Polynomial maps with strongly nilpotent Jacobian matrix and the Jacobian Conjecture

ARNO VAN DEN ESSEN and ENGELBERT HUBBERS
Department of Mathematics, University of Nijmegen, The Netherlands

Abstract

Let $H : k^n \to k^n$ be a polynomial map. It is shown that the Jacobian matrix JH is strongly nilpotent (definition 1.1) if and only if JH is linearly triangularizable if and only if the polynomial map $F = X+H$ is linearly triangularizable. Furthermore it is shown that for such maps F sF is linearizable for almost all $s \in k$ (except a finite number of roots of unity).

Introduction

In [1] Bass, Connell and Wright and in [7] Yagzhev showed that it suffices to prove the Jacobian Conjecture for polynomial maps $F : \mathbb{C}^n \to \mathbb{C}^n$ of the form $F = X+H$, where $H = (H_1, \ldots, H_n)$ is a cubic homogeneous polynomial map i.e. each H_i is either zero or homogeneous of degree three. Since $\det(JF) \in \mathbb{C}^*$ is equivalent to JH is nilpotent (cf [1, Lemma 4.1]) it follows that the Jacobian Conjecture is equivalent to: if $F = X + H$ with JH nilpotent, then F is invertible. Hence it is clear that understanding nilpotent Jacobian matrices is crucial for the study of the Jacobian Conjecture.

In [6], in an attempt to understand quadratic homogeneous polynomial maps, Meisters and Olech introduced the strongly nilpotent Jacobian matrices: a Jacobian matrix JH is strongly nilpotent if $JH(x_1) \ldots JH(x_n) = 0$ for all vectors $x_1, \ldots, x_n \in \mathbb{C}^n$. They showed in [6] that for quadratic homogeneous polynomial maps JH is strongly nilpotent if and only if JH is nilpotent, if $n \leq 4$. However for $n \geq 5$ there are counterexamples (cf [4] and [6]).

On the other hand the obvious question: is the Jacobian Conjecture true for arbitrary polynomial maps $F = X + H$ with JH is strongly nilpotent, remained open.

In this paper we give an affirmative answer to this question. In fact we obtain a much stronger result; in theorem 1.6 we show that the Jacobian matrix JH is strongly nilpotent if and only if JH is linearly triangularizable if and only if the polynomial map $F = X + H$ is linearly triangularizable. Furthermore we show that for such maps F the map sF is linearizable for almost all $s \in \mathbb{C}$ (except a finite number of roots of unity). So for such F the linearization conjecture of Meisters is true (it turned out to be false in general as was shown in [3]).

1. Definitions and formulation of the first main result

Throughout this paper k denotes an arbitrary field and $k[X] := k[X_1, \ldots, X_n]$ denotes the polynomial ring in n variables over k. Let $H = (H_1, \ldots, H_n) : k^n \to k^n$ be a polynomial map i.e. $H_i \in k[X]$ for all i. By JH or $JH(X)$ we denote its Jacobian matrix. So $JH(X) \in M_n(k[X])$.

Now let $Y_{(1)} = (Y_{(1)1}, \ldots, Y_{(1)n}), \ldots, Y_{(n)} = (Y_{(n)1}, \ldots, Y_{(n)n})$ be n sets of n new variables. So for each i $JH(Y_{(i)})$ belongs to the ring of $n \times n$ matrices with entries in the n^2 variable polynomial ring $k[Y_{(i)j}; 1 \leq i, j \leq n]$.

Definition 1.1. *The Jacobian matrix JH is called* strongly nilpotent *if and only if the matrix $JH(Y_{(1)}) \ldots JH(Y_{(n)})$ is the zero matrix.*

Example 1.2. *If JH is upper triangular with zeros on the main diagonal, then one readily verifies that JH is strongly nilpotent. In fact the main result of this paper (theorem 1.6 below) asserts that a matrix JH is strongly nilpotent if and only if it is uppertriangular with zeros on the main diagonal after a suitable linear change of coordinates!*

Remark 1.3. *One easily verifies that if k is an infinite field, then definition 1.1 is equivalent to $JH(x_1) \ldots JH(x_n) = 0$ for all $x_1, \ldots, x_n \in k^n$. So for $k = \mathbb{R}$ and H homogeneous of degree two we obtain the strong nilpotence property introduced by Meisters and Olech in [6]. See also [4].*

To formulate the first main result of this paper we need one more definition.

