Cubic Similarity in Dimension Five

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Abstract

In this paper we shall give a complete classification of all Drużkowski maps $F := X + (AX)^3 : k^5 \to k^5$ for which $J((AX)^3)$ is nilpotent. Furthermore we use this classification to find all representatives of Meisters' cubic similarity relation in dimension five.

1 Introduction

Throughout this paper k denotes an algebraically closed field of characteristic zero.

The inspiration to start with research on this particular topic originated mostly from Arno van den Essen and Gary Meisters.

1.1 Drużkowski's paper

After having described cubic homogeneous maps and cubic-linear homogeneous automorphisms, also known as Drużkowski forms, in dimension four [5], we now present some results in this area in dimension five. Since [5] was written, two years have passed. The main reason it took so long before the results of this paper were found, was the complexity of the five dimensional case. It was only very recently we rediscovered that the paper [2] by Drużkowski in conjunction with a theorem of [5], offers a great reduction to the triangular cubic-linear case.

Definition 1.1 Let A be a linear matrix over k. Then the map $F = X + (AX)^3$ is called *cubic-linear* or *in Drużkowski form.*

Theorem 1.2 *Let $r \in \mathbb{N}$. If the Jacobian Conjecture holds for every polynomial map $F : k^r \to k^r$ where F has the special form*

$$F = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_r \end{pmatrix} + \begin{pmatrix} H_1(x_1, \ldots, x_r) \\ H_2(x_1, \ldots, x_r) \\ \vdots \\ H_r(x_1, \ldots, x_r) \end{pmatrix}$$

with $H_i = 0$ *or* $\deg(H_i) = 3$ *(H_i homogeneous for all $i \in \{1, \ldots, r\}$) then for all $n \geq r$ and all $A \in \mathrm{Mat}_{n,n}(k)$ the Jacobian Conjecture holds for all Drużkowski forms*

$$G = X + (AX)^3$$

with $\mathrm{rank}(A) = r$ *and* $X = (x_1, \ldots, x_n)$.

Proof. See [5]. □

Because [5] cannot be accessed easily, most of the important theorems are reprinted in [3].

Lemma 1.3 *Let $F = X + (AX)^3$ with $A \in \mathrm{Mat}_{5,5}(k)$ and $J((AX)^3)$ is nilpotent. Then there exists linear invertible T such that $T^{-1}FT = X + (BX)^3$ where the last row of B is a null row.*

Proof. It is well known that the hypothesis on A implies that $r := \mathrm{rank}(A) \leq 4$. Let $AX = \left(\ell_i(X) \right)$ for $i = 1, \ldots, 5$. Therefore there exists some $T \in \mathrm{GL}_5(k)$ such that AT is on column Echelon form. Now define $G = T^{-1}FT$. Then

$$G = X + T^{-1}(ATX)^3 = \begin{pmatrix} x_1 + h_1(x_1, \ldots, x_r) \\ x_2 + h_2(x_1, \ldots, x_r) \\ \vdots \\ x_5 + h_5(x_1, \ldots, x_r) \end{pmatrix}$$

where h_i is homogeneous of degree three. Since $r \leq 4$ $h_i(x_1, \ldots, x_r)$ does not contain x_5. It follows that $J_{x_1,\ldots,x_4}(h_1, \ldots, h_4)$ is nilpotent. But then by [5, Corollary 2.8] we have that

$$\dim_k[h_1(X), h_2(X), h_3(X), h_4(X)] < 4$$

but then also

$$\dim_k[\ell_1^3(TX), \ell_2^3(TX), \ell_3^3(TX), \ell_4^3(TX), \ell_5^3(TX)] < 5.$$

And substituting $X := T^{-1}X$ gives $\dim_k[\ell_1^3(X), \ell_2^3(X), \ell_3^3(X), \ell_4^3(X), \ell_5^3(X)] < 5$. But this implies that there exists $S \in \mathrm{GL}_5(k)$ such that $S^{-1}FS$ is on Drużkowski form and has the last row equal to zero. □

We now present an improvement of [2, Theorem 2.1] for the case $n = 5$.

Theorem 1.4 *If a polynomial map $F = X + (AX)^3 : k^5 \to k^5$ has $\det(JF) = 1$ and $\mathrm{rank}(A) < 3$ or $\mathrm{corank}(A) < 3$, then there exists an invertible linear map L such that $L \circ F \circ L^{-1} = X + (BX)^3$, with B is upper triangular with null diagonal.*

Proof. Though the original theorem in [2] only claims that F is a tame automorphism, we can almost copy the proof in that paper. Simply because in three of the four cases it is shown that LFL^{-1} has the desired form (and hence F tame).

- $\text{rank}(A) = 1$. The proof is exactly the same as in [2].
- $\text{corank}(A) = 1$. From lemma 1.3 it follows that we are always in case (i) of Drużkowski's paper.
- $\text{corank}(A) = 2$. From lemma 1.3 it now follows that we are always in case (iii) of Drużkowski's paper.
- $\text{rank}(A) = 2$. This is the only part where Drużkowski doesn't show that F can be transformed to the desired form. To prove this case we use the lemmata 1.5, 1.6 and 1.8.

□

Lemma 1.5 *Assume* $\text{rank}(A) = 2$*. By lemma 1.3 we have that the last row is equal to zero. Now if we write*

$$A = \begin{pmatrix} a_1 & a_2 & a_3 & a_4 & a_5 \\ b_1 & b_2 & b_3 & b_4 & b_5 \\ c_1 & c_2 & c_3 & c_4 & c_5 \\ d_1 & d_2 & d_3 & d_4 & d_5 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} & & & & a_5 \\ & A' & & & b_5 \\ & & & & c_5 \\ & & & & d_5 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

and we consider the Drużkowski form $X' + (A'X')^3$ *(where* $X' = (x_1, \ldots, x_4)$*) we may assume that* A' *equals*

$$\begin{pmatrix} a_1 & a_2 & a_3 & a_4 \\ \lambda_2 a_1 & \lambda_2 a_2 & \lambda_2 a_3 & \lambda_2 a_4 \\ \lambda_3 a_1 & \lambda_3 a_2 & \lambda_3 a_3 & \lambda_3 a_4 \\ \lambda_4 a_1 & \lambda_4 a_2 & \lambda_4 a_3 & \lambda_4 a_4 \end{pmatrix} \text{ or } \begin{pmatrix} a_1 & a_2 & a_3 & a_4 \\ b_1 & b_2 & b_3 & b_4 \\ \lambda a_1 + \mu b_1 & \lambda a_2 + \mu b_2 & \lambda a_3 + \mu b_3 & \lambda a_4 + \mu b_4 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

and $d_5 = 0$.

Proof. This lemma is again based on Drużkowski's paper. Naturally if $\text{rank}(A) = 2$ we have $\text{rank}(A') = 1$ or $\text{rank}(A') = 2$. The first case obviously coincides with the first matrix. Now if $\text{rank}(A') = 2$ we know that we can transform the matrix such that the first two rows are independent. But since we deal with a 4×4 matrix, $\text{rank}(A') = 2$ means also $\text{corank}(A') = 2$ and here we can use Drużkowski's proof, since $J((A'X')^3)$ is nilpotent if $J((AX)^3)$ is nilpotent, where he states that at least one of the rows of this 4×4 matrix is parallel to another row. Say

$$(d_1, \ldots, d_4) = \lambda(a_1, \ldots, a_4) \tag{1}$$

Also $(d_1, \ldots, d_5) = \mu_1(a_1, \ldots, a_5) + \mu_2(b_1, \ldots, b_5)$ (since $\text{rank}(A) = 2$ and $(a_1, \ldots, a_5)$, $(b_1, \ldots, b_5)$ are independent). So in particular

$$(d_1, \ldots, d_4) = \mu_1(a_1, \ldots, a_4) + \mu_2(b_1, \ldots, b_4) \tag{2}$$

Since $(a_1, \ldots, a_4)$ and $(b_1, \ldots, b_4)$ are independent it follows from (1) and (2) that $\mu_1 = \lambda$ and $\mu_2 = 0$. So $(d_1, \ldots, d_5) = \lambda(a_1, \ldots, a_5)$. Then making a change of coordinates, we may assume that $d_1 = \cdots = d_5 = 0$, which proves the lemma. □

Lemma 1.6 *Let A and A' be as in lemma 1.5. Assume*

$$A' = \begin{pmatrix} a_1 & a_2 & a_3 & a_4 \\ \lambda_2 a_1 & \lambda_2 a_2 & \lambda_2 a_3 & \lambda_2 a_4 \\ \lambda_3 a_1 & \lambda_3 a_2 & \lambda_3 a_3 & \lambda_3 a_4 \\ \lambda_4 a_1 & \lambda_4 a_2 & \lambda_4 a_3 & \lambda_4 a_4 \end{pmatrix}$$

Then there exists a linear invertible map $T \in k[X]$ and $B \in \mathrm{GL}_5(k)$ such that $T^{-1} \circ (X + (AX)^3) \circ T = X + (BX)^3$ with B is upper triangular with null diagonal.

Proof. Note that if either λ_2, λ_3 or λ_4 equals zero, we are in a special case of lemma 1.8. Hence we may assume $\lambda_2\lambda_3\lambda_4 \neq 0$. If we now look at A itself, we see that we are done if A is triangularizable or if we have that two rows of A are parallel to each other. After these observations we now start by showing we may assume $a_5 = 0$. Take $T = (x_1 - \frac{a_5}{a_1} x_5, x_2, x_3, x_4, x_5)$. (Of course we may assume $a_1 \neq 0$.) Then $T^{-1}FT$ is on Drużkowski form with the matrix:

$$\begin{pmatrix} a_1 & a_2 & a_3 & a_4 & 0 \\ \lambda_2 a_1 & \lambda_2 a_2 & \lambda_2 a_3 & \lambda_2 a_4 & -\lambda_2 a_5 + b_5 \\ \lambda_3 a_1 & \lambda_3 a_2 & \lambda_3 a_3 & \lambda_3 a_4 & -\lambda_3 a_5 + c_5 \\ \lambda_4 a_1 & \lambda_4 a_2 & \lambda_4 a_3 & \lambda_4 a_4 & -\lambda_4 a_5 + d_5 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

and by putting $b_5' = -\lambda_2 a_5 + b_5$, $c_5' = -\lambda_3 a_5 + c_5$ and $d_5' = -\lambda_4 a_5 + d_5$ we get the same structure as our original A, only with $a_5 = 0$.
Now put $Y_1 = a_1x_1 + a_2x_2 + a_3x_3 + a_4x_4$. Using the nilpotency of the corresponding Jacobian matrix we obtain the following polynomial in Y_1 and x_5 by looking at the 1×1 principal minors:[1]

$$M_1 := a_1 {Y_1}^2 + \lambda_2\, a_2\, (\lambda_2\, Y_1 + b_5\, x_5)^2 + \lambda_3\, a_3\, (\lambda_3\, Y_1 + c_5\, x_5)^2 + \lambda_4\, a_4\, (\lambda_4\, Y_1 + d_5\, x_5)^2$$

Collecting the coefficients of the monomials x_5^2, Y_1^2 and x_5Y_1 we get three equations:[2]

$$\Big\{\lambda_3\, a_3\, {c_5}^2 + \lambda_2\, a_2\, {b_5}^2 + \lambda_4\, a_4\, {d_5}^2, {\lambda_3}^3\, a_3 + {\lambda_2}^3\, a_2 + a_1 + {\lambda_4}^3\, a_4,$$
$$2\, {\lambda_2}^2\, a_2\, b_5 + 2\, {\lambda_4}^2\, a_4\, d_5 + 2\, {\lambda_3}^2\, a_3\, c_5\Big\}$$

We will show how this system can be solved completely. It is common knowledge that Maple's solve doesn't always give all solutions. So we solve this system by hand. Sometimes we have to make assumptions. This leads to a tree of solutions. During this process, the set given directly after $aa*$ is the remaining set of equations after substitution of $aa*$ in the original system. At top level we start with solving the second equation:[3]

$$aa := \Big\{a_1 = -{\lambda_3}^3\, a_3 - {\lambda_2}^3\, a_2 - {\lambda_4}^3\, a_4\Big\}$$

[1] It is obvious we don't have to look at the $n \times n$ principal minors with $n \geq 2$ for they are always zero because of the dependency relation.

[2] The '$= 0$' part in the equations is omitted.

[3] The first two letters in the name $aa*$ are completely arbitrary; the letters following these two show to which branch a certain solution belongs.

$$\left\{\lambda_3\,a_3\,{c_5}^2+\lambda_2\,a_2\,{b_5}^2+\lambda_4\,a_4\,{d_5}^2,0,2\,{\lambda_2}^2\,a_2\,b_5+2\,{\lambda_4}^2\,a_4\,d_5+2\,{\lambda_3}^2\,a_3\,c_5\right\}$$

Unfortunately this is the last place where we have a unique solution. Already here we have to make assumptions. Choose between $a_2 = 0$ and $a_2 \neq 0$. Let's start with $a_2 = 0$ and take a leftmost depth first strategy.[4]

$$aaa := \left\{a_2 = 0, a_1 = -{\lambda_3}^3\,a_3 - {\lambda_4}^3\,a_4\right\}$$

$$\left\{\lambda_4\,a_4\,{d_5}^2+\lambda_3\,a_3\,{c_5}^2,0,2\,{\lambda_4}^2\,a_4\,d_5+2\,{\lambda_3}^2\,a_3\,c_5\right\}$$

Again two choices: $a_3 = 0$ or $a_3 \neq 0$.

$$aaaa := \left\{a_3 = 0, a_2 = 0, a_1 = -{\lambda_4}^3\,a_4\right\}$$

$$\left\{\lambda_4\,a_4\,{d_5}^2,0,2\,{\lambda_4}^2\,a_4\,d_5\right\}$$

Here we choose between $a_4 = 0$ and $a_4 \neq 0$.

$$aaaaa := \{a_3 = 0, a_2 = 0, a_4 = 0, a_1 = 0\}$$

$$\{0\}$$

So *aaaaa* is a solution of the original system. Now back to *aaaab*. Here we have $a_4 \neq 0$ and hence $d_5 = 0$.

$$aaaab := \left\{a_3 = 0, d_5 = 0, a_2 = 0, a_1 = -{\lambda_4}^3\,a_4\right\}$$

$$\{0\}$$

And also *aaaab* is a solution. Backtracking gives us that in *aaab* $a_3 \neq 0$ and hence we can solve for c_5.

$$aaab := \left\{a_2 = 0, a_1 = -{\lambda_3}^3\,a_3 - {\lambda_4}^3\,a_4, c_5 = -\frac{{\lambda_4}^2\,a_4\,d_5}{{\lambda_3}^2\,a_3}\right\}$$

$$\left\{\frac{\lambda_4\,a_4\,{d_5}^2\,\left({\lambda_4}^3\,a_4+{\lambda_3}^3\,a_3\right)}{{\lambda_3}^3\,a_3},0\right\}$$

Unfortunately we have to choose again, but this time we have three choices:

- $a_4 = 0$,
- $a_4 \neq 0$ and $d_5 = 0$ and
- $a_4 \neq 0$ and $d_5 \neq 0$.

[4] Figure 1 on page 8 explains why this is a leftmost depth first strategy.

Following these three branches we get:

$$aaaba := \left\{c_5 = 0, a_2 = 0, a_4 = 0, a_1 = -{\lambda_3}^3\, a_3\right\}$$

$$\{0\}$$

$$aaabb := \left\{c_5 = 0, d_5 = 0, a_2 = 0, a_1 = -{\lambda_3}^3\, a_3 - {\lambda_4}^3\, a_4\right\}$$

$$\{0\}$$

$$aaabc := \left\{a_3 = -\frac{{\lambda_4}^3\, a_4}{{\lambda_3}^3}, c_5 = \frac{d_5\, \lambda_3}{\lambda_4}, a_2 = 0, a_1 = 0\right\}$$

$$\{0\}$$

So also *aaaba*, *aaabb* and *aaabc* are solutions. Here at this moment the branch of $a_2 = 0$ is completely solved, so now we start with $a_2 \neq 0$. Here we can solve for b_5.

$$aab := \left\{b_5 = -\frac{1}{2}\,\frac{2\,{\lambda_4}^2\, a_4\, d_5 + 2\,{\lambda_3}^2\, a_3\, c_5}{{\lambda_2}^2\, a_2}, a_1 = -{\lambda_3}^3\, a_3 - {\lambda_2}^3\, a_2 - {\lambda_4}^3\, a_4\right\}$$

$$\left\{\frac{\lambda_3\, a_3\, {c_5}^2\, {\lambda_2}^3 + \lambda_4\, a_4\, {d_5}^2\, {\lambda_2}^3}{{\lambda_2}^3} + \frac{{\lambda_4}^4\, {a_4}^2\, {d_5}^2 + 2\,{\lambda_4}^2\, a_4\, d_5\, {\lambda_3}^2\, a_3\, c_5 + {\lambda_3}^4\, {a_3}^2\, {c_5}^2}{{\lambda_2}^3\, a_2}, 0\right\}$$

This equation may seem a bit complicated. However if we multiply by a_2 (and we already know $a_2 \neq 0$) we get a linear equation in a_2. So what we have to do is check whether the coefficient of a_2 in this linear equation equals zero. There are three solutions for which this coefficient equals zero:

- $c_5 = 0$ and $a_4 = 0$
- $c_5 = 0$, $a_4 \neq 0$ and $d_5 = 0$ and
- $c_5 \neq 0$ and $a_3 = -\dfrac{\lambda_4 a_4 d_5^2}{\lambda_3 c_5^2}$.

So these solutions provide the first three branches at this place. The fourth branch is of course the solution assuming that the coefficient of $a_2 \neq 0$:

- $a_2 = -\dfrac{(\lambda_4^2 a_4 d_5 + \lambda_3^2 a_3 c_5)^2}{\lambda_2^3(\lambda_4 a_4 d_5^2 + \lambda_3 a_3 c_5^2)}$

Exploring the first two branches we directly find solutions of the complete system.

$$aaba := \left\{a_4 = 0, c_5 = 0, b_5 = 0, a_1 = -{\lambda_3}^3\, a_3 - {\lambda_2}^3\, a_2\right\}$$

$$\{0\}$$

$$aabb := \left\{d_5 = 0, c_5 = 0, b_5 = 0, a_1 = -{\lambda_3}^3\, a_3 - {\lambda_2}^3\, a_2 - {\lambda_4}^3\, a_4\right\}$$

$$\{0\}$$

The third case is different:

$$aabc := \left\{a_3 = -\frac{\lambda_4\, a_4\, {d_5}^2}{\lambda_3\, {c_5}^2}, b_5 = \frac{\lambda_4\, a_4\, d_5\, (-\lambda_4\, c_5 + \lambda_3\, d_5)}{c_5\, {\lambda_2}^2\, a_2},\right.$$
$$\left. a_1 = -\frac{-{\lambda_3}^2\, \lambda_4\, a_4\, {d_5}^2 + {\lambda_2}^3\, a_2\, {c_5}^2 + {\lambda_4}^3\, a_4\, {c_5}^2}{{c_5}^2}\right\}$$

$$\left\{\frac{{\lambda_4}^2\, {a_4}^2\, {d_5}^2\, (-\lambda_4\, c_5 + \lambda_3\, d_5)^2}{{\lambda_2}^3\, a_2\, {c_5}^2}, 0\right\}$$

At this point we have again three choices:

- $a_4 = 0$,
- $a_4 \neq 0$ and $d_5 = 0$ and
- $a_4 \neq 0$ and $d_5 \neq 0$.

$$aabca := \left\{a_4 = 0, b_5 = 0, a_1 = -{\lambda_2}^3\, a_2, a_3 = 0\right\}$$

$$\{0\}$$

$$aabcb := \left\{a_1 = -\frac{{\lambda_2}^3\, a_2\, {c_5}^2 + {\lambda_4}^3\, a_4\, {c_5}^2}{{c_5}^2}, d_5 = 0, b_5 = 0, a_3 = 0\right\}$$

$$\{0\}$$

$$aabcc := \left\{d_5 = \frac{\lambda_4\, c_5}{\lambda_3}, a_3 = -\frac{{\lambda_4}^3\, a_4}{{\lambda_3}^3}, b_5 = 0, a_1 = -{\lambda_2}^3\, a_2\right\}$$

$$\{0\}$$

So again we have found three solutions: *aabca*, *aabcb* and *aabcc*. Now we can go back to *aabd*.

$$aabd := \left\{b_5 = \frac{\%1\, \lambda_2}{{\lambda_4}^2\, a_4\, d_5 + {\lambda_3}^2\, a_3\, c_5}, a_2 = -\frac{\left({\lambda_4}^2\, a_4\, d_5 + {\lambda_3}^2\, a_3\, c_5\right)^2}{{\lambda_2}^3\, \%1},\right.$$
$$\left. a_1 = -\frac{\lambda_4\, a_4\, \lambda_3\, a_3\, (-\lambda_4\, c_5 + d_5\, \lambda_3)^2}{\%1}\right\}$$
$$\%1 := \lambda_4\, a_4\, {d_5}^2 + \lambda_3\, a_3\, {c_5}^2$$

$$\{0\}$$

So finally we have found all solutions of the original system: *aaaaa*, *aaaab*, *aaaba*, *aaabb*, *aaabc*, *aaba*, *aabb*, *aabca*, *aabcb*, *aabcc* and *aabd*. The corresponding tree of solutions is shown in figure 1.

