the cases with nilpotence index ≥ 2 . Furthermore, we represent all solutions by their matrix form and we explicitly show the assumptions we had to make to find each solution.

The following subsections provide the proof for this theorem:

Theorem 8.12

*Let $F : k^5 \rightarrow k^5$ be a cubic-linear map such that $\det(JF) = 1$. Then there exists $T \in \mathfrak{A}_n$ such that $T^{-1}FT$ is of the form $X + (A_i X)^{*3}$ where A_i are the corresponding A 's with label i in the sections 8.3.1, 8.3.2, 8.3.3 and 8.3.4.*

8.3.1 Nilpotence index two

Assuming $J((AX)^{*3})^2 = 0$ gives a system of 119 equations. We get the solution graph of figure 8.5. As usual the boxed numbers coincide with the numbers of the matrices below. For technical reasons (available slopes) this graph doesn't reflect our regular leftmost depth-first strategy to find all solutions. However if one looks at the labels

²The fact that we choose the nilpotence index of the corresponding Jacobian matrix as the invariant, and for instance not the rank of the matrix A , is based on the observation that it is easier to compute $J((AX)^{*3})^n$ and see which conditions must be fulfilled in order to get n as the nilpotence index than to choose a general matrix of a certain rank and compute $J((AX)^{*3})^5$ and solve the system.

at the branches one can reconstruct this strategy. Furthermore because we ordered the solutions afterwards by rank, the order of the boxed numbers may seem a bit strange.

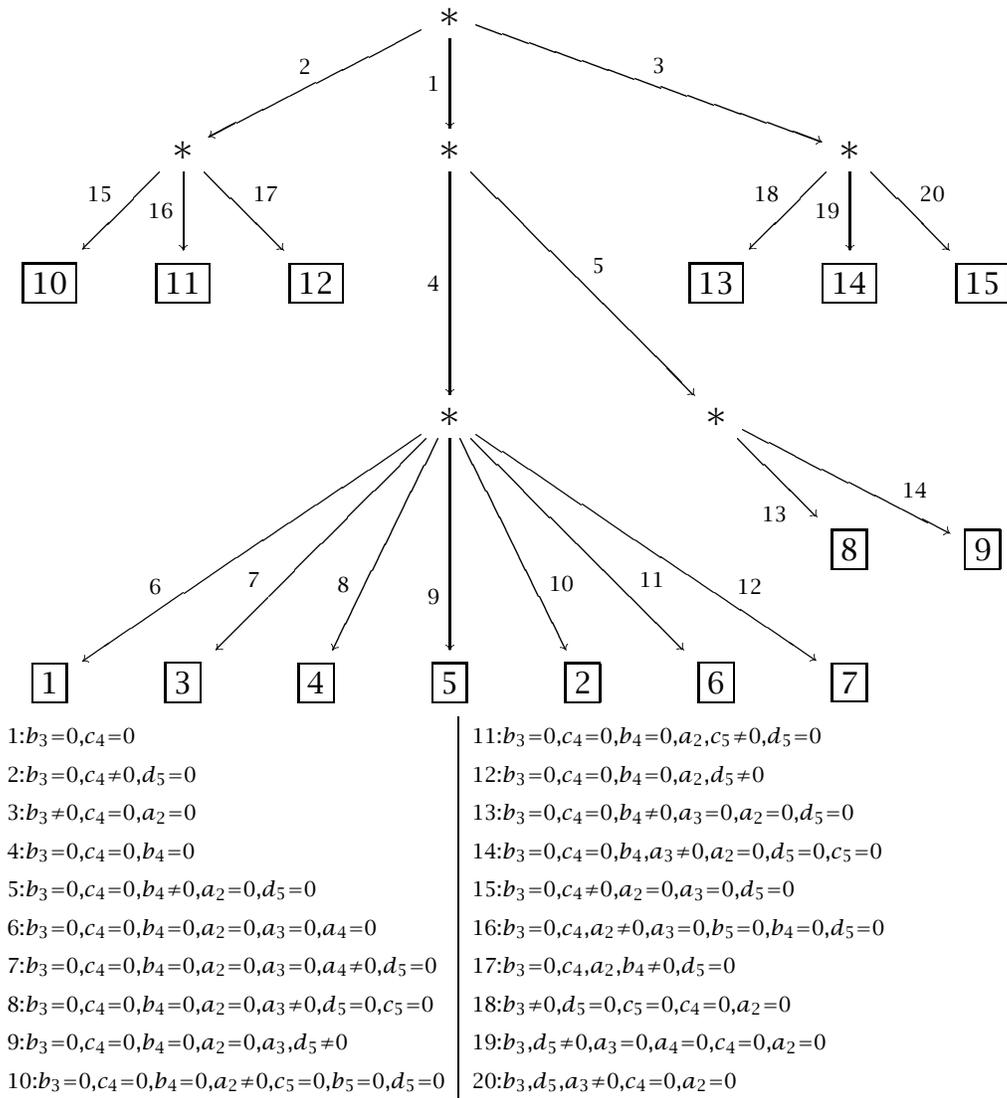


Figure 8.5: Solution graph for nilpotence index two.

If we substitute these solutions we get:

$$1. \begin{pmatrix} 0 & 0 & 0 & 0 & a_5 \\ 0 & 0 & 0 & 0 & b_5 \\ 0 & 0 & 0 & 0 & c_5 \\ 0 & 0 & 0 & 0 & d_5 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \text{rank } 1.$$

$$2. \begin{pmatrix} 0 & a_2 & a_3 & a_4 & a_5 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \text{rank } 1, a_2 \neq 0.$$

$$3. \begin{pmatrix} 0 & 0 & 0 & a_4 & a_5 \\ 0 & 0 & 0 & 0 & b_5 \\ 0 & 0 & 0 & 0 & c_5 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \text{rank } 2, a_4 \neq 0.$$

$$4. \begin{pmatrix} 0 & 0 & a_3 & a_4 & a_5 \\ 0 & 0 & 0 & 0 & b_5 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \text{rank } 2, a_3 \neq 0.$$

$$5. \begin{pmatrix} 0 & 0 & a_3 & -\frac{a_3 c_5^3}{d_5^3} & a_5 \\ 0 & 0 & 0 & 0 & b_5 \\ 0 & 0 & 0 & 0 & c_5 \\ 0 & 0 & 0 & 0 & d_5 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \text{rank } 2, a_3 \neq 0, d_5 \neq 0.$$

$$6. \begin{pmatrix} 0 & a_2 & -\frac{a_2 b_5^3}{c_5^3} & a_4 & a_5 \\ 0 & 0 & 0 & 0 & b_5 \\ 0 & 0 & 0 & 0 & c_5 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \text{rank } 2, a_2 \neq 0, c_5 \neq 0.$$

$$7. \begin{pmatrix} 0 & a_2 & a_3 & \frac{-a_2 b_5^3 - a_3 c_5^3}{d_5^3} & a_5 \\ 0 & 0 & 0 & 0 & b_5 \\ 0 & 0 & 0 & 0 & c_5 \\ 0 & 0 & 0 & 0 & d_5 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \text{rank } 2, a_2 \neq 0, d_5 \neq 0.$$

$$8. \begin{pmatrix} 0 & 0 & 0 & a_4 & a_5 \\ 0 & 0 & 0 & b_4 & b_5 \\ 0 & 0 & 0 & 0 & c_5 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \text{rank } 2, b_4 \neq 0.$$

$$9. \begin{pmatrix} 0 & 0 & a_3 & a_4 & a_5 \\ 0 & 0 & 0 & b_4 & b_5 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \text{rank } 2, a_3 \neq 0, b_4 \neq 0.$$

$$10. \begin{pmatrix} 0 & 0 & 0 & a_4 & a_5 \\ 0 & 0 & 0 & b_4 & b_5 \\ 0 & 0 & 0 & c_4 & c_5 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \text{rank } 2, c_4 \neq 0.$$

$$11. \begin{pmatrix} 0 & a_2 & 0 & a_4 & a_5 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & c_4 & c_5 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \text{rank } 2, a_2 \neq 0, c_4 \neq 0.$$

$$12. \begin{pmatrix} 0 & a_2 & -\frac{a_2 b_4^3}{c_4^3} & a_4 & a_5 \\ 0 & 0 & 0 & b_4 & \frac{b_4 c_5}{c_4} \\ 0 & 0 & 0 & c_4 & c_5 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \text{rank } 2, a_2 \neq 0, b_4 \neq 0, c_4 \neq 0.$$

$$13. \begin{pmatrix} 0 & 0 & a_3 & a_4 & a_5 \\ 0 & 0 & b_3 & b_4 & b_5 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \text{rank } 2, b_3 \neq 0.$$

$$14. \begin{pmatrix} 0 & 0 & 0 & 0 & a_5 \\ 0 & 0 & b_3 & -\frac{b_3 c_5^3}{d_5^3} & b_5 \\ 0 & 0 & 0 & 0 & c_5 \\ 0 & 0 & 0 & 0 & d_5 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \text{rank } 2, b_3 \neq 0, d_5 \neq 0.$$

$$15. \begin{pmatrix} 0 & 0 & a_3 & -\frac{a_3 c_5^3}{d_5^3} & a_5 \\ 0 & 0 & b_3 & -\frac{b_3 c_5^3}{d_5^3} & b_5 \\ 0 & 0 & 0 & 0 & c_5 \\ 0 & 0 & 0 & 0 & d_5 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \text{rank } 2, a_3 \neq 0, b_3 \neq 0, d_5 \neq 0.$$

8.3.2 Nilpotence index three

This case gives a system of 123 equations. The solution graph is a little bit simpler as one can see in figure 8.6. Ordered by rank the solutions are:

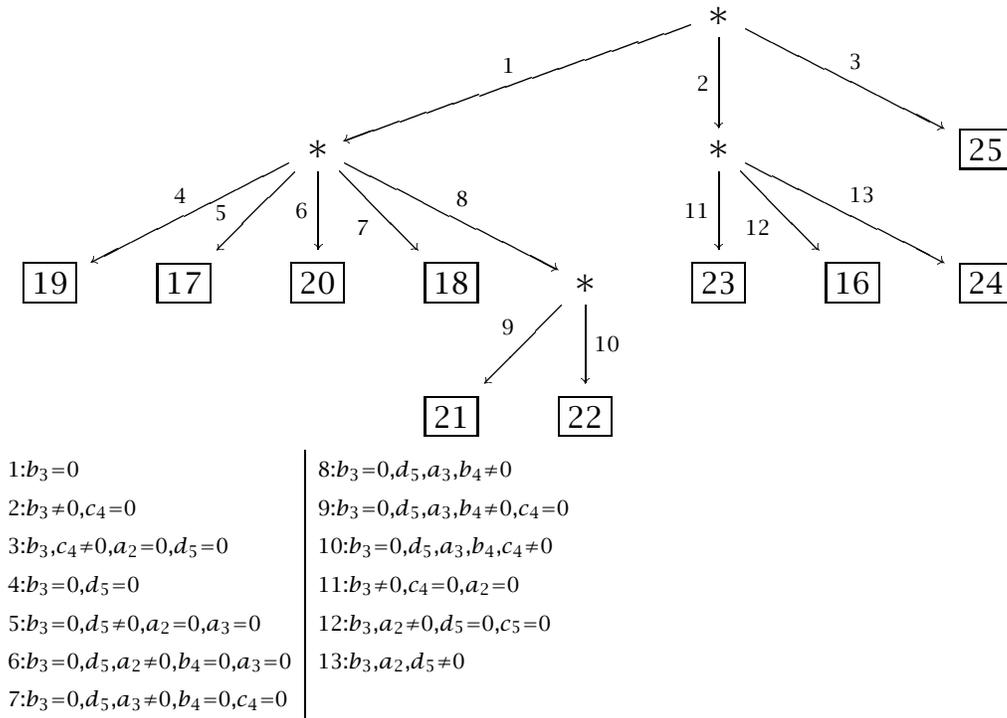


Figure 8.6: Solution graph for nilpotence index three.

$$16. \begin{pmatrix} 0 & a_2 & a_3 & a_4 & a_5 \\ 0 & 0 & b_3 & b_4 & b_5 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \text{rank } 2, a_2 \neq 0, b_3 \neq 0.$$

$$17. \begin{pmatrix} 0 & 0 & 0 & a_4 & a_5 \\ 0 & 0 & 0 & b_4 & b_5 \\ 0 & 0 & 0 & c_4 & c_5 \\ 0 & 0 & 0 & 0 & d_5 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \text{rank } 2, d_5 \neq 0.$$

$$18. \begin{pmatrix} 0 & a_2 & a_3 & a_4 & a_5 \\ 0 & 0 & 0 & 0 & b_5 \\ 0 & 0 & 0 & 0 & c_5 \\ 0 & 0 & 0 & 0 & d_5 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \text{rank } 2, a_3 \neq 0, d_5 \neq 0.$$

$$19. \begin{pmatrix} 0 & a_2 & a_3 & a_4 & a_5 \\ 0 & 0 & 0 & b_4 & b_5 \\ 0 & 0 & 0 & c_4 & c_5 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \text{rank } 3.$$

$$20. \begin{pmatrix} 0 & a_2 & 0 & a_4 & a_5 \\ 0 & 0 & 0 & 0 & b_5 \\ 0 & 0 & 0 & c_4 & c_5 \\ 0 & 0 & 0 & 0 & d_5 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \text{rank } 3, a_2 \neq 0, d_5 \neq 0.$$

$$21. \begin{pmatrix} 0 & 0 & a_3 & a_4 & a_5 \\ 0 & 0 & 0 & b_4 & b_5 \\ 0 & 0 & 0 & 0 & c_5 \\ 0 & 0 & 0 & 0 & d_5 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \text{rank } 3, a_3 \neq 0, b_4 \neq 0, d_5 \neq 0.$$

$$22. \begin{pmatrix} 0 & -\frac{a_3 c_4^3}{b_4^3} & a_3 & a_4 & a_5 \\ 0 & 0 & 0 & b_4 & \frac{b_4 c_5}{c_4} \\ 0 & 0 & 0 & c_4 & c_5 \\ 0 & 0 & 0 & 0 & d_5 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \text{rank } 3, a_3 \neq 0, b_4 \neq 0, c_4 \neq 0, d_5 \neq 0.$$

$$23. \begin{pmatrix} 0 & 0 & a_3 & a_4 & a_5 \\ 0 & 0 & b_3 & b_4 & b_5 \\ 0 & 0 & 0 & 0 & c_5 \\ 0 & 0 & 0 & 0 & d_5 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \text{rank } 3, b_3 \neq 0.$$

$$24. \begin{pmatrix} 0 & a_2 & a_3 & a_4 & a_5 \\ 0 & 0 & b_3 & -\frac{b_3 c_5^3}{d_5^3} & b_5 \\ 0 & 0 & 0 & 0 & c_5 \\ 0 & 0 & 0 & 0 & d_5 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \text{rank } 3, a_2 \neq 0, d_5 \neq 0.$$

$$25. \begin{pmatrix} 0 & 0 & a_3 & a_4 & a_5 \\ 0 & 0 & b_3 & b_4 & b_5 \\ 0 & 0 & 0 & c_4 & c_5 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \text{rank } 3, b_3 \neq 0, c_4 \neq 0.$$

8.3.3 Nilpotence index four

This case gives a system of 56 equations. We get a very simple solution graph (see figure 8.7). Ordered by rank the solutions are:

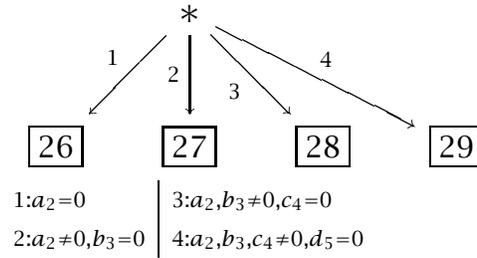


Figure 8.7: Solution graph for nilpotence index four.

$$26. \begin{pmatrix} 0 & 0 & a_3 & a_4 & a_5 \\ 0 & 0 & b_3 & b_4 & b_5 \\ 0 & 0 & 0 & c_4 & c_5 \\ 0 & 0 & 0 & 0 & d_5 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \text{rank } 3.$$

$$27. \begin{pmatrix} 0 & a_2 & a_3 & a_4 & a_5 \\ 0 & 0 & 0 & b_4 & b_5 \\ 0 & 0 & 0 & c_4 & c_5 \\ 0 & 0 & 0 & 0 & d_5 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \text{rank } 3, a_2 \neq 0.$$

$$28. \begin{pmatrix} 0 & a_2 & a_3 & a_4 & a_5 \\ 0 & 0 & b_3 & b_4 & b_5 \\ 0 & 0 & 0 & 0 & c_5 \\ 0 & 0 & 0 & 0 & d_5 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \text{rank } 3, a_2 \neq 0, b_3 \neq 0.$$

$$29. \begin{pmatrix} 0 & a_2 & a_3 & a_4 & a_5 \\ 0 & 0 & b_3 & b_4 & b_5 \\ 0 & 0 & 0 & c_4 & c_5 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \text{rank } 3, a_2 \neq 0, b_3 \neq 0, c_4 \neq 0.$$

8.3.4 Nilpotence index five

As was stated before, all triangular forms have a nilpotent Jacobian matrix, so there is only one matrix with nilpotence index five for the corresponding Jacobian matrix, namely the general map:

$$30. \begin{pmatrix} 0 & a_2 & a_3 & a_4 & a_5 \\ 0 & 0 & b_3 & b_4 & b_5 \\ 0 & 0 & 0 & c_4 & c_5 \\ 0 & 0 & 0 & 0 & d_5 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \text{rank } 4, a_2 \neq 0, b_3 \neq 0, c_4 \neq 0, d_5 \neq 0.$$

8.4 More cubic similarity

With the thirty matrices presented in section 8.3 we are now looking for representatives of Meisters' cubic similarity relation. In particular we try to complete Meisters' list. Because we classified all Drużkowski maps, a secure examination of these forms will give a complete list of representatives. We organise our search by grouping the matrices by rank, one of the invariants of the cubic similarity relation. It may seem strange that we changed from the nilpotence index invariant to this rank invariant, but we have our reasons. The nilpotence index is handsome to find the matrices as explained earlier. The rank is handsome to investigate the forms that we have found before. Simply because this investigation has to do with making assumptions on the parameters which appear in the forms. And normally the effect on the rank of e.g. $a_2 = 0$ is easier to predict than its effect on the nilpotence index. After ordering by rank, we try to find linear invertible T such that $T^{-1} \circ F \circ T$ is as simple as possible, i.e. has as little parameters as possible. Since these T 's involve parameters we must be very cautious regarding the invertibility of these T 's. We'll explain the basic approach we have taken. Let A be a matrix of the list in section 8.3.

1. Look at the assumptions we had to make in order to find this specific A .
2. Try to reduce A to cases already known by use of permutation matrices.
3. Take a general linear map T containing parameters.
4. Compute B where B is defined by $X + (BX)^{*3} = T^{-1} \circ (X + (AX)^{*3}) \circ T$.
5. Compare B with the already known representatives.
6. Guess which one of the known representatives can be identified with B by assigning smart values to the free variables in T . Call this matrix M .
7. Solve the system of equations generated by $B = M$ in the free variables of T .
8. If this system has no solutions:
 - Guess another M .
 - If all known representatives have been tried, one probably has found a matrix which is not cubic similar to any of the known representatives.
 - Reduce B as much as possible to M' , i.e. solve $B_{i,j} = 0$ or $B_{i,j} = 1$ for as many entries $B_{i,j}$ as possible.

- Prove that the new M' is indeed not cubic similar to all the old representatives of the same rank.

9. If the system of $B = M$ has at least one solution:

- Try to simplify this solution by setting free variables equal to zero or to one in case they cannot be set to zero.
- Check if this T implies some new assumptions on the original parameters in the matrix A in order to have that T is invertible.
 - If it does not, one can conclude that $A \not\sim M$.
 - If it does, assume that these extra assumptions *do not hold* and apply this information to A and call the result of these assumptions A' . (If for instance T is invertible only if $a_2 \neq 0$, substitute $a_2 = 0$ in A and call the result A' .) Now repeat all steps on this A' instead.

In section A.2.1 in appendix A this process is described for all thirty cases. And in section A.2.2 we present a list with all the actual T 's we used. Here we'll only present one example.

Example 8.13

(See case 8 in section 8.3.) Consider $F = X + (AX)^{*3}$ where

$$A = \begin{pmatrix} 0 & 0 & 0 & a_4 & a_5 \\ 0 & 0 & 0 & b_4 & b_5 \\ 0 & 0 & 0 & 0 & c_5 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

We already know that $b_4 \neq 0$. If we compute $T_1^{-1}FT_1 = X + (BX)^{*3}$ for a general map T_1 and try to solve the cases $B = J(2, 2)$, $B = J(2, 3)$ and $B = N(2, 3a)$ we don't get any solution at all. So most probably we have found a new representative. If we try to reduce this B , we see that we can find T_1 such that $B_{1,4} = 1$, $B_{2,4} = 1$, $B_{2,5} = 1$ and $B_{3,5} = 1$ and all other $B_{i,j} = 0$. We call this M' . Looking carefully at the definition of cubic similarity shows that this M' is indeed not cubic similar to the known representatives with rank two. We call this new representative $N(2, 2a)$. The T_1 we used is

$$\left(\frac{(b_5 a_4 - b_4 a_5)^3}{b_4^3} x_1, \frac{(b_5 a_4 - b_4 a_5)^3}{a_4^3} x_2, c_5^3 x_3, \frac{(b_5 a_4 - b_4 a_5) x_4}{b_4 a_4} - \frac{a_5 x_5}{a_4}, x_5 \right)$$

If we look at this T_1 we see that it is invertible only if $a_4 \neq 0$, $c_5 \neq 0$ and $b_5 a_4 - b_4 a_5 \neq 0$. (We already know that $b_4 \neq 0$.)

Now assume that $a_4 \neq 0$ and $c_5 \neq 0$ but $b_5 a_4 - b_4 a_5 = 0$ and start the process again. After taking a new T_2 and compute $T_2^{-1}FT_2$, we get a matrix B that can be identified with $J(2, 2)$. Solving this system yields that T_2 is

$$\left(x_5 + a_4^3 x_3, b_4^3 x_3, x_1, x_4 - \frac{b_5 x_2}{b_4 c_5}, \frac{x_2}{c_5} \right)$$

Looking at T_2 we note that we don't need any new assumptions. From T_1 it already follows that we have to look at the cases where $a_4 = 0$ and $c_5 = 0$.

Now assume $a_4 \neq 0$ and $b_5 a_4 - b_4 a_5 \neq 0$ but $c_5 = 0$. In this case the map T_3 gives $T_3^{-1}FT_3$ which is cubic similar to $J(2, 2)$ where T_3 is given by

$$\left(-\frac{(b_5 a_4 - b_4 a_5)^3 x_1}{b_4^3}, -\frac{(b_5 a_4 - b_4 a_5)^3 x_3}{a_4^3}, x_5, \frac{a_5 x_4}{a_4} - \frac{x_2 b_5}{b_4}, x_2 - x_4 \right)$$

Note that this map T_3 does not imply any new assumptions.

Now assume $a_4 \neq 0$ but $b_5 a_4 - b_4 a_5 = 0$ and $c_5 = 0$. We can immediately skip this case since it gives a matrix A with $\text{rank}(A) = 1$.

So the next case is $a_4 = 0$. In order to remain in a rank two case we must have that either $a_5 \neq 0$ or $c_5 \neq 0$. We may assume $c_5 \neq 0$ since a simple permutation $P = (x_3, x_2, x_1, x_4, x_5)$ swaps the first and third row. So now we can use T_4 is

$$\left(x_5 + a_5^3 x_3, b_4^3 x_1, c_5^3 x_3, x_2 - \frac{b_5 x_4}{b_4}, x_4 \right)$$

to get that $T_4^{-1}FT_4$ is cubic similar to $J(2, 2)$.

Because T_4 doesn't imply new assumptions, this last case solves the question concerning cubic similarity completely for this matrix A .

