$\mathcal{D}_n(A)$ for a class of polynomial automorphisms and stably tameness

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Abstract

In this paper we introduce a set, denoted by $\mathcal{D}_n(A)$, for every commutative ring A and every positive integer n. It is shown that the elements of this set can be used to give an explicit description of the class $\mathcal{H}_n(A)$ introduced in [5]. We deduce that each polynomial map of the form $F = X + H$ with $H \in \mathcal{H}_n(A)$ can be written as a finite product of automorphisms of the form $\exp(D)$, where each D is a locally nilpotent derivation satisfying $D^2(X_i) = 0$ for all i. Furthermore we deduce that all such F's are stably tame.

1 Notations, definitions and an explicit description of the class $\mathcal{H}_n(A)$

1.1 Notations

Throughout this paper A denotes an arbitrary commutative ring and $A[X] := A[X_1, \ldots, X_n]$ denotes the polynomial ring in n variables over A. Furthermore if $G = (G_1, \ldots, G_n) \in A[X]^n$ and $S = (S_{ij}(X)) \in M_{p,q}(A[X])$ then $S(G)$ or $S_{|G}$ denotes the $p \times q$ matrix $(S_{ij}(G_1, \ldots, G_n))_{i,j}$. In particular if $F \in A[X]^n$ ($= M_{n,1}(A[X])$) then the composition of the polynomial maps F and G, denoted $F \circ G$, is equal to $F(G)$.

Matrix multiplication will be denoted by the symbol '$*$'. So if $S, T \in M_n(A[X])$ then the matrix product of S and T is denoted by $S * T$. By X we denote the column vector $(X_1, \ldots, X_n)^t$. In the sequel we also need another multiplication in $M_n(A[X])$, which we denote by '$\triangle$'. This multiplication is defined as follows:

$$S \triangle T := S(T * X) * T$$

for all $S, T \in M_n(A[X])$.

One easily verifies that this multiplication is associative, so it makes sense to write

$$S_1 \triangle S_2 \triangle \cdots \triangle S_n$$

for each n-tuple $S_1, \ldots, S_n$ in $M_n(A[X])$. Sometimes we need to extend a vector of length $1 \leq p \leq n-1$ or a $p \times p$ matrix to a vector of length n respectively an $n \times n$ matrix. This is done as follows: let $1 \leq p \leq n-1$, $c \in A[X]^p$ and $T \in M_p(A[X])$. Then $\tilde{c}^n$ denotes the vector

$$\tilde{c}^n = \begin{pmatrix} c \\ 0 \\ \vdots \\ 0 \end{pmatrix} \in A[X]^n,$$

obtained by extending c by $n-p$ zeros and $\tilde{T}^n$ denotes the matrix

$$\tilde{T}^n = \begin{pmatrix} T & 0 \\ 0 & I_{n-p} \end{pmatrix} \in M_n(A[X]),$$

obtained by extending T with the $n-p \times n-p$ identity matrix. To simplify the notations we drop the superscript 'n' and write $\tilde{c}$ and $\tilde{T}$, even sometimes when it is clear from the context that we mean $\tilde{c}^{n-1}$ respectively $\tilde{T}^{n-1}$ instead of $\tilde{c}^n$ respectively $\tilde{T}^n$.

Finally the adjoint of a matrix T is denoted by $\mathrm{Adj}(T)$ and if $a_1, \ldots, a_p$ are elements of a (non-necessary commutative) ring then $\prod_{i=1}^{p} a_i$ denotes the element $a_1 \cdots a_p$.

1.2 $\mathcal{D}_n(A)$ and the class $\mathcal{H}_n(A)$

In [5] we introduced a new class of polynomial maps, denoted by $\mathcal{H}_n(A)$, and showed that for each $H \in \mathcal{H}_n(A)$ the Jacobian matrix JH is nilpotent and that the polynomial map $F = X + H$ is invertible over A with $\det(JF) = 1$.

Let us recall the definition of $\mathcal{H}_n(A)$.

Definition 1.1 First if $n = 1$ we define $\mathcal{H}_1(A) = A$. If $n \geq 2$ we define $\mathcal{H}_n(A)$ inductively as follows: let $H \in A[X]^n$, then $H \in \mathcal{H}_n(A)$ if and only if there exist $T \in M_n(A)$, $c \in A^n$ and $H_\star \in \mathcal{H}_{n-1}(A[X_n])$ such that

$$H = \mathrm{Adj}(T) * \begin{pmatrix} H_\star \\ 0 \end{pmatrix}_{|T*X} + c. \tag{1}$$

The main aim of this section is to give an explicit description of the elements of $\mathcal{H}_n(A)$. Therefore we introduce some useful objects.

Definition 1.2 Let $n \geq 2$. Then $\mathcal{D}_n(A)$ is the set of $(2n-1)$-tuples

$$(T, c) := (T_2, \ldots, T_n, c_1, \ldots, c_n)$$

where $T_n \in M_n(A)$, $T_i \in M_i(A[X_{i+1}, \ldots, X_n])$ for all $2 \leq i \leq n-1$, $c_n \in A^n(= M_{n,1}(A))$ and $c_i \in M_{i,1}(A[X_{i+1}, \ldots, X_n])$ for all $1 \leq i \leq n-1$.