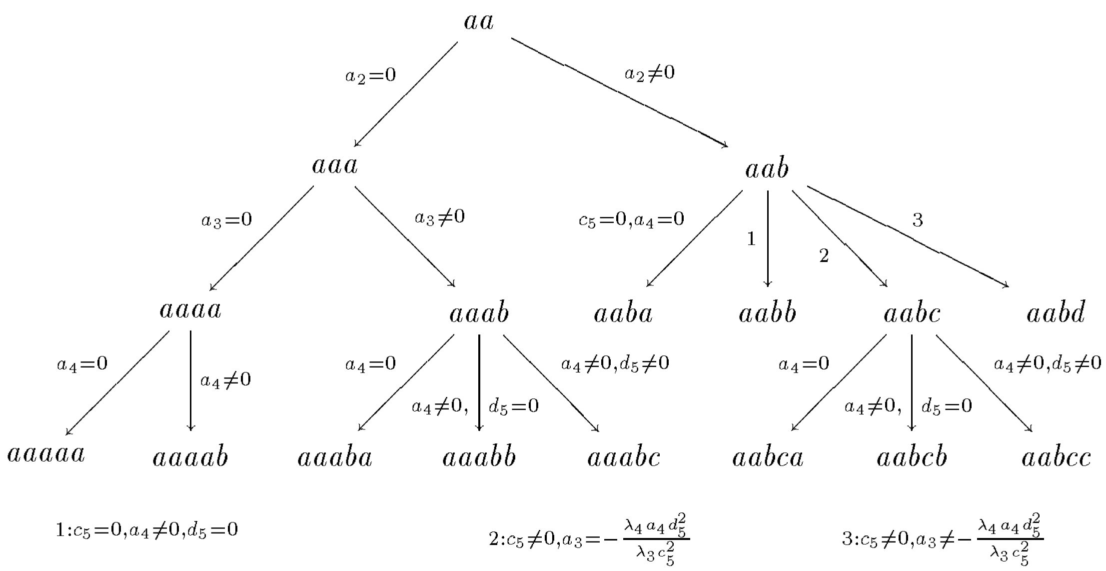


Figure 1: Solution tree of lemma 1.6.

Now if we substitute these solutions into our original A, together with $a_5 = 0$, we get eleven matrices. The first ten all have two rows which are parallel to each other and hence we are done with these ten cases. The eleventh matrix is not of this type. It is given by:

$$B := \begin{pmatrix} -\frac{\lambda_3\, a_3\, \lambda_4\, a_4\, \%2^2}{\%1} & -\frac{\%3^2}{{\lambda_2}^3\, \%1} & a_3 & a_4 & 0 \\ -\frac{\lambda_2\, \lambda_3\, a_3\, \lambda_4\, a_4\, \%2^2}{\%1} & -\frac{\%3^2}{{\lambda_2}^2\, \%1} & \lambda_2\, a_3 & \lambda_2\, a_4 & \frac{\%1\, \lambda_2}{\%3} \\ -\frac{{\lambda_3}^2\, a_3\, \lambda_4\, a_4\, \%2^2}{\%1} & -\frac{\lambda_3\, \%3^2}{{\lambda_2}^3\, \%1} & \lambda_3\, a_3 & \lambda_3\, a_4 & c_5 \\ -\frac{{\lambda_4}^2\, \lambda_3\, a_3\, a_4\, \%2^2}{\%1} & -\frac{\lambda_4\, \%3^2}{{\lambda_2}^3\, \%1} & \lambda_4\, a_3 & \lambda_4\, a_4 & d_5 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\%1 := \lambda_3\, a_3\, {c_5}^2 + \lambda_4\, a_4\, {d_5}^2$$
$$\%2 := -\lambda_4\, c_5 + \lambda_3\, d_5$$
$$\%3 := {\lambda_4}^2\, a_4\, d_5 + {\lambda_3}^2\, a_3\, c_5$$

However if we define

$$T := \Big(x_1, x_2, x_3, \Big(-2\, {\lambda_3}^2\, a_3\, {\lambda_4}^2\, a_4\, x_1\, {\lambda_2}^3\, c_5\, d_5 + \lambda_3\, a_3\, {\lambda_4}^3\, a_4\, x_1\, {\lambda_2}^3\, {c_5}^2$$
$$+ {\lambda_3}^3\, a_3\, \lambda_4\, a_4\, x_1\, {\lambda_2}^3\, {d_5}^2 + x_2\, {\lambda_3}^4\, {a_3}^2\, {c_5}^2 + x_2\, {\lambda_4}^4\, {a_4}^2\, {d_5}^2$$
$$+ 2\, x_2\, {\lambda_4}^2\, a_4\, d_5\, {\lambda_3}^2\, a_3\, c_5 - a_3\, x_3\, {\lambda_2}^3\, \lambda_4\, a_4\, {d_5}^2 - {a_3}^2\, x_3\, {\lambda_2}^3\, \lambda_3\, {c_5}^2$$

$$- a_4{}^2\, x_4\, \lambda_2{}^3\, \lambda_4\, d_5{}^2 - a_4\, x_4\, \lambda_2{}^3\, \lambda_3\, a_3\, c_5{}^2\Big) \Big/ \Big(a_4 \lambda_2{}^3 \, \left(\lambda_4\, a_4\, d_5{}^2 + \lambda_3\, a_3\, c_5{}^2 \right) \Big), x_5 \Big)$$

we get $T^{-1} \circ (X + (BX)^3) \circ T = X + (CX)^3$ where C is given by

$$\begin{pmatrix} 0 & 0 & 0 & -a_4 & 0 \\ 0 & 0 & 0 & -\lambda_2\, a_4 & \dfrac{\left(\lambda_4\, a_4\, d_5{}^2 + \lambda_3\, a_3\, c_5{}^2\right) \lambda_2}{\lambda_4{}^2\, a_4\, d_5 + \lambda_3{}^2\, a_3\, c_5} \\ 0 & 0 & 0 & -\lambda_3\, a_4 & c_5 \\ 0 & 0 & 0 & 0 & -\dfrac{\left(-\lambda_4\, c_5 + d_5\, \lambda_3\right)^{2/3} c_5{}^{1/3}\, a_3{}^{1/3}\, d_5{}^{1/3}}{\left(\lambda_4{}^2\, a_4\, d_5 + \lambda_3{}^2\, a_3\, c_5\right)^{1/3}} \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

and in particular we see that C is on triangular form. Before we can finish the proof we must add a minor remark: this eleventh solution *aabd* excludes the cases *aaba*, *aabb* and *aabc* hence the factor $\lambda_4 a_4 d_5^2 + \lambda_3 a_3 c_5^2 \neq 0$. So the only way this T might be undefined is if $a_4 = 0$. But if we substitute $a_4 = 0$ into the matrix B we get a matrix where the second and the third row are parallel to each other, so the case $a_4 = 0$ is also no problem. Hence the proof is finally completed. □

Remark 1.7 At first the given T may seem to appear out of nowhere, but in fact it doesn't. It is of the form:

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ -x_4 \\ x_5 \end{pmatrix} - \begin{pmatrix} 0 \\ 0 \\ 0 \\ \frac{B_{1,1}}{B_{1,4}} x_1 + \frac{B_{1,2}}{B_{1,4}} x_2 + \frac{B_{1,3}}{B_{1,4}} x_3 + \frac{B_{1,5}}{B_{1,4}} x_5 \\ 0 \end{pmatrix}$$

which is a natural choice.

Lemma 1.8 *Let A and A' be as in lemma 1.5. Assume*

$$A' = \begin{pmatrix} a_1 & a_2 & a_3 & a_4 \\ b_1 & b_2 & b_3 & b_4 \\ \lambda a_1 + \mu b_1 & \lambda a_2 + \mu b_2 & \lambda a_3 + \mu b_3 & \lambda a_4 + \mu b_4 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Then there exists a linear invertible map $T \in k[X]$ and $B \in \mathrm{GL}_5(k)$ such that $T^{-1} \circ (X + (AX)^3) \circ T = X + (BX)^3$ with B is upper triangular with null diagonal.

Proof. Obviously, since the first two rows of A' are independent, also the first two rows of A are independent. But then it follows that $c_5 = \lambda a_5 + \mu b_5$. Furthermore if $d_5 \neq 0$ we have that the fourth row is parallel to either the first or the second row. It cannot be a non-trivial combination of these rows since a_1, a_2, a_3, a_4 and b_1, b_2, b_3, b_4 are independent. But in that case we can conjugate with a suitable transformation and get the complete fourth row equal to zero. Hence we may assume $d_5 = 0$. We also assume both $\lambda \neq 0$ and $\mu \neq 0$. Put $Y_1 := a_1x_1 + a_2x_2 + a_3x_3 + a_4x_4$ and $Y_2 := b_1x_1 + b_2x_2 + b_3x_3 + b_4x_4$. Now if we look at the principal minors, we get the following polynomials in Y_1, Y_2 and x_5. Here M_1 stands for the polynomial we get by looking at the 1×1 principal minors and M_2 for the 2×2 principal minors.

$$M_1 := a_1\,(Y_1 + a_5\,x_5)^2 + b_2\,(Y_2 + b_5\,x_5)^2 + (\lambda\,a_3 + \mu\,b_3)\,(\lambda\,Y_1 + \mu\,Y_2 + (a_5\,\lambda + b_5\,\mu)\,x_5)^2 \tag{3}$$

$$\begin{aligned} M_2 := {} & (a_1\,b_2 - a_2\,b_1)\,(Y_1 + a_5\,x_5)^2\,(Y_2 + b_5\,x_5)^2 \\ & + (a_1\,\mu\,b_3 - a_3\,\mu\,b_1)\,(Y_1 + a_5\,x_5)^2\,(\lambda\,Y_1 + \mu\,Y_2 + (a_5\,\lambda + b_5\,\mu)\,x_5)^2 \\ & + (b_2\,\lambda\,a_3 - b_3\,\lambda\,a_2)\,(Y_2 + b_5\,x_5)^2\,(\lambda\,Y_1 + \mu\,Y_2 + (a_5\,\lambda + b_5\,\mu)\,x_5)^2 \end{aligned} \tag{4}$$

Collecting the coefficients of the monomials in Y_1, Y_2 and x_5 gives us a set of 21 equations. We solve these equations by looking at the easy ones[5] and substitute their solutions in the original system. We start with:

$$\{2\,(\lambda\,a_3 + \mu\,b_3)\,\mu\,\lambda\}$$

Since $\lambda \neq 0$ and $\mu \neq 0$ we have

$$bb := \left\{b_3 = -\frac{\lambda\,a_3}{\mu}\right\}$$

Substituting this solution gives as new (very) easy equations

$$\{a_1, b_2\}$$

Adding this information to what we already know gives:

$$bb := \left\{a_1 = 0, b_2 = 0, b_3 = -\frac{\lambda\,a_3}{\mu}\right\}$$

By repeating this we get:

$$\left\{2\,\lambda^3\,a_3\,a_2\right\}$$

Here we have two solutions, depending on the fact whether $a_3 = 0$ or $a_3 \neq 0$.

$$bba := \{a_3 = 0, a_1 = 0, b_2 = 0, b_3 = 0\}$$

$$bbb := \left\{a_2 = 0, a_1 = 0, b_2 = 0, b_3 = -\frac{\lambda\,a_3}{\mu}\right\}$$

Substituting the first set gives as easy equations:

$$\{-a_2\,b_1\}$$

and again we have two solutions:

$$bbaa := \{a_3 = 0, a_2 = 0, a_1 = 0, b_2 = 0, b_3 = 0\}$$

[5]'Easy ones' should be read as: with the smallest number of variables.

$$bbab := \{a_3 = 0, a_1 = 0, b_1 = 0, b_2 = 0, b_3 = 0\}$$

If we substitute these two solutions we see that in both cases the complete system vanishes. So now we must go back to the branch given by the set bbb. Substituting this one gives:

$$\left\{-\mu^3\, a_3\, b_1\right\}$$

But here we use the fact that $a_3 \neq 0$ in this branch so we only get the solution $b_1 = 0$:

$$bbb := \left\{a_2 = 0, a_1 = 0, b_1 = 0, b_2 = 0, b_3 = -\frac{\lambda\, a_3}{\mu}\right\}$$

And if we substitute this solution into the original system, we see that the complete system vanishes.
Hence we have found three solutions to the equations given by (3) and (4). Substituting these three solutions in our original A gives:

$$bbaa \ : \ \begin{pmatrix} 0 & 0 & 0 & a_4 & a_5 \\ b_1 & 0 & 0 & b_4 & b_5 \\ \mu\, b_1 & 0 & 0 & \lambda\, a_4 + \mu\, b_4 & a_5\, \lambda + b_5\, \mu \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$bbab \ : \ \begin{pmatrix} 0 & a_2 & 0 & a_4 & a_5 \\ 0 & 0 & 0 & b_4 & b_5 \\ 0 & \lambda\, a_2 & 0 & \lambda\, a_4 + \mu\, b_4 & a_5\, \lambda + b_5\, \mu \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$bbb \ : \ \begin{pmatrix} 0 & 0 & a_3 & a_4 & a_5 \\ 0 & 0 & -\dfrac{\lambda\, a_3}{\mu} & b_4 & b_5 \\ 0 & 0 & 0 & \lambda\, a_4 + \mu\, b_4 & a_5\, \lambda + b_5\, \mu \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

The third matrix is already on triangular form. Swapping the first and the third row and column brings the first matrix on triangular form and swapping the second and third row and column brings the second matrix on triangular form. In particular we have that in all cases A is triangularizable.
What if $\lambda = 0$ or (exclusive or) $\mu = 0$? Without loss of generality we may assume $\lambda = 0$.
In the same way as before we now derive:

$$\begin{aligned} M_1 := {} & a_1\, (Y_1 + a_5\, x_5)^2 + b_2\, (Y_2 + b_5\, x_5)^2 \\ & + \mu\, b_3\, (\mu\, Y_2 + b_5\, \mu\, x_5)^2 \end{aligned} \tag{5}$$

$$\begin{aligned} M_2 := {} & (a_1\, b_2 - a_2\, b_1)\, (Y_1 + a_5\, x_5)^2\, (Y_2 + b_5\, x_5)^2 \\ & + (a_1\, \mu\, b_3 - a_3\, \mu\, b_1)\, (Y_1 + a_5\, x_5)^2\, (\mu\, Y_2 + b_5\, \mu\, x_5)^2 \end{aligned} \tag{6}$$

After collecting the coefficients of the monomials in Y_1, Y_2 and x_5 we get as easy equations and their solutions:

$$\begin{aligned}
\{a_1\} &\rightarrow cc := \{a_1 = 0\} \\
\left\{b_2 + \mu^3 b_3\right\} &\rightarrow cc := \left\{b_2 = -\mu^3 b_3, a_1 = 0\right\} \\
\left\{-b_1\left(\mu^3 a_3 + a_2\right)\right\} &\rightarrow cca := \left\{a_1 = 0, b_2 = -\mu^3 b_3, b_1 = 0\right\} \\
&\rightarrow ccb := \left\{a_1 = 0, b_2 = -\mu^3 b_3, a_2 = -\mu^3 a_3\right\}
\end{aligned}$$

Substituting these solutions gives:

$$cca : \begin{pmatrix} 0 & a_2 & a_3 & a_4 & a_5 \\ 0 & -\mu^3 b_3 & b_3 & b_4 & b_5 \\ 0 & -\mu^4 b_3 & \mu b_3 & \mu b_4 & b_5 \mu \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \quad ccb : \begin{pmatrix} 0 & -\mu^3 a_3 & a_3 & a_4 & a_5 \\ b_1 & -\mu^3 b_3 & b_3 & b_4 & b_5 \\ \mu b_1 & -\mu^4 b_3 & \mu b_3 & \mu b_4 & b_5 \mu \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

And here we see that in both cases the third row is μ times the second row and due to this parallelnes we can transform this case to the case where $\lambda = 0$ and $\mu = 0$, which we will describe now:

$$M_1 := a_1 (Y_1 + a_5 x_5)^2 + b_2 (Y_2 + b_5 x_5)^2 \tag{7}$$
$$M_2 := (a_1 b_2 - a_2 b_1)(Y_1 + a_5 x_5)^2 (Y_2 + b_5 x_5)^2 \tag{8}$$

These polynomials results in the following chain of easy equations and solutions.

$$\begin{aligned}
\{a_1, b_2\} &\rightarrow dd := \{a_1 = 0, b_2 = 0\} \\
\{-a_2 b_1\} &\rightarrow dda := \{a_1 = 0, b_1 = 0, b_2 = 0\} \\
&\rightarrow ddb := \{a_1 = 0, b_2 = 0, a_2 = 0\}
\end{aligned}$$

And this results in the matrices:

$$dda : \begin{pmatrix} 0 & a_2 & a_3 & a_4 & a_5 \\ 0 & 0 & b_3 & b_4 & b_5 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \quad ddb : \begin{pmatrix} 0 & 0 & a_3 & a_4 & a_5 \\ b_1 & 0 & b_3 & b_4 & b_5 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

And indeed these are the matrices given by *cca* and *ccb* where $\mu = 0$. Furthermore one easily verifies that *dda* is already on triangular form and *ddb* is on triangular form after swapping the first and the second row and column. And in particular the case of lemma 1.8 is always triangularizable. □

1.2 Cubic similarity

With the reduction of theorem 1.4 it was possible to give a complete classification of all cubic-linear automorphisms in dimension five. Now with this new classification in dimension five we did an analogous thing as in [5]: use it to find representatives with respect to Meisters' cubic similarity equivalence relation. (See [6].)

Definition 1.9 Let $F = X + (AX)^3$ and $G = X + (BX)^3$ be two polynomial automorphisms on Drużkowski form. Then the matrices $A, B \in \mathrm{Mat}_{n,n}(k)$ are called *cubic similar* ($A \overset{3}{\sim} B$) if there exists a linear invertible polynomial map T with $T^{-1}FT = G$.

The idea behind this definition is that it is rather special that if T is a linear invertible map and F is a Drużkowski form one has that $T^{-1}FT$ is again on Drużkowski form.

Definition 1.9 is in terms of maps. For computational use however it is often preferable to work in terms of matrices.

Lemma 1.10 *Let $F = X + (AX)^3$ and $G = X + (BX)^3$ be two polynomial maps on Drużkowski form. Then $A \overset{3}{\sim} B$ if and only if there exists $T \in \mathrm{GL}_n(k)$ with $(ATX)^3 = T(BX)^3$.*

Proof. The following statements are equivalent:

- $A \overset{3}{\sim} B$.
- There exists an invertible map T with $T^{-1}FT = G$.
- There exists an invertible map T with $T^{-1}(TX + (ATX)^3) = X + (BX)^3$.
- There exists an invertible map T with $X + T^{-1}(ATX)^3 = X + (BX)^3$.
- There exists an invertible map T with $T^{-1}(ATX)^3 = (BX)^3$.
- There exists an invertible matrix T with $T^{-1}(ATX)^3 = (BX)^3$.
- There exists an invertible matrix T with $(ATX)^3 = T(BX)^3$.

This proves the lemma. □

1.3 Meisters' matrices

In [8] Meisters gives a list of eighteen mutually inequivalent representatives in dimension five. His naming convention is based on the following notions. As stated in [7], the rank of A and the nilpotence index of $J((AX)^3)$ are invariants.[6] So both these numbers are mentioned in the name of the matrix: the first number is the rank, the second number is the nilpotence index. Furthermore we have J's, N's and P's. The J means that the matrix is on Jordan normal form. The N means that it is nilpotent but not on Jordan form and has no parameters in it. The P means that there are some parameters left in the matrices which cannot be reduced to a complex number. The small letters at the end of the name are simply added as an index, since the N and P classes have more than one element with the same rank and nilpotence index. Finally for some P matrices an integer is appended to show the number of parameters in it.

[6] Note that the nilpotence index of A itself is *not* an invariant. In fact A doesn't need to be nilpotent in general.