Definition 1.4. *i) Let $F = X + H$ be a polynomial map. We say that F* is in (upper) triangular form *if $H_i \in k[X_{i+1}, \ldots, X_n]$ for all $1 \leq i \leq n-1$ and $H_n \in k$.*

ii) We say that F is linearly triangularizable *if there exists $T \in GL_n(k)$ such that $T^{-1}FT$ is in upper triangular form.*

One easily verifies the following lemma:

Lemma 1.5. *Let $F = X + H$ be a polynomial map. Then F is in upper triangular form if and only if JH is upper triangular with zeros on the main diagonal.*

Now we are ready to formulate the first main result of this paper:

Theorem 1.6. *Let $H = (H_1, \ldots, H_n) : k^n \to k^n$ be a polynomial map. Then there is equivalence between*

i) *JH is strongly nilpotent.*
ii) *There exists $T \in GL_n(k)$ such that $J(T^{-1}HT)$ is upper triangular with zeros on the main diagonal.*
iii) *$F := X + H$ is linearly triangularizable.*

From this theorem it immediately follows that:

Corollary 1.7. *If $F = X + H$ with JH strongly nilpotent, then F is invertible.*

2. The proof of theorem 1.6

The proof of theorem 1.6 is based on the following two results.

Lemma 2.1. *Let $JH = \sum_{|\alpha| \le d} A_\alpha X^\alpha$, where $d = \max_i(\deg(H_i)) - 1$ and $A_\alpha \in M_n(k)$ for all α. Then JH is strongly nilpotent if and only if $A_{\alpha_{(1)}} \ldots A_{\alpha_{(n)}} = 0$, for all multi-indices $\alpha_{(i)}$ with $|\alpha_{(i)}| \le d$.*

Proof. By definition 1.1 we obtain

$$\left(\sum_{|\alpha_{(1)}| \le d} A_{\alpha_{(1)}} Y_{(1)}^{\alpha_{(1)}} \right) \ldots \left(\sum_{|\alpha_{(n)}| \le d} A_{\alpha_{(n)}} Y_{(n)}^{\alpha_{(n)}} \right) = 0.$$

The result then follows by looking at the coefficients of $Y_{(1)}^{\alpha_{(1)}} \ldots Y_{(n)}^{\alpha_{(n)}}$. □

Proposition 2.2. *Let V be a finite dimensional k-vectorspace and $\ell_1, \ldots, \ell_p$ k-linear maps from V to V. Let $r \in \mathbb{N}$, $r \ge 1$. If $\ell_{i_1} \circ \ldots \circ \ell_{i_r} = 0$ for each r-tuple $\ell_{i_1}, \ldots, \ell_{i_r}$ with $1 \le i_1, \ldots, i_r \le p$, then there exists a basis (v) of V such that $Mat(\ell_i, (v)) = D_i$ where D_i is an upper triangular matrix with zeros on the main diagonal.*

Proof. Let $d := \dim(V)$. We use induction on d. First let $d = 1$. Then the hypothesis implies that $\ell_i^r = 0$ for each i. So $\ell_i = 0$ for each i and we are done. So let $d > 1$ and assume that the assertion is proved for all $d-1$ dimensional vectorspaces. Now we (also) use induction on r. If $r = 1$ then each $\ell_i = 0$. So let $r \ge 2$. Then for each $(r-1)$-tuple $\ell_{i_2} \ldots \ell_{i_r}$ with $1 \le i_2, \ldots, i_r \le p$ we have

$$\ell_1 \ell_{i_2} \ldots \ell_{i_r} = 0, \ldots, \ell_p \ell_{i_2} \ldots \ell_{i_r} = 0. \tag{2.1}$$

If $\ell_{i_2} \ldots \ell_{i_r} = 0$ for each such $(r-1)$-tuple we are done by the induction hypothesis on r. So we may assume that for some $(r-1)$-tuple $\ell_{i_2} \ldots \ell_{i_r}$ the map $\ell_{i_2} \ldots \ell_{i_r} \ne 0$. So there exists $v \ne 0$, $v \in V$ with $v_1 := \ell_{i_2} \ldots \ell_{i_r} v \ne 0$. From (2.1) we deduce that $\ell_i v_1 = 0$ for all i. Then consider $\bar{V} := V/kv_1$. Since $l_i v_1 = 0$ for all i we get induced k-linear maps $\bar{\ell}_i : \bar{V} \to \bar{V}$. Since $\dim(\bar{V}) = d-1$ the induction hypothesis implies that there exist $v_2, \ldots, v_r$ in V such that $(\bar{v}_2, \ldots, \bar{v}_r)$ is a k-basis of $\bar{V}$ and $\mathrm{Mat}(\bar{\ell}_i, (\bar{v}_2, \ldots, \bar{v}_r))$ is on upper triangular form. Then $(v) = (v_1, v_2, \ldots, v_r)$ is as desired. □

Corollary 2.3. *Let $A_1, \ldots, A_p \in M_n(k)$. Let $r \in \mathbb{N}$, $r \geq 1$. If $A_{i_1} \ldots A_{i_r} = 0$ for each r-tuple $A_{i_1}, \ldots, A_{i_r}$ with $1 \leq i_1, \ldots, i_r \leq p$, then there exists $T \in GL_n(k)$ such that $T^{-1}A_iT = D_i$, where each D_i is an upper triangular matrix with zeros on the main diagonal.*

Now we are able to present the proof of theorem 1.6.