If $n \geq 3$ we get a natural map $\pi : \mathcal{D}_n(A) \to \mathcal{D}_{n-1}(A[X_n])$ defined by

$$\pi((T_2, \ldots, T_n, c_1, \ldots, c_n)) = (T_2, \ldots, T_{n-1}, c_1, \ldots, c_{n-1}).$$

Instead of $\pi((T, c))$ we often write (T', c').

Definition 1.3 Let $n \geq 2$ and $0 \leq p \leq n-2$. Then

$$E_{n,p} : \mathcal{D}_n(A) \to A[X]^n$$

is given by

1. $E_{n,0}((T, c)) := \mathrm{Adj}(T_n) * \tilde{c}_{n-1}\ {}_{|T_n * X}$ for all $(T, c) \in \mathcal{D}_n(A)$.
2. If $n \geq 3$ and $1 \leq p \leq n-2$, then inductively (with respect to n)

$$E_{n,p}((T, c)) := \mathrm{Adj}(T_n) * \begin{pmatrix} E_{n-1,p-1}((T', c')) \\ 0 \end{pmatrix}_{|T_n * X}$$

Instead of $E_{n,p}((T, c))$ we simply write $E_{n,p}(T, c)$.

Now we are able to give the main result of this section.

Proposition 1.4 *Let $n \geq 2$ and $H \in A[X]^n$. Then $H \in \mathcal{H}_n(A)$ if and only if there exists $(T, c) \in \mathcal{D}_n(A)$ such that*

$$H = \sum_{p=0}^{n-2} E_{n,p}(T, c) + c_n.$$

Proof. By induction on n. The case $n = 2$ is obvious, so let $n \geq 3$. Then

$$H = \mathrm{Adj}(T_n) * \begin{pmatrix} H_\star \\ 0 \end{pmatrix}_{|T_n * X} + c_n$$

where $T_n \in M_n(A)$, $c_n \in A^n$ and $H_\star \in \mathcal{H}_{n-1}(A[X_n])$. So by the induction hypothesis we have

$$H_\star = \sum_{p=0}^{n-3} E_{n-1,p}(T^*, c^*) + c^*_{n-1}$$

for some $(T^*, c^*) \in \mathcal{D}_{n-1}(A[X_n])$. Put $(T, c) := (T^*, T_n, C^*, c_n)$ and observe that $(T, c) \in \mathcal{D}_n(A)$ and $(T', c') = (T^*, c^*)$. So

$$\begin{aligned} H &= \sum_{p=0}^{n-3} \mathrm{Adj}(T_n) * E_{n-1,p}(T', c')_{|T_n * X} + \mathrm{Adj}(T_n) * \begin{pmatrix} c^*_{n-1} \\ 0 \end{pmatrix}_{|T_n * X} + c_n \\ &= \sum_{p=1}^{n-2} E_{n,p}(T, c) + E_{n,0}(T, c) + c_n \\ &= \sum_{p=0}^{n-2} E_{n,p}(T, c) + c_n. \end{aligned}$$

□

Proposition 1.5 *Let $n \geq 2$, $0 \leq p \leq n-2$ and $(T,c) \in \mathcal{D}_n(A)$. Then*

$$E_{n,p}(T,c) = \mathrm{Adj}(\tilde{T}_{n-p} \Delta \cdots \Delta \tilde{T}_{n-1} \Delta T_n) * \tilde{c}_{n-p-1}\ |_{(\tilde{T}_{n-p}\Delta\cdots\Delta\tilde{T}_{n-1}\Delta T_n)*X}$$

Proof. By induction on p. The case $p=0$ is obvious. So let $p \geq 1$. Then

$$\begin{aligned}
& E_{n,p}(T,c) \\
= \;& \mathrm{Adj}(T_n) * \begin{pmatrix} E_{n-1,p-1}(T',c') \\ 0 \end{pmatrix}_{|T_n * X} \\
= \;& \text{(by the induction hypothesis)} \\
& \mathrm{Adj}(T_n) * \left[\mathrm{Adj}(\tilde{T}_{n-p} \Delta \cdots \Delta \tilde{T}_{n-1})_{|T_n*X} * \left(\tilde{c}_{n-p-1}\ |_{(\tilde{T}_{n-p}\Delta\cdots\Delta\tilde{T}_{n-1})*X} \right)_{|T_n*X} \right] \\
= \;& \mathrm{Adj}((\tilde{T}_{n-p} \Delta \cdots \Delta \tilde{T}_{n-1})_{|T_n*X} * T_n) * \tilde{c}_{n-p-1}((\tilde{T}_{n-p} \Delta \cdots \Delta \tilde{T}_{n-1})_{|T_n*X}) * T_n * X \\
= \;& \mathrm{Adj}(\tilde{T}_{n-p} \Delta \cdots \Delta \tilde{T}_{n-1} \Delta T_n) * \tilde{c}_{n-p-1}\ |_{(\tilde{T}_{n-p}\Delta\cdots\Delta\tilde{T}_{n-1}\Delta T_n)*X}
\end{aligned}$$

□

Example 1.6 Consider the polynomial map $F := X + H : \mathbb{C}^4 \to \mathbb{C}^4$ where H equals

$$\begin{pmatrix} -X_2X_4^2 - e_4X_3^2X_4 - 2\frac{m_4}{g_4}X_2X_3X_4 - g_4X_1X_3X_4 - k_4X_3^3 - \frac{m_4^2}{g_4^2}X_2X_3^2 - m_4X_1X_3^2 \\ -X_3X_4^2 - e_3X_3^2X_4 + g_4X_2X_3X_4 - k_3X_3^3 + m_4X_2X_3^2 + g_4^2X_1X_3^2 \\ -\frac{1}{3}X_4^3 \\ 0 \end{pmatrix}$$

and $e_3, k_3, e_4, g_4, k_4, m_4 \in \mathbb{C}$ and $g_4 \neq 0$. This F is invertible. In fact if we take $P = P^{-1} = (X_4, X_3, X_2, X_1)$, we have that PFP is one of the eight representatives of the cubic homogeneous maps in dimension four as given by the second author in [6], also published in [4, Theorem 2.10].