- There is one J-matrix of rank one:

$$\underset{J(1,2)}{\begin{pmatrix} 0&1&0&0&0\\ 0&0&0&0&0\\ 0&0&0&0&0\\ 0&0&0&0&0\\ 0&0&0&0&0 \end{pmatrix}}$$

- There are two J-matrices in rank two:

$$\underset{J(2,2)}{\begin{pmatrix} 0&1&0&0&0\\ 0&0&0&0&0\\ 0&0&0&1&0\\ 0&0&0&0&0\\ 0&0&0&0&0 \end{pmatrix}} \quad \underset{J(2,3)}{\begin{pmatrix} 0&1&0&0&0\\ 0&0&1&0&0\\ 0&0&0&0&0\\ 0&0&0&0&0\\ 0&0&0&0&0 \end{pmatrix}}$$

  and one N-matrix:

$$\underset{N(2,3a)}{\begin{pmatrix} 0&0&1&0&0\\ 0&0&1&1&0\\ 0&0&0&1&0\\ 0&0&0&0&0\\ 0&0&0&0&0 \end{pmatrix}}$$

- In rank three we have two J-matrices

$$\underset{J(3,3)}{\begin{pmatrix} 0&1&0&0&0\\ 0&0&1&0&0\\ 0&0&0&0&0\\ 0&0&0&0&1\\ 0&0&0&0&0 \end{pmatrix}} \quad \underset{J(3,4)}{\begin{pmatrix} 0&1&0&0&0\\ 0&0&1&0&0\\ 0&0&0&1&0\\ 0&0&0&0&0\\ 0&0&0&0&0 \end{pmatrix}}$$

  and five N-matrices:

$$\underset{N(3,3a)}{\begin{pmatrix} 0&0&1&0&0\\ 0&0&1&1&0\\ 0&0&0&1&1\\ 0&0&0&0&0\\ 0&0&0&0&0 \end{pmatrix}} \quad \underset{N(3,4a)}{\begin{pmatrix} 0&1&1&0&0\\ 0&0&1&0&0\\ 0&0&0&1&0\\ 0&0&0&0&0\\ 0&0&0&0&0 \end{pmatrix}} \quad \underset{N(3,4b)}{\begin{pmatrix} 0&0&1&0&0\\ 0&0&1&1&0\\ 0&0&0&1&0\\ 0&0&0&0&1\\ 0&0&0&0&0 \end{pmatrix}}$$

$$\underset{N(3,4c)}{\begin{pmatrix} 0&0&1&0&0\\ 0&0&1&1&0\\ 0&0&0&1&1\\ 0&0&0&0&1\\ 0&0&0&0&0 \end{pmatrix}} \quad \underset{N(3,4d)}{\begin{pmatrix} 0&1&0&1&0\\ 0&0&1&0&0\\ 0&0&0&1&0\\ 0&0&0&0&0\\ 0&0&0&0&0 \end{pmatrix}}$$

- In rank four we have one J-matrix,

$$\underset{J(4,5)}{\begin{pmatrix} 0&1&0&0&0\\ 0&0&1&0&0\\ 0&0&0&1&0\\ 0&0&0&0&1\\ 0&0&0&0&0 \end{pmatrix}}$$

four N-matrices

$$\begin{pmatrix} 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \quad \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$N(4,5a)$ $\qquad$ $N(4,5b)$

$$\begin{pmatrix} 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \quad \begin{pmatrix} 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$N(4,5c)$ $\qquad$ $N(4,5d)$

and two P-matrices:

$$\begin{pmatrix} 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & a & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \quad \begin{pmatrix} 0 & 1 & 1 & b & 0 \\ 0 & 0 & 1 & a & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$P(4,5c)$ $\qquad$ $P(4,5c2)$

Remark 1.11 Note the following points:

- $P(4,5c)$ is not called $P(4,5a)$, which should be natural if one uses the small letter just as an index as with the N-matrices. However in this case the c is used because $P(4,5c)_{|a=1} = N(4,5c)$, where $P(4,5c)_{|a=1}$ means substitute $a = 1$ in $P(4,5c)$.
- Note also that $P(4,5c)_{|a=0} = N(4,5a)$. Hence we add the restriction that $a \notin \{0,1\}$ for $P(4,5c)$.
- $P(4,5c)_{|a=a_1} \overset{3}{\not\sim} P(4,5c)_{|a=a_2}$ if $a_1 \neq a_2$.
- $P(4,5c2)_{|b=0} = P(4,5c)$, hence we add the restriction $b \neq 0$ for $P(4,5c2)$. Note that there are no restrictions on the a in $P(4,5c2)$.

Meisters already stated that these matrices were not a complete set of representatives. But due to lack of time he hasn't found more matrices. However, apart from this incompleteness, there is also an incorrectness in this set of matrices. In fact we have:

Lemma 1.12 *The matrices $J(3,4)$ and $N(3,4d)$ are cubic similar.*

Proof. Let $T := (x_1, x_2 - x_4, x_3, x_4, x_5)$. Then it is easy to see that

$$T^{-1} \circ \left(X + (N(3,4d)X)^3\right) \circ T = X + (J(3,4)X)^3$$

which implies $N(3,4d) \overset{3}{\sim} J(3,4)$. □

2 Classification in dimension five

As we have seen in theorem 1.4, we may assume that the Drużkowski map is on triangular form. So the most general Drużkowski map in dimension five is $F = X + (AX)^3$ where A is the matrix:

$$A := \begin{pmatrix} 0 & a_2 & a_3 & a_4 & a_5 \\ 0 & 0 & b_3 & b_4 & b_5 \\ 0 & 0 & 0 & c_4 & c_5 \\ 0 & 0 & 0 & 0 & d_5 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

If one computes $J((AX)^3)$ one sees that this Jacobian matrix is always nilpotent, independent of the choices of the ten parameters. In fact it is even strongly nilpotent. However since our goal is a classification with respect to cubic similarity and the nilpotence index of this Jacobian matrix is an invariant of this relation (see [7]), we divide the general case into the five possible values for the nilpotence index.[7] For each of these values n we compute the matrix $(J((AX)^3))^n$ and assume it is equal to the null matrix. This gives each time a set of equations which turns out to be easy to solve.

In order to be sure one has found all solutions, one cannot trust the standard 'solve' mechanism of Maple. So the method used to solve these systems consists of a lot of, very simple, hand-work. Just start with the small equations, like single variables or products of a few variables, solve them and substitute the solutions in the original system and look again for the simple, small equations. The best situation is of course if such small equations have a unique solution. Unfortunately this doesn't happen very often. Most of the time one has to make assumptions that either a variable equals zero or that it doesn't, which gives you two branches to examine. One branch is simplified because all terms this variable appears in, vanish. And the other is simplified because one knows that one can safely divide through this variable.[8] But since the systems are pretty small (the largest one consists of 123 equations) this branching is not a big problem. Using Maple on a computer with over 128 Mb of internal memory, all systems were solved on one single day.

Before we give the results of this process, we remark that $J((AX)^3)$ has nilpotence index one if and only if A equals the null matrix itself. So we only consider the cases with nilpotence index ≥ 2. Furthermore, we represent all solutions by their matrix form and we explicitly show the assumptions we had to make to find each solution.

[7]The fact that we choose the nilpotence index of the corresponding Jacobian matrix as the invariant, and for instance not the rank of the matrix A, is based on the observation that it is easier to compute $J((AX)^3)^n$ and see which conditions must be fulfilled in order to get n as the nilpotence index than to choose a general matrix of a certain rank and compute $J((AX)^3)^5$ and solve the system.

[8]See the proof of lemma 1.6 and lemma 1.8 for examples of this process.

2.1 Nilpotence index two

Assuming $J((AX)^3)^2 = 0$ gives a system of 119 equations. We get the solution tree of figure 2 on page 18. The boxed numbers coincide with the numbers of the matrices below. Because we ordered the solutions afterwards by rank, the order of the boxed numbers may seem a bit strange. Furthermore we swapped branch 1 and 2 in order to get a nicer diagram, so the leftmost approach is visually a bit disturbed. However the numbered assumptions still follow this strategy.

If we substitute these solutions we get:

1. $\begin{pmatrix} 0 & 0 & 0 & 0 & a_5 \\ 0 & 0 & 0 & 0 & b_5 \\ 0 & 0 & 0 & 0 & c_5 \\ 0 & 0 & 0 & 0 & d_5 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$, rank 1.

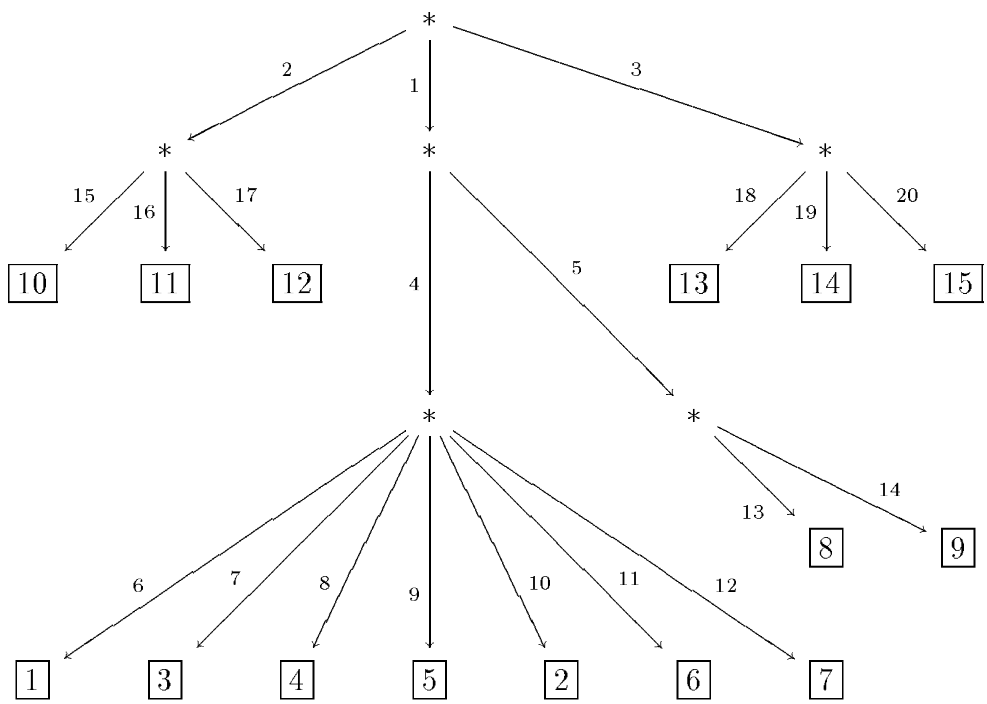
2. $\begin{pmatrix} 0 & a_2 & a_3 & a_4 & a_5 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$, rank 1, $a_2 \neq 0$.

3. $\begin{pmatrix} 0 & 0 & 0 & a_4 & a_5 \\ 0 & 0 & 0 & 0 & b_5 \\ 0 & 0 & 0 & 0 & c_5 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$, rank 2, $a_4 \neq 0$.

4. $\begin{pmatrix} 0 & 0 & a_3 & a_4 & a_5 \\ 0 & 0 & 0 & 0 & b_5 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$, rank 2, $a_3 \neq 0$.

5. $\begin{pmatrix} 0 & 0 & a_3 & -\frac{a_3 c_5^3}{d_5^3} & a_5 \\ 0 & 0 & 0 & 0 & b_5 \\ 0 & 0 & 0 & 0 & c_5 \\ 0 & 0 & 0 & 0 & d_5 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$, rank 2, $a_3 \neq 0$, $d_5 \neq 0$.

6. $\begin{pmatrix} 0 & a_2 & -\frac{a_2 b_5^3}{c_5^3} & a_4 & a_5 \\ 0 & 0 & 0 & 0 & b_5 \\ 0 & 0 & 0 & 0 & c_5 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$, rank 2, $a_2 \neq 0$, $c_5 \neq 0$.



1:$b_3=0,c_4=0$

2:$b_3=0,c_4\neq0,d_5=0$

3:$b_3\neq0,c_4=0,a_2=0$

4:$b_3=0,c_4=0,b_4=0$

5:$b_3=0,c_4=0,b_4\neq0,a_2=0,d_5=0$

6:$b_3=0,c_4=0,b_4=0,a_2=0,a_3=0,a_4=0$

7:$b_3=0,c_4=0,b_4=0,a_2=0,a_3=0,a_4\neq0,d_5=0$

8:$b_3=0,c_4=0,b_4=0,a_2=0,a_3\neq0,d_5=0,c_5=0$

9:$b_3=0,c_4=0,b_4=0,a_2=0,a_3,d_5\neq0$

10:$b_3=0,c_4=0,b_4=0,a_2\neq0,c_5=0,b_5=0,d_5=0$

11:$b_3=0,c_4=0,b_4=0,a_2,c_5\neq0,d_5=0$

12:$b_3=0,c_4=0,b_4=0,a_2,d_5\neq0$

13:$b_3=0,c_4=0,b_4\neq0,a_3=0,a_2=0,d_5=0$

14:$b_3=0,c_4=0,b_4,a_3\neq0,a_2=0,d_5=0,c_5=0$

15:$b_3=0,c_4\neq0,a_2=0,a_3=0,d_5=0$

16:$b_3=0,c_4,a_2\neq0,a_3=0,b_5=0,b_4=0,d_5=0$

17:$b_3=0,c_4,a_2,b_4\neq0,d_5=0$

18:$b_3\neq0,d_5=0,c_5=0,c_4=0,a_2=0$

19:$b_3,d_5\neq0,a_3=0,a_4=0,c_4=0,a_2=0$

20:$b_3,d_5,a_3\neq0,c_4=0,a_2=0$

Figure 2: Solution tree for nilpotence index two.

7. $\begin{pmatrix} 0 & a_2 & a_3 & \dfrac{-a_2b_5^3 - a_3c_5^3}{d_5^3} & a_5 \\ 0 & 0 & 0 & 0 & b_5 \\ 0 & 0 & 0 & 0 & c_5 \\ 0 & 0 & 0 & 0 & d_5 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$, rank 2, $a_2 \neq 0$, $d_5 \neq 0$.

8. $\begin{pmatrix} 0 & 0 & 0 & a_4 & a_5 \\ 0 & 0 & 0 & b_4 & b_5 \\ 0 & 0 & 0 & 0 & c_5 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$, rank 2, $b_4 \neq 0$.

9. $\begin{pmatrix} 0 & 0 & a_3 & a_4 & a_5 \\ 0 & 0 & 0 & b_4 & b_5 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$, rank 2, $a_3 \neq 0$, $b_4 \neq 0$.

10. $\begin{pmatrix} 0 & 0 & 0 & a_4 & a_5 \\ 0 & 0 & 0 & b_4 & b_5 \\ 0 & 0 & 0 & c_4 & c_5 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$, rank 2, $c_4 \neq 0$.

11. $\begin{pmatrix} 0 & a_2 & 0 & a_4 & a_5 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & c_4 & c_5 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$, rank 2, $a_2 \neq 0$, $c_4 \neq 0$.

12. $\begin{pmatrix} 0 & a_2 & -\dfrac{a_2b_4^3}{c_4^3} & a_4 & a_5 \\ 0 & 0 & 0 & b_4 & \dfrac{b_4c_5}{c_4} \\ 0 & 0 & 0 & c_4 & c_5 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$, rank 2, $a_2 \neq 0$, $b_4 \neq 0$, $c_4 \neq 0$.

13. $\begin{pmatrix} 0 & 0 & a_3 & a_4 & a_5 \\ 0 & 0 & b_3 & b_4 & b_5 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$, rank 2, $b_3 \neq 0$.

14. $$\begin{pmatrix} 0 & 0 & 0 & 0 & a_5 \\ 0 & 0 & b_3 & -\frac{b_3 c_5^3}{d_5^3} & b_5 \\ 0 & 0 & 0 & 0 & c_5 \\ 0 & 0 & 0 & 0 & d_5 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$, rank 2, $b_3 \neq 0$, $d_5 \neq 0$.

15. $$\begin{pmatrix} 0 & 0 & a_3 & -\frac{a_3 c_5^3}{d_5^3} & a_5 \\ 0 & 0 & b_3 & -\frac{b_3 c_5^3}{d_5^3} & b_5 \\ 0 & 0 & 0 & 0 & c_5 \\ 0 & 0 & 0 & 0 & d_5 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$, rank 2, $a_3 \neq 0$, $b_3 \neq 0$, $d_5 \neq 0$.

2.2 Nilpotence index three

This case gives a system of 123 equations. The solution tree is a little bit simpler as one can see in figure 3 on page 22. Ordered by rank the solutions are:

16. $$\begin{pmatrix} 0 & a_2 & a_3 & a_4 & a_5 \\ 0 & 0 & b_3 & b_4 & b_5 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$, rank 2, $a_2 \neq 0$, $b_3 \neq 0$.

17. $$\begin{pmatrix} 0 & 0 & 0 & a_4 & a_5 \\ 0 & 0 & 0 & b_4 & b_5 \\ 0 & 0 & 0 & c_4 & c_5 \\ 0 & 0 & 0 & 0 & d_5 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$, rank 2, $d_5 \neq 0$.

18. $$\begin{pmatrix} 0 & a_2 & a_3 & a_4 & a_5 \\ 0 & 0 & 0 & 0 & b_5 \\ 0 & 0 & 0 & 0 & c_5 \\ 0 & 0 & 0 & 0 & d_5 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$, rank 2, $a_3 \neq 0$, $d_5 \neq 0$.

19. $$\begin{pmatrix} 0 & a_2 & a_3 & a_4 & a_5 \\ 0 & 0 & 0 & b_4 & b_5 \\ 0 & 0 & 0 & c_4 & c_5 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$, rank 3.

20. $$\begin{pmatrix} 0 & a_2 & 0 & a_4 & a_5 \\ 0 & 0 & 0 & 0 & b_5 \\ 0 & 0 & 0 & c_4 & c_5 \\ 0 & 0 & 0 & 0 & d_5 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \text{rank } 3,\ a_2 \neq 0,\ d_5 \neq 0.$$

21. $$\begin{pmatrix} 0 & 0 & a_3 & a_4 & a_5 \\ 0 & 0 & 0 & b_4 & b_5 \\ 0 & 0 & 0 & 0 & c_5 \\ 0 & 0 & 0 & 0 & d_5 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \text{rank } 3,\ a_3 \neq 0,\ b_4 \neq 0,\ d_5 \neq 0.$$

22. $$\begin{pmatrix} 0 & -\frac{a_3 c_4^3}{b_4^3} & a_3 & a_4 & a_5 \\ 0 & 0 & 0 & b_4 & \frac{b_4 c_5}{c_4} \\ 0 & 0 & 0 & c_4 & c_5 \\ 0 & 0 & 0 & 0 & d_5 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \text{rank } 3,\ a_3 \neq 0,\ b_4 \neq 0,\ c_4 \neq 0,\ d_5 \neq 0.$$

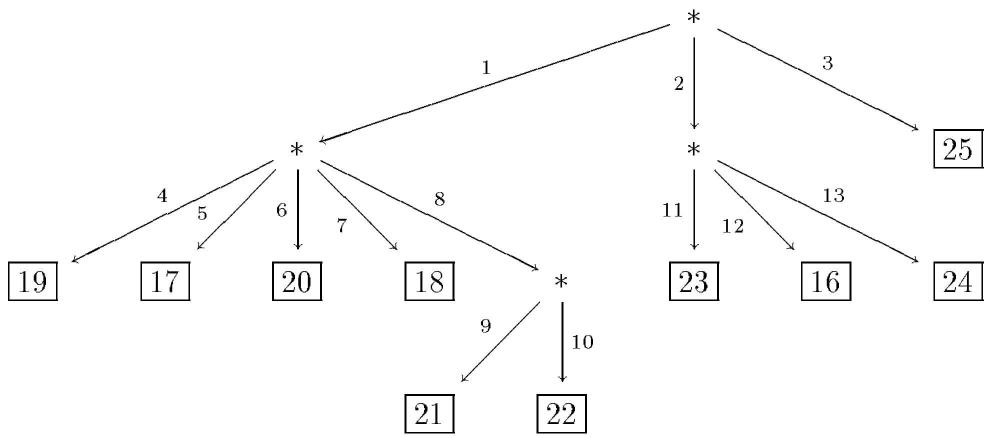
23. $$\begin{pmatrix} 0 & 0 & a_3 & a_4 & a_5 \\ 0 & 0 & b_3 & b_4 & b_5 \\ 0 & 0 & 0 & 0 & c_5 \\ 0 & 0 & 0 & 0 & d_5 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \text{rank } 3,\ b_3 \neq 0.$$

24. $$\begin{pmatrix} 0 & a_2 & a_3 & a_4 & a_5 \\ 0 & 0 & b_3 & -\frac{b_3 c_5^3}{d_5^3} & b_5 \\ 0 & 0 & 0 & 0 & c_5 \\ 0 & 0 & 0 & 0 & d_5 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \text{rank } 3,\ a_2 \neq 0,\ d_5 \neq 0.$$

25. $$\begin{pmatrix} 0 & 0 & a_3 & a_4 & a_5 \\ 0 & 0 & b_3 & b_4 & b_5 \\ 0 & 0 & 0 & c_4 & c_5 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \text{rank } 3,\ b_3 \neq 0,\ c_4 \neq 0.$$

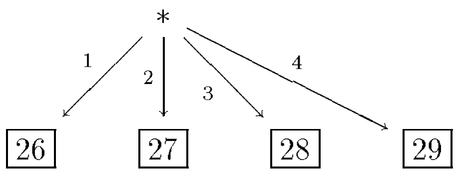
2.3 Nilpotence index four

This case gives a system of 56 equations. We get a very simple solution tree (see figure 4 on page 22). Ordered by rank the solutions are:



1:b_3=0
2:b_3≠0,c_4=0
3:b_3,c_4≠0,a_2=0,d_5=0
4:b_3=0,d_5=0
5:b_3=0,d_5≠0,a_2=0,a_3=0
6:b_3=0,d_5,a_2≠0,b_4=0,a_3=0
7:b_3=0,d_5,a_3≠0,b_4=0,c_4=0
8:b_3=0,d_5,a_3,b_4≠0
9:b_3=0,d_5,a_3,b_4≠0,c_4=0
10:b_3=0,d_5,a_3,b_4,c_4≠0
11:b_3≠0,c_4=0,a_2=0
12:b_3,a_2≠0,d_5=0,c_5=0
13:b_3,a_2,d_5≠0

Figure 3: Solution tree for nilpotence index three.