Applying the method described above to all 30 matrices of section 8.3 gives us nineteen new representatives. The new matrices are:

$$\begin{pmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \quad N(2, 2a) \quad \begin{pmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \quad N(2, 3b) \quad \begin{pmatrix} 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \quad N(3, 3b)$$

$$\begin{pmatrix} 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \quad N(3, 4e) \quad \begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \quad N(3, 4f) \quad \begin{pmatrix} 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \quad N(3, 4g)$$

$$\begin{pmatrix} 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \quad N(3, 4h) \quad \begin{pmatrix} 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \quad N(3, 4i) \quad \begin{pmatrix} 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \quad N(3, 4j)$$

$$\begin{pmatrix} 0 & 0 & 1 & a & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \quad P(3, 4a) \quad \begin{pmatrix} 0 & 0 & 1 & 0 & a \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \quad P(3, 4c) \quad \begin{pmatrix} 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & a \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \quad P(3, 4g)$$

$$\begin{array}{ccc}
\begin{pmatrix} 0 & 0 & 1 & 1 & a \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 1 & a & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & a \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \\
P(3,4h) & P(3,4i) & P(3,4j) \\
\\
\begin{pmatrix} 0 & 0 & 1 & a & b \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 1 & a & 1 & 0 \\ 0 & 0 & 0 & 1 & b \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \\
P(3,4a2) & P(3,4j2) & N(4,5e) \\
\\
& \begin{pmatrix} 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & a & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \\
& P(4,5e)
\end{array}$$

Together with Meisters' matrices brings this a total of 36 representatives for the cubic similarity relation.

Remark 8.14

The matrix $N(2,3b)$ deserves some special attention. The -1 seems a bit strange: why isn't it $P(2,3a)$ with a parameter a on the place of the -1 . The answer is in fact pretty simple. As long as $a \notin \{0, 1\}$, $P(2,3a) \stackrel{3}{\sim} N(2,3b)$. Furthermore $P(2,3a)|_{a=0} \stackrel{3}{\sim} P(2,3a)|_{a=1} \stackrel{3}{\sim} N(2,3a)$. So independent of the value of the parameter a , $P(2,3a)$ can be reduced to a matrix with no parameters left in it. So there's no need to add a P -matrix.

Remark 8.15

The names of the P -matrices are based on the observations that:

- $P(3,4a)|_{a=1} \stackrel{3}{\sim} N(3,4a)$ and $P(3,4a)|_{a=0} = N(3,4b)$.
- $P(3,4c)|_{a=1} \stackrel{3}{\sim} N(3,4c)$ and $P(3,4c)|_{a=0} = N(3,4b)$.
- $P(3,4g)|_{a=1} = N(3,4g)$ and $P(3,4g)|_{a=0} \stackrel{3}{\sim} N(3,4a)$.
- $P(3,4h)|_{a=1} = N(3,4h)$ and $P(3,4h)|_{a=0} \stackrel{3}{\sim} N(3,4b)$.
- $P(3,4i)|_{a=1} = N(3,4i)$ and $P(3,4i)|_{a=0} \stackrel{3}{\sim} N(3,4a)$.
- $P(3,4j)|_{a=1} = N(3,4j)$ and $P(3,4j)|_{a=0} \stackrel{3}{\sim} N(3,4a)$.
- $P(3,4a2)|_{a=0} = P(3,4c)$ and $P(3,4a2)|_{b=0} = P(3,4a)$, hence $P(3,4c2)$ would have been a correct name also.
- $P(3,4j2)|_{a=0} = P(3,4j)$. Furthermore we have $P(3,4j2)|_{b=0,a=-1} \stackrel{3}{\sim} N(3,3a)$ and $P(3,4j2)|_{b=0,a \neq 0, a \neq -1} \stackrel{3}{\sim} N(3,4a)$.

-
- $P(4, 5e)|_{a=1} = N(4, 5e)$ and $P(4, 5e)|_{a=0} = N(4, 5d)$.

So we add for $P(3, 4a)$, $P(3, 4c)$, $P(3, 4g)$, $P(3, 4h)$, $P(3, 4i)$, $P(3, 4j)$ and $P(4, 5e)$ the restriction that $a \notin \{0, 1\}$. For $P(3, 4a2)$ and $P(3, 4j2)$ we add $a, b \neq 0$.

Chapter 9

Conclusions and suggestions for future research

9.1 Conclusions

At the time the author started working as a PhD-student, September 1994, the official subject was ‘Research on (invertible) polynomial maps using computer algebra’. The unofficial task was ‘Find a counterexample to the Jacobian Conjecture’. The reason for this distinction between the official and unofficial task comes from the fact that we were fully aware that chances of finding such a counterexample were not very good. And in that case this thesis would have been a description of a negative result. Now that you have read the previous chapters, you know that indeed we were not able to find this counterexample. However during this search we did find various interesting things concerning polynomial maps. Let us recall the most important results from this thesis:

- We have found a connection between linearly triangularisable polynomial maps and strongly nilpotent matrices. (Theorem 6.12.) This last property can be tested very easily using computer algebra.
- We have introduced the class $\mathcal{H}_n(A)$ of invertible polynomial maps. It turns out to be a large class. And we were able to prove the Jacobian Conjecture for this class. (Corollary 3.24.) Furthermore we were able to show that the elements of this class are all stably tame automorphisms. (Theorem 5.36.)
- The introduction of $\mathcal{D}_n(A)$ crosses a bridge from the theoretical research to the experimental research using computer algebra. One doesn’t need to know anything about $\mathcal{H}_n(A)$ to find numerous examples of Keller maps. One only has to write down a set of matrices and vectors of a certain type and a computer can build a (possibly very complex) polynomial map for which we know a priori already that the Jacobian Conjecture holds. So this is a very handsome tool to generate examples of various kinds.

- Computations have shown that there exist basically two normal forms for quadratic homogeneous maps in dimension five. One which is linearly triangulable and one which is not. (Theorem 7.11.)
- Huge computations have provided a classification of all cubic-linear Keller maps in dimension five. (Theorem 8.12.) After more tedious work, this classification has led to a complete set of generators with respect to Meisters' cubic similarity relation. (Page 172.)
- Last but not least, using the theory build in this thesis the Jacobian Conjecture for differential equations –better known as Markus-Yamabe Conjecture– has been solved after 35 years. (Theorem 2.27.) It turned out to be false by an almost trivial example!

So even though the ultimate goal –a counterexample to the Jacobian Conjecture– was not reached, the research of the last four years is far from useless.

9.2 Suggestions for future work

In addition to the results we mentioned in the previous section, we can also point out some gaps in this thesis.

- We know a lot about $\mathcal{H}_n(A)$. Unfortunately we don't know that much about $\overline{\mathcal{H}_n(A)}$. For instance we have no $\overline{\mathcal{D}_n(A)}$ equivalent. And as a result of this, we don't know whether the elements of $\overline{\mathcal{H}_n(A)}$ are stably tame. (Note that the proof for the $\mathcal{H}_n(A)$ case heavily depends on the features of $\mathcal{D}_n(A)$.)
- We do have a good algorithm to build a polynomial map given a $\mathcal{D}_n(A)$ -tuple. The other way round doesn't work very well. First as we have seen there is no uniqueness. Second only up to $\mathcal{H}_4(\mathbb{C})$ we can arrange that we get $\mathcal{D}_4(\mathbb{C})$ tuples where $\det(T_i) = 1$ for all i . At the moment this is not implemented yet, but it can be done using an argument by Suslin. For $n > 4$ we haven't got a good algorithm yet that will find a $\mathcal{D}_4(\mathbb{C})$ structure even if we know that it exists.
- It should be interesting if one finds a cubic homogeneous counterexample to the Dependence Problem.
- A related problem to this Dependence Problem is the following conjecture:

Conjecture 9.1

Consider a set of n matrices $A_i \in \text{Mat}_n(k)$ such that

$$A_{i(j)} = A_{j(i)}$$

i.e. the j -th column of A_i is equal to the i -th column of A_j . Now if $A_1 x_1 + \dots + A_n x_n$ is nilpotent then the rows of

$$A = \begin{pmatrix} A_1 & A_2 & \cdots & A_n \end{pmatrix}$$

are linearly dependent over k .

This condition on the columns appears naturally if one takes for instance $F = X + Q$ where Q is quadratic homogeneous and one puts $JQ = A_1x_1 + \cdots + A_nx_n$.

- Chapter 7 is more or less based upon the assumption that problem DPLQ (problem 7.1) has an affirmative answer. If this could be proved the main result of chapter 7 becomes more valuable.
- The proof of theorem 8.3 is for a small part based upon computations. Although the resulting systems that have to be solved are pretty simple and by means of the solution graphs it is easy to verify that the solutions given in this thesis are indeed all solutions as claimed, it would be nice if this computational proof could be replaced by a more theoretical proof.

Appendix A

Tables, listings, computations etc.

Introduction

The title of this appendix already describes its meaning. In this appendix we present 'boring' details which are necessary to provide some ways of verification of all the things claimed by computation in the body of this thesis. Mainly this means that we give explanations for chapter 7 and chapter 8.

A.1 Classification of quadratic homogeneous maps

A.1.1 Command lists

The method used to solve the systems makes it possible to repeat the computations easily. Most of these listings can be used exactly as below. However there are some points where we solve a specific equation which is taken out of the large system by explicitly referring to its index in this set. Unfortunately this index need not be the same in different Maple runs. So at these places one has to be careful.

The file `part4` contains the so-called Jacobian package. For instance the procedure `strongnilpotent` comes from this package. Furthermore the procedures `es3`, `es4`, `es7` and `cleanupsubs` are defined in `part4`. These `es?` procedures are used to extract simple equations, and substitute these partial solutions into the original system. Their function is: `es3` extracts all equations with a specified list of variables; `es4` substitutes a solution in a set of equations and tries to simplify the result; `es7` extracts all equations in a set with the lowest number of variables. This `es7` procedure turns out to be very helpful in solving large systems of equations. Finally the procedure `cleanupsubs` deletes the free entries of a solution, i.e. the entries like $a_1 = a_1$.

Case (7.3)

```
> read part4:
> X:=[x[1],x[2],x[3],x[4],x[5]]:
> HH:=[mu*x[2]*x[3]+nu*x[2]^2+tau[1]*x[3]^2+(a[1]*x[1]+b[1]*x[2]+c[1]*x[
> 3])*x[4]+(d[1]*x[1]+e[1]*x[2]+f[1]*x[3]+g[1]*x[4])*x[5],
```

```

> tau[2]*x[3]^2+(a[2]*x[1]+b[2]*x[2]+c[2]*x[3])*x[4]+(d[2]*x[1]+e[2]*x[2
> ]+f[2]*x[3]+g[2]*x[4])*x[5],(a[3]*x[1]+b[3]*x[2]+c[3]*x[3])*x[4]+(d[3]
> *x[1]+e[3]*x[2]+f[3]*x[3]+g[3]*x[4])*x[5],0,0];
> H:=subs({d[1]=0,e[1]=0,f[1]=0,g[1]=0,g[2]=0,g[3]=0},HH);
> H:=subs({a[3]=0,b[3]=0,c[3]=0},H);
> H:=subs({b[2]=-a[1]},H);
> H:=subs({a[2]=0,a[1]=0},H);
> H:=subs({f[3]=-e[2]},H);
> JH:=jacobian(H,X);
> JH5:=evalm(JH^5);
> sys:={'coeffs(collect(JH5[i,j],X,distributed),X)','$i'=1..5','$j'=1..5}
> :nops("");
> infolevel[es3]:=3;infolevel[es4]:=3;infolevel[es7]:=3;
> ss7:=es7(sys);
> ss3:=es3(ss7,{},{e[2],f[2],e[3]});
> oo:=cleanupsubs([solve(ss3)]);
> aaa:={e[2] = 0, e[3] = 0};
> aab:={f[2] = -e[2]^2/e[3]}:#e[3]<>0
> ss:=es4(aaa,sys);
> ss7:=es7(ss);
> ss3:=es3(ss7,{},{d[3],b[1],f[2]});
> oo:=cleanupsubs([solve(ss3)]);
> aaaa:=subun({d[3] = 0},aaa);
> aaab:=subun({f[2] = 0},aaa):#d[3]<>0
> aaac:=subun({b[1] = 0},aaa):#d[3],f[2]<>0
> ss:=es4(aaaa,ss);
> ss7:=es7(ss);
> oo:=cleanupsubs([solve(ss7)]);
> aaaaa:=subun({d[2] = 0},aaaa);
> aaaab:=subun({nu = 0, b[1] = 0, mu = 0},aaaa):#d[2]<>0
> ss:=es4(aaaaa,ss);
> subs(aaaaa,H);
> strongnilpotent(jacobian(",X),X);
> sols:=[aaaaa];
> ss:=es4(aaaab,sys):#d[2]<>0
> subs(aaaab,H);
> strongnilpotent(jacobian(",X),X);
> sols:=[op(sols),aaaab];
> ss:=es4(aaab,sys):#d[3]<>0
> ss7:=es7(ss);
> ss3:=es3(ss7,{},{d[3],nu,c[2]});
> oo:=cleanupsubs([solve(ss3)]);
> aaaba:=subun({nu = 0},aaab):#d[3]<>0
> aaabb:=subun({c[2] = 0},aaab):#d[3],nu<>0
> ss:=es4(aaaba,ss):#d[3]<>0
> ss7:=es7(ss);
> oo:=cleanupsubs([solve(ss7)]);
> aaaba:=subun({mu = 0},aaaba):#d[3]<>0
> ss:=es4(aaaba,ss):#d[3]<>0
> ss7:=es7(ss);
> oo:=cleanupsubs([solve(ss7)]);
> aaaba:=subun({tau[1] = 0},aaaba):#d[3]<>0
> ss:=es4(aaaba,ss):#d[3]<>0
> ss7:=es7(ss);
> oo:=cleanupsubs([solve(ss7)]);
> aaabaa:=subun({b[1] = 0},aaaba):#d[3]<>0
> aaabab:=subun({c[2] = 0, tau[2] = 0},aaaba):#d[3],b[1]<>0

```

```

> ss:=es4(aaabaa,ss):#d[3]<>0
> ss7:=es7(ss);
> oo:=cleanupsubs([solve(ss7)]);
> aaabaa:=subun({c[1] = 0},aaabaa):#d[3]<>0
> ss:=es4(aaabaa,ss):#d[3]<>0
> ss;
> subs(aaabaa,H);
> strongnilpotent(jacobian(",X),X);
> sols:=[op(sols),aaabaa]:nops("");
> ss:=es4(aaabab,sys):#d[3],b[1]<>0
> ss7:=es7(ss);
> oo:=cleanupsubs([solve(ss7,{c[1]})]);
> aaabab:=subun({c[1] = -b[1]*d[2]/d[3]},aaabab):#d[3],b[1]<>0
> ss:=es4(aaabab,sys):#d[3],b[1]<>0
> subs(aaabab,H);strongnilpotent(jacobian(",X),X);
> sols:=[op(sols),aaabab]:nops("");
> ss:=es4(aaabb,sys):#d[3],nu<>0
> ss7:=es7(ss);
> oo:=cleanupsubs([solve(ss7)]);
> aaabb:=subun({tau[2] = 0},aaabb):#d[3],nu<>0
> ss:=es4(aaabb,ss):#d[3],nu<>0
> ss7:=es7(ss);
> ss3:=es3(ss7,{},{d[3],b[1],d[2],c[1],nu,mu});
> oo:=cleanupsubs([solve(ss3,{c[1],mu})]);
> aaabb:=subun({c[1] = -b[1]*d[2]/d[3], mu =
> -2*nu*d[2]/d[3]},aaabb):#d[3],nu<>0
> ss:=es4(aaabb,ss):#d[3],nu<>0
> ss7:=es7(ss);
> oo:=cleanupsubs([solve(ss7,{tau[1]})]);
> aaabb:=subun({tau[1] = d[2]^2*nu/d[3]^2},aaabb):#d[3],nu<>0
> ss:=es4(aaabb,ss):#d[3],nu<>0
> subs(aaabb,H);strongnilpotent(jacobian(",X),X);
> sols:=[op(sols),aaabb]:nops("");
> ss:=es4(aaac,sys):#d[3],f[2]<>0
> ss7:=es7(ss);
> oo:=cleanupsubs([solve(ss7)]);
> aaac:=subun({c[1] = 0},aaac):#d[3],f[2]<>0
> ss:=es4(aaac,ss):#d[3],f[2]<>0
> ss7:=es7(ss);
> oo:=cleanupsubs([solve(ss7)]);
> aaac:=subun({nu = 0, mu = 0},aaac):#d[3],f[2]<>0
> ss:=es4(aaac,ss):#d[3],f[2]<>0
> ss7:=es7(ss);
> oo:=cleanupsubs([solve(ss7)]);
> aaac:=subun({tau[1] = 0},aaac):#d[3],f[2]<>0
> ss:=es4(aaac,ss):#d[3],f[2]<>0
> subs(aaac,H);strongnilpotent(jacobian(",X),X);
> sols:=[op(sols),aaac]:nops("");
> ss:=es4(aab,sys):#e[3]<>0
> ss7:=es7(ss);
> ss3:=es3(ss7,{},{d[3],c[2],b[1]});
> oo:=cleanupsubs([solve(ss3)]);
> aaba:=subun({d[3] = 0},aab):#e[3]<>0
> aabb:=subun({c[2] = 0},aab):#e[3],d[3]<>0
> aabc:=subun({b[1] = 0},aab):#e[3],d[3],c[2]<>0
> ss:=es4(aaba,ss):#e[3]<>0
> ss7:=es7(ss);

```

```

> oo:=cleanupsubs([solve(ss7)]);
> aabaa:=subun({d[2] = 0},aabaa):#e[3]<>0
> aabab:=subun({nu = 0},aabaa):#e[3],d[2]<>0
> ss:=es4(aabaa,ss):#e[3]<>0
> ss7:=es7(ss);
> oo:=cleanupsubs([solve(ss7)]);
> aabaa:=subun({c[2] = 0, tau[2] = 0},aabaa):#e[3]<>0
> ss:=es4(aabaa,ss):#e[3]<>0
> subs(aabaa,H);strongnilpotent(jacobian(",X),X);
> sols:=[op(sols),aabaa]:nops("");
> ss:=es4(aabab,sys):#e[3],d[2]<>0
> ss7:=es7(ss);
> oo:=cleanupsubs([solve(ss7)]);
> aabab:=subun({mu = 0},aabab):#e[3],d[2]<>0
> ss:=es4(aabab,ss):#e[3],d[2]<>0
> ss7:=es7(ss);
> oo:=cleanupsubs([solve(ss7)]);
> aabab:=subun({tau[2] = 0},aabab):#e[3],d[2]<>0
> ss:=es4(aabab,ss):#e[3],d[2]<>0
> ss7:=es7(ss);
> oo:=cleanupsubs([solve(ss7)]);
> aabab:=subun({tau[1] = 0},aabab):#e[3],d[2]<>0
> ss:=es4(aabab,ss):#e[3],d[2]<>0
> ss7:=es7(ss);
> oo:=cleanupsubs([solve(ss7,{c[2]})]);
> aabab:=subun({c[2] = -b[1]*d[2]/e[3]},aabab):#e[3],d[2]<>0
> ss:=es4(aabab,ss):#e[3],d[2]<>0
> ss7:=es7(ss);
> oo:=cleanupsubs([solve(ss7,{c[1]})]);
> aabab:=subun({c[1] = -b[1]*e[2]/e[3]},aabab):#e[3],d[2]<>0
> ss:=es4(aabab,ss):#e[3],d[2]<>0
> subs(aabab,H);strongnilpotent(jacobian(",X),X);
> sols:=[op(sols),aabab]:nops("");
> ss:=es4(aabb,sys):#e[3],d[3]<>0
> ss7:=es7(ss);
> oo:=cleanupsubs([solve(ss7)]);
> aabba:=subun({tau[2] = 0},aabba):#e[3],d[3]<>0
> aabbb:=subun({nu = 0, b[1] = 0, mu = 0},aabba):#e[3],d[3],tau[2]<>0
> ss:=es4(aabba,ss):#e[3],d[3]<>0
> ss7:=es7(ss);
> ss3:=es3(ss7,{},{d[3],b[1],d[2],c[1],nu,mu});
> oo:=cleanupsubs([solve(ss3,{c[1],mu})]);
> aabba:=subun({c[1] = -b[1]*d[2]/d[3], mu =
-2*nu*d[2]/d[3]},aabba):#e[3],d[3]<>0
> ss:=es4(aabba,ss):#e[3],d[3]<>0
> ss7:=es7(ss);
> oo:=cleanupsubs([solve(ss7,{tau[1]})]);
> aabba:=subun({tau[1] = d[2]^2*nu/d[3]^2},aabba):#e[3],d[3]<>0
> ss:=es4(aabba,ss):#e[3],d[3]<>0
> ss7:=es7(ss);
> oo:=cleanupsubs([solve(ss7)]);
> aabbaa:=subun({d[2] = e[2]*d[3]/e[3]},aabba):#e[3],d[3]<>0
> aabbab:=subun({nu = 0, b[1] =
0},aabba):#e[3],d[3],e[3]*d[2]-e[2]*d[3]<>0
> ss:=es4(aabbaa,ss):#e[3],d[3]<>0
> subs(aabbba,H);strongnilpotent(jacobian(",X),X);
> sols:=[op(sols),aabbba]:nops("");

```

```

> ss:=es4(aabbab,sys):#e[3],d[3],e[3]*d[2]-e[2]*d[3]<>0
> subs(aabbab,H);strongnilpotent(jacobian(",X),X);
> sols:=op(sols),aabbab]:nops("");
> ss:=es4(aabbb,sys):#e[3],d[3],tau[2]<>0
> ss7:=es7(ss);
> oo:=cleanupsubs([solve(ss7)]);
> aabbb:=subun({c[1] = 0},aabbb):#e[3],d[3],tau[2]<>0
> ss:=es4(aabbb,ss):#e[3],d[3],tau[2]<>0
> ss7:=es7(ss);
> oo:=cleanupsubs([solve(ss7)]);
> aabbb:=subun({tau[1] = -tau[2]*e[3]/d[3]},aabbb):#e[3],d[3],tau[2]<>0
> ss:=es4(aabbb,ss):#e[3],d[3],tau[2]<>0
> ss7:=es7(ss);
> oo:=cleanupsubs([solve(ss7,{e[2]})]);
> aabbb:=subun({e[2] = e[3]*d[2]/d[3]},aabbb):#e[3],d[3],tau[2]<>0
> ss:=es4(aabbb,ss):#e[3],d[3],tau[2]<>0
> subs(aabbb,H);strongnilpotent(jacobian(",X),X);
> sols:=op(sols),aabbb]:nops("");
> ss:=es4(aabc,sys):#e[3],d[3],c[2]<>0
> ss7:=es7(ss);
> oo:=cleanupsubs([solve(ss7)]);
> aabc:=subun({nu = 0, mu = 0},aabc):#e[3],d[3],c[2]<>0
> ss:=es4(aabc,ss):#e[3],d[3],c[2]<>0
> ss7:=es7(ss);
> oo:=cleanupsubs([solve(ss7)]);
> aabc:=subun({c[1] = -c[2]*e[3]/d[3], tau[1] =
> -tau[2]*e[3]/d[3]},aabc):#e[3],d[3],c[2]<>0
> ss:=es4(aabc,ss):#e[3],d[3],c[2]<>0
> ss7:=es7(ss);
> oo:=cleanupsubs([solve(ss7,{e[2]})]);
> aabc:=subun({e[2] = e[3]*d[2]/d[3]},aabc):#e[3],d[3],c[2]<>0
> ss:=es4(aabc,ss):#e[3],d[3],c[2]<>0
> subs(aabc,H);strongnilpotent(jacobian(",X),X);
> sols:=op(sols),aabc]:nops("");
>