Proof. *ii)* $\to$ *iii)* follows from lemma 1.5. So let's prove *iii)* $\to$ *i)*. If $F = X + H$ is linearly triangularizable, then by lemma 1.5 $J(T^{-1}HT)$ is an upper triangular matrix with zeros on the main diagonal. So as remarked in example 1.2 this implies that $J(T^{-1}HT)$ is strongly nilpotent. Finally observe that $J(T^{-1}HT) = T^{-1}JH(TX)T$. So the strong nilpotency of $J(T^{-1}HT)$ implies that $JH(TY_{(1)}) \ldots JH(TY_{(n)}) = 0$, which implies in turn that JH is strongly nilpotent.

Finally we prove *i)* $\to$ *ii)*. So let JH be strongly nilpotent. Now if we write $JH = \sum_{|\alpha| \leq d} A_\alpha X^\alpha$, then by lemma 2.1 $A_{\alpha_{(1)}} \ldots A_{\alpha_{(n)}} = 0$ for all n-tuples with $|\alpha_{(i)}| \leq d$. So by corollary 2.3 there exists $T \in GL_n(k)$ such that $T^{-1}A_\alpha T = D_\alpha$ for all α with $|\alpha| \leq d$, where D_α is an upper triangular matrix with zeros on the main diagonal. Consequently so is $T^{-1}JH(X)T$ $(= \sum T^{-1}A_\alpha TX^\alpha)$ and hence so is $J(T^{-1}HT) = T^{-1}JH(TX)T$, which is obtained by replacing X by TX in $T^{-1}JH(X)T$. □

3. Strongly nilpotent Jacobian matrices and Meisters linearization conjecture

In [2] Deng, Meisters and Zampieri studied dilations of polynomial maps with $\det(JF) \in \mathbb{C}^*$. They were able to prove that for large enough $s \in \mathbb{C}$ the map sF is locally linearizable to $sJF(0)X$ by means of an analytic map φ_s, the so-called Schröder map, which inverse is an entire function and satisfies some nice properties.

Their original aim was to show that φ_s is entire analytic, which would imply that sF and hence F is injective, which in turn would imply the Jacobian Conjecture. Although they were not able to prove the 'entireness' of φ_s, calculations of many examples of polynomial maps of the form $X + H$ with H cubic homogeneous showed that in all these cases the Schröder map was even much better than expected, namely it was a polynomial automorphism! (cf [5]) This lead Meisters to the following conjecture:

Conjecture 3.1. (Linearization Conjecture, Meisters [5])
Let $F = X + H$ be a cubic homogeneous polynomial map with JH nilpotent. Then

for almost all $s \in \mathbb{C}$ (except a finite number of roots of unity) there exists a polynomial automorphism φ_s such that $\varphi_s^{-1} sF\varphi = sX$.

Recently in [3] it was shown by the first author that the conjecture is false if $n \geq 5$ and true if $n \leq 4$.

In this section we show that Meisters linearization conjecture is true for all $n \geq 1$ if we replace 'JH is nilpotent' by 'JH is strongly nilpotent'. In fact we even don't need the assumption that this H is cubic homogeneous. More precisely we have:

Theorem 3.2. *Let k be a field, $k(s)$ the field of rational functions in one variable and $F : k^n \to k^n$ a polynomial map of the form $F = X + H$ with $F(0) = 0$ and JH strongly nilpotent. Then there exists an over k linearly triangularizable polynomial automorphism $\varphi_s \in Aut_{k(s)}(k(s)[X])$ such that*

$$\varphi_s^{-1} sF\varphi_s = sJF(0)X.$$

Furthermore, the zeros of the denominators of the coefficients of the X-monomials appearing in φ_s are roots of unity.

Before we can prove this result we need one definition and some lemmas.