1:a_2=0
2:a_2≠0,b_3=0
3:a_2,b_3≠0,c_4=0
4:a_2,b_3,c_4≠0,d_5=0

Figure 4: Solution tree for nilpotence index four.

26. $$\begin{pmatrix} 0 & 0 & a_3 & a_4 & a_5 \\ 0 & 0 & b_3 & b_4 & b_5 \\ 0 & 0 & 0 & c_4 & c_5 \\ 0 & 0 & 0 & 0 & d_5 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \text{ rank } 3.$$

27. $$\begin{pmatrix} 0 & a_2 & a_3 & a_4 & a_5 \\ 0 & 0 & 0 & b_4 & b_5 \\ 0 & 0 & 0 & c_4 & c_5 \\ 0 & 0 & 0 & 0 & d_5 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \text{ rank } 3,\ a_2 \neq 0.$$

28. $$\begin{pmatrix} 0 & a_2 & a_3 & a_4 & a_5 \\ 0 & 0 & b_3 & b_4 & b_5 \\ 0 & 0 & 0 & 0 & c_5 \\ 0 & 0 & 0 & 0 & d_5 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \text{ rank } 3,\ a_2 \neq 0,\ b_3 \neq 0.$$

29. $$\begin{pmatrix} 0 & a_2 & a_3 & a_4 & a_5 \\ 0 & 0 & b_3 & b_4 & b_5 \\ 0 & 0 & 0 & c_4 & c_5 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \text{ rank } 3,\ a_2 \neq 0,\ b_3 \neq 0,\ c_4 \neq 0.$$

2.4 Nilpotence index five

As was stated before, all triangular forms have a nilpotent Jacobian matrix, so there is only one matrix with nilpotence index five for the corresponding Jacobian matrix, namely the general map:

30. $$\begin{pmatrix} 0 & a_2 & a_3 & a_4 & a_5 \\ 0 & 0 & b_3 & b_4 & b_5 \\ 0 & 0 & 0 & c_4 & c_5 \\ 0 & 0 & 0 & 0 & d_5 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \text{ rank } 4,\ a_2 \neq 0,\ b_3 \neq 0,\ c_4 \neq 0,\ d_5 \neq 0.$$

3 Power similarity

With the thirty matrices presented in section 2 we are now looking for representatives of Meisters' cubic similarity relation. We do this by grouping those matrices by their rank. In particular we choose the rank invariant at this point, because this is easier now that we already have the thirty matrices. It is easier to see what the effect on the rank will be in case a variable is assumed to be zero than what the effect on the nilpotence index of the corresponding map will be.

Ordered by rank we try to find transformations T such that $T^{-1} \circ (X + (AX)^3) \circ T$ is as easy as possible. In other words, we try to reduce the number of parameters in the thirty matrices. As we have already seen with $P(4,5c)$ and $P(4,5c2)$ it sometimes isn't possible to reduce all parameters.

Because all transformations must be linear *invertible* maps we have to be careful with these general transformations still containing the parameters. Essentially we are looking for maps T of the form:

$$T_i := \lambda_i x_{\sigma(i)} + \mu_i(\widehat{x_{\sigma(i)}}) \tag{9}$$

where σ is an element of the permutation group S_5 and $\mu_i(\widehat{x_{\sigma(i)}})$ is a linear function in $\{x_1, \ldots, x_5\} \setminus \{x_{\sigma(i)}\}$. Since these are linear maps, they are invertible as soon as $\det(J(T_i)) \neq 0$. This can only be the case if $\lambda_i \neq 0$ for all i and μ_i is not involved in divisions by zero.

Our approach to find these transformations depends heavily on this observation. We use the assumptions we already have as a result of the process to find those thirty matrices (i.e. the ones listed in section 2), define a general map T as in (9), conjugate with this T, and solve for λ_i and μ_i in order to get a simple matrix. If we substitute such a solution into T, we compute $\det(JT)$, and see which extra assumptions we need to be sure that this determinant is never zero. Furthermore we sometimes have to add extra assumptions to avoid divisions by zero.

In fact this almost completely describes how we get to the list of separate cases in section 3.2. In a way these cases can be read best from bottom to top: the last case for each map is the first transformation we find. We see what extra conditions we need, and climb to the first case of a map by assuming that these extra conditions are not fulfilled. The only reason why we put them in the opposite order is that the first case of a map contains more 'a=0' assumptions and is hence shorter.

3.1 New matrices

As stated before, Meisters' set of matrices is not complete. During the process described before we found the following new matrices, which are not cubic similar pairwise and also not cubic similar to Meisters' matrices.

- Rank two: two N-matrices:

$$\underset{N(2,2a)}{\begin{pmatrix} 0&0&0&1&0\\ 0&0&0&1&1\\ 0&0&0&0&1\\ 0&0&0&0&0\\ 0&0&0&0&0 \end{pmatrix}} \quad \underset{N(2,3b)}{\begin{pmatrix} 0&0&0&1&0\\ 0&0&0&1&1\\ 0&0&0&1&-1\\ 0&0&0&0&1\\ 0&0&0&0&0 \end{pmatrix}}$$

  The last matrix deserves some special attention. The -1 seems a bit strange: why isn't it $P(2,3a)$ with a parameter a on the place of the -1. The answer is in fact pretty simple. As long as $a \notin \{0,1\}$, $P(2,3a) \overset{3}{\sim} N(2,3b)$. Furthermore $P(2,3a)_{|a=0} \overset{3}{\sim} P(2,3a)_{|a=1} \overset{3}{\sim} N(2,3a)$. So independent of the value of the parameter a, $P(2,3a)$ can be reduced to a matrix with no parameters left in it. So there's no need to add a P-matrix.

- Rank three: this is the most difficult case.[9] We found one new representative for which the nilpotence index of the corresponding Jacobian matrix equals three:

$$\underset{N(3,3b)}{\begin{pmatrix} 0&1&1&0&0\\ 0&0&0&1&0\\ 0&0&0&0&1\\ 0&0&0&0&0\\ 0&0&0&0&0 \end{pmatrix}}$$

With nilpotence index four we have found six N-matrices:

$$\underset{N(3,4e)}{\begin{pmatrix} 0&0&1&0&1\\ 0&0&1&0&0\\ 0&0&0&1&0\\ 0&0&0&0&1\\ 0&0&0&0&0 \end{pmatrix}} \quad \underset{N(3,4f)}{\begin{pmatrix} 0&0&1&0&0\\ 0&0&0&1&1\\ 0&0&0&1&0\\ 0&0&0&0&1\\ 0&0&0&0&0 \end{pmatrix}} \quad \underset{N(3,4g)}{\begin{pmatrix} 0&0&1&1&0\\ 0&0&0&1&1\\ 0&0&0&1&0\\ 0&0&0&0&1\\ 0&0&0&0&0 \end{pmatrix}}$$

$$\underset{N(3,4h)}{\begin{pmatrix} 0&0&1&1&1\\ 0&0&1&0&0\\ 0&0&0&1&0\\ 0&0&0&0&1\\ 0&0&0&0&0 \end{pmatrix}} \quad \underset{N(3,4i)}{\begin{pmatrix} 0&1&1&0&0\\ 0&0&0&1&1\\ 0&0&0&1&0\\ 0&0&0&0&1\\ 0&0&0&0&0 \end{pmatrix}} \quad \underset{N(3,4j)}{\begin{pmatrix} 0&1&0&1&0\\ 0&0&0&1&1\\ 0&0&0&1&0\\ 0&0&0&0&1\\ 0&0&0&0&0 \end{pmatrix}}$$

And six P-matrices with one parameter:

$$\underset{P(3,4a)}{\begin{pmatrix} 0&0&1&a&0\\ 0&0&1&1&0\\ 0&0&0&1&0\\ 0&0&0&0&1\\ 0&0&0&0&0 \end{pmatrix}} \quad \underset{P(3,4c)}{\begin{pmatrix} 0&0&1&0&a\\ 0&0&1&1&0\\ 0&0&0&1&0\\ 0&0&0&0&1\\ 0&0&0&0&0 \end{pmatrix}} \quad \underset{P(3,4g)}{\begin{pmatrix} 0&0&1&1&0\\ 0&0&0&1&a\\ 0&0&0&1&0\\ 0&0&0&0&1\\ 0&0&0&0&0 \end{pmatrix}}$$

$$\underset{P(3,4h)}{\begin{pmatrix} 0&0&1&1&a\\ 0&0&1&0&0\\ 0&0&0&1&0\\ 0&0&0&0&1\\ 0&0&0&0&0 \end{pmatrix}} \quad \underset{P(3,4i)}{\begin{pmatrix} 0&1&a&0&0\\ 0&0&0&1&1\\ 0&0&0&1&0\\ 0&0&0&0&1\\ 0&0&0&0&0 \end{pmatrix}} \quad \underset{P(3,4j)}{\begin{pmatrix} 0&1&0&1&0\\ 0&0&0&1&a\\ 0&0&0&1&0\\ 0&0&0&0&1\\ 0&0&0&0&0 \end{pmatrix}}$$

And also two P-matrices with two parameters

$$\underset{P(3,4a2)}{\begin{pmatrix} 0&0&1&a&b\\ 0&0&1&1&0\\ 0&0&0&1&0\\ 0&0&0&0&1\\ 0&0&0&0&0 \end{pmatrix}} \quad \underset{P(3,4j2)}{\begin{pmatrix} 0&1&a&1&0\\ 0&0&0&1&b\\ 0&0&0&1&0\\ 0&0&0&0&1\\ 0&0&0&0&0 \end{pmatrix}}$$

[9] It is not a surprise that this is the difficult case. It is quite usual that a large or a small rank doesn't give as much freedom as a rank somewhere in the middle.

- Rank four is relatively easy: we found one N-matrix and one P-matrix with one variable:

$$\underset{N(4,5e)}{\begin{pmatrix} 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}} \quad \underset{P(4,5e)}{\begin{pmatrix} 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & a & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}}$$

Together with Meisters' matrices brings this a total of 36 representatives for the cubic similarity relation.
The names of the P-matrices are based on the observations that:

- $P(3,4a)_{|a=1} \overset{3}{\sim} N(3,4a)$ and $P(3,4a)_{|a=0} = N(3,4b)$.
- $P(3,4c)_{|a=1} \overset{3}{\sim} N(3,4c)$ and $P(3,4c)_{|a=0} = N(3,4b)$.
- $P(3,4g)_{|a=1} = N(3,4g)$ and $P(3,4g)_{|a=0} \overset{3}{\sim} N(3,4a)$.
- $P(3,4h)_{|a=1} = N(3,4h)$ and $P(3,4h)_{|a=0} \overset{3}{\sim} N(3,4b)$.
- $P(3,4i)_{|a=1} = N(3,4i)$ and $P(3,4i)_{|a=0} \overset{3}{\sim} N(3,4a)$.
- $P(3,4j)_{|a=1} = N(3,4j)$ and $P(3,4j)_{|a=0} \overset{3}{\sim} N(3,4a)$.
- $P(3,4a2)_{|a=0} = P(3,4c)$ and $P(3,4a2)_{|b=0} = P(3,4a)$, hence $P(3,4c2)$ would have been a correct name also.
- $P(3,4j2)_{|a=0} = P(3,4j)$. Furthermore we have $P(3,4j2)_{|b=0,a=-1} \overset{3}{\sim} N(3,3a)$ and $P(3,4j2)_{|b=0,a\neq 0,a\neq -1} \overset{3}{\sim} N(3,4a)$.
- $P(4,5e)_{|a=1} = N(4,5e)$ and $P(4,5e)_{|a=0} = N(4,5d)$.

So we add for $P(3,4a)$, $P(3,4c)$, $P(3,4g)$, $P(3,4h)$, $P(3,4i)$, $P(3,4j)$ and $P(4,5e)$ the restriction that $a \notin \{0,1\}$. For $P(3,4a2)$ and $P(3,4j2)$ we add $a, b \neq 0$.

3.2 Description of the cases

All maps are grouped by their rank. The numbers coincide with the numbers in front of the solutions in section 2. For each map F we see whether we can find a suitable transformation map T such that $T^{-1}FT$ is one of the representatives listed before. Most of the time this means that we have to make some assumptions on the parameters in F. At the beginning of section 3 we explained how we have come to this distinction between cases.

The proof that these assumptions lead to those representatives is given in section 4 by showing the concrete transformations.

3.3 Rank one

In section 2 we have seen that there are two maps of rank one: the cases 1 and 2.

1. Obviously at least one of the four variables should be unequal to zero. Because the first four columns are equal to zero, we can permute the first four rows without any consequences with respect to the cubic similarity relation. Hence we may assume $a_5 \neq 0$ and then this map is cubic similar to $J(1,2)$.

2. We are in a case where we already know that $a_2 \neq 0$. This gives that this map is cubic similar to $J(1,2)$.

3.4 Rank two

Here we have sixteen matrices to examine.

3. We know $a_4 \neq 0$. Note that either b_5 or $c_5 \neq 0$, otherwise the rank is one. We may assume $c_5 \neq 0$. Then this map is cubic similar to $J(2,2)$.

4. We know $a_3 \neq 0$. Because of the rank we must have $b_5 \neq 0$. Now also this map is cubic similar to $J(2,2)$.

5. We know $a_3 \neq 0$ and $d_5 \neq 0$. Cubic similar to $J(2,2)$.

6. Here we have $a_2 \neq 0$ and $c_5 \neq 0$. Cubic similar to $J(2,2)$.

7. In this case we have $a_2 \neq 0$ and $d_5 \neq 0$. Again cubic similar to $J(2,2)$.

8. We know $b_4 \neq 0$.

   (a) Assume $a_4 = 0$. We may assume $c_5 \neq 0$, because if $c_5 = 0$ we must have $a_5 \neq 0$ and we can safely permute the first and third rows, since the first and third columns are completely zero. Then cubic similar to $J(2,2)$.

   (b) Assume $a_4 \neq 0$ and $c_5 = 0$, hence $a_5b_4 - a_4b_5 \neq 0$. Then cubic similar to $J(2,2)$.

   (c) Assume $a_4 \neq 0$ and $c_5 \neq 0$ and $a_5b_4 - a_4b_5 = 0$. Then cubic similar to $J(2,2)$.

   (d) Assume $a_4 \neq 0$ and $c_5 \neq 0$ and $a_5b_4 - a_4b_5 \neq 0$. Then cubic similar to $N(2,2a)$.

9. We have $a_3 \neq 0$ and $b_4 \neq 0$. Hence cubic similar to $J(2,2)$. After permutation of the first two rows this is basically the same map as map 13b.

10. We have $c_4 \neq 0$. If either $b_4 = 0$ or $a_4 = 0$, we can permute the rows such that $c_4 \neq 0$ appears on the fourth place in the second row, and a zero (either a_4 or b_4) appears on the fourth place in the third row. But then we have the same map as map 8. Hence we may assume that $a_4 \neq 0$, $b_4 \neq 0$ and $c_4 \neq 0$.

Furthermore we also have that either a_5, b_5 or $c_5 \neq 0$. But since we can also swap the fourth and the fifth column, we know that if $a_5b_5c_5 = 0$ we can permute this map such that we get a zero on the fourth place in the third row and a non-zero element on the fourth place in the second row. Or in other words, we can reduce this case to map 8. So we may even assume that none of the appearing variables is equal to zero.

(a) Assuming $b_4a_5 - a_4b_5 = 0$ and $c_4a_5 - a_4c_5 = 0$ gives a rank one case, so let's assume $b_4a_5 - a_4b_5 = 0$ and $c_4a_5 - a_4c_5 \neq 0$. Then cubic similar to $J(2,2)$.

(b) Assume $b_4a_5 - a_4b_5 \neq 0$ and $c_4a_5 - a_4c_5 = 0$. Then cubic similar to $J(2,2)$.

(c) Assume $b_4a_5 - a_4b_5 \neq 0$, $c_4a_5 - a_4c_5 \neq 0$ and $b_4c_5 - c_4b_5 = 0$. Then cubic similar to $J(2,2)$.

(d) Finally assume $b_4a_5 - a_4b_5 \neq 0$, $c_4a_5 - a_4c_5 \neq 0$ and $b_4c_5 - c_4b_5 \neq 0$. Then cubic similar to $N(2,2a)$.

11. We have $a_2 \neq 0$ and $c_4 \neq 0$. Basically the same as map 9: permute second and third rows and columns and substitute $a_2 = a_3$, $c_4 = b_4$ and $c_5 = b_5$. Hence also cubic similar to $J(2,2)$, just like map 9.

12. We have $a_2 \neq 0$, $b_4 \neq 0$ and $c_4 \neq 0$. Cubic similar to $J(2,2)$.

13. Here we know $b_3 \neq 0$.

    (a) Assume $a_3 = 0$ and $a_4 = 0$. Then $a_5 \neq 0$. Cubic similar to $J(2,2)$.

    (b) Assume $a_3 = 0$ and $a_4 \neq 0$. Cubic similar to $J(2,2)$.

    (c) Assume $a_3 \neq 0$, $a_3b_4 - b_3a_4 \neq 0$ and $a_3b_5 - b_3a_5 = 0$. Cubic similar to $J(2,2)$.

    (d) Assume $a_3 \neq 0$, $a_3b_4 - b_3a_4 = 0$ and $a_3b_5 - b_3a_5 \neq 0$. Cubic similar to $J(2,2)$.

    (e) Assuming $a_3 \neq 0$, $a_3b_4 - b_3a_4 = 0$ and $a_3b_5 - b_3a_5 = 0$ gives a rank one case, hence the only case left is $a_3 \neq 0$, $a_3b_4 - b_3a_4 \neq 0$ and $a_3b_5 - b_3a_5 \neq 0$. Cubic similar to $J(2,2)$.

14. We have $b_3 \neq 0$ and $d_5 \neq 0$. Cubic similar to $J(2,2)$.

15. We already know $a_3 \neq 0$, $b_3 \neq 0$ and $d_5 \neq 0$.

    (a) Assume $a_3b_5 - b_3a_5 = 0$. Cubic similar to $J(2,2)$.

    (b) Assume $a_3b_5 - b_3a_5 \neq 0$. Cubic similar to $N(2,2a)$.

16. We know $a_2 \neq 0$ and $b_3 \neq 0$. So this map is cubic similar to $J(2,3)$.

17. We know $d_5 \neq 0$. Of course at least one of a_4, b_4 or $c_4 \neq 0$. Since the first three columns are equal to zero, we can change the order of the first three rows without disturbing the structure of the matrix. Hence we may assume that $a_4 \neq 0$.

(a) Assume $b_4 = 0$ and $c_4 = 0$. Then cubic similar to $J(2,3)$.

(b) Assume $b_4 = 0$, $c_4 \neq 0$ and $a_4c_5 - c_4a_5 = 0$. Then cubic similar to $J(2,3)$.

(c) Assume $b_4 = 0$, $c_4 \neq 0$ and $a_4c_5 - c_4a_5 \neq 0$. Then cubic similar to $N(2,3a)$.

(d) Assume $b_4 \neq 0$, $c_4 = 0$ and $b_4a_5 - a_4b_5 = 0$. Then cubic similar to $J(2,3)$.

(e) Assume $b_4 \neq 0$, $c_4 = 0$ and $b_4a_5 - a_4b_5 \neq 0$. Then cubic similar to $N(2,3a)$.

(f) Assume $b_4 \neq 0$, $c_4 \neq 0$, $b_4a_5 - a_4b_5 = 0$ and $a_4c_5 - c_4a_5 = 0$. Then cubic similar to $J(2,3)$.

(g) Assume $b_4 \neq 0$, $c_4 \neq 0$, $b_4a_5 - a_4b_5 \neq 0$ and $a_4c_5 - c_4a_5 = 0$. Then cubic similar to $N(2,3a)$.

(h) Assume $b_4 \neq 0$, $c_4 \neq 0$, $b_4a_5 - a_4b_5 = 0$ and $a_4c_5 - c_4a_5 \neq 0$. Then cubic similar to $N(2,3a)$.

(i) Assume $b_4 \neq 0$, $c_4 \neq 0$, $b_4a_5 - a_4b_5 \neq 0$, $a_4c_5 - c_4a_5 \neq 0$ and $b_4c_5 - c_4b_5 = 0$. Then cubic similar to $N(2,3a)$.