```

Case (7.4)

```

> read part4:
> X:=[x[1],x[2],x[3],x[4],x[5]]:
> HH:=[mu*x[2]*x[3]+nu*x[2]^2+tau[1]*x[3]^2+(a[1]*x[1]+b[1]*x[2]+c[1]*x[
> 3])*x[4]+(d[1]*x[1]+e[1]*x[2]+f[1]*x[3]+g[1]*x[4])*x[5],tau[2]*x[3]^2+
> (a[2]*x[1]+b[2]*x[2]+c[2]*x[3])*x[4]+(d[2]*x[1]+e[2]*x[2]+f[2]*x[3]+g[
> 2]*x[4])*x[5],(a[3]*x[1]+b[3]*x[2]+c[3]*x[3])*x[4]+(d[3]*x[1]+e[3]*x[2
> ]+f[3]*x[3]+g[3]*x[4])*x[5],0,0];
> H:=subs({d[2]=0,e[2]=0,f[2]=0,g[1]=0,g[2]=0,g[3]=0},HH);
> H:=subs({a[3]=0,b[3]=0,c[3]=0},H);
> H:=subs({b[2]=-a[1],f[3]=-d[1]},H);
> H:=subs({a[2]=0,a[1]=0},H);
> JH:=jacobian(H,X);
> JH5:=evalm(JH^5):
> sys:={'coeffs(collect(JH5[i,j],X,distributed),X) '$i'=1..5 '$j'=1..5}
> :nops("");
> infolevel[es3]:=3:infolevel[es4]:=3:infolevel[es7]:=3:
> ss7:=es7(sys);
> oo:=cleanupsubs([solve(ss7)]);
> aaa:={d[3] = 0}:

```

```

> aab:={mu = 0}:#d[3]<>0
> ss:=es4(aaa,sys):
> ss7:=es7(ss);
> oo:=cleanupsubs([solve(ss7)]);
> aaa:=subun({d[1] = 0},aaa):
> ss:=es4(aaa,ss):
> ss7:=es7(ss);
> oo:=cleanupsubs([solve(ss7)]);
> aaaa:=subun({e[3] = 0},aaa):
> aaab:=subun({tau[2] = 0, c[2] = 0},aaa):#e[3]<>0
> ss:=es4(aaaa,ss):
> subs(aaaa,H);strongnilpotent(jacobian(",X),X);
> solsd:=[aaaa]:
> ss:=es4(aaab,sys):#e[3]<>0
> subs(aaab,H);strongnilpotent(jacobian(",X),X);
> solsd:=[op(solsd),aaab]:
> ss:=es4(aab,sys):#d[3]<>0
> ss7:=es7(ss);
> ss3:=es3(ss7,{},{d[1],f[1],d[3]});
> oo:=cleanupsubs([solve(ss3)]);
> aab:=subun({f[1] = -d[1]^2/d[3]},aab):#d[3]<>0
> ss:=es4(aab,ss):#d[3]<>0
> ss7:=es7(ss);
> ss3:=es3(ss7,{},{d[3],c[2],nu,b[1]});
> oo:=cleanupsubs([solve(ss3)]);
> aaba:=subun({c[2] = 0},aab):#d[3]<>0
> aabb:=subun({nu = 0, b[1] = 0},aab):#d[3],c[2]<>0
> ss:=es4(aaba,ss):#d[3]<>0
> ss7:=es7(ss);
> oo:=cleanupsubs([solve(ss7)]);
> aaba:=subun({c[1] = 0},aaba):#d[3]<>0
> ss:=es4(aaba,ss):#d[3]<>0
> ss7:=es7(ss);
> oo:=cleanupsubs([solve(ss7)]);
> aabaa:=subun({tau[2] = 0},aabaa):#d[3]<>0
> aabab:=subun({nu = 0, b[1] = 0},aabaa):#d[3],tau[2]<>0
> ss:=es4(aabaa,ss):#d[3]<>0
> ss7:=es7(ss);
> oo:=cleanupsubs([solve(ss7)]);
> aabaa:=subun({tau[1] = 0},aabaa):#d[3]<>0
> ss:=es4(aabaa,ss):#d[3]<>0
> subs(aabaa,H);strongnilpotent(jacobian(",X),X);
> solsd:=[op(solsd),aabaa]:nops("");
> ss:=es4(aabab,sys):#d[3],tau[2]<>0
> ss7:=es7(ss);
> oo:=cleanupsubs([solve(ss7,{tau[1]})]);
> aabab:=subun({tau[1] = -tau[2]*e[3]/d[3]},aabab):#d[3],tau[2]<>0
> ss:=es4(aabab,ss):#d[3],tau[2]<>0
> ss7:=es7(ss);
> oo:=cleanupsubs([solve(ss7,{e[1]})]);
> aabab:=subun({e[1] = d[1]*e[3]/d[3]},aabab):#d[3],tau[2]<>0
> ss:=es4(aabab,ss):#d[3],tau[2]<>0
> subs(aabab,H);strongnilpotent(jacobian(",X),X);
> solsd:=[op(solsd),aabab]:nops("");
> ss:=es4(aabb,sys):#d[3],c[2]<>0
> ss7:=es7(ss);
> oo:=cleanupsubs([solve(ss7,{c[1],tau[1]})]);

```

```

> aabb:=subun({tau[1] = -tau[2]*e[3]/d[3], c[1] =
> -c[2]*e[3]/d[3]},aabb):#d[3],c[2]<>0
> ss:=es4(aabb,ss):#d[3],c[2]<>0
> ss7:=es7(ss);
> oo:=cleanupsubs([solve(ss7,{e[1]})]);
> aabb:=subun({e[1] = d[1]*e[3]/d[3]},aabb):#d[3],c[2]<>0
> ss:=es4(aabb,ss):#d[3],c[2]<>0
> subs(aabb,H);strongnilpotent(jacobian(",X),X);
> solsd:=[op(solsd),aabb]:nops("");
> for i from 1 to nops(solsd) do subs(solsd[i],H) od;
>

```

Case (7.5)

```

> read part4:
> X:=[x[1],x[2],x[3],x[4],x[5]]:
> HH:=[mu*x[2]*x[3]+nu*x[2]^2+tau[1]*x[3]^2+(a[1]*x[1]+b[1]*x[2]+c[1]*x[
> 3])*x[4]+(d[1]*x[1]+e[1]*x[2]+f[1]*x[3]+g[1]*x[4])*x[5],tau[2]*x[3]^2+
> (a[2]*x[1]+b[2]*x[2]+c[2]*x[3])*x[4]+(d[2]*x[1]+e[2]*x[2]+f[2]*x[3]+g[
> 2]*x[4])*x[5],(a[3]*x[1]+b[3]*x[2]+c[3]*x[3])*x[4]+(d[3]*x[1]+e[3]*x[2
> ]+f[3]*x[3]+g[3]*x[4])*x[5],0,0];
> H:=subs({c[1]=0,c[2]=0,f[1]=0,f[2]=0,g[1]=0,g[2]=0,g[3]=0,d[3]=0,e[3]=
> 0,f[3]=0,tau[1]=0,tau[2]=0},HH);
> H:=subs({a[3]=0,b[3]=0,c[3]=0},H);
> H:=subs({b[2]=-a[1],d[1]=-e[2]},H);
> H:=subs({a[1]=0,a[2]=0},H);
> JH:=jacobian(H,X);
> JH5:=evalm(JH^5):
> sys:={'coeffs(collect(JH5[i,j],X,distributed),X) '$'i'=1..5 '$'j'=1..5}
> :nops("");
> infolevel[es3]:=3:infolevel[es4]:=3:infolevel[es7]:=3:
> ss7:=es7(sys);
> oo:=cleanupsubs([solve(ss7)]);
> aaa:={d[2] = 0}:
> aab:={nu = 0, mu = 0, b[1]=0}:#d[2]<>0
> ss:=es4(aaa,sys):
> ss7:=es7(ss);
> oo:=cleanupsubs([solve(ss7)]);
> aaa:=subun({e[2] = 0},aaa):
> ss:=es4(aaa,ss):
> solse:=[aaa]:
> ss:=es4(aab,sys):#d[2]<>0
> ss7:=es7(ss);
> oo:=cleanupsubs([solve(ss7)]);
> aab:=subun({e[1] = -e[2]^2/d[2]},aab):#d[2]<>0
> ss:=es4(aab,ss):#d[2]<>0
> solse:=[op(solse),aab]:
> for i from 1 to nops(solse) do subs(solse[i],H) od;
>

```

A.1.2 The actual transformations

In this subsection we list the 19 transformations one can use to reduce the $19 X + Q_i + E_1$ (or $X + Q_i + E_1 + E_2$) to either a triangular map or in case 8 the non-triangularisable normal form N of (7.6). Furthermore, by looking at the solution graphs in section 7.2

one can check that in order to find a specific solution Q_i , we had to assume that all factors which appear here in the corresponding T_i are all non-zero.

The list here is generated automatically by Maple. Therefore the order of the elements specified is not always the same as one would write by hand

1. $(x_1, x_2, x_3, x_4, x_5)$
2. $(x_2, x_1, x_3, x_4, x_5)$
3. $(x_3, x_1, x_2, x_4, x_5)$
4. $(x_2, x_3 + \frac{d_2 x_1}{d_3}, x_1, x_4, x_5)$
5. $(x_2, x_3 + \frac{d_2 x_1}{d_3}, x_1, x_4, x_5)$
6. $(x_3, x_1, x_2, x_4, x_5)$
7. $(x_1, x_3 + \frac{e_2 x_2}{e_3}, x_2, x_4, x_5)$
8. $(x_1, x_2 + \frac{e_2 x_3}{e_3}, x_3, x_4, x_5)$
9. $(x_2, x_3 + \frac{e_2 x_1}{e_3}, x_1, x_4, x_5)$
10. $(x_3, x_2 + \frac{e_2 x_1}{e_3}, x_1, x_4, x_5)$
11. $(x_1, x_3 + \frac{d_2 x_2}{d_3} - \frac{d_3 x_1}{e_3}, x_2, x_4, x_5)$
12. $(x_1, x_3 + \frac{d_2 x_2}{d_3} - \frac{d_3 x_1}{e_3}, x_2, x_4, x_5)$
13. $(x_1, x_2, x_3, x_4, x_5)$
14. $(x_1, x_3, x_2, x_4, x_5)$
15. $(x_2 + \frac{d_1 x_1}{d_3}, x_3, x_1, x_4, x_5)$
16. $(x_3 + \frac{d_1 x_2}{d_3} - \frac{e_3 x_1}{d_3}, x_1, x_2, x_4, x_5)$
17. $(x_3 + \frac{d_1 x_2}{d_3} - \frac{e_3 x_1}{d_3}, x_1, x_2, x_4, x_5)$
18. $(x_1, x_2, x_3, x_4, x_5)$
19. $(x_2 - \frac{e_2 x_1}{d_2}, x_1, x_3, x_4, x_5)$

A.2 Finding cubic similarity generators

A.2.1 Description of the cases

As stated in section 8.4, all maps are grouped by their rank. The numbers coincide with the numbers in front of the solutions in section 8.3. For each map F we see whether we can find a suitable transformation map T such that $T^{-1}FT$ is one of the representatives listed before. Most of the time this means that we have to make some assumptions on the parameters in F . At the beginning of section 8.4 we explained how we have come to this distinction between cases.

The proof that these assumptions lead to those representatives is given in section A.2.2 by showing the concrete transformations.

Rank one

In section 8.3 we have seen that there are two maps of rank one: the cases 1 and 2.

1. Obviously at least one of the four variables should be unequal to zero. Because the first four columns are equal to zero, we can permute the first four rows without any consequences with respect to the cubic similarity relation. Hence we may assume $a_5 \neq 0$ and then this map is cubic similar to $J(1, 2)$.
2. We are in a case where we already know that $a_2 \neq 0$. This gives that this map is cubic similar to $J(1, 2)$.

Rank two

Here we have sixteen matrices to examine.

3. We know $a_4 \neq 0$. Note that either b_5 or $c_5 \neq 0$, otherwise the rank is one. We may assume $c_5 \neq 0$. Then this map is cubic similar to $J(2, 2)$.
4. We know $a_3 \neq 0$. Because of the rank we must have $b_5 \neq 0$. Now also this map is cubic similar to $J(2, 2)$.
5. We know $a_3 \neq 0$ and $d_5 \neq 0$. Cubic similar to $J(2, 2)$.
6. Here we have $a_2 \neq 0$ and $c_5 \neq 0$. Cubic similar to $J(2, 2)$.
7. In this case we have $a_2 \neq 0$ and $d_5 \neq 0$. Again cubic similar to $J(2, 2)$.
8. We know $b_4 \neq 0$.
 - (a) Assume $a_4 = 0$. We may assume $c_5 \neq 0$, because if $c_5 = 0$ we must have $a_5 \neq 0$ and we can safely permute the first and third rows, since the first and third columns are completely zero. Then cubic similar to $J(2, 2)$.
 - (b) Assume $a_4 \neq 0$ and $c_5 = 0$, hence $a_5b_4 - a_4b_5 \neq 0$. Then cubic similar to $J(2, 2)$.

- (c) Assume $a_4 \neq 0$ and $c_5 \neq 0$ and $a_5b_4 - a_4b_5 = 0$. Then cubic similar to $J(2, 2)$.
- (d) Assume $a_4 \neq 0$ and $c_5 \neq 0$ and $a_5b_4 - a_4b_5 \neq 0$. Then cubic similar to $N(2, 2a)$.
9. We have $a_3 \neq 0$ and $b_4 \neq 0$. Hence cubic similar to $J(2, 2)$. After permutation of the first two rows this is basically the same map as map 13b.
10. We have $c_4 \neq 0$. If either $b_4 = 0$ or $a_4 = 0$, we can permute the rows such that $c_4 \neq 0$ appears on the fourth place in the second row, and a zero (either a_4 or b_4) appears on the fourth place in the third row. But then we have the same map as map 8. Hence we may assume that $a_4 \neq 0$, $b_4 \neq 0$ and $c_4 \neq 0$.
- Furthermore we also have that either a_5 , b_5 or $c_5 \neq 0$. But since we can also swap the fourth and the fifth column, we know that if $a_5b_5c_5 = 0$ we can permute this map such that we get a zero on the fourth place in the third row and a non-zero element on the fourth place in the second row. Or in other words, we can reduce this case to map 8. So we may even assume that none of the appearing variables is equal to zero.
- (a) Assuming $b_4a_5 - a_4b_5 = 0$ and $c_4a_5 - a_4c_5 = 0$ gives a rank one case, so let's assume $b_4a_5 - a_4b_5 = 0$ and $c_4a_5 - a_4c_5 \neq 0$. Then cubic similar to $J(2, 2)$.
- (b) Assume $b_4a_5 - a_4b_5 \neq 0$ and $c_4a_5 - a_4c_5 = 0$. Then cubic similar to $J(2, 2)$.
- (c) Assume $b_4a_5 - a_4b_5 \neq 0$, $c_4a_5 - a_4c_5 \neq 0$ and $b_4c_5 - c_4b_5 = 0$. Then cubic similar to $J(2, 2)$.
- (d) Finally assume $b_4a_5 - a_4b_5 \neq 0$, $c_4a_5 - a_4c_5 \neq 0$ and $b_4c_5 - c_4b_5 \neq 0$. Then cubic similar to $N(2, 2a)$.
11. We have $a_2 \neq 0$ and $c_4 \neq 0$. Basically the same as map 9: permute second and third rows and columns and substitute $a_2 = a_3$, $c_4 = b_4$ and $c_5 = b_5$. Hence also cubic similar to $J(2, 2)$, just like map 9.
12. We have $a_2 \neq 0$, $b_4 \neq 0$ and $c_4 \neq 0$. Cubic similar to $J(2, 2)$.
13. Here we know $b_3 \neq 0$.
- (a) Assume $a_3 = 0$ and $a_4 = 0$. Then $a_5 \neq 0$. Cubic similar to $J(2, 2)$.
- (b) Assume $a_3 = 0$ and $a_4 \neq 0$. Cubic similar to $J(2, 2)$.
- (c) Assume $a_3 \neq 0$, $a_3b_4 - b_3a_4 \neq 0$ and $a_3b_5 - b_3a_5 = 0$. Cubic similar to $J(2, 2)$.
- (d) Assume $a_3 \neq 0$, $a_3b_4 - b_3a_4 = 0$ and $a_3b_5 - b_3a_5 \neq 0$. Cubic similar to $J(2, 2)$.
- (e) Assuming $a_3 \neq 0$, $a_3b_4 - b_3a_4 = 0$ and $a_3b_5 - b_3a_5 = 0$ gives a rank one case, hence the only case left is $a_3 \neq 0$, $a_3b_4 - b_3a_4 \neq 0$ and $a_3b_5 - b_3a_5 \neq 0$. Cubic similar to $J(2, 2)$.

14. We have $b_3 \neq 0$ and $d_5 \neq 0$. Cubic similar to $J(2, 2)$.
15. We already know $a_3 \neq 0$, $b_3 \neq 0$ and $d_5 \neq 0$.
- (a) Assume $a_3b_5 - b_3a_5 = 0$. Cubic similar to $J(2, 2)$.
 - (b) Assume $a_3b_5 - b_3a_5 \neq 0$. Cubic similar to $N(2, 2a)$.
16. We know $a_2 \neq 0$ and $b_3 \neq 0$. So this map is cubic similar to $J(2, 3)$.
17. We know $d_5 \neq 0$. Of course at least one of a_4 , b_4 or $c_4 \neq 0$. Since the first three columns are equal to zero, we can change the order of the first three rows without disturbing the structure of the matrix. Hence we may assume that $a_4 \neq 0$.
- (a) Assume $b_4 = 0$ and $c_4 = 0$. Then cubic similar to $J(2, 3)$.
 - (b) Assume $b_4 = 0$, $c_4 \neq 0$ and $a_4c_5 - c_4a_5 = 0$. Then cubic similar to $J(2, 3)$.
 - (c) Assume $b_4 = 0$, $c_4 \neq 0$ and $a_4c_5 - c_4a_5 \neq 0$. Then cubic similar to $N(2, 3a)$.
 - (d) Assume $b_4 \neq 0$, $c_4 = 0$ and $b_4a_5 - a_4b_5 = 0$. Then cubic similar to $J(2, 3)$.
 - (e) Assume $b_4 \neq 0$, $c_4 = 0$ and $b_4a_5 - a_4b_5 \neq 0$. Then cubic similar to $N(2, 3a)$.
 - (f) Assume $b_4 \neq 0$, $c_4 \neq 0$, $b_4a_5 - a_4b_5 = 0$ and $a_4c_5 - c_4a_5 = 0$. Then cubic similar to $J(2, 3)$.
 - (g) Assume $b_4 \neq 0$, $c_4 \neq 0$, $b_4a_5 - a_4b_5 \neq 0$ and $a_4c_5 - c_4a_5 = 0$. Then cubic similar to $N(2, 3a)$.
 - (h) Assume $b_4 \neq 0$, $c_4 \neq 0$, $b_4a_5 - a_4b_5 = 0$ and $a_4c_5 - c_4a_5 \neq 0$. Then cubic similar to $N(2, 3a)$.
 - (i) Assume $b_4 \neq 0$, $c_4 \neq 0$, $b_4a_5 - a_4b_5 \neq 0$, $a_4c_5 - c_4a_5 \neq 0$ and $b_4c_5 - c_4b_5 = 0$. Then cubic similar to $N(2, 3a)$.
 - (j) Assume $b_4 \neq 0$, $c_4 \neq 0$, $b_4a_5 - a_4b_5 \neq 0$, $a_4c_5 - c_4a_5 \neq 0$ and $b_4c_5 - c_4b_5 \neq 0$. Then cubic similar to $N(2, 3b)$.
18. Here we have $a_3 \neq 0$ and $d_5 \neq 0$.
- (a) Assume $a_2b_5^3 + a_3c_5^3 + a_4d_5^3 = 0$. Cubic similar to $J(2, 2)$.
 - (b) Assume $a_2b_5^3 + a_3c_5^3 + a_4d_5^3 \neq 0$. Cubic similar to $J(2, 3)$.

Rank three

We have eleven matrices to examine.

19. Obviously we must have that either a_2 or $a_3 \neq 0$. Since swapping columns two and three also swaps rows two and three and we have no restrictions on b_4 , b_5 and c_4 , c_5 , we may assume that $a_2 \neq 0$. Furthermore it is clear that in order to have a rank three case we must have $b_4c_5 - c_4b_5 \neq 0$.

- (a) Assume $b_4 = 0$ and $a_3 = 0$. Hence $c_4 \neq 0$ and $b_5 \neq 0$. Cubic similar to $J(3, 3)$.
- (b) Assume $b_4 = 0$ and $a_3 \neq 0$. Hence $c_4 \neq 0$ and $b_5 \neq 0$. Cubic similar to $N(3, 3b)$.
- (c) Assume $b_4 \neq 0$, $c_4 = 0$ and $a_3 = 0$. Hence $c_5 \neq 0$. Cubic similar to $J(3, 3)$.
- (d) Assume $b_4 \neq 0$, $c_4 = 0$ and $a_3 \neq 0$. Hence $c_5 \neq 0$. Cubic similar to $N(3, 3b)$.
- (e) Assume $b_4 \neq 0$, $c_4 \neq 0$ and $a_3 = 0$. Cubic similar to $J(3, 3)$.
- (f) Assume $b_4 \neq 0$, $c_4 \neq 0$, $a_3 \neq 0$ and $a_4 = 0$. Cubic similar to $N(3, 3b)$.
- (g) Assume $b_4 \neq 0$, $c_4 \neq 0$, $a_3 \neq 0$, $a_4 \neq 0$ and $a_5b_4 - b_5a_4 = 0$ and hence $a_5c_4 - c_5a_4 \neq 0$ (otherwise contradiction with $b_4c_5 - c_4b_5 \neq 0$). Cubic similar to $N(3, 3b)$.
- (h) Assume $b_4 \neq 0$, $c_4 \neq 0$, $a_3 \neq 0$, $a_4 \neq 0$ and $a_5b_4 - b_5a_4 \neq 0$. Cubic similar to $N(3, 3b)$.
20. We already know $a_2 \neq 0$ and $d_5 \neq 0$. However if $c_4 = 0$ we have a rank two case. Hence we may assume $c_4 \neq 0$.
- (a) Assume $a_4d_5^3 + a_2b_5^3 = 0$. Cubic similar to $J(3, 3)$.
- (b) Assume $a_4d_5^3 + a_2b_5^3 \neq 0$ and $b_5 = 0$, hence $a_4 \neq 0$. Cubic similar to $N(3, 3a)$.
- (c) Assume $a_4d_5^3 + a_2b_5^3 \neq 0$ and $b_5 \neq 0$, hence $a_4 \neq 0$. Cubic similar to $N(3, 3a)$.
21. We know $a_3 \neq 0$, $b_4 \neq 0$ and $d_5 \neq 0$.
- (a) Assume $a_3c_5^3 + a_4d_5^3 = 0$. Then cubic similar to $J(3, 3)$.
- (b) Assume $a_3c_5^3 + a_4d_5^3 \neq 0$. Then cubic similar to $N(3, 3a)$.
22. We know $a_3 \neq 0$, $b_4 \neq 0$, $c_4 \neq 0$ and $d_5 \neq 0$.
- (a) Assume $a_4 = 0$. Then cubic similar to $J(3, 3)$.
- (b) Assume $a_4 \neq 0$. Then cubic similar to $N(3, 3a)$.
23. We have $b_3 \neq 0$. Furthermore $c_5 \neq 0$ or $d_5 \neq 0$, and $a_3 \neq 0$ or $a_4 \neq 0$. It is also obvious that $a_4b_3 - b_4a_3 \neq 0$.
- (a) Assume $a_4 = 0$, hence $b_4 \neq 0$ and $a_3 \neq 0$. Now if $c_5 \neq 0$, we can conjugate with $(x_2, x_1, x_4, x_3, x_5)$ and we are back in case 21. So we may assume $c_5 = 0$ and hence $d_5 \neq 0$. Cubic similar to $J(3, 3)$.
- (b) Assume $a_4 \neq 0$ and $a_3 = 0$. If we now assume $d_5 \neq 0$ then we can conjugate with $(x_2, x_1, x_3, x_4, x_5)$ and we are again back in case 21. So we may assume $d_5 = 0$, and hence $c_5 \neq 0$. Cubic similar to $J(3, 3)$.