Now consider the following element (T,c) of $\mathcal{D}_4(\mathbb{C})$ where

$$T = \left(\begin{pmatrix} 1 & 0 \\ g_4^2X_3 & g_4X_4 + m_4X_3 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \right)$$

and

$$c = \left(\begin{pmatrix} \frac{-1}{g_4^2}X_2 \end{pmatrix}, \begin{pmatrix} -X_3^2(e_4X_4 + k_4X_3) \\ -X_3X_4^2 - e_3X_4X_3^2 - k_3X_3^3 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ \frac{-1}{3}X_4^3 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \right)$$

Our claim is that

$$H = \sum_{p=0}^{2} E_{4,p}(T,c) + c_4.$$

To prove this we will compute $E_{4,0}$, $E_{4,1}$ and $E_{4,2}$ by the method of proposition 1.5. Note that $c_4 = 0$. Since $T_4 = \tilde{T}_3 = I_4$, $E_{4,0}$ and $E_{4,1}$ are easy:

$$\begin{aligned} E_{4,0} &= \mathrm{Adj}(T_4) * \tilde{c}_3\,_{|T_4 * X} = \tilde{c}_3 = \begin{pmatrix} 0 \\ 0 \\ \frac{-1}{3}X_4^3 \\ 0 \end{pmatrix} \\ E_{4,1} &= \mathrm{Adj}(\tilde{T}_3 \triangle T_4) * \tilde{c}_2\,_{\left|(\tilde{T}_3 \triangle T_4) * X\right.} = \tilde{c}_2 = \begin{pmatrix} -X_3^2(e_4X_4 + k_4X_3) \\ -X_3X_4^2 - e_3X_4X_3^2 - k_3X_3^3 \\ 0 \\ 0 \end{pmatrix} \end{aligned}$$

Before we compute $E_{4,2}$ we present the following identities:

$$\begin{aligned} \tilde{T}_2 \triangle \tilde{T}_3 \triangle T_4 &= \tilde{T}_2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ g_4^2X_3 & g_4X_4 + m_4X_3 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \\ \mathrm{Adj}(\tilde{T}_2 \triangle \tilde{T}_3 \triangle T_4) &= \begin{pmatrix} g_4X_4 + m_4X_3 & 0 & 0 & 0 \\ -g_4^2X_3 & 1 & 0 & 0 \\ 0 & 0 & g_4X_4 + m_4X_3 & 0 \\ 0 & 0 & 0 & g_4X_4 + m_4X_3 \end{pmatrix} \\ (\tilde{T}_2 \triangle \tilde{T}_3 \triangle T_4) * X &= \begin{pmatrix} X_1 \\ g_4^2X_1X_3 + g_4X_2X_4 + m_4X_2X_3 \\ X_3 \\ X_4 \end{pmatrix} \\ \tilde{c}_1\,_{\left|(\tilde{T}_2 \triangle \tilde{T}_3 \triangle T_4) * X\right.} &= \begin{pmatrix} -X_1X_3 - \frac{1}{g_4}X_2X_4 - \frac{m_4}{g_4^2}X_2X_3 \\ 0 \\ 0 \\ 0 \end{pmatrix} \end{aligned}$$

and finally

$$\begin{aligned} E_{4,2} &= \mathrm{Adj}(\tilde{T}_2 \triangle \tilde{T}_3 \triangle T_4) * \tilde{c}_1\,_{\left|(\tilde{T}_2 \triangle \tilde{T}_3 \triangle T_4) * X\right.} \\ &= \begin{pmatrix} -\left(X_4 + \frac{m_4}{g_4}X_3\right)\left(g_4X_1X_3 + X_2X_4 + \frac{m_4}{g_4}X_2X_3\right) \\ X_3(g_4^2X_1X_3 + g_4X_2X_4 + m_4X_2X_3) \\ 0 \\ 0 \end{pmatrix} \end{aligned}$$

It is easy to verify that $H = E_{4,0} + E_{4,1} + E_{4,2} + c_4$, which was our claim.

2 Nice derivations

Let $B := A[x_1, \ldots, x_n]$ be a finitely generated A-algebra and D a subset of $\mathrm{Der}_A(B)$. By B^D we denote the set of all $b \in B$ such that $d(b) = 0$ for all $d \in D$.

Definition 2.1 Let $D \subset \mathrm{Der}_A(B)$ a finite subset and $\tau \in \mathrm{Der}_A(B)$.

1. We say that τ is *derived from D in at most one step* if τ is of the form $\tau = \sum_{d\in D} b_d d$, where $b_d \in B^D$ for all $d \in D$.