(j) Assume $b_4 \neq 0$, $c_4 \neq 0$, $b_4a_5 - a_4b_5 \neq 0$, $a_4c_5 - c_4a_5 \neq 0$ and $b_4c_5 - c_4b_5 \neq 0$. Then cubic similar to $N(2,3b)$.

18. Here we have $a_3 \neq 0$ and $d_5 \neq 0$.

(a) Assume $a_2b_5^3 + a_3c_5^3 + a_4d_5^3 = 0$. Cubic similar to $J(2,2)$.

(b) Assume $a_2b_5^3 + a_3c_5^3 + a_4d_5^3 \neq 0$. Cubic similar to $J(2,3)$.

3.5 Rank three

We have eleven matrices to examine.

19. Obviously we must have that either a_2 or $a_3 \neq 0$. Since swapping columns two and three also swaps rows two and three and we have no restrictions on b_4, b_5 and c_4, c_5, we may assume that $a_2 \neq 0$. Furthermore it is clear that in order to have a rank three case we must have $b_4c_5 - c_4b_5 \neq 0$.

(a) Assume $b_4 = 0$ and $a_3 = 0$. Hence $c_4 \neq 0$ and $b_5 \neq 0$. Cubic similar to $J(3,3)$.

(b) Assume $b_4 = 0$ and $a_3 \neq 0$. Hence $c_4 \neq 0$ and $b_5 \neq 0$. Cubic similar to $N(3,3b)$.

(c) Assume $b_4 \neq 0$, $c_4 = 0$ and $a_3 = 0$. Hence $c_5 \neq 0$. Cubic similar to $J(3,3)$.

(d) Assume $b_4 \neq 0$, $c_4 = 0$ and $a_3 \neq 0$. Hence $c_5 \neq 0$. Cubic similar to $N(3,3b)$.

(e) Assume $b_4 \neq 0$, $c_4 \neq 0$ and $a_3 = 0$. Cubic similar to $J(3,3)$.

(f) Assume $b_4 \neq 0$, $c_4 \neq 0$, $a_3 \neq 0$ and $a_4 = 0$. Cubic similar to $N(3,3b)$.

(g) Assume $b_4 \neq 0$, $c_4 \neq 0$, $a_3 \neq 0$, $a_4 \neq 0$ and $a_5b_4 - b_5a_4 = 0$ and hence $a_5c_4 - c_5a_4 \neq 0$ (otherwise contradiction with $b_4c_5 - c_4b_5 \neq 0$). Cubic similar to $N(3,3b)$.

(h) Assume $b_4 \neq 0$, $c_4 \neq 0$, $a_3 \neq 0$, $a_4 \neq 0$ and $a_5b_4 - b_5a_4 \neq 0$. Cubic similar to $N(3, 3b)$.

20. We already know $a_2 \neq 0$ and $d_5 \neq 0$. However if $c_4 = 0$ we have a rank two case. Hence we may assume $c_4 \neq 0$.

    (a) Assume $a_4d_5^3 + a_2b_5^3 = 0$. Cubic similar to $J(3, 3)$.

    (b) Assume $a_4d_5^3 + a_2b_5^3 \neq 0$ and $b_5 = 0$, hence $a_4 \neq 0$. Cubic similar to $N(3, 3a)$.

    (c) Assume $a_4d_5^3 + a_2b_5^3 \neq 0$ and $b_5 \neq 0$, hence $a_4 \neq 0$. Cubic similar to $N(3, 3a)$.

21. We know $a_3 \neq 0$, $b_4 \neq 0$ and $d_5 \neq 0$.

    (a) Assume $a_3c_5^3 + a_4d_5^3 = 0$. Then cubic similar to $J(3, 3)$.

    (b) Assume $a_3c_5^3 + a_4d_5^3 \neq 0$. Then cubic similar to $N(3, 3a)$.

22. We know $a_3 \neq 0$, $b_4 \neq 0$, $c_4 \neq 0$ and $d_5 \neq 0$.

    (a) Assume $a_4 = 0$. Then cubic similar to $J(3, 3)$.

    (b) Assume $a_4 \neq 0$. Then cubic similar to $N(3, 3a)$.

23. We have $b_3 \neq 0$. Furthermore $c_5 \neq 0$ or $d_5 \neq 0$, and $a_3 \neq 0$ or $a_4 \neq 0$. It is also obvious that $a_4b_3 - b_4a_3 \neq 0$.

    (a) Assume $a_4 = 0$, hence $b_4 \neq 0$ and $a_3 \neq 0$. Now if $c_5 \neq 0$, we can conjugate with $(x_2, x_1, x_4, x_3, x_5)$ and we are back in case 21. So we may assume $c_5 = 0$ and hence $d_5 \neq 0$. Cubic similar to $J(3, 3)$.

    (b) Assume $a_4 \neq 0$ and $a_3 = 0$. If we now assume $d_5 \neq 0$ then we can conjugate with $(x_2, x_1, x_3, x_4, x_5)$ and we are again back in case 21. So we may assume $d_5 = 0$, and hence $c_5 \neq 0$. Cubic similar to $J(3, 3)$.

    (c) Assuming $a_4 \neq 0$, $a_3 \neq 0$, $a_3c_5^3 + a_4d_5^3 = 0$, $b_3c_5^3 + b_4d_5^3 = 0$ and $a_3b_5 - b_3a_5 \neq 0$ gives a rank two map, so we may assume $a_4 \neq 0$, $a_3 \neq 0$, $a_3c_5^3 + a_4d_5^3 = 0$, $b_3c_5^3 + b_4d_5^3 \neq 0$ and $a_3b_5 - b_3a_5 = 0$. Hence $c_5 \neq 0$ and $d_5 \neq 0$. Cubic similar to $J(3, 3)$.

    (d) Assume $a_4 \neq 0$, $a_3 \neq 0$, $a_3c_5^3 + a_4d_5^3 = 0$, $b_3c_5^3 + b_4d_5^3 \neq 0$ and $a_3b_5 - b_3a_5 \neq 0$, hence $c_5 \neq 0$ and $d_5 \neq 0$. Cubic similar to $J(3, 3)$.

    (e) Assume $a_4 \neq 0$, $a_3 \neq 0$, $a_3c_5^3 + a_4d_5^3 \neq 0$ and $b_3c_5^3 + b_4d_5^3 = 0$, hence $d_5 \neq 0$. Cubic similar to $J(3, 3)$.

    (f) Assume $a_4 \neq 0$, $a_3 \neq 0$, $a_3c_5^3 + a_4d_5^3 \neq 0$, $b_3c_5^3 + b_4d_5^3 \neq 0$ and $a_3b_5 - b_3a_5 = 0$. Cubic similar to $N(3, 3a)$.

    (g) Assume $a_4 \neq 0$, $a_3 \neq 0$, $a_3c_5^3 + a_4d_5^3 \neq 0$, $b_3c_5^3 + b_4d_5^3 \neq 0$, $a_3b_5 - b_3a_5 \neq 0$ and $d_5 = 0$, hence $c_5 \neq 0$. Cubic similar to $N(3, 3a)$.

(h) Assume $a_4 \neq 0$, $a_3 \neq 0$, $a_3c_5^3 + a_4d_5^3 \neq 0$, $b_3c_5^3 + b_4d_5^3 \neq 0$, $a_3b_5 - b_3a_5 \neq 0$ and $d_5 \neq 0$. Cubic similar to $N(3,3a)$.

24. Here we have $a_2 \neq 0$ and $d_5 \neq 0$. But it is obvious that $b_3 = 0$ gives a rank two case, so we also have $b_3 \neq 0$.

    (a) Assume $a_3c_5^3 + a_4d_5^3 = 0$. Then cubic similar to $J(3,3)$.

    (b) Assume $a_3c_5^3 + a_4d_5^3 \neq 0$. Then cubic similar to $N(3,3b)$.

25. We have $b_3 \neq 0$ and $c_4 \neq 0$.

    (a) Assume $a_3 = 0$. If we assume $a_5c_4 - c_5a_4 = 0$ we get a rank two case, so we may assume $a_5c_4 - c_5a_4 \neq 0$. Assume furthermore $a_4 = 0$, hence $a_5 \neq 0$. Cubic similar to $J(3,3)$.

    (b) Assume $a_3 = 0$, $a_5c_4 - c_5a_4 \neq 0$ and $a_4 \neq 0$. Cubic similar to $J(3,3)$.

    (c) Assume $a_3 \neq 0$ and $a_3b_4 - b_3a_4 = 0$. If in addition $a_3b_5 - b_3a_5 = 0$ then we have a rank two case, so we may assume $a_3b_5 - b_3a_5 \neq 0$. Cubic similar to $N(3,3a)$.

    (d) Assume $a_3 \neq 0$ and $a_3b_4 - b_3a_4 \neq 0$. If $c_5(a_3b_4 - b_3a_4) - c_4(a_3b_5 - b_3a_5) = 0$ we have a rank two case, so we may assume $c_5(a_3b_4 - b_3a_4) - c_4(a_3b_5 - b_3a_5) \neq 0$. Cubic similar to $N(3,3a)$.

26. If $d_5 = 0$ we are back in case 25, hence we may assume $d_5 \neq 0$. Furthermore if $c_4 = 0$ and $b_3 \neq 0$ we are back in case 23. And if both $c_4 = 0$ and $b_3 = 0$, we must have $a_3 \neq 0$ and $b_4 \neq 0$ to remain in a rank three case. Since we already knew that $d_5 \neq 0$ we are back in case 21. Hence we may assume $c_4 \neq 0$. Furthermore, the case $a_3 \neq 0$ and $b_3 = 0$ is equivalent with $b_3 \neq 0$ and $a_3 = 0$ since we can swap the first two rows.

    (a) Assume $a_3 = 0$, $b_3 \neq 0$, $a_4 = 0$ and $b_4 = 0$. Cubic similar to $J(3,4)$.

    (b) Assume $a_3 = 0$, $b_3 \neq 0$, $a_4 = 0$ and $b_4 \neq 0$. Cubic similar to $N(3,4a)$.

    (c) Assume $a_3 = 0$, $b_3 \neq 0$, $a_4 \neq 0$, $b_4 = 0$ and $a_5c_4 - c_5a_4 = 0$. Cubic similar to $J(3,4)$.

    (d) Assume $a_3 = 0$, $b_3 \neq 0$, $a_4 \neq 0$, $b_4 = 0$ and $a_5c_4 - c_5a_4 \neq 0$. Cubic similar to $N(3,4f)$.

    (e) Assume $a_3 = 0$, $b_3 \neq 0$, $a_4 \neq 0$ and $b_4 \neq 0$. Cubic similar to $P(3,4g)$. However, for specific choices we have

    - $N(3,4a)$ if $a_5 = \dfrac{a_4c_5}{c_4}$.
    - $N(3,4g)$ if $a_5 = \dfrac{a_4c_5}{c_4} + \dfrac{a_4d_5\sqrt[3]{b_4}}{c_4\sqrt[3]{b_3}}$. (Since $a_4b_4d_5 \neq 0$ these two cases really exclude each other.)

(f) Assume $a_3 \neq 0$, $b_3 \neq 0$, $b_4 = 0$, $a_4 = 0$ and $a_3b_5 - b_3a_5 = 0$. Cubic similar to $J(3,4)$.

(g) Assume $a_3 \neq 0$, $b_3 \neq 0$, $b_4 = 0$, $a_4 = 0$ and $a_3b_5 - b_3a_5 \neq 0$. Cubic similar to $N(3,4e)$.

(h) Assume $a_3 \neq 0$, $b_3 \neq 0$, $b_4 = 0$ and $a_4 \neq 0$. Cubic similar to $P(3,4h)$. For some specific choices we get a different matrix:

- $N(3,4b)$ if $b_5 = \dfrac{b_3(a_5c_4 - c_5a_4)}{a_3c_4}$.
- $N(3,4h)$ if $b_5 = \dfrac{b_3(a_5c_4 - c_5a_4)}{a_3c_4} - \dfrac{b_3d_5\sqrt[3]{a_4^4}}{c_4\sqrt[3]{a_3^4}}$. (Note that the last fraction is never zero, so these two cases really exclude each other.)

(i) Assume $a_3 \neq 0$, $b_3 \neq 0$ and $b_4 \neq 0$. Cubic similar to $P(3,4a2)$. For some specific choices we get a different matrix:

- $N(3,4a)$ if $a_4 = \dfrac{a_3b_4}{b_3}$ and $b_5 = \dfrac{a_5b_3}{a_3}$.
- $N(3,4b)$ if $a_4 = 0$ and $a_5 = -\dfrac{a_3(b_4c_5 - c_4b_5)}{b_3c_4}$.
- $N(3,4c)$ if $a_4 = 0$ and $b_5 = -\dfrac{-a_3b_4d_5\sqrt[3]{b_3^3b_4} + a_5b_3^2c_4 + a_3b_3b_4c_5}{a_3b_3c_4}$.
- $P(3,4a)$ if $a_5 = -\dfrac{c_5(a_3b_4 - b_3a_4) - a_3b_5c_4}{b_3c_4}$ and $a_4 \notin \{0, \frac{a_3b_4}{b_3}\}$.
- $P(3,4c)$ if $a_4 = 0$ and $b_5 \neq -\dfrac{-a_3b_4d_5\sqrt[3]{b_3^3b_4} + a_5b_3^2c_4 + a_3b_3b_4c_5}{a_3b_3c_4}$.

27. Now we have $a_2 \neq 0$. If $d_5 = 0$ we are in case 19, so $d_5 \neq 0$. Furthermore if $b_4 = 0$ and $c_4 = 0$, we have a rank two case. So at least one of them should be unequal to zero. Note also that if $b_4 = 0$ and $a_3 = 0$ we are back in case 20. So if $b_4 = 0$ we may assume $a_3 \neq 0$.

(a) Assume $b_4 = 0$ and $a_2b_5^3 + a_4d_5^3 = 0$, hence $c_4 \neq 0$ and $a_3 \neq 0$. Then cubic similar to $J(3,4)$.

(b) Assume $c_4 = 0$ and $a_3c_5^3 + a_4d_5^3 = 0$, hence $b_4 \neq 0$. Then cubic similar to $J(3,4)$.

(c) Assume $b_4 = 0$ and $a_2b_5^3 + a_4d_5^3 \neq 0$, hence $c_4 \neq 0$ and $a_3 \neq 0$. Then cubic similar to $N(3,4a)$.

(d) Assume $c_4 = 0$ and $a_3c_5^3 + a_4d_5^3 \neq 0$, hence $b_4 \neq 0$. Then cubic similar to $N(3,4a)$.

(e) Assume $b_4 \neq 0$, $c_4 \neq 0$, $a_4 = 0$, $b_4c_5 - c_4b_5 = 0$ and $a_2b_4^3 + a_3c_4^3 = 0$. Then cubic similar to $J(3,3)$.

(f) Assume $b_4 \neq 0$, $c_4 \neq 0$, $a_4 = 0$, $b_4c_5 - c_4b_5 = 0$, $a_2b_4^3 + a_3c_4^3 \neq 0$ and $a_3 = 0$. Then cubic similar to $J(3,4)$.

(g) Assume $b_4 \neq 0$, $c_4 \neq 0$, $a_4 = 0$, $b_4c_5 - c_4b_5 = 0$, $a_2b_4^3 + a_3c_4^3 \neq 0$ and $a_3 \neq 0$. Then cubic similar to $J(3,4)$.

(h) Assume $b_4 \neq 0$, $c_4 \neq 0$, $a_4 = 0$ and $b_4c_5 - c_4b_5 \neq 0$. Then cubic similar to $P(3,4i)$.

- $N(3,4f)$ if $a_3 = 0$.
- $N(3,4i)$ if $a_3 = -\dfrac{a_2b_4^3}{c_4^3}$.

(i) Assume $b_4 \neq 0$, $c_4 \neq 0$ and $a_4 \neq 0$. Then cubic similar to $P(3,4j2)$.

- $N(3,3a)$ if $c_5 = \dfrac{b_5c_4}{b_4}$ and $a_3 = -\dfrac{a_2b_4^3}{c_4^3}$.
- $N(3,4a)$ if $c_5 = \dfrac{b_5c_4}{b_4}$ and $a_3 \neq -\dfrac{a_2b_4^3}{c_4^3}$.
- $P(3,4j)$ if $c_5 \neq \dfrac{b_5c_4}{b_4}$ and $a_3 = 0$.
- $N(3,4j)$ if $c_5 = \dfrac{b_5c_4}{b_4} - \dfrac{c_4d_5\sqrt[3]{a_4}}{b_4\sqrt[3]{a_2}}$ and $a_3 = 0$.

28. We have $a_2 \neq 0$ and $b_3 \neq 0$. Note that if $c_5 = 0$ and $d_5 = 0$, we are back in case 16.

(a) Assume $d_5 = 0$ and $a_3 = 0$. Hence $c_5 \neq 0$. Cubic similar to $J(3,4)$.

(b) Assume $d_5 = 0$ and $a_3 \neq 0$. Hence $c_5 \neq 0$. Cubic similar to $N(3,4a)$.

(c) Assume $d_5 \neq 0$, $a_3c_5^3 + a_4d_5^3 = 0$ and $b_3c_5^3 + b_4d_5^3 = 0$. Cubic similar to $J(3,3)$.

(d) Assume $d_5 \neq 0$, $a_3c_5^3 + a_4d_5^3 = 0$ and $b_3c_5^3 + b_4d_5^3 \neq 0$. Cubic similar to $J(3,4)$.

(e) Assume $d_5 \neq 0$, $a_3c_5^3 + a_4d_5^3 \neq 0$ and $b_3c_5^3 + b_4d_5^3 = 0$. Cubic similar to $N(3,3b)$.

(f) Assume $d_5 \neq 0$, $a_3c_5^3 + a_4d_5^3 \neq 0$ and $b_3c_5^3 + b_4d_5^3 \neq 0$. Cubic similar to $N(3,4a)$.

29. We have $a_2 \neq 0$, $b_3 \neq 0$ and $c_4 \neq 0$.

(a) Assume $a_3 = 0$. Cubic similar to $J(3,4)$.

(b) Assume $a_3 \neq 0$. Cubic similar to $N(3,4a)$.

3.6 Rank four

In the rank four case we have only one matrix to examine.

30. In this case we know $a_2 \neq 0$, $b_3 \neq 0$, $c_4 \neq 0$ and $d_5 \neq 0$.
    (a) Assume $a_3 = 0$, $a_4 = 0$ and $b_4 = 0$. Cubic similar to $J(4,5)$.
    (b) Assume $a_3 = 0$, $a_4 = 0$ and $b_4 \neq 0$. Cubic similar to $N(4,5b)$.
    (c) Assume $a_3 = 0$ and $a_4 \neq 0$. Cubic similar to $P(4,5e)$.
    - $N(4,5d)$ if $b_4 = 0$.
    - $N(4,5e)$ if $b_4 = \dfrac{\sqrt[4]{a_4 b_3 c_4^3}}{\sqrt[4]{a_2}}$.

    (d) Assume $a_3 \neq 0$. Cubic similar to $P(4,5c2)$.
    - $N(4,5a)$ if $a_4 = 0$ and $b_4 = 0$.
    - $N(4,5c)$ if $a_4 = 0$ and $b_4 = b_3 c_4^3 d_5^6$.
    - $P(4,5c)$ if $a_4 = 0$ and $b_4 \notin \{0, b_3 c_4^3 d_5^6\}$.

Remark 3.1 In the description given above it sometimes happens that we start with a matrix A where $J((AX)^3)$ has a certain nilpotence index, but after applying some assumptions it has a smaller nilpotence index. (See for instance the cases 18a, 27e, 28c and 28e.) In fact we could have deleted these cases from the list because it must be equivalent to one of the other cases done before, because this nilpotence index is also an invariant of the cubic similarity relation, but because this way it is easier to verify that we really have a *complete* description of all cases, we left them in.

4 Transformations

In this section we present the actual transformations used in the cases of section 3. Since this is the only 'proof' we can give, it must be clear that there are no typing errors in this list. Therefore we used the Maple to LaTeX feature from version 5.3. Unfortunately this has as disadvantage that the transformation mappings are not always in their nicest form. Furthermore, Maple uses %n in the expressions as an abbreviation. The actual value of %n is given directly below the map.