- (c) Assuming $a_4 \neq 0, a_3 \neq 0, a_3c_5^3 + a_4d_5^3 = 0, b_3c_5^3 + b_4d_5^3 = 0$ and $a_3b_5 - b_3a_5 \neq 0$ gives a rank two map, so we may assume $a_4 \neq 0, a_3 \neq 0, a_3c_5^3 + a_4d_5^3 = 0, b_3c_5^3 + b_4d_5^3 \neq 0$ and $a_3b_5 - b_3a_5 = 0$. Hence $c_5 \neq 0$ and $d_5 \neq 0$. Cubic similar to $J(3, 3)$.
- (d) Assume $a_4 \neq 0, a_3 \neq 0, a_3c_5^3 + a_4d_5^3 = 0, b_3c_5^3 + b_4d_5^3 \neq 0$ and $a_3b_5 - b_3a_5 \neq 0$, hence $c_5 \neq 0$ and $d_5 \neq 0$. Cubic similar to $J(3, 3)$.
- (e) Assume $a_4 \neq 0, a_3 \neq 0, a_3c_5^3 + a_4d_5^3 \neq 0$ and $b_3c_5^3 + b_4d_5^3 = 0$, hence $d_5 \neq 0$. Cubic similar to $J(3, 3)$.
- (f) Assume $a_4 \neq 0, a_3 \neq 0, a_3c_5^3 + a_4d_5^3 \neq 0, b_3c_5^3 + b_4d_5^3 \neq 0$ and $a_3b_5 - b_3a_5 = 0$. Cubic similar to $N(3, 3a)$.
- (g) Assume $a_4 \neq 0, a_3 \neq 0, a_3c_5^3 + a_4d_5^3 \neq 0, b_3c_5^3 + b_4d_5^3 \neq 0, a_3b_5 - b_3a_5 \neq 0$ and $d_5 = 0$, hence $c_5 \neq 0$. Cubic similar to $N(3, 3a)$.
- (h) Assume $a_4 \neq 0, a_3 \neq 0, a_3c_5^3 + a_4d_5^3 \neq 0, b_3c_5^3 + b_4d_5^3 \neq 0, a_3b_5 - b_3a_5 \neq 0$ and $d_5 \neq 0$. Cubic similar to $N(3, 3a)$.
24. Here we have $a_2 \neq 0$ and $d_5 \neq 0$. But it is obvious that $b_3 = 0$ gives a rank two case, so we also have $b_3 \neq 0$.
- (a) Assume $a_3c_5^3 + a_4d_5^3 = 0$. Then cubic similar to $J(3, 3)$.
- (b) Assume $a_3c_5^3 + a_4d_5^3 \neq 0$. Then cubic similar to $N(3, 3b)$.
25. We have $b_3 \neq 0$ and $c_4 \neq 0$.
- (a) Assume $a_3 = 0$. If we assume $a_5c_4 - c_5a_4 = 0$ we get a rank two case, so we may assume $a_5c_4 - c_5a_4 \neq 0$. Assume furthermore $a_4 = 0$, hence $a_5 \neq 0$. Cubic similar to $J(3, 3)$.
- (b) Assume $a_3 = 0, a_5c_4 - c_5a_4 \neq 0$ and $a_4 \neq 0$. Cubic similar to $J(3, 3)$.
- (c) Assume $a_3 \neq 0$ and $a_3b_4 - b_3a_4 = 0$. If in addition $a_3b_5 - b_3a_5 = 0$ then we have a rank two case, so we may assume $a_3b_5 - b_3a_5 \neq 0$. Cubic similar to $N(3, 3a)$.
- (d) Assume $a_3 \neq 0$ and $a_3b_4 - b_3a_4 \neq 0$. If $c_5(a_3b_4 - b_3a_4) - c_4(a_3b_5 - b_3a_5) = 0$ we have a rank two case, so we may assume $c_5(a_3b_4 - b_3a_4) - c_4(a_3b_5 - b_3a_5) \neq 0$. Cubic similar to $N(3, 3a)$.
26. If $d_5 = 0$ we are back in case 25, hence we may assume $d_5 \neq 0$. Furthermore if $c_4 = 0$ and $b_3 \neq 0$ we are back in case 23. And if both $c_4 = 0$ and $b_3 = 0$, we must have $a_3 \neq 0$ and $b_4 \neq 0$ to remain in a rank three case. Since we already knew that $d_5 \neq 0$ we are back in case 21. Hence we may assume $c_4 \neq 0$. Furthermore, the case $a_3 \neq 0$ and $b_3 = 0$ is equivalent with $b_3 \neq 0$ and $a_3 = 0$ since we can swap the first two rows.
- (a) Assume $a_3 = 0, b_3 \neq 0, a_4 = 0$ and $b_4 = 0$. Cubic similar to $J(3, 4)$.
- (b) Assume $a_3 = 0, b_3 \neq 0, a_4 = 0$ and $b_4 \neq 0$. Cubic similar to $N(3, 4a)$.

- (c) Assume $a_3 = 0, b_3 \neq 0, a_4 \neq 0, b_4 = 0$ and $a_5c_4 - c_5a_4 = 0$. Cubic similar to $J(3,4)$.
- (d) Assume $a_3 = 0, b_3 \neq 0, a_4 \neq 0, b_4 = 0$ and $a_5c_4 - c_5a_4 \neq 0$. Cubic similar to $N(3,4f)$.
- (e) Assume $a_3 = 0, b_3 \neq 0, a_4 \neq 0$ and $b_4 \neq 0$. Cubic similar to $P(3,4g)$. However, for specific choices we have

- $N(3,4a)$ if $a_5 = \frac{a_4c_5}{c_4}$.
- $N(3,4g)$ if $a_5 = \frac{a_4c_5}{c_4} + \frac{a_4d_5\sqrt[3]{b_4}}{c_4\sqrt[3]{b_3}}$. (Since $a_4b_4d_5 \neq 0$ these two cases really exclude each other.)

- (f) Assume $a_3 \neq 0, b_3 \neq 0, b_4 = 0, a_4 = 0$ and $a_3b_5 - b_3a_5 = 0$. Cubic similar to $J(3,4)$.
- (g) Assume $a_3 \neq 0, b_3 \neq 0, b_4 = 0, a_4 = 0$ and $a_3b_5 - b_3a_5 \neq 0$. Cubic similar to $N(3,4e)$.
- (h) Assume $a_3 \neq 0, b_3 \neq 0, b_4 = 0$ and $a_4 \neq 0$. Cubic similar to $P(3,4h)$. For some specific choices we get a different matrix:

- $N(3,4b)$ if $b_5 = \frac{b_3(a_5c_4 - c_5a_4)}{a_3c_4}$.
- $N(3,4h)$ if $b_5 = \frac{b_3(a_5c_4 - c_5a_4)}{a_3c_4} - \frac{b_3d_5\sqrt[3]{a_4^4}}{c_4\sqrt[3]{a_3^4}}$. (Note that the last fraction is never zero, so these two cases really exclude each other.)

- (i) Assume $a_3 \neq 0, b_3 \neq 0$ and $b_4 \neq 0$. Cubic similar to $P(3,4a2)$. For some specific choices we get a different matrix:

- $N(3,4a)$ if $a_4 = \frac{a_3b_4}{b_3}$ and $b_5 = \frac{a_5b_3}{a_3}$.
- $N(3,4b)$ if $a_4 = 0$ and $a_5 = -\frac{a_3(b_4c_5 - c_4b_5)}{b_3c_4}$.
- $N(3,4c)$ if $a_4 = 0$ and $b_5 = -\frac{-a_3b_4d_5\sqrt[3]{b_3^3b_4} + a_5b_3^2c_4 + a_3b_3b_4c_5}{a_3b_3c_4}$.
- $P(3,4a)$ if $a_5 = -\frac{c_5(a_3b_4 - b_3a_4) - a_3b_5c_4}{b_3c_4}$ and $a_4 \notin \{0, \frac{a_3b_4}{b_3}\}$.
- $P(3,4c)$ if $a_4 = 0$ and $b_5 \neq -\frac{-a_3b_4d_5\sqrt[3]{b_3^3b_4} + a_5b_3^2c_4 + a_3b_3b_4c_5}{a_3b_3c_4}$.

27. Now we have $a_2 \neq 0$. If $d_5 = 0$ we are in case 19, so $d_5 \neq 0$. Furthermore if $b_4 = 0$ and $c_4 = 0$, we have a rank two case. So at least one of them should be unequal to zero. Note also that if $b_4 = 0$ and $a_3 = 0$ we are back in case 20. So if $b_4 = 0$ we may assume $a_3 \neq 0$.

- (a) Assume $b_4 = 0$ and $a_2b_5^3 + a_4d_5^3 = 0$, hence $c_4 \neq 0$ and $a_3 \neq 0$. Then cubic similar to $J(3, 4)$.
- (b) Assume $c_4 = 0$ and $a_3c_5^3 + a_4d_5^3 = 0$, hence $b_4 \neq 0$. Then cubic similar to $J(3, 4)$.
- (c) Assume $b_4 = 0$ and $a_2b_5^3 + a_4d_5^3 \neq 0$, hence $c_4 \neq 0$ and $a_3 \neq 0$. Then cubic similar to $N(3, 4a)$.
- (d) Assume $c_4 = 0$ and $a_3c_5^3 + a_4d_5^3 \neq 0$, hence $b_4 \neq 0$. Then cubic similar to $N(3, 4a)$.
- (e) Assume $b_4 \neq 0$, $c_4 \neq 0$, $a_4 = 0$, $b_4c_5 - c_4b_5 = 0$ and $a_2b_4^3 + a_3c_4^3 = 0$. Then cubic similar to $J(3, 3)$.
- (f) Assume $b_4 \neq 0$, $c_4 \neq 0$, $a_4 = 0$, $b_4c_5 - c_4b_5 = 0$, $a_2b_4^3 + a_3c_4^3 \neq 0$ and $a_3 = 0$. Then cubic similar to $J(3, 4)$.
- (g) Assume $b_4 \neq 0$, $c_4 \neq 0$, $a_4 = 0$, $b_4c_5 - c_4b_5 = 0$, $a_2b_4^3 + a_3c_4^3 \neq 0$ and $a_3 \neq 0$. Then cubic similar to $J(3, 4)$.
- (h) Assume $b_4 \neq 0$, $c_4 \neq 0$, $a_4 = 0$ and $b_4c_5 - c_4b_5 \neq 0$. Then cubic similar to $P(3, 4i)$.
- $N(3, 4f)$ if $a_3 = 0$.
 - $N(3, 4i)$ if $a_3 = -\frac{a_2b_4^3}{c_4^3}$.
- (i) Assume $b_4 \neq 0$, $c_4 \neq 0$ and $a_4 \neq 0$. Then cubic similar to $P(3, 4j2)$.
- $N(3, 3a)$ if $c_5 = \frac{b_5c_4}{b_4}$ and $a_3 = -\frac{a_2b_4^3}{c_4^3}$.
 - $N(3, 4a)$ if $c_5 = \frac{b_5c_4}{b_4}$ and $a_3 \neq -\frac{a_2b_4^3}{c_4^3}$.
 - $P(3, 4j)$ if $c_5 \neq \frac{b_5c_4}{b_4}$ and $a_3 = 0$.
 - $N(3, 4j)$ if $c_5 = \frac{b_5c_4}{b_4} - \frac{c_4d_5^3\sqrt[3]{a_4}}{b_4^3\sqrt[3]{a_2}}$ and $a_3 = 0$.

28. We have $a_2 \neq 0$ and $b_3 \neq 0$. Note that if $c_5 = 0$ and $d_5 = 0$, we are back in case 16.

- (a) Assume $d_5 = 0$ and $a_3 = 0$. Hence $c_5 \neq 0$. Cubic similar to $J(3, 4)$.
- (b) Assume $d_5 = 0$ and $a_3 \neq 0$. Hence $c_5 \neq 0$. Cubic similar to $N(3, 4a)$.
- (c) Assume $d_5 \neq 0$, $a_3c_5^3 + a_4d_5^3 = 0$ and $b_3c_5^3 + b_4d_5^3 = 0$. Cubic similar to $J(3, 3)$.
- (d) Assume $d_5 \neq 0$, $a_3c_5^3 + a_4d_5^3 = 0$ and $b_3c_5^3 + b_4d_5^3 \neq 0$. Cubic similar to $J(3, 4)$.
- (e) Assume $d_5 \neq 0$, $a_3c_5^3 + a_4d_5^3 \neq 0$ and $b_3c_5^3 + b_4d_5^3 = 0$. Cubic similar to $N(3, 3b)$.

- (f) Assume $d_5 \neq 0$, $a_3c_5^3 + a_4d_5^3 \neq 0$ and $b_3c_5^3 + b_4d_5^3 \neq 0$. Cubic similar to $N(3, 4a)$.

29. We have $a_2 \neq 0$, $b_3 \neq 0$ and $c_4 \neq 0$.

- (a) Assume $a_3 = 0$. Cubic similar to $J(3, 4)$.
 (b) Assume $a_3 \neq 0$. Cubic similar to $N(3, 4a)$.

Rank four

In the rank four case we have only one matrix to examine.

30. In this case we know $a_2 \neq 0$, $b_3 \neq 0$, $c_4 \neq 0$ and $d_5 \neq 0$.

- (a) Assume $a_3 = 0$, $a_4 = 0$ and $b_4 = 0$. Cubic similar to $J(4, 5)$.
 (b) Assume $a_3 = 0$, $a_4 = 0$ and $b_4 \neq 0$. Cubic similar to $N(4, 5b)$.
 (c) Assume $a_3 = 0$ and $a_4 \neq 0$. Cubic similar to $P(4, 5e)$.
 • $N(4, 5d)$ if $b_4 = 0$.
 • $N(4, 5e)$ if $b_4 = \frac{\sqrt[4]{a_4b_3c_4^3}}{\sqrt[4]{a_2}}$.
 (d) Assume $a_3 \neq 0$. Cubic similar to $P(4, 5c2)$.
 • $N(4, 5a)$ if $a_4 = 0$ and $b_4 = 0$.
 • $N(4, 5c)$ if $a_4 = 0$ and $b_4 = b_3c_4^3d_5^6$.
 • $P(4, 5c)$ if $a_4 = 0$ and $b_4 \notin \{0, b_3c_4^3d_5^6\}$.

Remark A.1

In the description given above it sometimes happens that we start with a matrix A where $J((AX)^{*3})$ has a certain nilpotence index, but after applying some assumptions it has a smaller nilpotence index. (See for instance the cases 18a, 27e, 28c and 28e.) In fact we could have deleted these cases from the list because it must be equivalent to one of the other cases done before, because this nilpotence index is also an invariant of the cubic similarity relation, but because this way it is easier to verify that we really have a *complete* description of all cases, we left them in.

A.2.2 Transformations

In this section we present the actual transformations used in the cases of section 8.4. Since this is the only 'proof' we can give, we used the Maple to \LaTeX feature from version 5.3. Unfortunately this has as disadvantage that the transformation mappings are not always in their nicest form. Furthermore, Maple uses %n in the expressions as an abbreviation. The actual value of %n is given directly below the map.

	T	$T^{-1}FT$
Rank one		
1	$\left(x_1, x_5 + \frac{b_5^3 x_1}{a_5^3}, x_3 + \frac{c_5^3 x_1}{a_5^3}, x_4 + \frac{d_5^3 x_1}{a_5^3}, \frac{x_2}{a_5} \right)$	$J(1, 2)$
2	$\left(x_1, \frac{x_2}{a_2} - \frac{a_5 x_5}{a_2} - \frac{a_4 x_4}{a_2} - \frac{a_3 x_3}{a_2}, x_3, x_4, x_5 \right)$	$J(1, 2)$
Rank two		
3	$\left(a_4^3 x_1, x_5 + b_5^3 x_3, c_5^3 x_3, x_2 - \frac{a_5 x_4}{a_4}, x_4 \right)$	$J(2, 2)$
4	$\left(a_3^3 x_1, b_5^3 x_3, x_2 - \frac{a_4 x_5}{a_3} - \frac{a_5 x_4}{a_3}, x_5, x_4 \right)$	$J(2, 2)$
5	$\left(a_3^3 x_1, x_5 + b_5^3 x_3, x_2 + c_5^3 x_3 - \frac{a_5 x_4}{a_3}, d_5^3 x_3, x_4 \right)$	$J(2, 2)$
6	$\left(a_2^3 x_1, x_2 + b_5^3 x_3 - \frac{a_4 x_5}{a_2} - \frac{a_5 x_4}{a_2}, c_5^3 x_3, x_5, x_4 \right)$	$J(2, 2)$
7	$\left(a_2^3 x_1, x_2 - \frac{a_3 x_5}{a_2} + b_5^3 x_3 - \frac{a_5 x_4}{a_2}, x_5 + c_5^3 x_3, d_5^3 x_3, x_4 \right)$	$J(2, 2)$
8a	$\left(x_5 + a_5^3 x_3, b_4^3 x_1, c_5^3 x_3, x_2 - \frac{b_5 x_4}{b_4}, x_4 \right)$	$J(2, 2)$
8b	$\left(-\frac{(b_5 a_4 - b_4 a_5)^3 x_1}{b_4^3}, -\frac{(b_5 a_4 - b_4 a_5)^3 x_3}{a_4^3}, x_5, \right.$ $\left. \frac{a_5 x_4}{a_4} - \frac{x_2 b_5}{b_4}, x_2 - x_4 \right)$	$J(2, 2)$
8c	$\left(x_5 + a_4^3 x_3, b_4^3 x_3, x_1, x_4 - \frac{b_5 x_2}{b_4 c_5}, \frac{x_2}{c_5} \right)$	$J(2, 2)$
8d	$\left(\frac{(b_5 a_4 - b_4 a_5)^3 x_1}{b_4^3}, \frac{(b_5 a_4 - b_4 a_5)^3 x_2}{a_4^3}, c_5^3 x_3, \right.$ $\left. \frac{(b_5 a_4 - b_4 a_5) x_4}{b_4 a_4} - \frac{a_5 x_5}{a_4}, x_5 \right)$	$N(2, 2a)$
9	$\left(a_3^3 x_1, b_4^3 x_3, x_2 - \frac{a_4 x_4}{a_3} + \frac{(b_5 a_4 - b_4 a_5) x_5}{b_4 a_3}, x_4 - \frac{b_5 x_5}{b_4}, x_5 \right)$	$J(2, 2)$
10a	$\left(-\frac{(-c_4 a_5 + c_5 a_4)^3 x_3}{c_4^3}, x_5 - \frac{b_5^3 (-c_4 a_5 + c_5 a_4)^3 x_3}{c_4^3 a_5^3}, \right.$ $\left. -\frac{(-c_4 a_5 + c_5 a_4)^3 x_1}{a_5^3}, x_2 - \frac{c_5 x_4}{c_4}, x_4 - \frac{a_4 x_2}{a_5} \right)$	$J(2, 2)$
10b	$\left(a_4^3 x_5 + a_4^3 x_1, b_4^3 x_3, \frac{c_5^3 a_4^3 x_1}{a_5^3}, \frac{b_5 a_4 x_2}{\%1} - \frac{a_5 b_4 x_4}{\%1}, \right.$ $\left. -\frac{b_4 a_4 x_2}{\%1} + \frac{b_4 a_4 x_4}{\%1} \right)$	$J(2, 2)$
	$\%1 := b_5 a_4 - b_4 a_5$	

	T	$T^{-1}FT$
10c	$\left(\frac{(-c_4 a_5 + c_5 a_4)^3 x_1}{c_5^3}, \frac{b_5^3 x_3}{c_5^3} + x_5, x_3, x_2 - \frac{a_5 x_4}{-c_4 a_5 + c_5 a_4}, \right.$ $\left. - \frac{c_4 x_2}{c_5} + \frac{a_4 x_4}{-c_4 a_5 + c_5 a_4} \right)$	$J(2, 2)$
10d	$\left(\frac{\%1^3 (-c_4 a_5 + c_5 a_4)^3 x_1}{a_5^3 (-b_4 c_5 + b_5 c_4)^3}, -\frac{\%1^3 x_2}{a_5^3}, -\frac{(-c_4 a_5 + c_5 a_4)^3 x_3}{a_5^3}, \right.$ $\left. \frac{\%1 c_5 x_4}{a_5 (-b_4 c_5 + b_5 c_4)} + x_5, -\frac{c_4 \%1 x_4}{a_5 (-b_4 c_5 + b_5 c_4)} - \frac{a_4 x_5}{a_5} \right)$	$N(2, 2a)$
	$\%1 := b_5 a_4 - b_4 a_5$	
11	$\left(a_2^3 x_1, x_2 - \frac{a_4 x_4}{a_2} + \frac{(-c_4 a_5 + c_5 a_4) x_5}{c_4 a_2}, c_4^3 x_3, x_4 - \frac{c_5 x_5}{c_4}, x_5 \right)$	$J(2, 2)$
12	$\left(a_2^3 x_1, x_2 + b_4^3 x_3 - \frac{a_4 x_4}{a_2} + \frac{(-c_4 a_5 + c_5 a_4) x_5}{c_4 a_2}, \right.$ $\left. c_4^3 x_3, x_4 - \frac{c_5 x_5}{c_4}, x_5 \right)$	$J(2, 2)$
13a	$\left(a_5^3 x_1, b_3^3 x_3, x_4 - \frac{b_4 x_5}{b_3} - \frac{b_5 x_2}{b_3}, x_5, x_2 \right)$	$J(2, 2)$
13b	$\left(a_4^3 x_1, b_3^3 x_3, x_4 - \frac{x_2 b_4}{b_3} - \frac{(b_5 a_4 - b_4 a_5) x_5}{a_4 b_3}, x_2 - \frac{a_5 x_5}{a_4}, x_5 \right)$	$J(2, 2)$
13c	$\left(a_3^3 x_1, b_3^3 x_3, -\frac{x_2 b_4 a_3}{\%1} + \frac{a_4 b_3 x_4}{\%1} - \frac{a_5 x_5}{a_3}, \right.$ $\left. -\frac{a_3 b_3 x_4}{\%1} + \frac{a_3 b_3 x_2}{\%1}, x_5 \right)$	$J(2, 2)$
	$\%1 := -b_4 a_3 + a_4 b_3$	
13d	$\left(a_3^3 x_1, b_3^3 x_3, \frac{x_2 b_5 a_3}{\%1} - \frac{a_5 b_3 x_4}{\%1} - \frac{a_4 x_5}{a_3}, x_5, -\frac{a_3 b_3 x_2}{\%1} + \right.$ $\left. \frac{a_3 b_3 x_4}{\%1} \right)$	$J(2, 2)$
	$\%1 := b_5 a_3 - a_5 b_3$	
13e	$\left(a_3^3 x_1, b_3^3 x_3, \frac{x_2 b_5 a_3}{\%1} - \frac{a_5 b_3 x_4}{\%1} - \frac{(b_5 a_4 - b_4 a_5) x_5}{\%1}, x_5, \right.$ $\left. -\frac{a_3 b_3 x_2}{\%1} + \frac{a_3 b_3 x_4}{\%1} + \frac{(-b_4 a_3 + a_4 b_3) x_5}{\%1} \right)$	$J(2, 2)$
	$\%1 := b_5 a_3 - a_5 b_3$	
14	$\left(x_5 + a_5^3 x_1, b_3^3 x_3, x_4 + c_5^3 x_1 - \frac{b_5 x_2}{b_3}, d_5^3 x_1, x_2 \right)$	$J(2, 2)$
15a	$\left(x_5 + a_3^3 x_1, b_3^3 x_1, x_2 + c_5^3 x_3 - \frac{a_5 x_4}{a_3}, d_5^3 x_3, x_4 \right)$	$J(2, 2)$
15b	$\left(a_3^3 x_2, b_3^3 x_1, x_4 - \frac{a_3^3 b_3^3 c_5^3 x_3}{\%1^3} + \frac{b_5 a_3 x_5}{\%1}, \right.$ $\left. -\frac{d_5^3 a_3^3 b_3^3 x_3}{\%1^3}, -\frac{a_3 b_3 x_5}{\%1} \right)$	$N(2, 2a)$
	$\%1 := b_5 a_3 - a_5 b_3$	