2. Let $m \geq 2$. We say that τ is *derived from D in at most m steps* if there exists a sequence of finite subsets

$$D = D_0, D_1, D_2, \ldots, D_m$$

of $\mathrm{Der}_A(B)$ such that $\tau \in D_m$ and all elements of D_i are derived from D_{i-1} in at most one step, for all $1 \leq i \leq m$. If furthermore the elements of D satisfy $d_1 d_2(x_i) = 0$ for all $d_1, d_2 \in D$ and all i, then τ is called *nice of order $\leq m$*, with respect to $x_1, \ldots, x_n$ and D.

Proposition 2.2 *Notations as in definition 2.1. If $d_1 d_2(x_i) = 0$ for all $d_1, d_2 \in D$ and all i, then $d_1 d_2(x_i) = 0$ for all $d_1, d_2 \in D_m$ and all i. In particular $d^2(x_i) = 0$ for every nice derivation.*

Proof. We use induction on m. The case $m = 0$ is obvious since $D_0 = D$. Now let $m \geq 1$. Then $d_1 = \sum_{d\in D_{m-1}} b_d d$, $d_2 = \sum_{d'\in D_{m-1}} b'_{d'} d'$ with $b_d, b_{d'} \in B^{D_{m-1}}$. Then

$$d_1 d_2(x_i) = \sum_{d,d'} bd(b'_{d'})d(x_i) + \sum_{d,d'} b_d b'_{d'} dd'(x_i). \tag{2}$$

Now observe that $d(b'_{d'}) = 0$ since $b'_{d'} \in B^{D_{m-1}}$ and $d \in D_{m-1}$. Finally the induction hypothesis gives $dd'(x_i) = 0$ for all $d, d' \in D_{m-1}$ and all i, so (2) implies $d_1 d_2(x_i) = 0$. □

We demonstrate these aspects by the so-called Winkelmann derivation. See [10].

Example 2.3 Let $\tau = (1 + X_4 X_2 - X_5 X_3)\partial_{X_1} + X_5 \partial_{X_2} + X_4 \partial_{X_3}$, a derivation on $B := A[X_1, X_2, X_3, X_4, X_5]$. Let $D = \{\partial_{X_1}, \partial_{X_2}, \partial_{X_3}\}$. Then τ is nice of order two with respect to X_1, X_2, X_3, X_4, X_5 and D. To show that this is true, we present a sequence of finite subsets of $\mathrm{Der}_A(B)$,

$$D = D_0, D_1, D_2$$

Take $D_1 := \{\partial_{X_1}, X_5\partial_{X_2} + X_4\partial_{X_3}\}$ and $D_2 := \{\tau\}$. Note that in definition 2.1 it is not demanded that the set D_i of this sequence is a subset of D_{i+1}. The only demand is that each D_i is a finite subset of $\mathrm{Der}_A(B)$. Since $X_4, X_5 \in B^D$ it follows immediately that ∂_{X_1} and $X_5\partial_{X_2} + X_4\partial_{X_3}$ are derived from D in one step. And from $1 + X_4X_2 - X_5X_3 \in B^{D_1}$ it follows that τ is derived from D_1 in one step. Obviously we have $d_1 d_2(X_i) = 0$ for all $d_1, d_2 \in D$ and hence with proposition 2.2 also $\tau^2(X_i) = 0$.

3 Derivations associated to polynomial maps

The main aim of this section is to show that for each $0 \leq p \leq n-2$ the polynomial map $X + E_{n,p}(T,c)$ (where $(T,c) \in \mathcal{D}_n(A)$) is of the form $\exp(d)$, for some nice A-derivation d of $A[X]$. Observe that d is locally nilpotent if d is nice with respect to $X_1, \ldots, X_n$ since $d^2(X_i) = 0$ for all i, by proposition 2.2.

In order to prove this result (see theorem 3.3), we need to generalise some of the notions of section 1 to arbitrary finitely generated A-algebras. So let $B := A[x_1, \ldots, x_n]$ be a finitely generated A-algebra and $\varphi : A[X_1, \ldots, X_n] \to B$ the A-ringhomomorphism defined by $\varphi(X_i) = x_i$ for all i. For each $p, q \geq 1$ consider the natural extension

$$\varphi : M_{p,q}(A[X_1, \ldots, X_n]) \to M_{p,q}(B).$$

Then for each $(T,c) \in \mathcal{D}_n(A)$ we define

$$E_{n,p}(T,c)(x) := \varphi(E_{n,p}(T,c)) \in B^n.$$

Now let $(\partial_1, \ldots, \partial_n)$ be an n-tuple of A-derivations of B. To each vector $b = (b_1, \ldots, b_n)^t \in B^n$ we associate the following A-derivation of B:

$$D(b; \partial_1, \ldots, \partial_n) := b_1\partial_1 + \cdots + b_n\partial_n \quad (= b^t * \begin{pmatrix} \partial_1 \\ \vdots \\ \partial_n \end{pmatrix}).$$

To formulate the next lemma we need some more notations: let $(T,c) \in \mathcal{D}_n(A)$. Put

$$\begin{aligned} (x'_1, \ldots, x'_n)^t &:= T_n * (x_1, \ldots, x_n)^t \\ (\partial'_1, \ldots, \partial'_n) &:= (\mathrm{Adj}(T_n))^t * (\partial_1, \ldots, \partial_n)^t \\ x'' &:= (x'_1, \ldots, x'_{n-1}) \\ (T'', c'') &:= (T'(X_n = x'_n), c'(X_n = x'_n)) \in \mathcal{D}_{n-1}(A[x'_n]) \end{aligned}$$