	T	$T^{-1}FT$
	Rank one	
1	$\left(x_1, x_5 + \dfrac{{b_5}^3 x_1}{{a_5}^3}, x_3 + \dfrac{{c_5}^3 x_1}{{a_5}^3}, x_4 + \dfrac{{d_5}^3 x_1}{{a_5}^3}, \dfrac{x_2}{a_5}\right)$	$J(1,2)$
2	$\left(x_1, \dfrac{x_2}{a_2} - \dfrac{a_5 x_5}{a_2} - \dfrac{a_4 x_4}{a_2} - \dfrac{a_3 x_3}{a_2}, x_3, x_4, x_5\right)$	$J(1,2)$

	T	$T^{-1}FT$
	Rank two	
3	$\left(a_4^3 x_1, x_5 + b_5^3 x_3, c_5^3 x_3, x_2 - \frac{a_5 x_4}{a_4}, x_4\right)$	$J(2,2)$
4	$\left(a_3^3 x_1, b_5^3 x_3, x_2 - \frac{a_4 x_5}{a_3} - \frac{a_5 x_4}{a_3}, x_5, x_4\right)$	$J(2,2)$
5	$\left(a_3^3 x_1, x_5 + b_5^3 x_3, x_2 + c_5^3 x_3 - \frac{a_5 x_4}{a_3}, d_5^3 x_3, x_4\right)$	$J(2,2)$
6	$\left(a_2^3 x_1, x_2 + b_5^3 x_3 - \frac{a_4 x_5}{a_2} - \frac{a_5 x_4}{a_2}, c_5^3 x_3, x_5, x_4\right)$	$J(2,2)$
7	$\left(a_2^3 x_1, x_2 - \frac{a_3 x_5}{a_2} + b_5^3 x_3 - \frac{a_5 x_4}{a_2}, x_5 + c_5^3 x_3, d_5^3 x_3, x_4\right)$	$J(2,2)$
8a	$\left(x_5 + a_5^3 x_3, b_4^3 x_1, c_5^3 x_3, x_2 - \frac{b_5 x_4}{b_4}, x_4\right)$	$J(2,2)$
8b	$\left(-\frac{(b_5 a_4 - b_4 a_5)^3 x_1}{b_4^3}, -\frac{(b_5 a_4 - b_4 a_5)^3 x_3}{a_4^3}, x_5, \frac{a_5 x_4}{a_4} - \frac{x_2 b_5}{b_4}, x_2 - x_4\right)$	$J(2,2)$
8c	$\left(x_5 + a_4^3 x_3, b_4^3 x_3, x_1, x_4 - \frac{b_5 x_2}{b_4 c_5}, \frac{x_2}{c_5}\right)$	$J(2,2)$
8d	$\left(\frac{(b_5 a_4 - b_4 a_5)^3 x_1}{b_4^3}, \frac{(b_5 a_4 - b_4 a_5)^3 x_2}{a_4^3}, c_5^3 x_3, \frac{(b_5 a_4 - b_4 a_5) x_4}{b_4 a_4} - \frac{a_5 x_5}{a_4} x_5\right)$	$N(2,2a)$
9	$\left(a_3^3 x_1, b_4^3 x_3, x_2 - \frac{a_4 x_4}{a_3} + \frac{(b_5 a_4 - b_4 a_5) x_5}{b_4 a_3}, x_4 - \frac{b_5 x_5}{b_4}, x_5\right)$	$J(2,2)$
10a	$\left(-\frac{(-c_4 a_5 + c_5 a_4)^3 x_3}{c_4^3}, x_5 - \frac{b_5^3 (-c_4 a_5 + c_5 a_4)^3 x_3}{c_4^3 a_5^3}, -\frac{(-c_4 a_5 + c_5 a_4)^3 x_1}{a_5^3}, x_2 - \frac{c_5 x_4}{c_4}, x_4 - \frac{a_4 x_2}{a_5}\right)$	$J(2,2)$
10b	$\left(a_4^3 x_5 + a_4^3 x_1, b_4^3 x_3, \frac{c_5^3 a_4^3 x_1}{a_5^3}, \frac{b_5 a_4 x_2}{\%1} - \frac{a_5 b_4 x_4}{\%1}, -\frac{b_4 a_4 x_2}{\%1} + \frac{b_4 a_4 x_4}{\%1}\right)$ $\%1 := b_5 a_4 - b_4 a_5$	$J(2,2)$
10c	$\left(\frac{(-c_4 a_5 + c_5 a_4)^3 x_1}{c_5^3}, \frac{b_5^3 x_3}{c_5^3} + x_5, x_3, x_2 - \frac{a_5 x_4}{-c_4 a_5 + c_5 a_4}, -\frac{c_4 x_2}{c_5} + \frac{a_4 x_4}{-c_4 a_5 + c_5 a_4}\right)$	$J(2,2)$

	T	$T^{-1}FT$
10d	$\left(\frac{\%1^3\,(-c_4\,a_5+c_5\,a_4)^3\,x_1}{a_5{}^3\,(-b_4\,c_5+b_5\,c_4)^3}, -\frac{\%1^3\,x_2}{a_5{}^3}, -\frac{(-c_4\,a_5+c_5\,a_4)^3\,x_3}{a_5{}^3}, \frac{\%1\,c_5\,x_4}{a_5\,(-b_4\,c_5+b_5\,c_4)}+x_5, -\frac{c_4\,\%1\,x_4}{a_5\,(-b_4\,c_5+b_5\,c_4)}-\frac{a_4\,x_5}{a_5}\right)$ $\%1 := b_5\,a_4-b_4\,a_5$	$N(2,2a)$
11	$\left(a_2{}^3\,x_1, x_2-\frac{a_4\,x_4}{a_2}+\frac{(-c_4\,a_5+c_5\,a_4)\,x_5}{c_4\,a_2}, c_4{}^3\,x_3, x_4-\frac{c_5\,x_5}{c_4}, x_5\right)$	$J(2,2)$
12	$\left(a_2{}^3\,x_1, x_2+b_4{}^3\,x_3-\frac{a_4\,x_4}{a_2}+\frac{(-c_4\,a_5+c_5\,a_4)\,x_5}{c_4\,a_2}, c_4{}^3\,x_3, x_4-\frac{c_5\,x_5}{c_4}, x_5\right)$	$J(2,2)$
13a	$\left(a_5{}^3\,x_1, b_3{}^3\,x_3, x_4-\frac{b_4\,x_5}{b_3}-\frac{b_5\,x_2}{b_3}, x_5, x_2\right)$	$J(2,2)$
13b	$\left(a_4{}^3\,x_1, b_3{}^3\,x_3, x_4-\frac{x_2\,b_4}{b_3}-\frac{(b_5\,a_4-b_4\,a_5)\,x_5}{a_4\,b_3}, x_2-\frac{a_5\,x_5}{a_4}, x_5\right)$	$J(2,2)$
13c	$\left(a_3{}^3\,x_1, b_3{}^3\,x_3, -\frac{x_2\,b_4\,a_3}{\%1}+\frac{a_4\,b_3\,x_4}{\%1}-\frac{a_5\,x_5}{a_3}, -\frac{a_3\,b_3\,x_4}{\%1}+\frac{a_3\,b_3\,x_2}{\%1}, x_5\right)$ $\%1 := -b_4\,a_3+a_4\,b_3$	$J(2,2)$
13d	$\left(a_3{}^3\,x_1, b_3{}^3\,x_3, \frac{x_2\,b_5\,a_3}{\%1}-\frac{a_5\,b_3\,x_4}{\%1}-\frac{a_4\,x_5}{a_3}, x_5, -\frac{a_3\,b_3\,x_2}{\%1}+\frac{a_3\,b_3\,x_4}{\%1}\right)$ $\%1 := b_5\,a_3-a_5\,b_3$	$J(2,2)$
13e	$\left(a_3{}^3\,x_1, b_3{}^3\,x_3, \frac{x_2\,b_5\,a_3}{\%1}-\frac{a_5\,b_3\,x_4}{\%1}-\frac{(b_5\,a_4-b_4\,a_5)\,x_5}{\%1}, x_5, -\frac{a_3\,b_3\,x_2}{\%1}+\frac{a_3\,b_3\,x_4}{\%1}+\frac{(-b_4\,a_3+a_4\,b_3)\,x_5}{\%1}\right)$ $\%1 := b_5\,a_3-a_5\,b_3$	$J(2,2)$
14	$\left(x_5+a_5{}^3\,x_1, b_3{}^3\,x_3, x_4+c_5{}^3\,x_1-\frac{b_5\,x_2}{b_3}, d_5{}^3\,x_1, x_2\right)$	$J(2,2)$
15a	$\left(x_5+a_3{}^3\,x_1, b_3{}^3\,x_1, x_2+c_5{}^3\,x_3-\frac{a_5\,x_4}{a_3}, d_5{}^3\,x_3, x_4\right)$	$J(2,2)$
15b	$\left(a_3{}^3\,x_2, b_3{}^3\,x_1, x_4-\frac{a_3{}^3\,b_3{}^3\,c_5{}^3\,x_3}{\%1^3}+\frac{b_5\,a_3\,x_5}{\%1}, -\frac{d_5{}^3\,a_3{}^3\,b_3{}^3\,x_3}{\%1^3}, -\frac{a_3\,b_3\,x_5}{\%1}\right)$ $\%1 := b_5\,a_3-a_5\,b_3$	$N(2,2a)$

	T	$T^{-1}FT$
16	$\left(a_2^3 b_3^9 x_1, b_3^3 x_2 - \frac{a_3 x_3}{a_2} - \frac{(-b_4 a_3 + a_4 b_3)\ x_4}{b_3 a_2} + \frac{(b_5 a_3 - a_5 b_3)\ x_5}{b_3 a_2}, x_3 - \frac{b_4 x_4}{b_3} - \frac{b_5 x_5}{b_3}, x_4, x_5\right)$	$J(2,3)$
17a	$\left(a_4^3 d_5^9 x_1, x_4 + b_5^3 x_2, x_5 + c_5^3 x_2, d_5^3 x_2 - \frac{a_5 x_3}{a_4}, x_3\right)$	$J(2,3)$
17b	$\left(a_4^3 d_5^9 x_1, x_4 + b_5^3 x_2, x_5 + d_5^9 c_4^3 x_1, d_5^3 x_2 - \frac{a_5 x_3}{a_4}, x_3\right)$	$J(2,3)$
17c	$\left(\frac{\%1^4 \sqrt{\%2}\, x_1}{c_4^4 d_5^4 a_4}, \frac{\%1\, b_5^3 \sqrt{\%2}\, x_3}{c_4 d_5^4 a_4} + x_5, \frac{\%1^4 \sqrt{\%2}\, x_2}{c_4 d_5^4 a_4^4}, \frac{\%1 \sqrt{\%2}\, x_3}{c_4 a_4 d_5} - \frac{a_5 \sqrt{\%2}\, x_4}{d_5 a_4}, \frac{\sqrt{\%2}\, x_4}{d_5}\right)$ $\%1 := -c_4 a_5 + c_5 a_4$ $\%2 := \frac{\%1}{c_4 d_5 a_4}$	$N(2,3a)$
17d	$\left(a_4^3 d_5^9 x_1, x_4 + b_4^3 d_5^9 x_1, x_5 + c_5^3 x_2, d_5^3 x_2 - \frac{a_5 x_3}{a_4}, x_3\right)$	$J(2,3)$
17e	$\left(\frac{\%1^4 \sqrt{\%2}\, x_1}{b_4^4 d_5^4 a_4}, \frac{\%1^4 \sqrt{\%2}\, x_2}{b_4 d_5^4 a_4^4}, \frac{\%1\, c_5^3 \sqrt{\%2}\, x_3}{b_4 d_5^4 a_4} + x_5, \frac{\%1 \sqrt{\%2}\, x_3}{b_4 a_4 d_5} - \frac{a_5 \sqrt{\%2}\, x_4}{d_5 a_4}, \frac{\sqrt{\%2}\, x_4}{d_5}\right)$ $\%1 := b_5 a_4 - b_4 a_5$ $\%2 := \frac{\%1}{b_4 d_5 a_4}$	$N(2,3a)$
17f	$\left(a_4^3 d_5^9 x_1, x_4 + b_4^3 d_5^9 x_1, x_5 + d_5^9 c_4^3 x_1, d_5^3 x_2 - \frac{a_5 x_3}{a_4}, x_3\right)$	$J(2,3)$
17g	$\left(\frac{\%1^4 \sqrt{\%2}\, x_1}{b_4^4 d_5^4 a_4}, \frac{\%1^4 \sqrt{\%2}\, x_2}{b_4 d_5^4 a_4^4}, \frac{\%1^4 c_4^3 \sqrt{\%2}\, x_1}{b_4^4 d_5^4 a_4^4} + x_5, \frac{\%1 \sqrt{\%2}\, x_3}{b_4 a_4 d_5} - \frac{a_5 \sqrt{\%2}\, x_4}{d_5 a_4}, \frac{\sqrt{\%2}\, x_4}{d_5}\right)$ $\%1 := b_5 a_4 - b_4 a_5$ $\%2 := \frac{\%1}{b_4 d_5 a_4}$	$N(2,3a)$
17h	$\left(\frac{\%1^4 \sqrt{\%2}\, x_1}{c_4^4 d_5^4 a_4}, \frac{\%1^4 b_4^3 \sqrt{\%2}\, x_1}{c_4^4 d_5^4 a_4^4} + x_5, \frac{\%1^4 \sqrt{\%2}\, x_2}{c_4 d_5^4 a_4^4}, \frac{\%1 \sqrt{\%2}\, x_3}{c_4 a_4 d_5} - \frac{a_5 \sqrt{\%2}\, x_4}{d_5 a_4}, \frac{\sqrt{\%2}\, x_4}{d_5}\right)$ $\%1 := -c_4 a_5 + c_5 a_4$ $\%2 := \frac{\%1}{c_4 d_5 a_4}$	$N(2,3a)$

	T	$T^{-1}FT$
17i	$\left(\frac{\%1^4\sqrt{\%2}\,x_1}{b_4{}^4\,d_5{}^4\,a_4}, \frac{\%1^4\sqrt{\%2}\,x_2}{b_4\,d_5{}^4\,a_4{}^4}+x_5, \frac{\%1^4\,c_4{}^3\sqrt{\%2}\,x_2}{b_4{}^4\,d_5{}^4\,a_4{}^4}, \frac{\%1\sqrt{\%2}\,x_3}{b_4\,a_4\,d_5}-\frac{a_5\sqrt{\%2}\,x_4}{d_5\,a_4}, \frac{\sqrt{\%2}\,x_4}{d_5}\right)$ $\%1 := b_5\,a_4 - b_4\,a_5$ $\%2 := \frac{\%1}{b_4\,d_5\,a_4}$	$N(2,3a)$
17j	$\left(\frac{\%1^4\sqrt{\%2}\,x_1}{b_4{}^4\,d_5{}^4\,a_4}, \frac{\%1^4\sqrt{\%2}\,x_2}{b_4\,d_5{}^4\,a_4{}^4}, \frac{\sqrt{\%2}\,c_4\,(-b_4\,c_5+b_5\,c_4)\,\%1^2\,\%3\,x_1}{b_4{}^4\,d_5{}^4\,a_4{}^3} + \frac{1}{2}\frac{\sqrt{\%2}\,c_4\,(-c_4\,a_5+c_5\,a_4)\,\%1^2\,\%3\,x_2}{b_4{}^3\,d_5{}^4\,a_4{}^4} - \frac{1}{2}\frac{\sqrt{\%2}\,c_4\,(-c_4\,a_5+c_5\,a_4)\,(-b_4\,c_5+b_5\,c_4)\,\%1^2\,x_3}{d_5{}^4\,a_4{}^3\,b_4{}^3} - \frac{\sqrt{\%2}\,(-c_4\,a_5+c_5\,a_4)\,(-b_4\,c_5+b_5\,c_4)\,\%1\,\%3\,x_4}{d_5{}^4\,a_4{}^3\,b_4{}^3}, \frac{\%1\sqrt{\%2}\,x_4}{b_4\,a_4\,d_5}-\frac{a_5\sqrt{\%2}\,x_5}{d_5\,a_4}, \frac{\sqrt{\%2}\,x_5}{d_5}\right)$ $\%1 := b_5\,a_4 - b_4\,a_5$ $\%2 := \frac{\%1}{b_4\,d_5\,a_4}$ $\%3 := b_5\,c_4\,a_4 + a_4\,b_4\,c_5 - 2\,c_4\,a_5\,b_4$	$N(2,3b)$
18a	$\left(a_3{}^3\,x_1, x_5+b_5{}^3\,x_3, x_2+c_5{}^3\,x_3-\frac{a_5\,x_4}{a_3}-\frac{a_2\,x_5}{a_3}, d_5{}^3\,x_3, x_4\right)$	$J(2,2)$
18b	$\left(\left(a_2\,b_5{}^3+c_5{}^3\,a_3+a_4\,d_5{}^3\right)^3\,x_1, b_5{}^3\,x_2+x_4-b_5{}^3\,x_5, c_5{}^3\,x_2-\frac{a_5\,x_3}{a_3}-\frac{a_2\,x_4}{a_3}+\frac{\left(a_2\,b_5{}^3+a_4\,d_5{}^3\right)\,x_5}{a_3}, d_5{}^3\,x_2-d_5{}^3\,x_5, x_3\right)$	$J(2,3)$
Rank three		
19a	$\left(a_2{}^3\,b_5{}^9\,x_1, b_5{}^3\,x_2+\frac{(-c_4\,a_5+c_5\,a_4)\,x_3}{a_2\,c_4}-\frac{a_4\,x_5}{a_2}, c_4{}^3\,x_4, -\frac{c_5\,x_3}{c_4}+x_5, x_3\right)$	$J(3,3)$

	T	$T^{-1}FT$
19b	$\left(a_2^3 b_5^9 x_1, b_5^3 x_2 + \frac{(-c_4 a_5 + c_5 a_4)\ x_4}{a_2 c_4} - \frac{a_4 \left(\frac{a_2 b_5^3}{a_3}\right)^{1/3} x_5}{a_2 c_4}, \frac{a_2 b_5^3 x_3}{a_3}, -\frac{c_5 x_4}{c_4} + \frac{\left(\frac{a_2 b_5^3}{a_3}\right)^{1/3} x_5}{c_4}, x_4\right)$	$N(3, 3b)$
19c	$\left(a_2^3 b_4^9 x_1, b_4^3 x_2 - \frac{a_4 x_3}{a_2} - \frac{(-b_5 a_4 + b_4 a_5)\ x_5}{a_2 b_4}, c_5^3 x_4, x_3 - \frac{b_5 x_5}{b_4}, x_5\right)$	$J(3, 3)$
19d	$\left(c_5^9 a_3^3 x_1, \frac{c_5^3 a_3 x_2}{a_2} - \frac{a_4 \left(\frac{c_5^3 a_3}{a_2}\right)^{1/3} x_4}{a_2 b_4} - \frac{(-b_5 a_4 + b_4 a_5)\ x_5}{a_2 b_4}, c_5^3 x_3, \frac{\left(\frac{c_5^3 a_3}{a_2}\right)^{1/3} x_4}{b_4} - \frac{b_5 x_5}{b_4}, x_5\right)$	$N(3, 3b)$
19e	$\left(\frac{a_2^3\ (b_4 c_5 - b_5 c_4)^9\ x_1}{c_4^9}, \frac{(b_4 c_5 - b_5 c_4)^3\ x_2}{c_4^3} - \frac{(-c_4 a_5 + c_5 a_4)\ x_3}{a_2 c_4} - \frac{(-b_5 a_4 + b_4 a_5)\ x_5}{a_2 b_4}, \frac{(b_4 c_5 - b_5 c_4)^3\ x_4}{b_4^3}, \frac{c_5 x_3}{c_4} - \frac{b_5 x_5}{b_4}, -x_3 + x_5\right)$	$J(3, 3)$
19f	$\left(\frac{a_2^3 \%1^9 x_1}{c_4^9}, \frac{\%1^3 x_2}{c_4^3} + \frac{a_5 x_4}{a_2} - \frac{a_5 \left(\frac{a_2 \%1^3}{a_3 c_4^3}\right)^{1/3} b_4 x_5}{\%1\, a_2}, \frac{a_2 \%1^3 x_3}{a_3 c_4^3}, \frac{c_5 x_4}{c_4} - \frac{b_5 \left(\frac{a_2 \%1^3}{a_3 c_4^3}\right)^{1/3} x_5}{\%1}, -x_4 + \frac{\left(\frac{a_2 \%1^3}{a_3 c_4^3}\right)^{1/3} b_4 x_5}{\%1}\right)$ $\%1 := b_4 c_5 - b_5 c_4$	$N(3, 3b)$
19g	$\left(\frac{a_2^3 b_4^9 \%1^9 x_1}{a_4^9 c_4^9}, \frac{b_4^3 \%1^3 x_2}{a_4^3 c_4^3} - \frac{\%1\, x_4}{a_2 c_4}, \frac{\%1^3 a_2 b_4^3 x_3}{a_3 a_4^3 c_4^3}, \frac{c_5 x_4}{c_4} - \frac{a_5 \left(\frac{\%1^3 a_2 b_4^3}{a_3 a_4^3 c_4^3}\right)^{1/3} x_5}{\%1}, -x_4 + \frac{\left(\frac{\%1^3 a_2 b_4^3}{a_3 a_4^3 c_4^3}\right)^{1/3} a_4 x_5}{\%1}\right)$	$N(3, 3b)$