	T	$T^{-1}FT$
16	$\left(a_2^3 b_3^9 x_1, b_3^3 x_2 - \frac{a_3 x_3}{a_2} - \frac{(-b_4 a_3 + a_4 b_3) x_4}{b_3 a_2} + \frac{(b_5 a_3 - a_5 b_3) x_5}{b_3 a_2}, \right.$ $\left. x_3 - \frac{b_4 x_4}{b_3} - \frac{b_5 x_5}{b_3}, x_4, x_5 \right)$	$J(2, 3)$
17a	$\left(a_4^3 d_5^9 x_1, x_4 + b_5^3 x_2, x_5 + c_5^3 x_2, d_5^3 x_2 - \frac{a_5 x_3}{a_4}, x_3 \right)$	$J(2, 3)$
17b	$\left(a_4^3 d_5^9 x_1, x_4 + b_5^3 x_2, x_5 + d_5^9 c_4^3 x_1, d_5^3 x_2 - \frac{a_5 x_3}{a_4}, x_3 \right)$	$J(2, 3)$
17c	$\left(\frac{\%1^4 \sqrt{\%2} x_1}{c_4^4 d_5^4 a_4}, \frac{\%1 b_5^3 \sqrt{\%2} x_3}{c_4 d_5^4 a_4} + x_5, \frac{\%1^4 \sqrt{\%2} x_2}{c_4 d_5^4 a_4^4}, \right.$ $\left. \frac{\%1 \sqrt{\%2} x_3}{c_4 a_4 d_5} - \frac{a_5 \sqrt{\%2} x_4}{d_5 a_4}, \frac{\sqrt{\%2} x_4}{d_5} \right)$	$N(2, 3a)$
	$\%1 := -c_4 a_5 + c_5 a_4$ $\%2 := \frac{\%1}{c_4 d_5 a_4}$	
17d	$\left(a_4^3 d_5^9 x_1, x_4 + b_4^3 d_5^9 x_1, x_5 + c_5^3 x_2, d_5^3 x_2 - \frac{a_5 x_3}{a_4}, x_3 \right)$	$J(2, 3)$
17e	$\left(\frac{\%1^4 \sqrt{\%2} x_1}{b_4^4 d_5^4 a_4}, \frac{\%1^4 \sqrt{\%2} x_2}{b_4 d_5^4 a_4^4}, \frac{\%1 c_5^3 \sqrt{\%2} x_3}{b_4 d_5^4 a_4} + x_5, \right.$ $\left. \frac{\%1 \sqrt{\%2} x_3}{b_4 a_4 d_5} - \frac{a_5 \sqrt{\%2} x_4}{d_5 a_4}, \frac{\sqrt{\%2} x_4}{d_5} \right)$	$N(2, 3a)$
	$\%1 := b_5 a_4 - b_4 a_5$ $\%2 := \frac{\%1}{b_4 d_5 a_4}$	
17f	$\left(a_4^3 d_5^9 x_1, x_4 + b_4^3 d_5^9 x_1, x_5 + d_5^9 c_4^3 x_1, d_5^3 x_2 - \frac{a_5 x_3}{a_4}, x_3 \right)$	$J(2, 3)$
17g	$\left(\frac{\%1^4 \sqrt{\%2} x_1}{b_4^4 d_5^4 a_4}, \frac{\%1^4 \sqrt{\%2} x_2}{b_4 d_5^4 a_4^4}, \frac{\%1^4 c_4^3 \sqrt{\%2} x_1}{b_4^4 d_5^4 a_4^4} + x_5, \right.$ $\left. \frac{\%1 \sqrt{\%2} x_3}{b_4 a_4 d_5} - \frac{a_5 \sqrt{\%2} x_4}{d_5 a_4}, \frac{\sqrt{\%2} x_4}{d_5} \right)$	$N(2, 3a)$
	$\%1 := b_5 a_4 - b_4 a_5$ $\%2 := \frac{\%1}{b_4 d_5 a_4}$	
17h	$\left(\frac{\%1^4 \sqrt{\%2} x_1}{c_4^4 d_5^4 a_4}, \frac{\%1^4 b_4^3 \sqrt{\%2} x_1}{c_4^4 d_5^4 a_4^4} + x_5, \frac{\%1^4 \sqrt{\%2} x_2}{c_4 d_5^4 a_4^4}, \right.$ $\left. \frac{\%1 \sqrt{\%2} x_3}{c_4 a_4 d_5} - \frac{a_5 \sqrt{\%2} x_4}{d_5 a_4}, \frac{\sqrt{\%2} x_4}{d_5} \right)$	$N(2, 3a)$
	$\%1 := -c_4 a_5 + c_5 a_4$ $\%2 := \frac{\%1}{c_4 d_5 a_4}$	

	T	$T^{-1}FT$
17i	$\left(\frac{\%1^4 \sqrt{\%2} x_1}{b_4^4 d_5^4 a_4}, \frac{\%1^4 \sqrt{\%2} x_2}{b_4 d_5^4 a_4^4} + x_5, \frac{\%1^4 c_4^3 \sqrt{\%2} x_2}{b_4^4 d_5^4 a_4^4}, \right.$ $\left. \frac{\%1 \sqrt{\%2} x_3}{b_4 a_4 d_5} - \frac{a_5 \sqrt{\%2} x_4}{d_5 a_4}, \frac{\sqrt{\%2} x_4}{d_5} \right)$ <p> $\%1 := b_5 a_4 - b_4 a_5$ $\%2 := \frac{\%1}{b_4 d_5 a_4}$ </p>	$N(2, 3a)$
17j	$\left(\frac{\%1^4 \sqrt{\%2} x_1}{b_4^4 d_5^4 a_4}, \frac{\%1^4 \sqrt{\%2} x_2}{b_4 d_5^4 a_4^4}, \frac{\sqrt{\%2} c_4 (-b_4 c_5 + b_5 c_4) \%1^2 \%3 x_1}{b_4^4 d_5^4 a_4^3} \right.$ $+ \frac{1}{2} \frac{\sqrt{\%2} c_4 (-c_4 a_5 + c_5 a_4) \%1^2 \%3 x_2}{b_4^3 d_5^4 a_4^4}$ $- \frac{1}{2} \frac{\sqrt{\%2} c_4 (-c_4 a_5 + c_5 a_4) (-b_4 c_5 + b_5 c_4) \%1^2 x_3}{d_5^4 a_4^3 b_4^3}$ $- \frac{\sqrt{\%2} (-c_4 a_5 + c_5 a_4) (-b_4 c_5 + b_5 c_4) \%1 \%3 x_4}{d_5^4 a_4^3 b_4^3},$ $\left. \frac{\%1 \sqrt{\%2} x_4}{b_4 a_4 d_5} - \frac{a_5 \sqrt{\%2} x_5}{d_5 a_4}, \frac{\sqrt{\%2} x_5}{d_5} \right)$ <p> $\%1 := b_5 a_4 - b_4 a_5$ $\%2 := \frac{\%1}{b_4 d_5 a_4}$ $\%3 := b_5 c_4 a_4 + a_4 b_4 c_5 - 2 c_4 a_5 b_4$ </p>	$N(2, 3b)$
18a	$\left(a_3^3 x_1, x_5 + b_5^3 x_3, x_2 + c_5^3 x_3 - \frac{a_5 x_4}{a_3} - \frac{a_2 x_5}{a_3}, d_5^3 x_3, x_4 \right)$	$J(2, 2)$
18b	$\left((a_2 b_5^3 + c_5^3 a_3 + a_4 d_5^3)^3 x_1, b_5^3 x_2 + x_4 - b_5^3 x_5, \right.$ $\left. c_5^3 x_2 - \frac{a_5 x_3}{a_3} - \frac{a_2 x_4}{a_3} + \frac{(a_2 b_5^3 + a_4 d_5^3) x_5}{a_3}, d_5^3 x_2 - d_5^3 x_5, x_3 \right)$	$J(2, 3)$
Rank three		
19a	$\left(a_2^3 b_5^9 x_1, b_5^3 x_2 + \frac{(-c_4 a_5 + c_5 a_4) x_3}{a_2 c_4} - \frac{a_4 x_5}{a_2}, \right.$ $\left. c_4^3 x_4, -\frac{c_5 x_3}{c_4} + x_5, x_3 \right)$	$J(3, 3)$
19b	$\left(a_2^3 b_5^9 x_1, b_5^3 x_2 + \frac{(-c_4 a_5 + c_5 a_4) x_4}{a_2 c_4} - \frac{a_4 \left(\frac{a_2 b_5^3}{a_3} \right)^{1/3} x_5}{a_2 c_4}, \right.$ $\left. \frac{a_2 b_5^3 x_3}{a_3}, -\frac{c_5 x_4}{c_4} + \frac{\left(\frac{a_2 b_5^3}{a_3} \right)^{1/3} x_5}{c_4}, x_4 \right)$	$N(3, 3b)$

	T	$T^{-1}FT$
19c	$\left(a_2^3 b_4^9 x_1, b_4^3 x_2 - \frac{a_4 x_3}{a_2} - \frac{(-b_5 a_4 + b_4 a_5) x_5}{a_2 b_4}, c_5^3 x_4, \right.$ $\left. x_3 - \frac{b_5 x_5}{b_4}, x_5 \right)$	$J(3, 3)$
19d	$\left(c_5^9 a_3^3 x_1, \frac{c_5^3 a_3 x_2}{a_2} - \frac{a_4 \left(\frac{c_5^3 a_3}{a_2} \right)^{1/3} x_4}{a_2 b_4} - \frac{(-b_5 a_4 + b_4 a_5) x_5}{a_2 b_4}, \right.$ $\left. c_5^3 x_3, \left(\frac{c_5^3 a_3}{a_2} \right)^{1/3} x_4 - \frac{b_5 x_5}{b_4}, x_5 \right)$	$N(3, 3b)$
19e	$\left(\frac{a_2^3 (b_4 c_5 - b_5 c_4)^9 x_1}{c_4^9}, \frac{(b_4 c_5 - b_5 c_4)^3 x_2}{c_4^3} - \frac{(-c_4 a_5 + c_5 a_4) x_3}{a_2 c_4} - \right.$ $\left. \frac{(-b_5 a_4 + b_4 a_5) x_5}{a_2 b_4}, \frac{(b_4 c_5 - b_5 c_4)^3 x_4}{b_4^3}, \frac{c_5 x_3}{c_4} - \frac{b_5 x_5}{b_4}, -x_3 + x_5 \right)$	$J(3, 3)$
19f	$\left(\frac{a_2^3 \%1^9 x_1}{c_4^9}, \frac{\%1^3 x_2}{c_4^3} + \frac{a_5 x_4}{a_2} - \frac{a_5 \left(\frac{a_2 \%1^3}{a_3 c_4^3} \right)^{1/3} b_4 x_5}{\%1 a_2}, \frac{a_2 \%1^3 x_3}{a_3 c_4^3}, \right.$ $\left. \frac{c_5 x_4}{c_4} - \frac{b_5 \left(\frac{a_2 \%1^3}{a_3 c_4^3} \right)^{1/3} x_5}{\%1}, -x_4 + \frac{\left(\frac{a_2 \%1^3}{a_3 c_4^3} \right)^{1/3} b_4 x_5}{\%1} \right)$	$N(3, 3b)$
	$\%1 := b_4 c_5 - b_5 c_4$	
19g	$\left(\frac{a_2^3 b_4^9 \%1^9 x_1}{a_4^9 c_4^9}, \frac{b_4^3 \%1^3 x_2}{a_4^3 c_4^3} - \frac{\%1 x_4}{a_2 c_4}, \frac{\%1^3 a_2 b_4^3 x_3}{a_3 a_4^3 c_4^3}, \frac{c_5 x_4}{c_4} - \right.$ $\left. \frac{a_5 \left(\frac{\%1^3 a_2 b_4^3}{a_3 a_4^3 c_4^3} \right)^{1/3} x_5}{\%1}, -x_4 + \frac{\left(\frac{\%1^3 a_2 b_4^3}{a_3 a_4^3 c_4^3} \right)^{1/3} a_4 x_5}{\%1} \right)$	$N(3, 3b)$
	$\%1 := -c_4 a_5 + c_5 a_4$	

	T	$T^{-1}FT$
19h	$\left(\frac{a_3^3 \%1^9 x_1}{b_4^9}, \frac{\%1^3 a_3 x_2}{a_2 b_4^3} - \frac{(-c_4 a_5 + c_5 a_4) \left(\frac{\%1^3 a_3}{a_2 b_4^3} \right)^{1/3} x_4}{a_2 \%1} - \right.$ $\left. \frac{(-b_5 a_4 + b_4 a_5) x_5}{a_2 b_4}, \frac{\%1^3 x_3}{b_4^3}, \frac{c_5 \left(\frac{\%1^3 a_3}{a_2 b_4^3} \right)^{1/3} x_4}{\%1} - \right.$ $\left. \frac{b_5 x_5}{b_4}, - \frac{\left(\frac{\%1^3 a_3}{a_2 b_4^3} \right)^{1/3} c_4 x_4}{\%1} + x_5 \right)$	$N(3, 3b)$
	$\%1 := b_4 c_5 - b_5 c_4$	
20a	$\left(a_2^3 x_4, b_5^3 x_2 - \frac{(a_2 c_5 b_5^3 + a_5 d_5^3 c_4) x_3}{a_2 d_5^3 c_4} + x_5, \right.$ $\left. d_5^9 c_4^3 x_1, d_5^3 x_2 - \frac{c_5 x_3}{c_4}, x_3 \right)$	$J(3, 3)$
20b	$\left(a_2^3 x_2, \frac{(a_2 d_5 c_4 a_4 - a_2^{1/3} a_4^{2/3} c_4 a_5 + a_2^{1/3} a_4^{5/3} c_5) x_4}{a_2 d_5 c_4 a_4} + \right.$ $\left. \frac{(-c_4 a_5 + c_5 a_4) x_5}{a_2^{2/3} d_5 c_4 a_4^{1/3}}, \frac{c_4^3 a_2^3 x_1}{a_4^3}, \frac{a_2 x_3}{a_4} - \frac{a_2^{1/3} c_5 x_4}{c_4 a_4^{1/3} d_5} - \right.$ $\left. \frac{a_2^{1/3} c_5 x_5}{c_4 a_4^{1/3} d_5}, \frac{a_2^{1/3} x_4}{a_4^{1/3} d_5} + \frac{a_2^{1/3} x_5}{a_4^{1/3} d_5} \right)$	$N(3, 3a)$
20c	$\left((a_2 b_5^3 + a_4 d_5^3)^3 x_2, b_5^3 x_3 + \right.$ $\left. \frac{(c_5 a_4 - c_4 a_5 + a_2 b_5^3 c_4 + a_4 d_5^3 c_4) x_4}{a_2 c_4} + \right.$ $\left. \frac{(-c_4 a_5 + c_5 a_4) x_5}{c_4 a_2}, d_5^9 c_4^3 x_1, d_5^3 x_3 - \frac{c_5 x_4}{c_4} - \frac{c_5 x_5}{c_4}, x_4 + x_5 \right)$	$N(3, 3a)$
21a	$\left(a_3^3 x_4, b_4^3 d_5^9 x_1, c_5^3 x_2 - \frac{(b_5 c_5^3 a_3 + b_4 a_5 d_5^3) x_3}{a_3 d_5^3 b_4} + x_5, \right.$ $\left. d_5^3 x_2 - \frac{b_5 x_3}{b_4}, x_3 \right)$	$J(3, 3)$
21b	$\left((c_5^3 a_3 + a_4 d_5^3)^3 x_1 + x_5, b_4^3 d_5^9 x_2, \right.$ $c_5^3 x_3 - \frac{(b_4 a_5 - b_5 a_4 + a_4 b_4 d_5^3) x_4}{b_4 a_3} - \frac{(-b_5 a_4 + b_4 a_5) x_5}{b_4 a_3},$ $\left. d_5^3 x_3 + \frac{(-b_5 + b_4 d_5^3) x_4}{b_4} - \frac{b_5 x_5}{b_4}, x_4 + x_5 \right)$	$N(3, 3a)$
22a	$\left(a_3^3 x_4, b_4^3 d_5^9 x_1, d_5^9 c_4^3 x_1 - \frac{a_5 x_3}{a_3} + x_5, d_5^3 x_2 - \frac{c_5 x_3}{c_4}, x_3 \right)$	$J(3, 3)$

	T	$T^{-1}FT$
22b	$\left(a_4^3 d_5^9 x_2, b_4^3 d_5^9 x_1, d_5^9 c_4^3 x_1 + \frac{(a_4 d_5^3 c_4 - c_4 a_5 + c_5 a_4) x_4}{a_3 c_4} - \right.$	$N(3, 3a)$
23a	$\left. \frac{(c_4 a_5 - c_5 a_4) x_5}{a_3 c_4}, d_5^3 x_3 - \frac{c_5 x_4}{c_4} - \frac{c_5 x_5}{c_4}, x_4 + x_5 \right)$ $\left(\frac{a_3^3 b_4^3 x_4}{b_3^3}, b_4^3 x_1, -\frac{a_5 x_3}{a_3 d_5} + \frac{b_4 x_5}{b_3}, \right.$ $\left. x_2 - \frac{(b_5 a_3 - a_5 b_3) x_3}{a_3 b_4 d_5} - x_5, \frac{x_3}{d_5} \right)$	$J(3, 3)$
23b	$\left(a_4^3 x_4, b_3^3 c_5^9 x_1, c_5^3 x_2 + \frac{(-b_5 a_4 + b_4 a_5) x_3}{a_4 b_3} - \frac{b_4 x_5}{b_3}, \right.$ $\left. -\frac{a_5 x_3}{a_4} + x_5, x_3 \right)$	$J(3, 3)$
23c	$\left(-\frac{a_3^3 (b_3 c_5^3 + b_4 d_5^3)^3 x_4}{d_5^9 b_3^3}, (b_3 c_5^3 + b_4 d_5^3)^3 x_1, \right.$ $\left. c_5^3 x_2 - \frac{a_5 x_3}{a_3} - \frac{b_4 x_5}{b_3}, d_5^3 x_2 + x_5, x_3 \right)$	$J(3, 3)$
23d	$\left(-\frac{a_3^3 \%2^3 x_4}{d_5^9 b_3^3}, \frac{a_3^9 \%2^{12} x_1}{\%1^9 d_5^{27}}, \frac{c_5^3 a_3^3 \%2^3 x_2}{d_5^9 \%1^3} - \right.$ $\left. \frac{(b_5 c_5^3 a_3 + b_4 a_5 d_5^3) x_3}{d_5^3 \%1} - \frac{b_4 x_5}{b_3}, \frac{a_3^3 \%2^3 x_2}{\%1^3 d_5^6} - x_3 + x_5, \frac{a_3 \%2 x_3}{d_5^3 \%1} \right)$ <p> $\%1 := b_5 a_3 - a_5 b_3$ $\%2 := b_3 c_5^3 + b_4 d_5^3$ </p>	$J(3, 3)$
23e	$\left(\%1^3 x_1, \frac{b_3^3 \%1^3 x_4}{d_5^9 a_3^3}, c_5^3 x_2 - \frac{(a_5 b_3 c_5^3 + a_4 d_5^3 b_5) x_3}{b_3 \%1} + \right.$ $\left. \frac{a_4 x_5}{a_3}, d_5^3 x_2 + \frac{d_5^3 (b_5 a_3 - a_5 b_3) x_3}{b_3 \%1} - x_5, x_3 \right)$ <p> $\%1 := c_5^3 a_3 + a_4 d_5^3$ </p>	$J(3, 3)$
23f	$\left((c_5^3 a_3 + a_4 d_5^3)^3 x_1, (b_3 c_5^3 + b_4 d_5^3)^3 x_2, c_5^3 x_3 - \right.$ $\left. \frac{(a_4 a_3 b_3 c_5^3 + a_4 a_3 b_4 d_5^3 + a_5 b_4 a_3 - a_5 a_4 b_3) x_4}{(b_4 a_3 - a_4 b_3) a_3} - \right.$ $\left. \frac{a_5 x_5}{a_3}, d_5^3 x_3 + \frac{a_3 (b_3 c_5^3 + b_4 d_5^3) x_4}{b_4 a_3 - a_4 b_3}, x_4 + x_5 \right)$	$N(3, 3a)$
23g	$\left(c_5^9 a_3^3 x_1, b_3^3 c_5^9 x_2, c_5^3 x_3 - \frac{(b_4 a_5 - b_5 a_4 + a_4 b_3 c_5^3) x_4}{\%1} - \right.$ $\left. \frac{(-b_5 a_4 + b_4 a_5) x_5}{\%1}, \frac{(-b_5 a_3 + a_5 b_3 + c_5^3 a_3 b_3) x_4}{\%1} - \right.$ $\left. \frac{(b_5 a_3 - a_5 b_3) x_5}{\%1}, x_4 + x_5 \right)$	$N(3, 3a)$

	T	$T^{-1}FT$
23h	$\begin{aligned} & \%1 := b_4 a_3 - a_4 b_3 \\ & \left(\frac{\%2^4 \sqrt{\frac{d_5^2 \%2}{b_3 \%1}} x_2}{b_3^4 d_5 \%1}, \frac{(b_3 c_5^3 + b_4 d_5^3)^3 \%2^4 \sqrt{\frac{d_5^2 \%2}{b_3 \%1}} x_1}{b_3^4 d_5 \%1^4}, \right. \\ & \quad \frac{\%2 c_5^3 \sqrt{\frac{d_5^2 \%2}{b_3 \%1}} x_3}{b_3 d_5 \%1} + \\ & \quad \frac{(a_3 b_4 b_5 + a_4 b_3 b_5 - 2 a_5 b_3 b_4) \sqrt{\frac{d_5^2 \%2}{b_3 \%1}} x_4}{b_3 d_5 \%3} - \\ & \quad \frac{(-b_5 a_4 + b_4 a_5) \sqrt{\frac{d_5^2 \%2}{b_3 \%1}} x_5}{d_5 \%3}, \frac{d_5^2 \%2 \sqrt{\frac{d_5^2 \%2}{b_3 \%1}} x_3}{b_3 \%1} - \\ & \left. 2 \frac{\%2 \sqrt{\frac{d_5^2 \%2}{b_3 \%1}} x_4}{d_5 \%3} - \frac{\%2 \sqrt{\frac{d_5^2 \%2}{b_3 \%1}} x_5}{d_5 \%3}, \frac{\sqrt{\frac{d_5^2 \%2}{b_3 \%1}} x_4}{d_5} + \frac{\sqrt{\frac{d_5^2 \%2}{b_3 \%1}} x_5}{d_5} \right) \end{aligned}$	$N(3, 3a)$
24a	$\begin{aligned} & \%1 := c_5^3 a_3 + a_4 d_5^3 \\ & \%2 := b_5 a_3 - a_5 b_3 \\ & \%3 := b_4 a_3 - a_4 b_3 \\ & \left(a_2^3 b_3^9 x_1, b_3^3 x_2 - \frac{a_3 x_3}{a_2} + \frac{(b_5 a_3 - a_5 b_3) x_5}{b_3 a_2}, \right. \\ & \quad \left. x_3 + c_5^3 x_4 - \frac{b_5 x_5}{b_3}, d_5^3 x_4, x_5 \right) \end{aligned}$	$J(3, 3)$
24b	$\begin{aligned} & \left(\%1^3 x_1, \frac{\%1 x_3}{a_2} + \frac{(b_5 a_3 - a_5 b_3) x_4}{a_2 b_3} - \frac{a_3 \left(\frac{\%1}{a_2}\right)^{1/3} x_5}{a_2 b_3}, c_5^3 x_2 - \right. \\ & \quad \left. \frac{b_5 x_4}{b_3} + \frac{\left(\frac{\%1}{a_2}\right)^{1/3} x_5}{b_3}, d_5^3 x_2, x_4 \right) \end{aligned}$	$N(3, 3b)$
25a	$\begin{aligned} & \%1 := c_5^3 a_3 + a_4 d_5^3 \\ & \left(a_5^3 x_4, b_3^3 c_4^9 x_1, c_4^3 x_2 - \frac{b_4 x_3}{b_3} + \frac{(b_4 c_5 - b_5 c_4) x_5}{b_3 c_4}, \right. \\ & \quad \left. x_3 - \frac{c_5 x_5}{c_4}, x_5 \right) \end{aligned}$	$J(3, 3)$
25b	$\begin{aligned} & \left(-\frac{(-c_4 a_5 + c_5 a_4)^3 x_4}{c_4^3}, -\frac{b_3^3 (-c_4 a_5 + c_5 a_4)^9 x_1}{a_4^9}, \right. \\ & \quad -\frac{(-c_4 a_5 + c_5 a_4)^3 x_2}{a_4^3} - \frac{(-b_5 a_4 + b_4 a_5) x_3}{a_4 b_3} + \\ & \quad \left. \frac{(b_4 c_5 - b_5 c_4) x_5}{b_3 c_4}, \frac{a_5 x_3}{a_4} - \frac{c_5 x_5}{c_4}, -x_3 + x_5 \right) \end{aligned}$	$J(3, 3)$