Lemma 3.1 *Let $n \geq 3$ and $1 \leq p \leq n-2$. Then*

$$D(E_{n,p}(T,c)(x); \partial_1, \ldots, \partial_n) = D(E_{n-1,p-1}(T'', c'')(x''); \partial'_1, \ldots, \partial'_{n-1}).$$

Proof.

$$\begin{aligned} & D(E_{n,p}(T,c)(x); \partial_1, \ldots, \partial_n) \\ = \; & (E_{n,p}(T,c)(x))^t * \begin{pmatrix} \partial_1 \\ \vdots \\ \partial_n \end{pmatrix} \\ = \; & \begin{pmatrix} (E_{n-1,p-1}(T',c')_{|T_n * x})^t & 0 \end{pmatrix} * (\mathrm{Adj}(T_n))^t * \begin{pmatrix} \partial_1 \\ \vdots \\ \partial_n \end{pmatrix} \end{aligned}$$

$$\begin{aligned} &= \begin{pmatrix} (E_{n-1,p-1}(T'',c'')(x''))^t & 0 \end{pmatrix} * \begin{pmatrix} \partial_1' \\ \vdots \\ \partial_n' \end{pmatrix} \\ &= D(E_{n-1,p-1}(T'',c'')(x''); \partial_1', \ldots, \partial_{n-1}') \end{aligned}$$

□

Lemma 3.2 *Notations as above. Let $a \in A$ and let $\partial_1, \ldots, \partial_n$ be A-derivations of B such that $\partial_i(x_j) = a\delta_{ij}$ for all i, j. Then*

$$\partial_i'(x_j') = a \det(T_n)\delta_{ij}$$

for all i, j.

Proof. Denote the i-th column of $\mathrm{Adj}(T_n)$ by $(t_{1i}^*, \ldots, t_{ni}^*)^t$ and the j-th row of T_n by $(t_{j1}, \ldots, t_{jn})$. Then

$$\begin{aligned} \partial_i'(x_j') &= \left(\sum_{s=1}^n t_{si}^* \partial_s\right)\left(\sum_{s=1}^n t_{js} x_s\right) \\ &= \sum_{s=1}^n a t_{si}^* t_{js} \\ &= a(T_n * \mathrm{Adj}(T_n))_{ji} \\ &= a \det(T_n)\delta_{ij} \end{aligned}$$

□

Now we are able to prove:

Theorem 3.3 *Let $\partial_1, \ldots, \partial_n$ be A-derivations on $A[x_1, \ldots, x_n]$ such that there exists an element $a \in A$ such that $\partial_i(x_j) = a\delta_{ij}$ for all i, j. Let $(T, c) \in \mathcal{D}_n(A)$. Then the A-derivation $d := D(E_{n,p}(T,c)(x); \partial_1, \ldots, \partial_n)$ is nice with respect to $x_1, \ldots, x_n$ and $D_0 := \{\partial_1, \ldots, \partial_n\}$, for all $n \geq 2$ and all $0 \leq p \leq n-2$.*

Proof.

1. The hypothesis on the ∂_i imply that $dd'(x_i) = 0$ for all $d, d' \in D_0$ and all i.

2. First we consider the case $p = 0$. Then

   $$E_{n,0}(T,c) = \mathrm{Adj}(T_n) * \tilde{c}_{n-1\ |T_n * X}.$$

   So

   $$d = (\tilde{c}_{n-1\ |T_n * X})^t * (\mathrm{Adj}(T_n))^t * \begin{pmatrix} \partial_1 \\ \vdots \\ \partial_n \end{pmatrix}.$$

Write $\tilde{c}_{n-1}^t = (\gamma_1(X_n), \ldots, \gamma_{n-1}(X_n), 0)$. Then the definition of x'_n and the ∂'_j imply that

$$d = (\gamma_1(x'_n), \ldots, \gamma_{n-1}(x'_n), 0) * (\partial'_1, \ldots, \partial'_n)^t = \sum_{i=1}^{n-1} \gamma_i(x'_n)\partial'_i \tag{3}$$

Put $D_1 := \{\partial'_1, \ldots, \partial'_{n-1}\}$ and observe that $D_1 \subset \mathrm{Der}_A(B)$ and that each element of D_1 is derived from D_0 in at most one step. Finally since $\partial'_i(x'_n) = 0$ for all $1 \le i \le n-1$ (by lemma 3.2) we get that $\gamma_i(x'_n) \in B^{D_1}$ for all $1 \le i \le n-1$. So (3) implies that d is derived from D_1 in at most one step. Consequently d is derived from D_0 in at most two steps. So d is nice with respect to $x_1, \ldots, x_n$ and D_0 by case 1.