	T	$T^{-1}FT$
19h	$\%1 := -c_4\,a_5 + c_5\,a_4$ $\left(\frac{a_3^{\,3}\,\%1^9\,x_1}{b_4^{\,9}}, \frac{\%1^3\,a_3\,x_2}{a_2\,b_4^{\,3}} - \frac{(-c_4\,a_5 + c_5\,a_4)\left(\frac{\%1^3\,a_3}{a_2\,b_4^{\,3}}\right)^{1/3} x_4}{a_2\,\%1} - \frac{(-b_5\,a_4 + b_4\,a_5)\,x_5}{a_2\,b_4}, \frac{\%1^3\,x_3}{b_4^{\,3}}, \frac{c_5\left(\frac{\%1^3\,a_3}{a_2\,b_4^{\,3}}\right)^{1/3} x_4}{\%1} - \frac{b_5\,x_5}{b_4}, -\frac{\left(\frac{\%1^3\,a_3}{a_2\,b_4^{\,3}}\right)^{1/3} c_4\,x_4}{\%1} + x_5\right)$	$N(3,3b)$
20a	$\%1 := b_4\,c_5 - b_5\,c_4$ $\left(a_2^{\,3}\,x_4, b_5^{\,3}\,x_2 - \frac{\left(a_2\,c_5\,b_5^{\,3} + a_5\,d_5^{\,3}\,c_4\right)x_3}{a_2\,d_5^{\,3}\,c_4} + x_5, d_5^{\,9}\,c_4^{\,3}\,x_1, d_5^{\,3}\,x_2 - \frac{c_5\,x_3}{c_4}, x_3\right)$	$J(3,3)$
20b	$\left(a_2^{\,3}\,x_2, \frac{\left(a_2\,d_5\,c_4\,a_4 - a_2^{\,1/3}\,a_4^{\,2/3}\,c_4\,a_5 + a_2^{\,1/3}\,a_4^{\,5/3}\,c_5\right)x_4}{a_2\,d_5\,c_4\,a_4} + \frac{(-c_4\,a_5 + c_5\,a_4)\,x_5}{a_2^{\,2/3}\,d_5\,c_4\,a_4^{\,1/3}}, \frac{c_4^{\,3}\,a_2^{\,3}\,x_1}{a_4^{\,3}}, \frac{a_2\,x_3}{a_4} - \frac{a_2^{\,1/3}\,c_5\,x_4}{c_4\,a_4^{\,1/3}\,d_5} - \frac{a_2^{\,1/3}\,c_5\,x_5}{c_4\,a_4^{\,1/3}\,d_5}, \frac{a_2^{\,1/3}\,x_4}{a_4^{\,1/3}\,d_5} + \frac{a_2^{\,1/3}\,x_5}{a_4^{\,1/3}\,d_5}\right)$	$N(3,3a)$
20c	$\left(\left(a_2\,b_5^{\,3} + a_4\,d_5^{\,3}\right)^3 x_2, b_5^{\,3}\,x_3 + \frac{\left(c_5\,a_4 - c_4\,a_5 + a_2\,b_5^{\,3}\,c_4 + a_4\,d_5^{\,3}\,c_4\right)x_4}{a_2\,c_4} + \frac{(-c_4\,a_5 + c_5\,a_4)\,x_5}{c_4\,a_2}, d_5^{\,9}\,c_4^{\,3}\,x_1, d_5^{\,3}\,x_3 - \frac{c_5\,x_4}{c_4} - \frac{c_5\,x_5}{c_4}, x_4 + x_5\right)$	$N(3,3a)$
21a	$\left(a_3^{\,3}\,x_4, b_4^{\,3}\,d_5^{\,9}\,x_1, c_5^{\,3}\,x_2 - \frac{\left(b_5\,c_5^{\,3}\,a_3 + b_4\,a_5\,d_5^{\,3}\right)x_3}{a_3\,d_5^{\,3}\,b_4} + x_5, d_5^{\,3}\,x_2 - \frac{b_5\,x_3}{b_4}, x_3\right)$	$J(3,3)$

	T	$T^{-1}FT$
21b	$\left(\left(c_5^3 a_3 + a_4 d_5^3\right)^3 x_1 + x_5, b_4^3 d_5^9 x_2, c_5^3 x_3 - \frac{\left(b_4 a_5 - b_5 a_4 + a_4 b_4 d_5^3\right) x_4}{b_4 a_3} - \frac{(-b_5 a_4 + b_4 a_5)\ x_5}{b_4 a_3}, d_5^3 x_3 + \frac{\left(-b_5 + b_4 d_5^3\right) x_4}{b_4} - \frac{b_5 x_5}{b_4}, x_4 + x_5\right)$	$N(3,3a)$
22a	$\left(a_3^3 x_4, b_4^3 d_5^9 x_1, d_5^9 c_4^3 x_1 - \frac{a_5 x_3}{a_3} + x_5, d_5^3 x_2 - \frac{c_5 x_3}{c_4}, x_3\right)$	$J(3,3)$
22b	$\left(a_4^3 d_5^9 x_2, b_4^3 d_5^9 x_1, d_5^9 c_4^3 x_1 + \frac{\left(a_4 d_5^3 c_4 - c_4 a_5 + c_5 a_4\right) x_4}{a_3 c_4} - \frac{(c_4 a_5 - c_5 a_4)\ x_5}{a_3 c_4}, d_5^3 x_3 - \frac{c_5 x_4}{c_4} - \frac{c_5 x_5}{c_4}, x_4 + x_5\right)$	$N(3,3a)$
23a	$\left(\frac{a_3^3 b_4^3 x_4}{b_3^3}, b_4^3 x_1, -\frac{a_5 x_3}{a_3 d_5} + \frac{b_4 x_5}{b_3}, x_2 - \frac{(b_5 a_3 - a_5 b_3)\ x_3}{a_3 b_4 d_5} - x_5, \frac{x_3}{d_5}\right)$	$J(3,3)$
23b	$\left(a_4^3 x_4, b_3^3 c_5^9 x_1, c_5^3 x_2 + \frac{(-b_5 a_4 + b_4 a_5)\ x_3}{a_4 b_3} - \frac{b_4 x_5}{b_3}, -\frac{a_5 x_3}{a_4} + x_5, x_3\right)$	$J(3,3)$
23c	$\left(-\frac{a_3^3 \left(b_3 c_5^3 + b_4 d_5^3\right)^3 x_4}{d_5^9 b_3^3}, \left(b_3 c_5^3 + b_4 d_5^3\right)^3 x_1, c_5^3 x_2 - \frac{a_5 x_3}{a_3} - \frac{b_4 x_5}{b_3}, d_5^3 x_2 + x_5, x_3\right)$	$J(3,3)$
23d	$\left(-\frac{a_3^3 \%2^3 x_4}{d_5^9 b_3^3}, \frac{a_3^9 \%2^{12} x_1}{\%1^9 d_5^{27}}, \frac{c_5^3 a_3^3 \%2^3 x_2}{d_5^9 \%1^3} - \frac{\left(b_5 c_5^3 a_3 + b_4 a_5 d_5^3\right) x_3}{d_5^3 \%1} - \frac{b_4 x_5}{b_3}, \frac{a_3^3 \%2^3 x_2}{\%1^3 d_5^6} - x_3 + x_5, \frac{a_3 \%2 x_3}{d_5^3 \%1}\right)$ $\%1 := b_5 a_3 - a_5 b_3$ $\%2 := b_3 c_5^3 + b_4 d_5^3$	$J(3,3)$
23e	$\left(\%1^3 x_1, \frac{b_3^3 \%1^3 x_4}{d_5^9 a_3^3}, c_5^3 x_2 - \frac{\left(a_5 b_3 c_5^3 + a_4 d_5^3 b_5\right) x_3}{b_3 \%1} + \frac{a_4 x_5}{a_3}, d_5^3 x_2 + \frac{d_5^3\ (b_5 a_3 - a_5 b_3)\ x_3}{b_3 \%1} - x_5, x_3\right)$ $\%1 := c_5^3 a_3 + a_4 d_5^3$	$J(3,3)$

	T	$T^{-1}FT$
23f	$\left(\left({c_5}^3\,a_3+a_4\,{d_5}^3\right)^3\,x_1,\left(b_3\,{c_5}^3+b_4\,{d_5}^3\right)^3\,x_2,{c_5}^3\,x_3-\frac{\left(a_4\,a_3\,b_3\,{c_5}^3+a_4\,a_3\,b_4\,{d_5}^3+a_5\,b_4\,a_3-a_5\,a_4\,b_3\right)\,x_4}{(b_4\,a_3-a_4\,b_3)\,a_3}-\frac{a_5\,x_5}{a_3},{d_5}^3\,x_3+\frac{a_3\,\left(b_3\,{c_5}^3+b_4\,{d_5}^3\right)\,x_4}{b_4\,a_3-a_4\,b_3},x_4+x_5\right)$	$N(3,3a)$
23g	$\left({c_5}^9\,{a_3}^3\,x_1,{b_3}^3\,{c_5}^9\,x_2,{c_5}^3\,x_3-\frac{(b_4\,a_5-b_5\,a_4+a_4\,b_3\,{c_5}^3)\,x_4}{\%1}-\frac{(-b_5\,a_4+b_4\,a_5)\,x_5}{\%1},\frac{(-b_5\,a_3+a_5\,b_3+{c_5}^3\,a_3\,b_3)\,x_4}{\%1}-\frac{(b_5\,a_3-a_5\,b_3)\,x_5}{\%1},x_4+x_5\right)$ $\%1 := b_4\,a_3-a_4\,b_3$	$N(3,3a)$
23h	$\left(\frac{\%2^4\,\sqrt{\frac{{d_5}^2\,\%2}{b_3\,\%1}}\,x_2}{{b_3}^4\,d_5\,\%1},\frac{\left(b_3\,{c_5}^3+b_4\,{d_5}^3\right)^3\,\%2^4\,\sqrt{\frac{{d_5}^2\,\%2}{b_3\,\%1}}\,x_1}{{b_3}^4\,d_5\,\%1^4},\frac{\%2\,{c_5}^3\,\sqrt{\frac{{d_5}^2\,\%2}{b_3\,\%1}}\,x_3}{b_3\,d_5\,\%1}+\frac{(a_3\,b_4\,b_5+a_4\,b_3\,b_5-2\,a_5\,b_3\,b_4)\,\sqrt{\frac{{d_5}^2\,\%2}{b_3\,\%1}}\,x_4}{b_3\,d_5\,\%3}-\frac{(-b_5\,a_4+b_4\,a_5)\,\sqrt{\frac{{d_5}^2\,\%2}{b_3\,\%1}}\,x_5}{d_5\,\%3},\frac{{d_5}^2\,\%2\,\sqrt{\frac{{d_5}^2\,\%2}{b_3\,\%1}}\,x_3}{b_3\,\%1}-2\,\frac{\%2\,\sqrt{\frac{{d_5}^2\,\%2}{b_3\,\%1}}\,x_4}{d_5\,\%3}-\frac{\%2\,\sqrt{\frac{{d_5}^2\,\%2}{b_3\,\%1}}\,x_5}{d_5\,\%3},\frac{\sqrt{\frac{{d_5}^2\,\%2}{b_3\,\%1}}\,x_4}{d_5}+\frac{\sqrt{\frac{{d_5}^2\,\%2}{b_3\,\%1}}\,x_5}{d_5}\right)$ $\%1 := {c_5}^3\,a_3+a_4\,{d_5}^3$ $\%2 := b_5\,a_3-a_5\,b_3$ $\%3 := b_4\,a_3-a_4\,b_3$	$N(3,3a)$
24a	$\left({a_2}^3\,{b_3}^9\,x_1,{b_3}^3\,x_2-\frac{a_3\,x_3}{a_2}+\frac{(b_5\,a_3-a_5\,b_3)\,x_5}{b_3\,a_2},x_3+{c_5}^3\,x_4-\frac{b_5\,x_5}{b_3},{d_5}^3\,x_4,x_5\right)$	$J(3,3)$

	T	$T^{-1}FT$
24b	$\left(\%1^3\,x_1, \frac{\%1\,x_3}{a_2} + \frac{(b_5\,a_3 - a_5\,b_3)\,x_4}{a_2\,b_3} - \frac{a_3\left(\frac{\%1}{a_2}\right)^{1/3} x_5}{a_2\,b_3}, c_5^3\,x_2 - \frac{b_5\,x_4}{b_3} + \frac{\left(\frac{\%1}{a_2}\right)^{1/3} x_5}{b_3}, d_5^3\,x_2, x_4\right)$ $\%1 := c_5^3\,a_3 + a_4\,d_5^3$	$N(3,3b)$
25a	$\left(a_5^3\,x_4, b_3^3\,c_4^9\,x_1, c_4^3\,x_2 - \frac{b_4\,x_3}{b_3} + \frac{(b_4\,c_5 - b_5\,c_4)\,x_5}{b_3\,c_4}, x_3 - \frac{c_5\,x_5}{c_4}, x_5\right)$	$J(3,3)$
25b	$\left(-\frac{(-c_4\,a_5 + c_5\,a_4)^3\,x_4}{c_4^3}, -\frac{b_3^3\,(-c_4\,a_5 + c_5\,a_4)^9\,x_1}{a_4^9}, -\frac{(-c_4\,a_5 + c_5\,a_4)^3\,x_2}{a_4^3} - \frac{(-b_5\,a_4 + b_4\,a_5)\,x_3}{a_4\,b_3} + \frac{(b_4\,c_5 - b_5\,c_4)\,x_5}{b_3\,c_4}, \frac{a_5\,x_3}{a_4} - \frac{c_5\,x_5}{c_4}, -x_3 + x_5\right)$	$J(3,3)$
25c	$\left(a_3^3\,c_4^9\,x_2, b_3^3\,c_4^9\,x_1, c_4^3\,x_3 + \frac{(-c_5\,a_4\,a_3\,b_3\,c_4^2 + a_5\,a_4\,b_3 - a_4\,b_5\,a_3 + a_3^2\,c_4^3\,b_5)\,x_4}{(b_5\,a_3 - a_5\,b_3)\,a_3} - \frac{a_4\,x_5}{a_3}, \frac{(a_3\,b_3\,c_5\,c_4^2 - a_5\,b_3 + b_5\,a_3)\,x_4}{b_5\,a_3 - a_5\,b_3} + x_5, -\frac{b_3\,a_3\,c_4^3\,x_4}{b_5\,a_3 - a_5\,b_3}\right)$	$N(3,3a)$
25d	$\left(\frac{\%2^4\sqrt{\%3}\,x_2}{c_4^4\,a_3\,b_3^4}, \frac{\%2^4\sqrt{\%3}\,x_1}{c_4^4\,a_3^4\,b_3}, -\frac{\%2\sqrt{\%3}\,x_3}{a_3\,b_3\,c_4} - \frac{\sqrt{\%3}\,b_4\,x_4}{c_4\,b_3} - \frac{\sqrt{\%3}\,(-b_5\,a_4 + b_4\,a_5)\,x_5}{\%1}, \frac{\sqrt{\%3}\,x_4}{c_4} - \frac{\sqrt{\%3}\,(b_5\,a_3 - a_5\,b_3)\,x_5}{\%1}, \frac{\sqrt{\%3}\,\%2\,x_5}{\%1}\right)$ $\%1 := -b_3\,a_4\,c_5 + b_3\,a_5\,c_4 + b_4\,c_5\,a_3 - b_5\,c_4\,a_3$ $\%2 := b_4\,a_3 - a_4\,b_3$ $\%3 := -\frac{\%2}{c_4\,a_3\,b_3}$	$N(3,3a)$
26a	$\left(x_5 + a_5^3\,x_3, c_4^9\,d_5^{27}\,b_3^3\,x_1, c_4^3\,d_5^9\,x_2 - \frac{b_5\,x_4}{b_3}, d_5^3\,x_3 - \frac{c_5\,x_4}{c_4}, x_4\right)$	$J(3,4)$

	T	$T^{-1}FT$
26b	$\left(\frac{{a_5}^3\sqrt{\%1}\,x_3}{b_3\,{c_4}^2\,{d_5}^3}+x_5,\frac{{b_4}^4\sqrt{\%1}\,x_1}{{b_3}^2\,{c_4}^5},\frac{b_4\sqrt{\%1}\,x_2}{{b_3}^2\,{c_4}^2}+\frac{(b_4\,c_5-b_5\,c_4)\left({b_3}^2\,c_4\sqrt{\%1}\right)^{1/3}x_4}{{b_3}^2\,{c_4}^2\,d_5},\frac{\sqrt{\%1}\,x_3}{b_3\,{c_4}^2}-\frac{c_5\left({b_3}^2\,c_4\sqrt{\%1}\right)^{1/3}x_4}{b_3\,{c_4}^2\,d_5},\frac{\left({b_3}^2\,c_4\sqrt{\%1}\right)^{1/3}x_4}{b_3\,c_4\,d_5}\right)$ $\%1:=b_3\,b_4\,c_4$	$N(3,4a)$
26c	$\left({a_4}^3\,{d_5}^9\,x_2+\frac{x_5}{{b_3}^3\,{c_4}^{12}\,{d_5}^{39}},{c_4}^9\,{d_5}^{27}\,{b_3}^3\,x_1,{c_4}^3\,{d_5}^9\,x_2-\frac{b_5\,x_4}{b_3},{d_5}^3\,x_3-\frac{a_5\,x_4}{a_4},x_4\right)$	$J(3,4)$
26d	$\left(\frac{\%1^4\sqrt{\%2}\,x_2}{{a_4}^2\,{d_5}^5\,{c_4}^5},-\frac{{b_3}^3\,\%1^{13}\sqrt{\%2}\,x_1}{{a_4}^{14}\,{d_5}^{14}\,{c_4}^5},\frac{\%1^4\sqrt{\%2}\,x_3}{{a_4}^5\,{d_5}^5\,{c_4}^2}-\frac{b_5\sqrt{\%2}\,x_5}{a_4\,{d_5}^2\,c_4\,b_3},-\frac{\%1\sqrt{\%2}\,x_4}{{a_4}^2\,{d_5}^2\,{c_4}^2}-\frac{c_5\sqrt{\%2}\,x_5}{a_4\,{d_5}^2\,{c_4}^2},\frac{\sqrt{\%2}\,x_5}{a_4\,{d_5}^2\,c_4}\right)$ $\%1:=-c_4\,a_5+c_5\,a_4$ $\%2:=-a_4\,d_5\,c_4\,\%1$	$N(3,4f)$
26e	$\left(\frac{\sqrt{\%1}\,b_4\,{a_4}^3\,x_2}{{b_3}^2\,{c_4}^5},\frac{{b_4}^4\sqrt{\%1}\,x_1}{{b_3}^2\,{c_4}^5},\frac{\sqrt{\%1}\,b_4\,x_3}{{b_3}^2\,{c_4}^2}+\frac{\left({b_3}^2\,c_4\sqrt{\%1}\right)^{1/3}(b_4\,c_5-b_5\,c_4)\,x_5}{{b_3}^2\,{c_4}^2\,d_5},\frac{\sqrt{\%1}\,x_4}{b_3\,{c_4}^2}-\frac{c_5\left({b_3}^2\,c_4\sqrt{\%1}\right)^{1/3}x_5}{b_3\,{c_4}^2\,d_5},\frac{\left({b_3}^2\,c_4\sqrt{\%1}\right)^{1/3}x_5}{b_3\,c_4\,d_5}\right)$ $\%1:=b_3\,b_4\,c_4$	$P(3,4g)$
26f	$\left({a_3}^3\,{c_4}^9\,{d_5}^{27}\,x_1,{c_4}^9\,{d_5}^{27}\,{b_3}^3\,x_1-\frac{x_5}{{a_3}^3\,{c_4}^{12}\,{d_5}^{39}},{c_4}^3\,{d_5}^9\,x_2-\frac{a_5\,x_4}{a_3},{d_5}^3\,x_3-\frac{c_5\,x_4}{c_4},x_4\right)$	$J(3,4)$
26g	$\left(-\frac{(b_5\,a_3-a_5\,b_3)^3\,\%1^{3/8}\,x_1}{{b_3}^6\,{a_3}^3\,{c_4}^3\,{d_5}^6},-\frac{(b_5\,a_3-a_5\,b_3)^3\,\%1^{3/8}\,x_2}{{b_3}^3\,{a_3}^6\,{c_4}^3\,{d_5}^6},-\frac{(b_5\,a_3-a_5\,b_3)\,\%1^{1/8}\,x_3}{{b_3}^2\,{a_3}^2\,c_4\,{d_5}^2}-\frac{b_5\,\%1^{1/8}\,x_5}{{b_3}^2\,a_3\,c_4\,{d_5}^2},\frac{\%1^{3/8}\,x_4}{{d_5}^3\,{b_3}^3\,{a_3}^3\,{c_4}^3}-\frac{c_5\,\%1^{1/8}\,x_5}{b_3\,a_3\,{c_4}^2\,{d_5}^2},\frac{\%1^{1/8}\,x_5}{b_3\,a_3\,c_4\,{d_5}^2}\right)$ $\%1:=(-b_5\,a_3+a_5\,b_3)\,{b_3}^7\,{a_3}^7\,{c_4}^5\,{d_5}^7$	$N(3,4e)$