	T	$T^{-1}FT$
25c	$\left(a_3^3 c_4^9 x_2, b_3^3 c_4^9 x_1, c_4^3 x_3 + \frac{(-c_5 a_4 a_3 b_3 c_4^2 + a_5 a_4 b_3 - a_4 b_5 a_3 + a_3^2 c_4^3 b_5) x_4 - (b_5 a_3 - a_5 b_3) a_3}{a_3} \right)$	$N(3, 3a)$
25d	$\left(\frac{a_4 x_5}{a_3}, \frac{(a_3 b_3 c_5 c_4^2 - a_5 b_3 + b_5 a_3) x_4}{b_5 a_3 - a_5 b_3} + x_5, -\frac{b_3 a_3 c_4^3 x_4}{b_5 a_3 - a_5 b_3} \right)$ $\left(\frac{\%2^4 \sqrt{\%3} x_2}{c_4^4 a_3 b_3^4}, \frac{\%2^4 \sqrt{\%3} x_1}{c_4^4 a_3^4 b_3}, -\frac{\%2 \sqrt{\%3} x_3}{a_3 b_3 c_4} - \frac{\sqrt{\%3} b_4 x_4}{c_4 b_3} - \frac{\sqrt{\%3} (-b_5 a_4 + b_4 a_5) x_5}{\%1}, \frac{\sqrt{\%3} x_4}{c_4} - \frac{\sqrt{\%3} (b_5 a_3 - a_5 b_3) x_5}{\%1}, \frac{\sqrt{\%3} \%2 x_5}{\%1} \right)$ <p> $\%1 := -b_3 a_4 c_5 + b_3 a_5 c_4 + b_4 c_5 a_3 - b_5 c_4 a_3$ $\%2 := b_4 a_3 - a_4 b_3$ $\%3 := -\frac{\%2}{c_4 a_3 b_3}$ </p>	$N(3, 3a)$
26a	$\left(x_5 + a_5^3 x_3, c_4^9 d_5^{27} b_3^3 x_1, c_4^3 d_5^9 x_2 - \frac{b_5 x_4}{b_3}, d_5^3 x_3 - \frac{c_5 x_4}{c_4}, x_4 \right)$	$J(3, 4)$
26b	$\left(\frac{a_5^3 \sqrt{\%1} x_3}{b_3 c_4^2 d_5^3} + x_5, \frac{b_4^4 \sqrt{\%1} x_1}{b_3^2 c_4^5}, \frac{b_4 \sqrt{\%1} x_2}{b_3^2 c_4^2} + \frac{(b_4 c_5 - b_5 c_4) (b_3^2 c_4 \sqrt{\%1})^{1/3} x_4}{b_3^2 c_4^2 d_5}, \frac{\sqrt{\%1} x_3}{b_3 c_4^2} - \frac{c_5 (b_3^2 c_4 \sqrt{\%1})^{1/3} x_4}{b_3 c_4^2 d_5}, \frac{(b_3^2 c_4 \sqrt{\%1})^{1/3} x_4}{b_3 c_4 d_5} \right)$ <p> $\%1 := b_3 b_4 c_4$ </p>	$N(3, 4a)$
26c	$\left(a_4^3 d_5^9 x_2 + \frac{x_5}{b_3^3 c_4^{12} d_5^{39}}, c_4^9 d_5^{27} b_3^3 x_1, c_4^3 d_5^9 x_2 - \frac{b_5 x_4}{b_3}, d_5^3 x_3 - \frac{a_5 x_4}{a_4}, x_4 \right)$	$J(3, 4)$
26d	$\left(\frac{\%1^4 \sqrt{\%2} x_2}{a_4^2 d_5^5 c_4^5}, -\frac{b_3^3 \%1^{13} \sqrt{\%2} x_1}{a_4^{14} d_5^{14} c_4^5}, \frac{\%1^4 \sqrt{\%2} x_3}{a_4^5 d_5^5 c_4^2} - \frac{b_5 \sqrt{\%2} x_5}{a_4 d_5^2 c_4 b_3}, -\frac{\%1 \sqrt{\%2} x_4}{a_4^2 d_5^2 c_4^2} - \frac{c_5 \sqrt{\%2} x_5}{a_4 d_5^2 c_4^2}, \frac{\sqrt{\%2} x_5}{a_4 d_5^2 c_4} \right)$ <p> $\%1 := -c_4 a_5 + c_5 a_4$ $\%2 := -a_4 d_5 c_4 \%1$ </p>	$N(3, 4f)$

	T	$T^{-1}FT$
26e	$\left(\frac{\sqrt{\%1} b_4 a_4^3 x_2}{b_3^2 c_4^5}, \frac{b_4^4 \sqrt{\%1} x_1}{b_3^2 c_4^5}, \frac{\sqrt{\%1} b_4 x_3}{b_3^2 c_4^2} + \right.$ $\frac{(b_3^2 c_4 \sqrt{\%1})^{1/3} (b_4 c_5 - b_5 c_4) x_5}{b_3^2 c_4^2 d_5}, \frac{\sqrt{\%1} x_4}{b_3 c_4^2} -$ $\left. \frac{c_5 (b_3^2 c_4 \sqrt{\%1})^{1/3} x_5}{b_3 c_4^2 d_5}, \frac{(b_3^2 c_4 \sqrt{\%1})^{1/3} x_5}{b_3 c_4 d_5} \right)$	$P(3, 4g)$
26f	$\%1 := b_3 b_4 c_4$ $\left(a_3^3 c_4^9 d_5^{27} x_1, c_4^9 d_5^{27} b_3^3 x_1 - \frac{x_5}{a_3^3 c_4^{12} d_5^{39}}, \right.$ $\left. c_4^3 d_5^9 x_2 - \frac{a_5 x_4}{a_3}, d_5^3 x_3 - \frac{c_5 x_4}{c_4}, x_4 \right)$	$J(3, 4)$
26g	$\left(- \frac{(b_5 a_3 - a_5 b_3)^3 \%1^{3/8} x_1}{b_3^6 a_3^3 c_4^3 d_5^6}, - \frac{(b_5 a_3 - a_5 b_3)^3 \%1^{3/8} x_2}{b_3^3 a_3^6 c_4^3 d_5^6}, \right.$ $\left. - \frac{(b_5 a_3 - a_5 b_3) \%1^{1/8} x_3}{b_3^2 a_3^2 c_4 d_5^2} - \frac{b_5 \%1^{1/8} x_5}{b_3^2 a_3 c_4 d_5^2}, \frac{\%1^{3/8} x_4}{d_5^3 b_3^3 a_3^3 c_4^3} - \right.$ $\left. \frac{c_5 \%1^{1/8} x_5}{b_3 a_3 c_4^2 d_5^2}, \frac{\%1^{1/8} x_5}{b_3 a_3 c_4 d_5^2} \right)$	$N(3, 4e)$
26h	$\%1 := (-b_5 a_3 + a_5 b_3) b_3^7 a_3^7 c_4^5 d_5^7$ $\left(\frac{\sqrt{\%1} a_4^4 x_1}{a_3^2 c_4^5}, \frac{a_4^4 b_3^3 \sqrt{\%1} x_2}{a_3^5 c_4^5}, \frac{\sqrt{\%1} a_4 x_3}{a_3^2 c_4^2} - \right.$ $\frac{b_5 (a_3^2 c_4 \sqrt{\%1})^{1/3} x_5}{a_3 c_4 b_3 d_5}, \frac{\sqrt{\%1} x_4}{a_3 c_4^2} -$ $\left. \frac{(a_3^2 c_4 \sqrt{\%1})^{1/3} c_5 x_5}{a_3 c_4^2 d_5}, \frac{(a_3^2 c_4 \sqrt{\%1})^{1/3} x_5}{a_3 c_4 d_5} \right)$	$P(3, 4h)$
26i	$\%1 := a_4 a_3 c_4$ $\left(\frac{a_3^3 b_4^4 \sqrt{\%1} x_1}{b_3^5 c_4^5}, \frac{\sqrt{\%1} b_4^4 x_2}{b_3^2 c_4^5}, \frac{\sqrt{\%1} b_4 x_3}{b_3^2 c_4^2} + \right.$ $\frac{(b_3^2 c_4 \sqrt{\%1})^{1/3} (b_4 c_5 - b_5 c_4) x_5}{b_3^2 c_4^2 d_5}, \frac{\sqrt{\%1} x_4}{b_3 c_4^2} -$ $\left. \frac{c_5 (b_3^2 c_4 \sqrt{\%1})^{1/3} x_5}{b_3 c_4^2 d_5}, \frac{(b_3^2 c_4 \sqrt{\%1})^{1/3} x_5}{b_3 c_4 d_5} \right)$	$P(3, 4a2)$
27a	$\left(a_3^3 c_4^9 d_5^{27} x_1, b_5^3 x_3 - \frac{(a_2 c_5 b_5^3 + a_5 d_5^3 c_4) x_4}{a_2 d_5^3 c_4} - \right.$ $\left. x_5, c_4^3 d_5^9 x_2 + \frac{a_2 x_5}{a_3}, d_5^3 x_3 - \frac{c_5 x_4}{c_4}, x_4 \right)$	$J(3, 4)$

	T	$T^{-1}FT$
27b	$\left(a_2^3 b_4^9 d_5^{27} x_1, b_4^3 d_5^9 x_2 - \frac{x_4 a_3 c_5^3 b_5}{a_2 b_4 d_5^3} - \frac{a_5 x_4}{a_2} - \frac{a_3 x_5}{a_2^4 b_4^{12} d_5^{39}}, c_5^3 x_3 + \frac{x_5}{a_2^3 b_4^{12} d_5^{39}}, d_5^3 x_3 - \frac{b_5 x_4}{b_4}, x_4 \right)$	$J(3, 4)$
27c	$\left(\frac{\sqrt{\%2} \%1^4 x_1}{a_3^2 c_4^5 d_5^{14}}, \frac{b_5^3 \%1 \%3^{1/3} x_3}{d_5^5 c_4^2 (\%1 a_3^2)^{2/3}} + \frac{(-c_4 a_5 + c_5 a_4) \sqrt{\%2} \%3^{2/3} x_4}{d_5^3 c_4^3 a_3 a_2 (\%1 a_3^2)^{2/3}} - \frac{a_3^5 c_4^8 d_5^{21} x_5}{(\%1 a_3^2)^{2/3} \%1^6}, \frac{\%1^2 \%3^{1/3} x_2}{d_5^5 c_4^2 a_3 (\%1 a_3^2)^{2/3}} + \frac{a_2 a_3^4 c_4^8 d_5^{21} x_5}{(\%1 a_3^2)^{2/3} \%1^6}, \frac{\sqrt{\%2} x_3}{a_3 c_4^2 d_5^2} - \frac{\%3^{1/3} c_5 x_4}{a_3 c_4^2 d_5^2}, \frac{\%3^{1/3} x_4}{a_3 c_4 d_5^2} \right)$ <p> $\%1 := a_2 b_5^3 + a_4 d_5^3$ $\%2 := \%1 a_3 c_4 d_5$ $\%3 := a_3^2 c_4 d_5 \sqrt{\%2}$ </p>	$N(3, 4a)$
27d	$\left(\frac{\sqrt{\%2} \%1^4 x_1}{a_2^2 d_5^{14} b_4^5}, \frac{\%1^2 \%3^{1/3} x_2}{d_5^5 b_4^2 a_2 (\%1 a_2^2)^{2/3}} - \frac{(-b_5 a_4 + b_4 a_5) \sqrt{\%2} \%3^{2/3} x_4}{d_5^3 b_4^3 a_2^2 (\%1 a_2^2)^{2/3}} - \frac{a_3 a_2^4 b_4^8 d_5^{21} x_5}{(\%1 a_2^2)^{2/3} \%1^6}, \frac{c_5^3 \%1 \%3^{1/3} x_3}{b_4^2 d_5^5 (\%1 a_2^2)^{2/3}} + \frac{a_2^5 b_4^8 d_5^{21} x_5}{(\%1 a_2^2)^{2/3} \%1^6}, \frac{\sqrt{\%2} x_3}{a_2 d_5^2 b_4^2} - \frac{\%3^{1/3} b_5 x_4}{a_2 d_5^2 b_4^2}, \frac{\%3^{1/3} x_4}{a_2 d_5^2 b_4} \right)$ <p> $\%1 := c_5^3 a_3 + a_4 d_5^3$ $\%2 := \%1 a_2 d_5 b_4$ $\%3 := a_2^2 d_5 b_4 \sqrt{\%2}$ </p>	$N(3, 4a)$
27e	$\left(a_2^3 x_4, b_4^3 d_5^9 x_1 + x_5, d_5^9 c_4^3 x_1 + \frac{x_3 a_5 c_4^3}{a_2 b_4^3}, d_5^3 x_2 - \frac{b_5 x_3}{b_4}, x_3 \right)$	$J(3, 3)$
27f	$\left(a_2^3 b_4^9 d_5^{27} x_1, b_4^3 d_5^9 x_2 - \frac{a_5 x_4}{a_2}, c_4^3 d_5^9 x_2 - x_5, d_5^3 x_3 - \frac{c_5 x_4}{c_4}, x_4 \right)$	$J(3, 4)$
27g	$\left(d_5^{27} (a_2 b_4^3 + a_3 c_4^3)^3 x_1, b_4^3 d_5^9 x_2 - \frac{a_5 x_4}{a_2} - x_5, c_4^3 d_5^9 x_2 + \frac{a_2 x_5}{a_3}, d_5^3 x_3 - \frac{b_5 x_4}{b_4}, x_4 \right)$	$J(3, 4)$
27h	$\left(-\frac{a_2^3 \%1^{13} \sqrt{\%2} x_1}{b_4^4 d_5^{13} c_4^{13}}, \frac{\%1^4 \sqrt{\%2} x_2}{b_4 d_5^4 c_4^4} - \frac{a_5 \sqrt{\%2} x_5}{d_5 a_2}, \frac{\%1^4 \sqrt{\%2} x_3}{b_4^4 d_5^4 c_4}, -\frac{\%1 \sqrt{\%2} x_4}{b_4 c_4 d_5} - \frac{\sqrt{\%2} c_5 x_5}{d_5 c_4}, \frac{\sqrt{\%2} x_5}{d_5} \right)$	$P(3, 4i)$

	T	$T^{-1}FT$
27i	$\begin{aligned} \%1 &:= b_4 c_5 - b_5 c_4 \\ \%2 &:= -\frac{\%1}{b_4 d_5 c_4} \\ &\left(\frac{a_4^4 \sqrt{\%1} x_1}{b_4^5 a_2^2}, \frac{a_4 \sqrt{\%1} x_2}{b_4^2 a_2^2} + \right. \\ &\frac{(b_4 a_2^2 \sqrt{\%1})^{1/3} (-c_4 a_5 + c_5 a_4) x_5}{a_2^2 b_4 d_5 c_4}, \frac{a_4 \sqrt{\%1} c_4^3 x_3}{b_4^5 a_2^2}, \frac{\sqrt{\%1} x_4}{b_4^2 a_2} - \\ &\left. \frac{(b_4 a_2^2 \sqrt{\%1})^{1/3} c_5 x_5}{a_2 b_4 c_4 d_5}, \frac{(b_4 a_2^2 \sqrt{\%1})^{1/3} x_5}{a_2 b_4 d_5} \right) \end{aligned}$	$P(3, 4j2)$
28a	$\begin{aligned} \%1 &:= a_4 a_2 b_4 \\ &\left(a_2^3 b_3^9 c_5^{27} x_1, b_3^3 c_5^9 x_2 - \frac{(a_5 + a_4) x_4}{a_2} + \frac{a_4 x_5}{a_2}, c_5^3 x_3 - \right. \\ &\left. \frac{(b_5 + b_4) x_4}{b_3} + \frac{b_4 x_5}{b_3}, x_4 - x_5, x_4 \right) \end{aligned}$	$J(3, 4)$
28b	$\begin{aligned} &\left(\frac{\sqrt{\%1} a_3^4 x_1}{a_2^2 b_3^5}, \frac{\sqrt{\%1} a_3 x_2}{a_2^2 b_3^2} + \right. \\ &\left. \frac{(a_3 b_5 \%2^{1/3} + b_4 c_5 a_3 a_2 b_3 - b_3^2 a_4 c_5 a_2 - a_5 \%2^{1/3} b_3) x_4}{a_2^2 b_3^2 c_5} - \right. \\ &\left. \frac{(b_4 a_3 - a_4 b_3) x_5}{a_2 b_3}, \frac{\sqrt{\%1} x_3}{a_2 b_3^2} - \frac{(b_5 \%2^{1/3} + b_4 c_5 a_2 b_3) x_4}{a_2 b_3^2 c_5} + \frac{b_4 x_5}{b_3}, \right. \\ &\left. x_4 - x_5, \frac{\%2^{1/3} x_4}{a_2 b_3 c_5} \right) \end{aligned}$	$N(3, 4a)$
28c	$\begin{aligned} \%1 &:= a_3 a_2 b_3 \\ \%2 &:= a_2^2 b_3 \sqrt{\%1} \\ &\left(a_2^3 b_3^9 x_1, b_3^3 x_2 - \frac{a_3 x_3}{a_2} + \frac{(b_5 a_3 - a_5 b_3) x_5}{b_3 a_2}, x_3 + c_5^3 x_4 - \right. \\ &\left. \frac{b_5 x_5}{b_3}, d_5^3 x_4, x_5 \right) \end{aligned}$	$J(3, 3)$
28d	$\begin{aligned} &\left(a_2^3 \%1^9 x_1, \%1^3 x_2 + \frac{(b_5 a_3 - a_5 b_3) x_4}{a_2 b_3} - \frac{a_3 x_5}{a_2^4 \%1^{12} d_5^3}, c_5^3 x_3 - \right. \\ &\left. \frac{b_5 x_4}{b_3} + \frac{b_4 x_5}{a_2^3 \%1^{13}}, d_5^3 x_3 - \frac{b_3 x_5}{a_2^3 \%1^{13}}, x_4 \right) \end{aligned}$	$J(3, 4)$
28e	$\begin{aligned} \%1 &:= b_3 c_5^3 + b_4 d_5^3 \\ &\left(a_2^3 b_3^9 x_1, b_3^3 x_2 - \frac{a_3 x_4}{a_2} + \right. \\ &\frac{(b_5 a_3 - a_5 b_3) (a_2 \%1^2)^{1/3} x_5}{a_2 \%1}, \frac{a_2 b_3^3 c_5^3 x_3}{\%1} + x_4 - \\ &\left. \frac{b_5 (a_2 \%1^2)^{1/3} x_5}{\%1}, \frac{a_2 b_3^3 d_5^3 x_3}{\%1}, \frac{b_3 (a_2 \%1^2)^{1/3} x_5}{\%1} \right) \end{aligned}$	$N(3, 3b)$

	T	$T^{-1}FT$
28f	$\begin{aligned} \%1 &:= c_5^3 a_3 + a_4 d_5^3 \\ &\left(\frac{\%2^4 \sqrt{\%2 \%1 a_2} x_1}{a_2^2 \%1^5}, \frac{\%2 \sqrt{\%2 \%1 a_2} x_2}{a_2^2 \%1^2} + \right. \\ &\quad \left. \frac{(\%1 a_2^2 \sqrt{\%2 \%1 a_2})^{1/3} (b_5 a_3 - a_5 b_3) x_4}{b_3 a_2^2 \%1} - \right. \\ &\quad \frac{\sqrt{\%2 \%1 a_2} d_5^3 (b_4 a_3 - a_4 b_3) x_5}{b_3 a_2^2 \%1^2}, \frac{c_5^3 \sqrt{\%2 \%1 a_2} x_3}{\%1^2 a_2} - \\ &\quad \frac{b_5 (\%1 a_2^2 \sqrt{\%2 \%1 a_2})^{1/3} x_4}{b_3 \%1 a_2} + \\ &\quad \frac{b_3 \%1 a_2}{\sqrt{\%2 \%1 a_2} b_4 d_5^3 x_5}, \frac{d_5^3 \sqrt{\%2 \%1 a_2} x_3}{\%1^2 a_2} - \\ &\quad \left. \frac{d_5^3 \sqrt{\%2 \%1 a_2} x_5}{\%1^2 a_2}, \frac{(\%1 a_2^2 \sqrt{\%2 \%1 a_2})^{1/3} x_4}{\%1 a_2} \right) \end{aligned}$	$N(3, 4a)$
29a	$\begin{aligned} \%1 &:= b_3 c_5^3 + b_4 d_5^3 \\ \%2 &:= c_5^3 a_3 + a_4 d_5^3 \\ &\left(a_2^3 b_3^9 c_4^{27} x_1, b_3^3 c_4^9 x_2 - \frac{a_4 x_4}{a_2} + \frac{(-c_4 a_5 + c_5 a_4) x_5}{c_4 a_2}, c_4^3 x_3 - \right. \\ &\quad \left. \frac{b_4 x_4}{b_3} + \frac{(b_4 c_5 - b_5 c_4) x_5}{b_3 c_4}, x_4 - \frac{c_5 x_5}{c_4}, x_5 \right) \end{aligned}$	$J(3, 4)$
29b	$\begin{aligned} &\left(\frac{\sqrt{\%1} a_3^4 x_1}{a_2^2 b_3^5}, \frac{\sqrt{\%1} a_3 x_2}{a_2^2 b_3^2} + \frac{(a_2^2 b_3 \sqrt{\%1})^{1/3} (b_4 a_3 - a_4 b_3) x_4}{a_2^2 b_3^2 c_4} - \right. \\ &\quad \frac{(-b_3 a_4 c_5 + b_3 a_5 c_4 + b_4 c_5 a_3 - b_5 c_4 a_3) x_5}{a_2 c_4 b_3}, \\ &\quad \frac{\sqrt{\%1} x_3}{a_2 b_3^2} + \frac{(b_4 c_5 - b_5 c_4) x_5}{b_3 c_4} - \frac{b_4 (a_2^2 b_3 \sqrt{\%1})^{1/3} x_4}{a_2 b_3^2 c_4}, \\ &\quad \left. \frac{(a_2^2 b_3 \sqrt{\%1})^{1/3} x_4}{a_2 b_3 c_4} - \frac{c_5 x_5}{c_4}, x_5 \right) \end{aligned}$	$N(3, 4a)$
	$\%1 := a_3 a_2 b_3$	
Rank four		
30a	$\begin{aligned} &\left(a_2^3 c_4^{27} d_5^{81} b_3^9 x_1, c_4^9 d_5^{27} b_3^3 x_2 - \frac{a_5 x_5}{a_2}, c_4^3 d_5^9 x_3 - \frac{b_5 x_5}{b_3}, \right. \\ &\quad \left. d_5^3 x_4 - \frac{c_5 x_5}{c_4}, x_5 \right) \end{aligned}$	$J(4, 5)$
30b	$\begin{aligned} &\left(\frac{b_4^{13} a_2^3 \sqrt{\%1} x_1}{b_3^5 c_4^{14}}, \frac{\sqrt{\%1} b_4^4 x_2}{b_3^2 c_4^5} - \frac{a_5 \%2^{1/3} x_5}{b_3 c_4 a_2 d_5}, \frac{\sqrt{\%1} b_4 x_3}{b_3^2 c_4^2} + \right. \\ &\quad \left. \frac{\%2^{1/3} (b_4 c_5 - b_5 c_4) x_5}{b_3^2 c_4^2 d_5}, \frac{\sqrt{\%1} x_4}{b_3 c_4^2} - \frac{c_5 \%2^{1/3} x_5}{b_3 c_4^2 d_5}, \frac{\%2^{1/3} x_5}{b_3 c_4 d_5} \right) \end{aligned}$	$N(4, 5b)$

	T	$T^{-1}FT$
30c	$\begin{aligned} \%1 &:= b_3 b_4 c_4 \\ \%2 &:= b_3^2 c_4 \sqrt{\%1} \\ &\left(\frac{a_4^3 \%1^{3/4} x_1}{a_2^3 c_4^6 b_3^3}, \frac{a_4 \%1^{1/4} x_2}{a_2^2 c_4^2 b_3} + \right. \\ &\quad \frac{\%2^{1/3} (-c_4 a_5 + c_5 a_4) x_5}{a_2^2 c_4^2 b_3 d_5}, \frac{\%1^{3/4} x_3}{a_2^3 c_4^3 b_3^3} + \\ &\quad \left. \frac{\%2^{1/3} (b_4 c_5 - b_5 c_4) x_5}{a_2 c_4^2 b_3^2 d_5}, \frac{\%1^{1/4} x_4}{a_2 c_4^2 b_3} - \frac{c_5 \%2^{1/3} x_5}{a_2 c_4^2 b_3 d_5}, \frac{\%2^{1/3} x_5}{a_2 c_4 b_3 d_5} \right) \\ \%1 &:= b_3^2 a_2^3 c_4^3 \sqrt{a_4 a_2 b_3 c_4} \\ \%2 &:= a_2^2 c_4 b_3^2 \%1^{1/4} \end{aligned}$	$P(4, 5e)$
30d	$\begin{aligned} &\left(\frac{\sqrt{\%1} a_3^4 x_1}{a_2^2 b_3^5}, \frac{\sqrt{\%1} a_3 x_2}{a_2^2 b_3^2} - \right. \\ &\quad \frac{\%2^{1/9} (-b_3 a_4 c_5 + b_3 a_5 c_4 + b_4 c_5 a_3 - b_5 c_4 a_3) x_5}{a_2^2 b_3^2 c_4^2 d_5}, \frac{\sqrt{\%1} x_3}{a_2 b_3^2} + \\ &\quad \frac{\%2^{1/9} (b_4 c_5 - b_5 c_4) x_5}{a_2 b_3^2 c_4^2 d_5}, \frac{(a_2^2 b_3 \sqrt{\%1})^{1/3} x_4}{a_2 b_3 c_4} - \\ &\quad \left. \frac{c_5 \%2^{1/9} x_5}{a_2 b_3 c_4^2 d_5}, \frac{\%2^{1/9} x_5}{a_2 b_3 c_4 d_5} \right) \\ \%1 &:= a_3 a_2 b_3 \\ \%2 &:= a_2^8 b_3^7 \sqrt{\%1} c_4^6 \end{aligned}$	$P(4, 5c2)$

Table A.1: Transformations for cubic similarity

Appendix B

The Jacobian package

Introduction

In this appendix we give a short description of the basic features which are available in the so-called Jacobian package. This package –or better collection of procedures because it is not a true package in the Maple sense– is developed at the University of Nijmegen. The main contributors have been Harm Derksen, Christian Eggermont, Mark van Hoeij, Peter van Rossum and the author of this thesis. The package was originally written for Maple 5.3. However there has been an update to Maple 5.4.