3. Now we prove the theorem by induction on n. If $n = 2$, then $p = 0$ and we are in case 2. So let $n \ge 3$. By case 2 we may assume that $p \ge 1$. Then by lemma 3.1 we have

$$d = D(E_{n-1,p-1}(T'', c'')(x''); \partial'_1, \ldots, \partial'_{n-1})$$

with $(T'', c'') \in \mathcal{D}_{n-1}(A[x'_n])$. By lemma 3.2 we can apply the induction hypothesis to the ring $A[x'_n]$ and the $(n-1)$-tuple of $A[x'_n]$-derivations $\partial'_1, \ldots, \partial'_{n-1}$ on the $A[x'_n]$-algebra $B' := A[x'_n][x'_1, \ldots, x'_{n-1}]$. So the $A[x'_n]$-derivation d on B' is nice with respect to $D'_0 := \{\partial'_1, \ldots, \partial'_{n-1}\}$ and $x'_1, \ldots, x'_{n-1}$. So there exists a sequence

$$D'_0, D'_1, \ldots, D'_m$$

of finite subsets of $\mathrm{Der}_{A[x'_n]}(B')$ such that $d \in D'_m$ and D'_i is derived from D'_{i-1} in at most one step for all $1 \le i \le m$. Now observe that $D'_0 \subset \mathrm{Der}_A(B)$ and that $B' \subset B$ since by definition obviously $x'_i \in B$ for all i. Consequently if d' is an $A[x'_n]$-derivation of B' derived from D'_0 in at most one step, then $d' \in \mathrm{Der}_A(B)$. Hence $D'_1 \subset \mathrm{Der}_A(B)$. Arguing in a similar way we conclude by induction on i that $D'_i \subset \mathrm{Der}_A(B)$ for all $0 \le i \le m$. Since as remarked in case 2 above, all elements of D'_0 ($= D_1$ in case 2) are derived from D_0 in at most one step we deduce that d is derived from D_0 in at most $m+1$ steps. Just define $D_i := D'_{i-1}$ for all $1 \le i \le m+1$. Hence d is nice with respect to $x_1, \ldots, x_n$ and D_0 by 1.

□

Corollary 3.4 *Let $(T, c) \in \mathcal{D}_n(A)$ and $0 \le p \le n-2$. Put*

$$D := D\left(E_{n,p}(T, c); \frac{\partial}{\partial X_1}, \ldots, \frac{\partial}{\partial X_n}\right).$$

Then D is nice with respect to $X_1, \ldots, X_n$ and $\{\frac{\partial}{\partial X_1}, \ldots, \frac{\partial}{\partial X_n}\}$. Furthermore we have $\exp(D) = X + E_{n,p}(T, c)$ and the inverse map is given by $\exp(-D) = X - E_{n,p}(T, c)$.

Proof. The first part is an immediate consequence of theorem 3.3. Furthermore $D^2(X_i) = 0$ by proposition 2.2. So $\exp(D)(X) = X + E_{n,p}(T, c)$ and the inverse map is given by $\exp(-D)(X) = X - E_{n,p}(T, c)$. □

4 The main theorem

In this section we show that for every $H \in \mathcal{H}_n(A)$ the polynomial map $F = X + H$ is a product of n polynomial automorphisms of the form $\exp(D)$, where each D is a nice derivation on $A[X]$. More precisely

Theorem 4.1 *Let $F = X + H$, where $H = \sum_{p=0}^{n-2} E_{n,p}(T,c) + c_n$, for some $(T,c) \in \mathcal{D}_n(A)$. Then*

$$F = \exp(D\left(c_n; \frac{\partial}{\partial X_1}, \ldots, \frac{\partial}{\partial X_n}\right)) \prod_{p=0}^{n-2} \exp(D\left(E_{n,p}(T,c); \frac{\partial}{\partial X_1}, \ldots, \frac{\partial}{\partial X_n}\right)).$$

Proof. Observe that

$$\exp(-D\left(c_n; \frac{\partial}{\partial X_1}, \ldots, \frac{\partial}{\partial X_n}\right)) \circ F = \sum_{p=0}^{n-2} E_{n,p}(T,c).$$

So the case $n = 2$ follows from corollary 3.4. Hence we may assume that $n \geq 3$. Now theorem 4.1 follows directly from proposition 4.2 below and corollary 3.4. □

Proposition 4.2 *Let $n \geq 3$, $0 \leq p \leq n-3$ and $(T,c) \in \mathcal{D}_n(A)$. Then*

$$\exp(-D(E_{n,p}(T,c))) \circ (X + \sum_{q=p}^{n-2} E_{n,q}(T,c)) = X + \sum_{q=p+1}^{n-2} E_{n,q}(T,c).$$

Proof. Put $G := \exp(-D(E_{n,p}(T,c)))$. So $G = X - E_{n,p}(T,c)$ (by corollary 3.4). Hence if we put

$$U := \tilde{T}_{n-p} \Delta \cdots \Delta \tilde{T}_{n-1} \Delta T_n$$

then by proposition 1.4 we get

$$G = X - \mathrm{Adj}(U) * \tilde{c}_{n-p-1\,|U*X}.$$

So if we put

$$f := X + \sum_{q=p}^{n-2} E_{n,q}(T,c)$$

then

$$G \circ f = f - \mathrm{Adj}(U(f)) * \tilde{c}_{n-p-1\,|U(f)*f}.$$

Since $U(f) = f$ (by corollary 4.4 below, with $j = 0$) we get

$$G \circ f = f - \mathrm{Adj}(U) * \tilde{c}_{n-p-1\,|U*f}.$$

Now observe that each component of $\tilde{c}_{n-p-1}$ belongs to $A[X_{n-p}, \ldots, X_n]$ and that for each $i \geq n-p$ $(U * f)_i = (U * X)_i$ (by lemma 4.3 below). So $\tilde{c}_{n-p-1\,|U*f} = \tilde{c}_{n-p-1\,|U*X}$ and hence

$$\begin{aligned} G \circ f &= f - \mathrm{Adj}(U) * \tilde{c}_{n-p-1\,|U*X} \\ &= f - E_{n,p}(T,c) \end{aligned}$$