	T	$T^{-1}FT$
26h	$\left(\frac{\sqrt{\%1}\,a_4{}^4\,x_1}{a_3{}^2\,c_4{}^5}, \frac{a_4{}^4\,b_3{}^3\,\sqrt{\%1}\,x_2}{a_3{}^5\,c_4{}^5}, \frac{\sqrt{\%1}\,a_4\,x_3}{a_3{}^2\,c_4{}^2} - \frac{b_5\,\left(a_3{}^2\,c_4\,\sqrt{\%1}\right)^{1/3}\,x_5}{a_3\,c_4\,b_3\,d_5}, \frac{\sqrt{\%1}\,x_4}{a_3\,c_4{}^2} - \frac{\left(a_3{}^2\,c_4\,\sqrt{\%1}\right)^{1/3}\,c_5\,x_5}{a_3\,c_4{}^2\,d_5}, \frac{\left(a_3{}^2\,c_4\,\sqrt{\%1}\right)^{1/3}\,x_5}{a_3\,c_4\,d_5}\right)$ $\%1 := a_4\,a_3\,c_4$	$P(3,4h)$
26i	$\left(\frac{a_3{}^3\,b_4{}^4\,\sqrt{\%1}\,x_1}{b_3{}^5\,c_4{}^5}, \frac{\sqrt{\%1}\,b_4{}^4\,x_2}{b_3{}^2\,c_4{}^5}, \frac{\sqrt{\%1}\,b_4\,x_3}{b_3{}^2\,c_4{}^2} + \frac{\left(b_3{}^2\,c_4\,\sqrt{\%1}\right)^{1/3}\,(b_4\,c_5 - b_5\,c_4)\,x_5}{b_3{}^2\,c_4{}^2\,d_5}, \frac{\sqrt{\%1}\,x_4}{b_3\,c_4{}^2} - \frac{c_5\,\left(b_3{}^2\,c_4\,\sqrt{\%1}\right)^{1/3}\,x_5}{b_3\,c_4{}^2\,d_5}, \frac{\left(b_3{}^2\,c_4\,\sqrt{\%1}\right)^{1/3}\,x_5}{b_3\,c_4\,d_5}\right)$ $\%1 := b_3\,b_4\,c_4$	$P(3,4a2)$
27a	$\left(a_3{}^3\,c_4{}^9\,d_5{}^{27}\,x_1, b_5{}^3\,x_3 - \frac{\left(a_2\,c_5\,b_5{}^3 + a_5\,d_5{}^3\,c_4\right)\,x_4}{a_2\,d_5{}^3\,c_4} - x_5, c_4{}^3\,d_5{}^9\,x_2 + \frac{a_2\,x_5}{a_3}, d_5{}^3\,x_3 - \frac{c_5\,x_4}{c_4}, x_4\right)$	$J(3,4)$
27b	$\left(a_2{}^3\,b_4{}^9\,d_5{}^{27}\,x_1, b_4{}^3\,d_5{}^9\,x_2 - \frac{x_4\,a_3\,c_5{}^3\,b_5}{a_2\,b_4\,d_5{}^3} - \frac{a_5\,x_4}{a_2} - \frac{a_3\,x_5}{a_2{}^4\,b_4{}^{12}\,d_5{}^{39}}, c_5{}^3\,x_3 + \frac{x_5}{a_2{}^3\,b_4{}^{12}\,d_5{}^{39}}, d_5{}^3\,x_3 - \frac{b_5\,x_4}{b_4}, x_4\right)$	$J(3,4)$
27c	$\left(\frac{\sqrt{\%2}\,\%1^4\,x_1}{a_3{}^2\,c_4{}^5\,d_5{}^{14}}, \frac{b_5{}^3\,\%1\,\%3^{1/3}\,x_3}{d_5{}^5\,c_4{}^2\,(\%1\,a_3{}^2)^{2/3}} + \frac{(-c_4\,a_5 + c_5\,a_4)\,\sqrt{\%2}\,\%3^{2/3}\,x_4}{d_5{}^3\,c_4{}^3\,a_3\,a_2\,(\%1\,a_3{}^2)^{2/3}} - \frac{a_3{}^5\,c_4{}^8\,d_5{}^{21}\,x_5}{(\%1\,a_3{}^2)^{2/3}\,\%1^6}, \frac{\%1^2\,\%3^{1/3}\,x_2}{d_5{}^5\,c_4{}^2\,a_3\,(\%1\,a_3{}^2)^{2/3}} + \frac{a_2\,a_3{}^4\,c_4{}^8\,d_5{}^{21}\,x_5}{(\%1\,a_3{}^2)^{2/3}\,\%1^6}, \frac{\sqrt{\%2}\,x_3}{a_3\,c_4{}^2\,d_5{}^2} - \frac{\%3^{1/3}\,c_5\,x_4}{a_3\,c_4{}^2\,d_5{}^2}, \frac{\%3^{1/3}\,x_4}{a_3\,c_4\,d_5{}^2}\right)$ $\%1 := a_2\,b_5{}^3 + a_4\,d_5{}^3$ $\%2 := \%1\,a_3\,c_4\,d_5$ $\%3 := a_3{}^2\,c_4\,d_5\,\sqrt{\%2}$	$N(3,4a)$

	T	$T^{-1}FT$
27d	$\left(\frac{\sqrt{\%2}\,\%1^4\,x_1}{a_2^2\,d_5^{14}\,b_4^5}, \frac{\%1^2\,\%3^{1/3}\,x_2}{d_5^5\,b_4^2\,a_2\,(\%1\,a_2^2)^{2/3}} - \frac{(-b_5\,a_4+b_4\,a_5)\,\sqrt{\%2}\,\%3^{2/3}\,x_4}{d_5^3\,b_4^3\,a_2^2\,(\%1\,a_2^2)^{2/3}} - \frac{a_3\,a_2^4\,b_4^8\,d_5^{21}\,x_5}{(\%1\,a_2^2)^{2/3}\,\%1^6}, \frac{c_5^3\,\%1\,\%3^{1/3}\,x_3}{b_4^2\,d_5^5\,(\%1\,a_2^2)^{2/3}} + \frac{a_2^5\,b_4^8\,d_5^{21}\,x_5}{(\%1\,a_2^2)^{2/3}\,\%1^6}, \frac{\sqrt{\%2}\,x_3}{a_2\,d_5^2\,b_4^2} - \frac{\%3^{1/3}\,b_5\,x_4}{a_2\,d_5^2\,b_4^2}, \frac{\%3^{1/3}\,x_4}{a_2\,d_5^2\,b_4}\right)$ $\%1 := c_5^3\,a_3 + a_4\,d_5^3$ $\%2 := \%1\,a_2\,d_5\,b_4$ $\%3 := a_2^2\,d_5\,b_4\,\sqrt{\%2}$	$N(3,4a)$
27e	$\left(a_2^3\,x_4, b_4^3\,d_5^9\,x_1 + x_5, d_5^9\,c_4^3\,x_1 + \frac{x_3\,a_5\,c_4^3}{a_2\,b_4^3}, d_5^3\,x_2 - \frac{b_5\,x_3}{b_4}, x_3\right)$	$J(3,3)$
27f	$\left(a_2^3\,b_4^9\,d_5^{27}\,x_1, b_4^3\,d_5^9\,x_2 - \frac{a_5\,x_4}{a_2}, c_4^3\,d_5^9\,x_2 - x_5, d_5^3\,x_3 - \frac{c_5\,x_4}{c_4}, x_4\right)$	$J(3,4)$
27g	$\left(d_5^{27}\,\left(a_2\,b_4^3 + a_3\,c_4^3\right)^3\,x_1, b_4^3\,d_5^9\,x_2 - \frac{a_5\,x_4}{a_2} - x_5, c_4^3\,d_5^9\,x_2 + \frac{a_2\,x_5}{a_3}, d_5^3\,x_3 - \frac{b_5\,x_4}{b_4}, x_4\right)$	$J(3,4)$
27h	$\left(-\frac{a_2^3\,\%1^{13}\,\sqrt{\%2}\,x_1}{b_4^4\,d_5^{13}\,c_4^{13}}, \frac{\%1^4\,\sqrt{\%2}\,x_2}{b_4\,d_5^4\,c_4^4} - \frac{a_5\,\sqrt{\%2}\,x_5}{d_5\,a_2}, \frac{\%1^4\,\sqrt{\%2}\,x_3}{b_4^4\,d_5^4\,c_4}, -\frac{\%1\,\sqrt{\%2}\,x_4}{b_4\,c_4\,d_5} - \frac{\sqrt{\%2}\,c_5\,x_5}{d_5\,c_4}, \frac{\sqrt{\%2}\,x_5}{d_5}\right)$ $\%1 := b_4\,c_5 - b_5\,c_4$ $\%2 := -\frac{\%1}{b_4\,d_5\,c_4}$	$P(3,4i)$
27i	$\left(\frac{a_4^4\,\sqrt{\%1}\,x_1}{b_4^5\,a_2^2}, \frac{a_4\,\sqrt{\%1}\,x_2}{b_4^2\,a_2^2} + \frac{\left(b_4\,a_2^2\,\sqrt{\%1}\right)^{1/3}\,(-c_4\,a_5 + c_5\,a_4)\,x_5}{a_2^2\,b_4\,d_5\,c_4}, \frac{a_4\,\sqrt{\%1}\,c_4^3\,x_3}{b_4^5\,a_2^2}, \frac{\sqrt{\%1}\,x_4}{b_4^2\,a_2} - \frac{\left(b_4\,a_2^2\,\sqrt{\%1}\right)^{1/3}\,c_5\,x_5}{a_2\,b_4\,c_4\,d_5}, \frac{\left(b_4\,a_2^2\,\sqrt{\%1}\right)^{1/3}\,x_5}{a_2\,b_4\,d_5}\right)$ $\%1 := a_4\,a_2\,b_4$	$P(3,4j2)$

	T	$T^{-1}FT$
28a	$\left(a_2{}^3\,b_3{}^9\,c_5{}^{27}\,x_1, b_3{}^3\,c_5{}^9\,x_2 - \frac{(a_5+a_4)\,x_4}{a_2} + \frac{a_4\,x_5}{a_2}, c_5{}^3\,x_3 - \frac{(b_5+b_4)\,x_4}{b_3} + \frac{b_4\,x_5}{b_3}, x_4 - x_5, x_4\right)$	$J(3,4)$
28b	$\left(\frac{\sqrt{\%1}\,a_3{}^4\,x_1}{a_2{}^2\,b_3{}^5}, \frac{\sqrt{\%1}\,a_3\,x_2}{a_2{}^2\,b_3{}^2} + \frac{\left(a_3\,b_5\,\%2^{1/3} + b_4\,c_5\,a_3\,a_2\,b_3 - b_3{}^2\,a_4\,c_5\,a_2 - a_5\,\%2^{1/3}\,b_3\right)\,x_4}{a_2{}^2\,b_3{}^2\,c_5} - \frac{(b_4\,a_3 - a_4\,b_3)\,x_5}{a_2\,b_3}, \frac{\sqrt{\%1}\,x_3}{a_2\,b_3{}^2} - \frac{\left(b_5\,\%2^{1/3} + b_4\,c_5\,a_2\,b_3\right)\,x_4}{a_2\,b_3{}^2\,c_5} + \frac{b_4\,x_5}{b_3}, x_4 - x_5, \frac{\%2^{1/3}\,x_4}{a_2\,b_3\,c_5}\right)$ $\%1 := a_3\,a_2\,b_3$ $\%2 := a_2{}^2\,b_3\,\sqrt{\%1}$	$N(3,4a)$
28c	$\left(a_2{}^3\,b_3{}^9\,x_1, b_3{}^3\,x_2 - \frac{a_3\,x_3}{a_2} + \frac{(b_5\,a_3 - a_5\,b_3)\,x_5}{b_3\,a_2}, x_3 + c_5{}^3\,x_4 - \frac{b_5\,x_5}{b_3}, d_5{}^3\,x_4, x_5\right)$	$J(3,3)$
28d	$\left(a_2{}^3\,\%1^9\,x_1, \%1^3\,x_2 + \frac{(b_5\,a_3 - a_5\,b_3)\,x_4}{a_2\,b_3} - \frac{a_3\,x_5}{a_2{}^4\,\%1^{12}\,d_5{}^3}, c_5{}^3\,x_3 - \frac{b_5\,x_4}{b_3} + \frac{b_4\,x_5}{a_2{}^3\,\%1^{13}}, d_5{}^3\,x_3 - \frac{b_3\,x_5}{a_2{}^3\,\%1^{13}}, x_4\right)$ $\%1 := b_3\,c_5{}^3 + b_4\,d_5{}^3$	$J(3,4)$
28e	$\left(a_2{}^3\,b_3{}^9\,x_1, b_3{}^3\,x_2 - \frac{a_3\,x_4}{a_2} + \frac{(b_5\,a_3 - a_5\,b_3)\,\left(a_2\,\%1^2\right)^{1/3}\,x_5}{a_2\,\%1}, \frac{a_2\,b_3{}^3\,c_5{}^3\,x_3}{\%1} + x_4 - \frac{b_5\,\left(a_2\,\%1^2\right)^{1/3}\,x_5}{\%1}, \frac{a_2\,b_3{}^3\,d_5{}^3\,x_3}{\%1}, \frac{b_3\,\left(a_2\,\%1^2\right)^{1/3}\,x_5}{\%1}\right)$ $\%1 := c_5{}^3\,a_3 + a_4\,d_5{}^3$	$N(3,3b)$

	T	$T^{-1}FT$
28f	$\left(\frac{\%2^4\sqrt{\%2\,\%1\,a_2}\,x_1}{{a_2}^2\,\%1^5}, \frac{\%2\sqrt{\%2\,\%1\,a_2}\,x_2}{{a_2}^2\,\%1^2} + \frac{\left(\%1\,{a_2}^2\sqrt{\%2\,\%1\,a_2}\right)^{1/3}(b_5\,a_3 - a_5\,b_3)\,x_4}{b_3\,{a_2}^2\,\%1} - \frac{\sqrt{\%2\,\%1\,a_2}\,{d_5}^3\,(b_4\,a_3 - a_4\,b_3)\,x_5}{b_3\,{a_2}^2\,\%1^2}, \frac{{c_5}^3\sqrt{\%2\,\%1\,a_2}\,x_3}{\%1^2\,a_2} - \frac{b_5\left(\%1\,{a_2}^2\sqrt{\%2\,\%1\,a_2}\right)^{1/3}x_4}{b_3\,\%1\,a_2} + \frac{\sqrt{\%2\,\%1\,a_2}\,b_4\,{d_5}^3\,x_5}{b_3\,\%1^2\,a_2}, \frac{{d_5}^3\sqrt{\%2\,\%1\,a_2}\,x_3}{\%1^2\,a_2} - \frac{{d_5}^3\sqrt{\%2\,\%1\,a_2}\,x_5}{\%1^2\,a_2}, \frac{\left(\%1\,{a_2}^2\sqrt{\%2\,\%1\,a_2}\right)^{1/3}x_4}{\%1\,a_2}\right)$ $\%1 := b_3\,{c_5}^3 + b_4\,{d_5}^3$ $\%2 := {c_5}^3\,a_3 + a_4\,{d_5}^3$	$N(3,4a)$
29a	$\left({a_2}^3\,{b_3}^9\,{c_4}^{27}\,x_1, {b_3}^3\,{c_4}^9\,x_2 - \frac{a_4\,x_4}{a_2} + \frac{(-c_4\,a_5 + c_5\,a_4)\,x_5}{c_4\,a_2}, {c_4}^3\,x_3 - \frac{b_4\,x_4}{b_3} + \frac{(b_4\,c_5 - b_5\,c_4)\,x_5}{b_3\,c_4}, x_4 - \frac{c_5\,x_5}{c_4}, x_5\right)$	$J(3,4)$
29b	$\left(\frac{\sqrt{\%1}\,{a_3}^4\,x_1}{{a_2}^2\,{b_3}^5}, \frac{\sqrt{\%1}\,a_3\,x_2}{{a_2}^2\,{b_3}^2} + \frac{\left({a_2}^2\,b_3\sqrt{\%1}\right)^{1/3}(b_4\,a_3 - a_4\,b_3)\,x_4}{{a_2}^2\,{b_3}^2\,c_4} - \frac{(-b_3\,a_4\,c_5 + b_3\,a_5\,c_4 + b_4\,c_5\,a_3 - b_5\,c_4\,a_3)\,x_5}{a_2\,c_4\,b_3}, \frac{\sqrt{\%1}\,x_3}{a_2\,{b_3}^2} + \frac{(b_4\,c_5 - b_5\,c_4)\,x_5}{b_3\,c_4} - \frac{b_4\left({a_2}^2\,b_3\sqrt{\%1}\right)^{1/3}x_4}{a_2\,{b_3}^2\,c_4}, \frac{\left({a_2}^2\,b_3\sqrt{\%1}\right)^{1/3}x_4}{a_2\,b_3\,c_4} - \frac{c_5\,x_5}{c_4}, x_5\right)$ $\%1 := a_3\,a_2\,b_3$	$N(3,4a)$
	Rank four	
30a	$\left({a_2}^3\,{c_4}^{27}\,{d_5}^{81}\,{b_3}^9\,x_1, {c_4}^9\,{d_5}^{27}\,{b_3}^3\,x_2 - \frac{a_5\,x_5}{a_2}, {c_4}^3\,{d_5}^9\,x_3 - \frac{b_5\,x_5}{b_3}, {d_5}^3\,x_4 - \frac{c_5\,x_5}{c_4}, x_5\right)$	$J(4,5)$

	T	$T^{-1}FT$
30b	$\left(\frac{{b_4}^{13}\,{a_2}^3\,\sqrt{\%1}\,x_1}{{b_3}^5\,{c_4}^{14}}, \frac{\sqrt{\%1}\,{b_4}^4\,x_2}{{b_3}^2\,{c_4}^5} - \frac{a_5\,\%2^{1/3}\,x_5}{b_3\,c_4\,a_2\,d_5}, \frac{\sqrt{\%1}\,b_4\,x_3}{{b_3}^2\,{c_4}^2} + \frac{\%2^{1/3}\,(b_4\,c_5 - b_5\,c_4)\,x_5}{{b_3}^2\,{c_4}^2\,d_5}, \frac{\sqrt{\%1}\,x_4}{b_3\,{c_4}^2} - \frac{c_5\,\%2^{1/3}\,x_5}{b_3\,{c_4}^2\,d_5}, \frac{\%2^{1/3}\,x_5}{b_3\,c_4\,d_5}\right)$ $\%1 := b_3\,b_4\,c_4$ $\%2 := {b_3}^2\,c_4\,\sqrt{\%1}$	$N(4,5b)$
30c	$\left(\frac{{a_4}^3\,\%1^{3/4}\,x_1}{{a_2}^3\,{c_4}^6\,{b_3}^3}, \frac{a_4\,\%1^{1/4}\,x_2}{{a_2}^2\,{c_4}^2\,b_3} + \frac{\%2^{1/3}\,(-c_4\,a_5 + c_5\,a_4)\,x_5}{{a_2}^2\,{c_4}^2\,b_3\,d_5}, \frac{\%1^{3/4}\,x_3}{{a_2}^3\,{c_4}^3\,{b_3}^3} + \frac{\%2^{1/3}\,(b_4\,c_5 - b_5\,c_4)\,x_5}{a_2\,{c_4}^2\,{b_3}^2\,d_5}, \frac{\%1^{1/4}\,x_4}{a_2\,{c_4}^2\,b_3} - \frac{c_5\,\%2^{1/3}\,x_5}{a_2\,{c_4}^2\,b_3\,d_5}, \frac{\%2^{1/3}\,x_5}{a_2\,c_4\,b_3\,d_5}\right)$ $\%1 := {b_3}^2\,{a_2}^3\,{c_4}^3\,\sqrt{a_4\,a_2\,b_3\,c_4}$ $\%2 := {a_2}^2\,c_4\,{b_3}^2\,\%1^{1/4}$	$P(4,5e)$
30d	$\left(\frac{\sqrt{\%1}\,{a_3}^4\,x_1}{{a_2}^2\,{b_3}^5}, \frac{\sqrt{\%1}\,a_3\,x_2}{{a_2}^2\,{b_3}^2} - \frac{\%2^{1/9}\,(-b_3\,a_4\,c_5 + b_3\,a_5\,c_4 + b_4\,c_5\,a_3 - b_5\,c_4\,a_3)\,x_5}{{a_2}^2\,{b_3}^2\,{c_4}^2\,d_5}, \frac{\sqrt{\%1}\,x_3}{a_2\,{b_3}^2} + \frac{\%2^{1/9}\,(b_4\,c_5 - b_5\,c_4)\,x_5}{a_2\,{b_3}^2\,{c_4}^2\,d_5}, \frac{\left({a_2}^2\,b_3\,\sqrt{\%1}\right)^{1/3}\,x_4}{a_2\,b_3\,c_4} - \frac{c_5\,\%2^{1/9}\,x_5}{a_2\,b_3\,{c_4}^2\,d_5}, \frac{\%2^{1/9}\,x_5}{a_2\,b_3\,c_4\,d_5}\right)$ $\%1 := a_3\,a_2\,b_3$ $\%2 := {a_2}^8\,{b_3}^7\,\sqrt{\%1}\,{c_4}^6$	$P(4,5c2)$

Table 1: Actual transformations

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