Without this package the computational results would never have been achieved.

Basically we can divide the procedures in this package into four parts: procedures concerning polynomial maps and matrices, procedures concerning derivations, procedures concerning ideals and procedures concerning miscellaneous topics. The list given here is not complete. We use a lot of other procedures which are very useful, but not very interesting at a theoretical level. In order to save some space we do not present examples here. They are available in the on-line help system.

B.1 Polynomial maps and matrices

composemap

Function: `composemap` - compose two polynomial maps

Calling sequence:

`composemap(map1,map2,var)`

Parameters:

`map1,map2` - list of algebraic expressions

`var` - list of names

Notes:

`composemap(map1,map2,var)` computes the composition of the polynomial maps `map1` and `map2`. `var` should be the list of indeterminates of `map1`. `composemap` does

the following: The indeterminates listed in var are substituted by the polynomials listed in map2 in the expression map1.

composemaps

Function: composemaps - compose two or more polynomial maps

Calling sequence:

composemaps(map1,map2,...,mapN,var)

Parameters:

map1,map2,...,mapN - list of algebraic expressions

var - list of names

Notes:

composemaps(map1,map2,...,mapsN,var) computes the composition of the maps map1,map2,...,mapN. var should be the list of indeterminates of map1.

composemaps calls N-1 times the composemap function. Hence the call composemaps(map1,map2,map3,var) and the call composemap(composemap(map1,map2,var),map3,var) are equivalent.

composemaps has a built in feature (or bug?) of coercion. If map1 has less elements than var, map1 will be extended to the dimension of var by adding the identity map of the right dimension.

conjugatemap

Function: conjugatemap - conjugate two polynomial maps

Calling sequence:

conjugatemap(map1,map2,var)

Parameters:

map1,map2 - list of algebraic expressions

var - list of names

Notes:

conjugatemap(map1,map2,var) computes the composition of the polynomial maps $(\text{map1})^{-1}$, map2 and map1. var should be the list of indeterminates of map1.

map1 must be invertible, map2 need not be invertible.

decomlinjac

Function: decomlinjac - decompose a Jacobian matrix of a quadratic polynomial map

Calling sequence:

decomlinjac(J, var)

Parameters:

J - the Jacobian matrix of a quadratic polynomial map

var - the variables of the polynomial map

Notes:

The command `decomlinjac(J,var)` decomposes this J into a list L of $n+1$ matrices, where n is the number of variables:

$$J = L[1] + L[2] \cdot \text{var}[1] + \cdots + L[n + 1] \cdot \text{var}[n]$$

This procedure also works for linear polynomial maps. It does not work for cubic or higher dimensional polynomial maps. Though it does return a list L in such a case, it is very unlikely that the equation above holds. (However this is not impossible!)

degreepart

Function: `degreepart` - extract part with a certain total degree

Calling sequence:

`degreepart(F, X, d)`
`degreepart(F, X, d, '>')`
`degreepart(F, X, d, '>=')`
`degreepart(F, X, d, '<')`
`degreepart(F, X, d, '<=')`

Parameters:

F - a polynomial or list of polynomials
 X - a list of variables
 d - the degree

Notes:

`degreepart(F,X,d)` extracts the part of the polynomial F which has total degree d with respect to the variables of X .

The operators '>', '>=', '<' and '<=' do what one would expect: `degreepart(F,X,d,'>=')` gives the part which has total degree $\geq d$.

If F is a polynomial map, i.e. a list of polynomials, the function is applied automatically to each entry.

druzmap

Function: `druzmap` - compute the cubic-linear map associated with a square matrix

Calling sequence:

`druzmap(A, X)`

Parameters:

A - a square matrix
 X - list of indeterminates

Notes:

The command `druzmap(A,X)` computes the Drużkowski or cubic-linear map $F = X + (AX)^*3$.

druzmat

Function: druzmat - compute the matrix associated with a cubic-linear map

Calling sequence:

druzmat(F, X)

Parameters:

F - a cubic-linear map
X - list of indeterminates

Notes:

The command druzmat(F,X) computes the matrix A if $F = X + (AX)^3$.

gbasisext

Function: gbasisext - compute reduced, minimal Gröbner basis

Calling sequence:

gbasisext(F, X, 'A','B')
gbasisext(F, X, 'A','B',termorder)

Parameters:

F - set or list of polynomials
X - list of indeterminates (not including parameters)
A,B - unevaluated name
termorder - (optional) term ordering: 'plex' (pure lexicographic) or 'tdeg' (total degree is default)

Notes:

The command gbasis(F,X,A,termorder) computes the reduced, minimal Gröbner basis G of the polynomials F with respect to the indeterminates X and the given term ordering.

The argument A will be assigned a matrix A such that $G = A * F$.

The argument B will be assigned a matrix A such that $F = B * G$.

This implementation is based upon the algorithm described in [12].

If X is a set an error message will be issued.

A list of indeterminates $X := [x_1, x_2, \dots, x_n]$ induces the ordering $x_1 > x_2 > \dots > x_n$.

homogeneousmap

Function: homogeneousmap - checks whether a map is homogeneous

Calling sequence:

homogeneousmap(H, X)
homogeneousmap(H, X, 'd')

Parameters:

H - a polynomial map
X - a list or set of indeterminates
deg - (optional) unevaluated name

Notes:

The command `homogeneousmap(H,X)` returns true if H is a homogeneous map of a certain degree d.

If the optional deg is present, this will be assigned the degree d. (Only in case it is indeed homogeneous.)

inversemap

Function: `inversemap` - inverse of a polynomial maps

Calling sequence:

`inversemap(map,var)`

Parameters:

map - list of algebraic expressions
var - list of names

Notes:

`inversemap(map,var)` computes the inverse of the polynomial map 'map'. var should be the list of indeterminates of map. If 'map' is not an invertible polynomial map, then `inversemap` returns FAIL.

invjacobian

Function: `invjacobian` - 'integrate' a Jacobian matrix to a polynomial map

Calling sequence:

`invjacobian(J,var)`

Parameters:

J - a square matrix
var - a vector or list of variables

Notes:

`invjacobian(J,var)` returns -if it exists- a polynomial map F such that `jacobian(F,var)` gives the matrix J.

If J is not a Jacobian matrix an error message is issued.

iteratemap

Function: `iteratemap` - iterate a polynomial map

Calling sequence:

`iteratemap(map,var,n)`

Parameters:

map - list of algebraic expressions
var - list of names
n - number of iterations

Notes:

iteratemap(map,var,n) computes the composition of the polynomial map map n times.
I.e. it computes map^n .

If $n < 2$ then map is returned without any compositions.

iterationindex

Function: iterationindex - compute the iteration index

Calling sequence:

iterationindex(H, X)
iterationindex(H, X, max)

Parameters:

H - a polynomial map
X - list of indeterminates
max - (optional) max number of iterations

Notes:

iterationindex(H,X) computes the smallest n such that $H^n = 0$. And with H^n we mean n times the composition of H with itself.

If a third argument max is present, the algorithm will automatically stop after max iterations. Default is that the algorithm will stop after 20 iterations.

maketri

Function: maketri - transform a polynomial map by permutation of the variables to triangular form

Calling sequence:

maketri(f,var)
maketri(f,var,'c')
maketri(f,var,'c',dummy)

Parameters:

f - algebraic expression
var - list of names
'c' - (optional) an unevaluated name
dummy - a dummy: i.e. any expression will do

Notes:

maketri(f,var) transforms the polynomial map f to triangular form, if possible by a permutation.

`maketri(f,var,'c')` does the same, but in addition it returns the permutation used to get to the triangular form in the 'c' parameter

`maketri(f,var,'c',dummy)` does the same, but in addition it shows intermediate results. The dummy parameter is only examined by the `nargs` argument; its type is not important.

If the error message 'Not triangularizable (2)' appears, the map `f` cannot be transformed to triangular form by a permutation. The '(2)' appears to show some distinction with the error message from the procedure 'trifom'.

Like `triform` also this procedure will be extended with the test from [45].

nilindex

Function: `nilindex` - compute the nilpotence index

Calling sequence:

`nilindex(M)`

Parameters:

`M` - a square matrix

Notes:

`nilindex(M)` returns the nilpotence index of `M`, i.e. the smallest positive integer `N` for which $M^N = 0$.

If `M` is not square or not nilpotent an error message is issued.

nilpotent

Function: `nilpotent` - checks whether a matrix is nilpotent

Calling sequence:

`nilpotent(M)`

Parameters:

`M` - a square matrix

Notes:

`nilpotent(M)` returns true if `M` is nilpotent, false if it is not.

normalfext

Function: `normalfext` - reduced form of a polynomial modulo an ideal

Calling sequence:

`normalfext(poly, F, X, 'A')`

`normalfext(poly, F, X, 'A', termorder)`

Parameters:

`F` - list of polynomials (normally, a Gröbner basis)

`X` - list of indeterminates (not including free parameters)

poly - the polynomial to be reduced

A - unevaluated name

termorder - (optional) term ordering: either 'plex' (pure lexicographic) or 'tdeg' (total degree — default)

Notes:

The command `normalfext(poly,F,X,termorder)` computes the fully reduced form `nf` of `poly` with respect to the ideal basis `F`, indeterminates `X`, and term ordering `termorder` (either 'plex' for pure lexicographic ordering, or 'tdeg' for total degree ordering).

The name `A` will be assigned the list of coefficients such that

$$poly = nf + A[1] * F[1] + A[2] * F[2] + \dots + A[n] * F[n]$$

Usually, one first computes a Gröbner basis for a set of polynomials (algebraic equations, or side relations) via `grobner[gbasis]`, for use as the reducing basis `F`. Note: `F` must be a Gröbner basis with respect to `X`, `termorder` (and not with respect to some permutation of `X`, for example).

paircubic

Function: `paircubic` - compute a paired cubic homogeneous map

Calling sequence:

`paircubic(F, X)`

`paircubic(F, X, 'B', 'C')`

Parameters:

`F` - cubic linear mapping

`X` - list of indeterminates

`B,C` - (optional) unevaluated names

Notes:

The command `paircubic(F,X)` computes a cubic homogeneous map which is 'paired to `F` through `B` and `C`' as defined by Gorni and Zampieri in [46].

A cubic homogeneous map $f : \mathbb{C}^n \rightarrow \mathbb{C}^n$ and a cubic linear mapping $F : \mathbb{C}^N \rightarrow \mathbb{C}^N$ where $F = X - (AX)^{*3}$ and $N > n$ are called paired through the $n \times N$ matrix `B` and the $N \times n$ matrix `C`, if $\ker(A)=\ker(B)$, $BC=I$ and $f(x)=BF(Cx)$ for all x in \mathbb{C}^n .

This new map will be a polynomial map in `X[1],...,X[n]`.

If one uses the long form `paircubic(F,X,'B','C')` the `B` and `C` will be assigned the matrices `B` and `C` mentioned above.

pairdruz

Function: `pairdruz` - compute a paired cubic-linear map

Calling sequence:

`pairdruz(F, X, Y)`

`pairdruz(F, X, Y, 'B', 'C')`

`pairdruz(F, X, Y, 'B', 'C', 'XX')`

Parameters:

F - cubic homogeneous mapping
X - list of indeterminates
Y - a name
B,C,XX - (optional) unevaluated names

Notes:

The command `pairdruz(F,X,Y)` computes a cubic-linear map which is 'paired to F through B and C' as defined by Gorni and Zampieri in [46].

It uses the same algorithm as `paircubic`, but goes the other way round.

This new map will be a polynomial map in $Y[1], \dots, Y[N]$.

If one uses the long form `pairdruz(F,X,Y,'B','C')` the B and C will be assigned the matrices B and C mentioned above.

The optional sixth argument will be assigned the new indeterminates: $[Y[1], \dots, Y[N]]$

reddegree

Function: `reddegree` - reduction to degree $C \leq 3$

Calling sequence:

`reddegree(map,var,y)`
`reddegree(map,var,y,lev)`

Parameters:

map - list of algebraic expressions
var - list of names
y - name
lev - (optional) level of reduction

Notes:

`reddegree(map,var,y)` gives the polynomial map in the indeterminates $y[1], y[2], \dots, y[n]$ for some n which one obtains after reducing 'map' to degree 3 as described in Bass, Connell and Wright, [8].

Currently the only possibilities for 'lev' are 'ext', 'hom' and 'druz'.

If no 'lev' is given, 'map' reduces to a map of the form $F = X + F_2 + F_3$, where F_i represents all monomials with total degree i.

If 'ext' is given, 'map' reduces to a map of the form $F = (X, Y) + (F_2 + Y, -F_3)$.

If 'hom' is given, 'map' reduces to a cubic homogeneous map of the form $F = (X, Y, T) + (F_2 * T + Y * T^2, -F_3, 0)$.

If 'druz' is given, 'map' reduces to a cubic linear map of the form $F = X + L^3$. This is done using the pairing mechanism introduced by Gorni and Zampieri in [46].

strongnilindex

Function: `strongnilindex` - compute the strong nilpotence index

Calling sequence:

strongnilindex(M, var)

Parameters:

M - a square matrix

var - a list or set of variables

Notes:

strongnilindex(M) returns the strong nilpotence index of M, i.e. the smallest positive integer N for which $M(X[1]) \cdot M(X[2]) \cdot \dots \cdot M(X[N]) = 0$. With M(X[1]) we mean the matrix one gets by substituting $\text{var}[i]=X[1][i]$ into M for i from 1 to n.

If M is not square or not strong nilpotent an error message is issued.

strongnilpotent

Function: strongnilpotent - checks whether a matrix is strong nilpotent

Calling sequence:

strongnilpotent(M, var)

Parameters:

M - a square matrix

var - a list or set of variables

Notes:

strongnilpotent(M) returns true if M is strong nilpotent with respect to the variables var, false if it is not.

triform

Function: triform - test whether a polynomial map is on triangular form

Calling sequence:

triform(f,var)

triform(f,var,'p')

Parameters:

f - algebraic expression

var - list of names

'p' - (optional) an unevaluated name

Notes:

triform(f,var) checks whether a polynomial map f is on lower triangular form.

triform(f,var,p) also checks whether a polynomial map f is on triangular form, but in addition returns the index of the first component of f that prevents f from being triangular. If in fact f is triangular, 'p' is assigned the value 0.

If the error message 'Not triangularizable (1)' appears, there exists some index i with

$$f[i] = a[i] \cdot \text{var}[i] + b[i](\text{var}[1], \dots, \text{var}[i-1], \text{var}[i])$$

and $\text{var}[i]$ really appears in $b[i]$. In this case f cannot be transformed to triangular form by permutations of the $\text{var}[i]$.

If the result 'false' appears, there exists some index i and j with

$$f[i] = a[i] \cdot \text{var}[i] + b[i](\text{var}[1], \dots, \text{var}[i-1], \text{var}[j])$$

where $j \geq i$. In this case permutation of the $\text{var}[i]$ might help to transform f to triangular form. This index i is set in the unevaluated name 'p' if present.

This procedure was implemented before the paper by Gorni, Tutaj and Zampieri [45] appeared in which they describe a method to verify whether a map is triangularizable by a permutation. It will be implemented in the future.

B.2 Derivations

Fder

Function: Fder - compute the derivations d/dF

Calling sequence:

Fder(F, X)

Parameters:

F - a polynomial map

X - a list of indeterminates

Notes:

Fder(F,X) computes the list of derivations

$$d/dF = [d/dF[1], d/dF[2], \dots, d/dF[n]]$$

where $n = \text{nops}(X)$.

The definition of this list of derivations is given by:

$$\begin{pmatrix} \frac{\partial}{\partial F[1]} \\ \frac{\partial}{\partial F[2]} \\ \vdots \\ \frac{\partial}{\partial F[n]} \end{pmatrix} = ((JH)^{-1})^T \begin{pmatrix} \frac{\partial}{\partial X[1]} \\ \frac{\partial}{\partial X[2]} \\ \vdots \\ \frac{\partial}{\partial X[n]} \end{pmatrix}$$

applyder

Function: applyder - apply a derivation on an expression

Calling sequence:

applyder(f,der,var)

Parameters:

f - algebraic expression

der - list of algebraic expressions

var - list of names

Notes:

applyder(f,der,var) gives the result of applying the derivation

$$\text{der}[1] \frac{\partial}{\partial \text{var}[1]} + \text{der}[2] \frac{\partial}{\partial \text{var}[2]} + \cdots + \text{der}[n] \frac{\partial}{\partial \text{var}[n]}$$

on f, where $n = \text{nops}(\text{der}) = \text{nops}(\text{var})$ is the number of variables.

applyders

Function: applyders - apply a list of derivations on an expression

Calling sequence:

applyder(f,ders,var)

Parameters:

f - algebraic expression
 ders - list of algebraic expressions
 var - list of names

Notes:

applyders(f,ders,var) gives the summation of applying the derivations ders[1],...,ders[n] to the expression f.

applyexpder

Function: applyexpder - apply exp(D) on an algebraic expression

Calling sequence:

applyexpder(f,der,var)
 applyexpder(f,der,var,t)

Parameters:

f - algebraic expression
 der - list of algebraic expressions
 var - list of names
 t - algebraic expression

Notes:

applyexpder(f,der,var,t) gives the result of applying the automorphism

$$\exp(tD) = \sum_{i=0}^{\infty} \frac{t^i D^i}{i}$$

on f where D is a locally nilpotent derivation defined by

$$D = \text{der}[1] \frac{\partial}{\partial \text{var}[1]} + \text{der}[2] \frac{\partial}{\partial \text{var}[2]} + \cdots + \text{der}[n] \frac{\partial}{\partial \text{var}[n]}$$

and $n = \text{nops}(\text{der}) = \text{nops}(\text{var})$ is the number of variables.

If this procedure is called with only 3 arguments, then applyexpder assumes $t=1$.

expder

Function: expder - compute the automorphism $\exp(D)$

Calling sequence:

expder(der,var)
expder(der,var,t)

Parameters:

der - list of algebraic expressions
var - list of names
t - algebraic expression

Notes:

expder(der,var,t) computes the automorphism

$$\exp(tD) = \sum_{i=0}^{\infty} \frac{t^i D^i}{i!}$$

where D is a locally nilpotent derivation defined by

$$D = \text{der}[1] \frac{\partial}{\partial \text{var}[1]} + \text{der}[2] \frac{\partial}{\partial \text{var}[2]} + \cdots + \text{der}[n] \frac{\partial}{\partial \text{var}[n]}$$

and $n = \text{nops}(\text{der}) = \text{nops}(\text{var})$ is the number of variables.

If this procedure is called with only 2 arguments, then expder assumes $t=1$.

ker

Function: ker - computes the kernel of a given locally nilpotent derivation, if the kernel as an algebra is finitely generated

Calling sequence:

ker(der,var)
ker(der,var,mxch)
ker(der,var,ps)
ker(der,var,ps,mxch)
ker(der,var,mxch,ps)

Parameters:

der - list of algebraic expressions
var - list of variables
ps - algebraic expression
mxch - integer (≥ 0)

Notes:

ker(der,var,ps) prints the chain length and returns a list of algebra generators of the kernel (if finitely generated as a \mathbb{Q} -algebra) of the derivation D . The (optional) element ps must be a preslice, i.e. $D(ps) \neq 0$ and $D(D(ps)) = 0$.

If this procedure is called without a preslice ps, then ker calls preslice to compute one.

In general the kernel of a locally nilpotent derivation does not have to be finitely generated as a \mathbb{Q} -algebra, so the algorithm possibly does not stop. The purpose of `mxch` (maximal chain length in the construction of the kernel) is to prevent this and the default of `mxch` is 1000. Furthermore, `ker` makes heavily use of the gröbnerbasis algorithm so don't expect it to be fast.

`D` is the locally nilpotent derivation defined by

$$D = \text{der}[1] \frac{\partial}{\partial \text{var}[1]} + \text{der}[2] \frac{\partial}{\partial \text{var}[2]} + \cdots + \text{der}[n] \frac{\partial}{\partial \text{var}[n]}$$

and $n = \text{nops}(\text{der}) = \text{nops}(\text{var})$ is the number of variables.

Cf. [25].

kerreduc

Function: `kerreduc` - try to reduce the number and size of the generators of the kernel of a locally nilpotent derivation

Calling sequence:

`kerreduc(xtgen,var)`
`kerreduc(xtgen,var,gen)`

Parameters:

`xtgen` - list of polynomials
`var` - list of variables
`gen` - list of polynomials

Notes:

logmap

Function: `logmap` - 'inverse' of `expder`

Calling sequence:

`logmap(H, X)`
`logmap(H, X, 'ind')`
`logmap(H, X, 'ind', max)`

Parameters:

`H` - a polynomial map
`X` - a list of indeterminates
`ind` - (optional) unevaluated name
`max` - (optional) max nr of iterations

Notes:

The command `logmap(H,X)` is in some sense the inverse of `expder(D,X)`. If `D` is a locally nilpotent derivation, `expder(D,X)` computes a polynomial map `H` such that $H = \exp(D)$. Now if `H` is a polynomial map, `logmap(H,X)` tries to compute a locally nilpotent derivation `D` such that $H = \exp(D)$.

Given H , the existence of a locally nilpotent derivation D with $H = \exp(D)$ is not guaranteed. Therefore `logmap` is not a real inverse of `expder`.

The optional third argument will be assigned the smallest n such that $(F^* - I)^n(X) = 0$. (Note that this is not the same as $(F - x)^n = 0$!) For explanation of F^* : see [26].