(by proposition 1.4). □

Lemma 4.3 *Let* $n \geq 3$, $0 \leq p \leq n-2$, $0 \leq j \leq p$ *and* $(T,c) \in \mathcal{D}_n(A)$. *Put* $f := X + \sum_{q=p}^{n-2} E_{n,q}(T,c)$. *Then*

$$[(\tilde{T}_{n-p+j}\Delta \cdots \Delta \tilde{T}_{n-1}\Delta T_n) * f]_i = [(\tilde{T}_{n-p+j}\Delta \cdots \Delta \tilde{T}_{n-1}\Delta T_n) * X]_i$$

for all $i \geq n-p+j$.

Proof. Put $U := \tilde{T}_{n-p+j}\Delta \cdots \Delta \tilde{T}_{n-1}\Delta T_n$. It suffices to show that for each $q \geq p$

$$[U * E_{n,q}(T,c)]_i = 0 \tag{4}$$

for all $i \geq n-p+j$. So let $q \geq p$, then $q \geq p-j$.

1. We first treat the case that $q = p-j$. Then $j = 0$ and $q = p$. Consequently $U = \tilde{T}_{n-p}\Delta \cdots \Delta \tilde{T}_{n-1}\Delta T_n$, $E_{n,q}(T,c) = E_{n,p}(T,c)$ and hence by proposition 1.4

$$\begin{aligned} U * E_{n,q}(T,c) &= U * \mathrm{Adj}(U) * \tilde{c}_{n-p-1\ |U*X} \\ &= \det(U) * \tilde{c}_{n-p-1\ |U*X} \end{aligned}$$

   Since the last $p+1$ coordinates of $\tilde{c}_{n-p-1}$ are zero, we obtain that

$$[U * E_{n,q}(T,c)]_i = 0$$

   for all $i \geq n-p$, which proves the case that $q = p-j$.

2. Now assume that $q \geq p-j+1$. So $n-q \leq n-p+j-1$. Put $V := \tilde{T}_{n-q}\Delta \cdots \Delta \tilde{T}_{n-p+j-1}$. Then by proposition 1.4 we can write

$$\begin{aligned} E_{n,q}(T,c) &= \mathrm{Adj}(V\Delta U) * \tilde{c}_{n-q-1\ |(V\Delta U)*X} \\ &= \mathrm{Adj}(V_{|U*X} * U) * \tilde{c}_{n-q-1\ |(V\Delta U)*X} \\ &= \mathrm{Adj}(U) * \mathrm{Adj}(V_{|U*X}) * \tilde{c}_{n-q-1\ |(V\Delta U)*X} \end{aligned}$$

   Consequently

$$U * E_{n,q}(T,c) = \det(U) * \mathrm{Adj}(V_{|U*X}) * \tilde{c}_{n-q-1\ |(V\Delta U)*X} \tag{5}$$

   Note that V, and hence $V_{|U*X}$, is of the form $\tilde{B}$ for some $B \in M_{n-p+j-1}(A[X])$. Furthermore $(\tilde{c}_{n-q-1})_i = 0$ if $i \geq n-q$ which implies that $(\tilde{c}_{n-q-1\ (V\Delta U)*X})_i = 0$ if $i \geq n-p+j$ (since $n-p+j > n-q$). Now the desired result (4) follows from (5).

□

Corollary 4.4 *Notations as in lemma 4.3. Then*

$$(\tilde{T}_{n-p+j}\Delta \cdots \Delta \tilde{T}_{n-1}\Delta T_n)(f) = \tilde{T}_{n-p+j}\Delta \cdots \Delta \tilde{T}_{n-1}\Delta T_n.$$

Proof. By induction on $N := p - j$. If $N = 0$ the result is obvious. So let $N \geq 1$. Then

$$\begin{aligned} & (\tilde{T}_{n-p+j} \Delta \cdots \Delta \tilde{T}_{n-1} \Delta T_n)(f) \\ = \; & \tilde{T}_{n-p+j \; |(\tilde{T}_{n-p+j+1} \Delta \cdots \Delta \tilde{T}_{n-1} \Delta T_n)(f) * f} * (\tilde{T}_{n-p+j+1} \Delta \cdots \Delta \tilde{T}_{n-1} \Delta T_n)(f) \\ = \; & \tilde{T}_{n-p+j \; |(\tilde{T}_{n-p+j+1} \Delta \cdots \Delta \tilde{T}_{n-1} \Delta T_n) * f} * (\tilde{T}_{n-p+j+1} \Delta \cdots \Delta \tilde{T}_{n-1} \Delta T_n)(f) \end{aligned}$$

by the induction hypothesis. Finally observe that the matrix elements of $\tilde{T}_{n-p+j}$ depend only on $X_{n-p+j+1}, \ldots, X_n$. The result follows immediately from lemma 4.3 (with $j+1$ instead of j). □

5 Stably tameness

With theorem 4.1 we are now able to prove the stably tame generators conjecture for all maps in our clas $\mathcal{H}_n(A)$. And will also show that this result is 'sharp': we give an example of an element of our class which is not tame, so in general we cannot get a better result than this stable tameness.