In order to guarantee termination, a maximum number of iterations is built in: 20. By specifying the fourth argument `max`, this number can be overwritten. Note that if one wants to do so, one also has to specify the third argument.

preslice

Function: `preslice` - compute a preslice of a locally nilpotent derivation

Calling sequence:

`preslice(der,var)`

Parameters:

`der` - list of algebraic expressions

`var` - list of names

Notes:

Compute a preslice of derivation `der` with respect to variables `var`.

slice

Function: `slice` - compute a slice of a locally nilpotent derivation

Calling sequence:

`slice(der,var,kernel)`

`slice(der,var,kernel,ps)`

Parameters:

`der` - list of algebraic expressions

`var` - list of names

`kernel` - kernel of `der`

`ps` - (optional) preslice of `der`

Notes:

Compute a slice of derivation `der` with respect to variables `var` and the kernel.

If the optional fourth argument `ps` is omitted, `slice` computes a preslice.

B.3 Ideals

dimension

Function: `dimension` - computes the maximal strongly algebraic independent sets modulo the given ideal I (= ideal of `alglst`), the dimension of I , and a maximal algebraic independent set modulo I

Calling sequence:

dimension(alglist,var)
dimension(alglist,var,orde)

Parameters:

alglist - list of polynomials
var - list of variables
orde - tdeg or plex

Notes:

Every maximal strongly independent set is also independent. It is proven that every maximal strongly independent set has the same cardinality modulo I iff I is prime

idimension

Function: idimension - calculate the dimension of a proper ideal

Calling sequence:

idimension(F,V)

Parameters:

F - list of algebraic expressions, denoting a proper ideal
V - list of variables

Notes:**iintersection**

Function: iintersection - compute the intersection of a sequence of ideals

Calling sequence:

iintersection(F1,...,Fn,V)

Parameters:

F1,...,Fn - lists of algebraic expressions, denoting ideals
V - list of variable names

Notes:**independentvars**

Function: independentvars - calculate all maximal strong independent sets modulo an ideal

Calling sequence:

independentvars(F,V)

Parameters:

F - list of algebraic expressions, denoting an ideal
V - list of variables

Notes:

ipower

Function: ipower - calculate a power of an ideal

Calling sequence:

ipower(F,m,V)

Parameters:

F - list of algebraic expressions, denoting an ideal

m - a natural number

V - list of variable names

Notes:

iproduct

Function: iproduct - calculate the product of two ideals

Calling sequence:

iproduct(F,G,V)

Parameters:

F - list of algebraic expressions, denoting an ideal

G - list of algebraic expressions, denoting an ideal

V - list of variable names

Notes:

iradical

Function: iradical - compute the radical of an ideal

Calling sequence:

iradical(F,V)

Parameters:

F - list of algebraic expressions, denoting a radical ideal

V - list of variables

Notes:

isimplify

Function: isimplify - simplify the basis of an ideal

Calling sequence:
isimplify(F,V)

Parameters:

F - list of algebraic expressions, denoting an ideal
V - list of variable names

Notes:

Returns a basis of the ideal (F) without obvious superfluous elements.

isradical

Function: isradical - decide whether a zero-dimensional ideal is radical

Calling sequence:
isradical(F,V)

Parameters:

F - list of algebraic expressions, denoting a zero-dimensional ideal.
V - list of variable names

Notes:

normalposition

Function: normalposition - extend a zero dimensional ideal to a radical ideal in normal position

Calling sequence:
normalposition(F,V,w)

Parameters:

F - list of algebraic expressions, denoting a zero dimensional ideal.
V - list of variable names
w - variable name (not appearing in V)

Notes:

quasifinite

Function: quasifinite - test if a polynomial map is quasifinite

Calling sequence:
quasifinite(P,X,Q,Y,F)

Parameters:

- P - list of algebraic expressions, denoting an ideal
- X - list of variables appearing in P
- Q - list of algebraic expressions, denoting an ideal
- Y - list of variables appearing in Q
- F - list of algebraic expressions, denoting a polynomial map from $V(P)$ to $V(Q)$

Notes:

Probably several bugs

radicalfactor

Function: radicalfactor - factorise a radical ideal

Calling sequence:

radicalfactor(F,V)

Parameters:

- F - list of algebraic expressions, denoting a radical ideal
- V - list of variables

Notes:

The function does not check if (F) is indeed a radical ideal.

univexp

Function: univexp - compute the univariate exponent of a zero-dimensional ideal

Calling sequence:

univexp(F,V)

Parameters:

- F - list of algebraic expressions, denoting a zero-dimensional ideal
- V - list of variables

Notes:

Let (F) be a zero-dimensional ideal in $k[V]$. For $1 \leq i \leq \text{nops}(V)$, define $f[i]$ as the unique monic generator of $(F) / k[V[i]]$ and let $m[i]$ be the highest exponent that occurs in the square-free decomposition of $f[i]$. The univariate exponent of (F) is then defined as

$$m = 1 + (m[1] - 1) + (m[2] - 1) + \cdots + (m[\text{nops}(V)] - 1)$$

It has the property that for every f in $\text{rad}(F)$, $f^m \in (F)$.

univexp does not check if (F) is indeed a zero-dimensional ideal

zfactor

Function: zfactor - factorise a zero dimensional ideal

Calling sequence:

zfactor(F,V)

Parameters:

F - list of algebraic expressions, denoting a zero dimensional ideal

V - list of variable names

Notes:

zfactor returns a list of pairs [P,Q]. Each Q is a primary component of F and P is the corresponding prime.

zfactor does not check if F indeed represents a zero dimensional ideal.

zradical

Function: zradical - compute the radical ideal of a zero-dimensional ideal

Calling sequence:

zradical(F,V)

Parameters:

F - list of algebraic expressions, denoting a zero dimensional ideal.

V - list of variable names

Notes:

zradical will not check if the ideal (F) is indeed a zero dimensional ideal. If it is not, an infinite computation might occur.

zradicalfactor

Function: zradicalfactor - factorise a zero dimensional radical ideal

Calling sequence:

zradicalfactor(F,V)

Parameters:

F - list of algebraic expressions, denoting a zero dimensional radical ideal

V - list of variable names

Notes:

zradicalfactor does not check if F indeed represents a zero dimensional radical ideal.

B.4 Miscellaneous

algdeg

Function: algdeg - degree of an algebraic extension

Calling sequence:

algdeg(list,var)

Parameters:

list - list of algebraic expressions

var - list of names

Notes:

algdeg(list,var) computes the degree of the algebraic extension

$$\mathbb{Q}(\text{var}[1], \text{var}[2], \dots, \text{var}[r]) : \mathbb{Q}(\text{list}[1], \text{list}[2], \dots, \text{list}[n])$$

where $r = \text{nops}(\text{list})$ and $n = \text{nops}(n)$. 'list' should be a list of polynomials in the indeterminates listed in var.

If $\mathbb{Q}(\text{var}[1], \text{var}[2], \dots, \text{var}[r]) : \mathbb{Q}(\text{list}[1], \text{list}[2], \dots, \text{list}[n])$ is a transcendent extension, then algdeg will give output 0.

eqdetJH

Function: eqdetJH - build system of equations for determinant JH equal 0

Calling sequence:

eqdetJH(H, X)
eqdetJH(H, X, n)

Parameters:

H - a polynomial map
X - a list of indeterminates
n - (optional) power of JH

Notes:

The command eqdetJH(H,X) builds the set of equations which must hold if $\det(JH) = 0$.

The optional argument n can be used to specify a different value for the determinant: eqdetJH(H,X,n) gives a set of equations for $\det(JH) = n$.

eqfinorder

Function: eqfinorder - build system of equations for finite order maps

Calling sequence:

eqfinorder(F, X, n)

Parameters:

F - a polynomial map
X - list of indeterminates
n - order

Notes:

The command eqfinorder(F,X,n) computes the system of equations that must hold if $F^n = X$.

eqnilpotentJH

Function: eqnilpotentJH - find equations equivalent to JH is nilpotent

Calling sequence:

eqnilpotentJH(H, X)
 eqnilpotentJH(H, X, start, end)

Parameters:

H - a homogeneous polynomial map
 X - a list or set of indeterminates
 start, end - (optional) integers

Notes:

The command eqnilpotentJH(H,X) computes a set of equations with the property that these equations hold if and only if JH is nilpotent.

The set of equations is build by calculating lots of determinants. These determinants vary from 1x1 to NxN where N is the dimension of the map. Using the start and end integers, one can supply the range of the dimensions of the determinants.

Cf. [88] by Wright.

eqtraceJH

Function: eqtraceJH - build system of equations for trace JH equal 0

Calling sequence:

eqtraceJH(H, X)
 eqtraceJH(H, X, n)

Parameters:

H - a polynomial map
 X - a list of indeterminates
 n - (optional) power of JH

Notes:

The command eqtraceJH(H,X) builds the set of equations which must hold if $\text{Tr}(JH) = 0$.

The optional argument n can be used to specify upper bound n for which $\text{Tr}(JH^n) = 0$. I.e. eqtraceJH(H,X,n) gives a system which means: $\text{Tr}(JH) = 0$, $\text{Tr}(JH^2) = 0$, ..., $\text{Tr}(JH^n) = 0$.

Basically this procedure does the same as the algorithm presented in [45].

trdeg

Function: trdeg - transcendence degree

Calling sequence:

trdeg(list,var)

Parameters:

list - list of algebraic expressions var - list of names

Notes:

`trdeg(list,var)` computes the transcendence degree of the field extension

$$\mathbb{Q}(\text{list}[1], \text{list}[2], \dots, \text{list}[n]) : \mathbb{Q}$$

where $n = \text{nops}(\text{list})$. 'list' should be a list of polynomials in the indeterminates listed in `var`.

Samenvatting

Het Jacobivermoeden

Het Jacobivermoeden stamt uit 1939. Het is voor het eerst genoemd door Ott-Heinrich Keller. Sinds die tijd hebben reeds vele wiskundigen zich op dit vermoeden gestort. Zo op het eerste gezicht lijkt dat echter nog niet veel gevolgen te hebben gehad. Voor $n = 1$ is het duidelijk dat het vermoeden juist is. Voor $n \geq 2$ weet men echter nog steeds niet of het nu waar is of niet. Dat wil zeggen, men weet niet of het algemene geval waar is. Men heeft sinds de uitspraak van Keller wel voor grote klassen van veeltermafbeeldingen kunnen bewijzen dat het vermoeden waar is. Alvorens aan te geven wat de bijdrage van dit proefschrift is noemen wij eerst enkele essentiële definities.

Definitie 1

Een veeltermafbeelding $F = (F_1, \dots, F_n) : k^n \rightarrow k^n$ is een afbeelding van de vorm

$$(x_1, \dots, x_n) \mapsto (F_1(x_1, \dots, x_n), \dots, F_n(x_1, \dots, x_n))$$

waar $F_i \in k[x_1, \dots, x_n]$. Zo'n veeltermafbeelding F is inverteerbaar over k als er een veeltermafbeelding $G : k^n \rightarrow k^n$ bestaat met $x_i = G_i(F_1, \dots, F_n)$.

Definitie 2

Als $F : k^n \rightarrow k^n$ een veeltermafbeelding is dan heet de matrix

$$JF = \left(\frac{\partial}{\partial x_j} F_i \right)_{i,j \leq n}$$

de Jacobiaan van F .

In het vervolg van deze samenvatting korten wij x_1, \dots, x_n af tot X .

Definitie 3

Verskillende klassen van veeltermafbeeldingen die bekeken worden zijn:

- Kwadratisch homogene afbeeldingen: afbeeldingen $F : k^n \rightarrow k^n$ van de vorm $F = X + H$ waarvoor geldt dat alle monomen in de veeltermafbeelding H graad 2 hebben.
- Cubisch homogene afbeeldingen: afbeeldingen $F : k^n \rightarrow k^n$ van de vorm $F = X + H$ waarvoor geldt dat alle monomen in de veeltermafbeelding H graad 3 hebben.

- Cubisch lineaire afbeeldingen: cubisch homogene afbeeldingen waarvoor geldt dat elke component van H te schrijven is als een derdemacht van een lineaire vorm in X .

De laatste definitie is niet nodig om het vermoeden te formuleren, wel om de rest van deze samenvatting te kunnen volgen. Het roemruchte vermoeden luidt nu:

Vermoeden 4 (Jacobivermoeden)

Zij k een lichaam met $\text{kar}(k) = 0$ en zij $F : k^n \rightarrow k^n$ een veeltermafbeelding. Als $\det(JF) \in k^$ dan is F inverteerbaar.*

In hoofdstuk 1 wordt verteld wat er zoal bekend is omtrent dit vermoeden. Zoals reeds eerder gezegd, is het vermoeden waar als $n = 1$. Verder is bijvoorbeeld bekend voor $n = 2$ dat het vermoeden waar is als $\deg(F) \leq 100$. Voor hogere dimensies is de situatie een stuk ingewikkelder. Desondanks zijn ook hier vrij goede resultaten bekend. Bekendste is wel de stelling (theorem 1.42) van Bass, Connell en Wright die zegt dat als het vermoeden waar is voor alle cubisch homogene afbeeldingen, dan is het Jacobivermoeden ook in zijn algemeenheid waar. Een soortgelijke reductiestelling is bewezen door Drużkowski: theorem 1.46. Alleen keek hij niet naar cubisch homogene afbeeldingen maar naar de cubisch lineaire afbeeldingen. In zijn afstudeerscriptie [51] heeft de auteur reeds aangetoond dat het Jacobivermoeden waar is voor cubisch homogene afbeeldingen met $n = 4$. In hetzelfde werk heeft hij ook laten zien dat voor de cubisch lineaire afbeeldingen het Jacobivermoeden waar is voor $n \leq 7$.

De resultaten in dit proefschrift zijn meerendeels een gevolg van een helaas niet geslaagde zoektocht naar een tegenvoorbeeld voor het Jacobivermoeden.

Het Markus-Yamabe vermoeden

Hoofdstuk 2 gaat over een vermoeden dat nauw verwant is aan het Jacobivermoeden: het zogenaamde Jacobivermoeden voor differentiaalvergelijkingen, beter bekend onder de naam Markus-Yamabe vermoeden.

Vermoeden 5 (Markus-Yamabe vermoeden)

Zij $F : \mathbb{R}^n \rightarrow \mathbb{R}^n$ een C^1 afbeelding zodanig dat voor elke $x \in \mathbb{R}^n$ alle eigenwaarden van $JF(x)$ een negatief reëel deel hebben. Als $F(p) = 0$ voor een $p \in \mathbb{R}^n$ dan is p een globale attractor van het stelsel $\dot{x}(t) = F(x(t))$, d.w.z. elke oplossing van het stelsel gaat naar 0 als t naar oneindig gaat.

Dit vermoeden werd in 1960 door Markus en Yamabe geformuleerd (en in speciale gevallen bewezen) in een poging de bekende stabiliteits-resultaten van Lyapunov te globaliseren. In 1993 werd het twee-dimensionale geval zowel door Feßler als door Gutierrez bewezen. Het geval $n > 2$ bleef open. Dit Markus-Yamabe vermoeden is van belang voor het onderzoek van dynamische systemen. Maar daarnaast is het ook van belang voor het Jacobivermoeden. Martelli en Fournier bewezen dat het Jacobivermoeden waar is als het Markus-Yamabe vermoeden waar is voor veeltermafbeeldingen. Echter in 1995 werd door samenwerking tussen Cima, van den Essen, Gasull, Mañosas

en de auteur het laatste stukje van de puzzel gevonden: een polynomiaal tegenvoorbeeld tegen het Markus-Yamabe vermoeden voor $n = 3$ (en daarmee tevens voor alle $n \geq 3$). Zie [15]. Het tegenvoorbeeld staat op de kaft van dit proefschrift. Het is zo kinderlijk eenvoudig dat het vreemd klinkt dat het meer dan 35 jaar heeft geduurd voor het werd gevonden.

De klasse $\mathcal{H}_n(A)$ en de structuur $\mathcal{D}_n(A)$

De vraag is nu natuurlijk: waar komt dit eenvoudige tegenvoorbeeld vandaan? Het antwoord wordt gegeven in hoofdstuk 3. Hierin definiëren wij een nieuwe klasse van veeltermafbeeldingen H met nilpotente Jacobiaan JH en laten onder andere zien dat elke $X + H$ inverteerbaar is. Precieser, voor iedere $n \geq 1$ en elke commutatieve ring A definiëren wij de klasse $\mathcal{H}_n(A)$ als volgt:

Definitie 6

Zij A een commutatieve ring. Dan

- $\mathcal{H}_1(A) = A$ en voor $n \geq 2$
- $H \in \mathcal{H}_n(A)$ dan en slechts dan als er bestaan
 1. $T \in \text{Mat}_n(A)$,
 2. $c \in A^n$ en
 3. $H_\star \in \mathcal{H}_{n-1}(A[x_n])$

zodanig dat

$$H = \text{Adj}(T) \begin{pmatrix} H_\star \\ 0 \end{pmatrix} \Big|_{TX} + c$$

waar $\text{Adj}(T)$ de geadjungeerde van T en $|_{TX}$ de ‘evaluatie in de vector TX ’ is.

De kracht van deze klasse $\mathcal{H}_n(A)$ is gelegen in het feit dat het een nieuwe grote klasse van inverteerbare veeltermafbeeldingen geeft, namelijk alle $F = X + H$ met $H \in \mathcal{H}_n(A)$, die als testcase voor allerlei problemen over inverteerbare veeltermafbeeldingen gebruikt kan worden.

Hoofdstuk 4 gaat verder met de klasse $\mathcal{H}_n(A)$. De inductieve definitie hier boven geeft al aan dat een afbeelding uit deze klasse veel van doen heeft met een serie matrices en vectoren. In dit hoofdstuk wordt dat geformaliseerd door de introductie van het begrip $\mathcal{D}_n(A)$. Dit is een structuur van matrices en vectoren die via een afbeelding $E_n : \mathcal{D}_n(A) \rightarrow k[X]^n$ een verband legt tussen de veeltermafbeeldingen enerzijds en de matrices en vectoren anderzijds. De kracht van deze $\mathcal{D}_n(A)$ -structuur is dat een eenvoudige beschrijving reeds een enorm ingewikkeld uitzijnde afbeelding tot gevolg kan hebben (example 4.10).

In hoofdstuk 5 wordt deze $\mathcal{D}_n(A)$ -structuur –en dus ook de $\mathcal{H}_n(A)$ -structuur– gekoppeld aan lokaal nilpotente derivaties. Er wordt bewezen dat elke veeltermafbeelding $F = X + H$ met H in de $\mathcal{H}_n(A)$ -klasse kan worden geschreven als een eindig product van $\exp(D_i)$'s waarbij de D_i 's lokaal nilpotente derivaties zijn. Deze stelling (theorem 5.12) wordt vervolgens gebruikt om aan te tonen dat deze veeltermafbeeldingen F stabiel tam zijn en wel zo dat toevoegen van $n - 1$ nieuwe variabelen gegarandeerd tot tamheid leidt.

Nilpotentie

Hoofdstuk 6 geeft het verband aan tussen sterk nilpotent zijn (definition 6.1) en op driehoeksvorm te brengen zijn met lineaire afbeeldingen (definition 1.9). Ook hier is sprake van een dan en slechts dan relatie tussen beide eigenschappen. Aangezien sterke nilpotentie met een computeralgebra pakket eenvoudig te testen is, is het nu dus ook eenvoudig om te bepalen of een veeltermafbeelding lineair op driehoeksvorm te brengen is.

Deze sterke nilpotentie is ook belangrijk in verband met lineariseerbaarheid van afbeeldingen. Afbeeldingen met een sterk nilpotente Jacobiaan blijken lineariseerbaar te zijn (theorem 6.18).

Verder wordt gebaseerd op onderzoek van Gorni, Tutaj en Zampieri nog een andere vorm van nilpotentie besproken: D-nilpotentie. Deze vorm is sterker dan sterke nilpotentie: elke D-nilpotente matrix is sterk nilpotent maar de omkering geldt niet. Deze D-nilpotentie is vooral handig in verband met het op driehoeksvorm brengen. Alleen nu gaat het niet via lineaire afbeeldingen maar slechts via permutaties. Ook hier krijgen wij een criterium dat een computeralgebra systeem redelijk eenvoudig in staat stelt om te beslissen of een veeltermafbeelding uitsluitend door permutaties reeds op driehoeksvorm is te brengen.

Berekeningen

De laatste twee echte hoofdstukken zijn gebaseerd op berekeningen uitgevoerd met Maple. In hoofdstuk 7 geven wij –onder de aanname dat een bepaalde afhankelijkheidsrelatie geldt (problem 7.1)– een complete klassificatie van alle kwadratisch homogene veeltermafbeeldingen in dimensie $n = 5$. Dat wil zeggen er wordt een lijst van 19 standaardvormen gegeven waarvoor geldt dat elke kwadratisch homogene afbeelding via een lineaire conjugatie op één van deze standaardvormen kan worden gebracht. Deze conjugaties zijn een gevolg van het op theoretische gronden reduceren van het aantal variabelen voordat de daadwerkelijke berekeningen gestart worden. Vervolgens wordt bekeken hoe het zit met de driehoeksvormen binnen deze lijst. Via de test op sterke nilpotentie is eenvoudig te achterhalen dat één van deze vormen niet op driehoeksvorm te brengen is via lineaire conjugaties. Alle andere vormen zijn wel op die manier op driehoeksvorm te brengen. Conclusie is dan ook dat er twee duidelijk verschillende klassen van kwadratisch homogene afbeeldingen bestaan in dimen-

sie vijf. In appendix A worden lijsten weergegeven die door Maple zijn gegenereerd tijdens deze berekeningen. Deze lijsten zorgen voor een mogelijkheid tot verificatie van de resultaten.

In hoofdstuk 8 wordt ook naar veeltermafbeeldingen in dimensie $n = 5$ gekeken. Alleen hier naar cubisch lineaire afbeeldingen in plaats van naar kwadratisch homogene afbeeldingen. De gebruikte methode is analoog. Eerst wordt de beginsituatie zo eenvoudig mogelijk gemaakt door op theoretisch niveau het aantal variabelen te beperken. Vervolgens worden de echte berekeningen uitgevoerd. Dit levert weer een lijst met standaardvormen. In hoofdstuk 7 werden deze standaardvormen met elkaar vergeleken ten opzichte van de trianguleerbaarheid. Nu worden deze standaardvormen met elkaar vergeleken ten opzichte van Meisters' cubic similarity relatie. Deze relatie is erop gebaseerd dat het niet vanzelfsprekend is dat cubisch lineaire afbeeldingen via conjugaties met lineaire afbeeldingen weer cubisch lineaire afbeeldingen oplevert. Het eindresultaat van dit hoofdstuk is nu dat er een verzameling normaalvormen van afbeeldingen wordt gegeven waarvoor geldt dat elke cubisch lineaire afbeelding in dimensie vijf via lineaire conjugatie tot één van deze normaalvormen is terug te voeren. Ook hier zijn de details van de berekeningen opgeborgen in appendix A.

De berekeningen voor deze twee hoofdstukken waren niet goed mogelijk geweest zonder gebruik te maken van het zogenaamde Jacobi pakket. Een collectie procedures die het mogelijk maakt om gemakkelijk met veeltermafbeeldingen te kunnen rekenen. In appendix B wordt een korte beschrijving gegeven van de belangrijkste mogelijkheden van dit pakket.

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This is probably the most difficult section for me to write. A lot of people have been important to me during the last four years. Where should I draw the line between being included or not? Fortunately there is no algorithm or theorem to verify that my choices are right or wrong.

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Engelbert Hubbers

Curriculum Vitae

The author was born on May 8, 1970 in Nijmegen, The Netherlands. He is the second of four sons of Martin Hubbers and Haijoke Tieks.

From 1981 through 1987 he received his secondary education at the Eckart College in Eindhoven. In 1987 he started his study Mathematics at the University of Nijmegen. After passing his propaedeutic exam in 1988, he started a second study in Computer Science. Together with B. van Linder he wrote his Master's thesis in Computer Science 'Default Ionic Logic: Its Syntax and Semantics' in 1992 (cf. [62], [63]). Supervisor Prof. J.-J.Ch. Meyer. Under the supervision of dr. A. van den Essen, he wrote his Master's thesis in Mathematics 'The Jacobian Conjecture: Cubic Homogeneous Maps in Dimension Four' (cf. [51]). February 1994 he received his Master's degree in Mathematics, in Computer Science and as a bonus in the combination of Mathematics and Computer Science.

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