First let us recall the conjecture (it was already mentioned in [1], [2], [3], [4] and [7]):

Conjecture 5.1 *For every invertible polynomial map $F : k^n \to k^n$ over a field k there exist $t_1, \ldots, t_m$ such that*

$$F^{[m]} = (F, t_1, \ldots, t_m) : k^{n+m} \to k^{n+m}$$

is tame, i.e. F is stably tame.

Theorem 5.2 *Let $F = X + H$ with $H \in \mathcal{H}_n(A)$. Then F is stably tame.*

To do this we use the following result due to Martha Smith in [9]:

Proposition 5.3 *Let D be a locally nilpotent derivation of $A[X]$. Let $a \in \ker(D)$. Extend D to $A[X][t]$ by setting $D(t) = 0$. Note that tD is locally nilpotent. Define $\rho \in \mathrm{Aut}_A\, A[X][t]$ by $\rho(X_i) = X_i, i = 1, \ldots, n$ and $\rho(t) = t + a$. Then*

$$(\exp(aD), t) = \rho^{-1} \exp(-tD) \rho \exp(tD).$$

Corollary 5.4 *Let D, a be as in proposition 5.3. If D is conjugate by a tame automorphism to a triangular derivation, then $(\exp(aD, t))$ is tame.*

Lemma 5.5 *Let τ be a nice derivation of order m with respect to $X_1, \ldots, X_n$ and $D := \{\frac{\partial}{\partial X_1}, \ldots, \frac{\partial}{\partial X_n}\}$ on $A[X]$. Then $\exp(a\tau)$ is stably tame for all $a \in \ker(\tau)$.*

Proof. We use induction on m. Consider the case that $m = 1$. Then $\tau = \sum_{d\in D} b_d d$ with $b_d \in A[X]^D = \cap_{d\in D}\ker(d) = A$. And hence $\tau(X_i) \in A$ and clearly τ is on triangular form. So now we can apply corollary 5.4 and find that $\exp(a\tau)$ is stably tame.
Now consider the case $m > 1$. We may assume that for all nice derivations $\sigma \in \mathrm{Der}_A(A[X])$ of order $m-1$ with respect to D and $X_1, \ldots, X_n$ and for any commutative ring A we have that $\exp(a\sigma)$ is stably tame for all $a \in \ker(\sigma)$. Let τ be nice of order m. Define ρ and extend τ to $A[X][t]$ as in proposition 5.3 (in fact we extend all derivations of D_i to $A[X][t]$ in this way). Now from

$$(\exp(a\tau), t) = \rho^{-1}\exp(-t\tau)\rho\exp(t\tau)$$

it follows that it suffices to see that $\exp(t\tau)$ is stably tame. Now we see that $t\tau = \sum_{d\in D_{m-1}} tb_d d$ with $tb_d \in A[X][t]^{D_{m-1}}$. But from this it follows that

$$\begin{aligned}\exp(t\tau) &= \exp(\sum_{d\in D_{m-1}} tb_d d)\\ &= \prod_{d\in D_{m-1}} \exp(tb_d d)\end{aligned}$$

This last equation follows from proposition 1.5. Obviously it suffices to prove that each $\exp(tb_d d)$ is stably tame to conclude that $\exp(t\tau)$ is stably tame. But d is a nice derivation of order $m-1$, $tb_d \in \ker(d)$ and hence we can apply the induction hypothesis to the ring $A[t]$ and find that $\exp(t\tau)$ is stably tame and hence $\exp(a\tau)$ is stably tame. □

Proof of theorem 5.2. Now if we look at theorem 4.1 we see that each $F = X + H$ with $H \in \mathcal{H}_n(A)$ can be written as the product of a finite number of $\exp(a_i D_i)$'s where each D_i is a nice derivation with respect to $X_1, \ldots, X_n$ and $\{\frac{\partial}{\partial X_1}, \ldots, \frac{\partial}{\partial X_n}\}$ and $a_i \in \ker(D_i)$. Applying lemma 5.5 n times gives us the desired result: F is stably tame. □

Remark 5.6 Note that we don't give an indication of the value of m in conjecture 5.1. As can be seen from the proof above, this m can be very high. At the highest level we have n $\exp(a_i D_i)$'s, but each of these factors can give rise to a great number of extra variables, depending on the 'order of niceness' of each D_i.

To conclude this paper we show that in general the automorphisms $F = X + H$ with $H \in \mathcal{H}_2(A)$ need not be tame. Actually, this idea was already presented by Nagata in [8].

Example 5.7 Let A be a domain, but not a principal ideal domain. Let $a, b \in A$ such that $Aa + Ab$ is not a principal ideal. Let $f(T) \in A[T]$ with $\deg(f) \geq 2$ and let $F = X + H$ with

$$H = \begin{pmatrix} bf(aX_1 + bX_2) \\ -af(aX_1 + bX_2) \end{pmatrix}$$

Since $H \in \mathcal{H}_2(A)$ F is an automorphism of $A[X_1, X_2]$. However, it is shown in [8] that F is not tame.

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