Definition 3.3. *We say that $X_1^{i_1} \dots X_n^{i_n} > X_1^{i'_1} \dots X_n^{i'_n}$ if and only if $\sum_{j=1}^n i_j > \sum_{j=1}^n i'_j$ or if $\sum_{j=1}^n i_j = \sum_{j=1}^n i'_j$ and there exists some $l \in \{1, 2, \dots, n\}$ such that $i_j = i'_j$ for all $j < l$ and $i_l > i'_l$.*

Furthermore we say that the rank *of the monomial $M := X_1^{i_1} \dots X_n^{i_n}$ is the index of this monomial in the ascending ordered list of all monomials M' in $X_1, \dots, X_n$ with* $\deg(M') \leq \deg(M)$ *(total degree).*

Example 3.4. *The rank of $X_1X_2X_3$ is 15, since the ascending ordered list of all monomials in X_1, X_2 and X_3 of total degree at most three is:*

$$X_3, X_2, X_1,$$

$$X_3^2, X_2X_3, X_2^2, X_1X_3, X_1X_2, X_1^2,$$

$$X_3^3, X_2X_3^2, X_2^2X_3, X_2^3, X_1X_3^2, X_1X_2X_3, X_1X_2^2, X_1^2X_3, X_1^2X_2, X_1^3$$

Lemma 3.5. *For each $2 \leq j \leq n-1$ let $\ell_j(X_{j+1}, \dots, X_n)$ be a linear form in $X_{j+1}, \dots, X_n$ and let $\mu \in k$. Then the leading monomial with respect to the order of definition 3.3 in the expansion of*

$$\mu \prod_{j=2}^{n} (sX_j + s\ell_j(X_{j+1}, \dots, X_n))^{i_j} \tag{3.1}$$

is

$$\mu s^{i_2 + \dots + i_n} x_2^{i_2} \dots X_n^{i_n}.$$

Proof. It is obvious that the monomial $\mu s^{i_2+\ldots+i_n} X_2^{i_2} \ldots X_n^{i_n}$ appears in the expansion of (3.1). Now we have to show that this is really the leading monomial. Note that all monomials in the expansion have the same (total) degree: $i_2+\ldots+i_n$. For each $j = 2, \ldots, n$ we get a contribution of $(sX_j + s\ell_j(X_{j+1}, \ldots, X_n))^{i_j}$ that is of the form

$$\sum_{k=0}^{i_j} \binom{i_j}{k} X_j^k (\ell_j(X_{j+1}, \ldots, X_n))^{i_j-k}$$

and since ℓ_j is a linear term that does not contain X_j it is obvious that we get the highest order monomial if we take $k = i_j$. So if we start with $j = 2$, we see that the highest X_2 power is i_2. And if we apply this result to $j = 3$ we see that the leading power product must begin with $X_2^{i_2} X_3^{i_3}$. If we do this for all j we see that it is obvious that the leading monomial is $\mu s^{i_2+\ldots+i_n} X_2^{i_2} \ldots X_n^{i_n}$. □

Lemma 3.6. *Let F be a polynomial map of the form:*

$$F = \begin{pmatrix} X_1 + a(X_2, \ldots, X_n) + \ell_1(X_2, \ldots, X_n) \\ X_2 + \ell_2(X_3, \ldots, X_n) \\ \vdots \\ X_{n-1} + \ell_{n-1}(X_n) \\ X_n \end{pmatrix}$$

where $a(X_2, \ldots, X_n)$ is a polynomial with leading monomial (with respect to the order of definition 3.3) $\lambda X_2^{i_2} \ldots X_n^{i_n}$ and $i_2+\ldots+i_n \geq 2$. Furthermore $\ell_i(X_{i+1}, \ldots, X_n)$ are some linear forms. Then there exists a polynomial map φ on triangular form such that

$$\varphi^{-1} sF\varphi = s \begin{pmatrix} X_1 + \tilde{a}(X_2, \ldots, X_n) + \ell_1(X_2, \ldots, X_n) \\ X_2 + \ell_2(X_3, \ldots, X_n) \\ \vdots \\ X_{n-1} + \ell_{n-1}(X_n) \\ X_n \end{pmatrix} \tag{3.2}$$

where the leading monomial of $\tilde{a}(X_2, \ldots, X_n)$, say $\tilde{\lambda} X_2^{j_2} \ldots X_n^{j_n}$, is of strict lower order than the leading monomial of $a(X_2, \ldots, X_n)$, i.e.:

$$X_2^{j_2} \ldots X_n^{j_n} < X_2^{i_2} \ldots X_n^{i_n}.$$

Proof. Let

$$\varphi = \begin{pmatrix} X_1 + \mu X_2^{i_2} \ldots X_n^{i_n} \\ X_2 \\ \vdots \\ X_n \end{pmatrix}$$

for some $\mu \in k$. It is obvious that φ is on triangular form. Proving that the equation

(3.2) is valid is equivalent with showing that

$$sF\varphi = \varphi(s \begin{pmatrix} X_1 + \tilde{a}(X_2,\ldots,X_n) + \ell_1(X_2,\ldots,X_n) \\ X_2 + \ell_2(X_3,\ldots,X_n) \\ \vdots \\ X_{n-1} + \ell_{n-1}(X_n) \\ X_n \end{pmatrix}) \tag{3.3}$$

is valid. We do this by looking at the n components. For $i \geq 2$ it is easy to see that the i-th component of the lefthandside of (3.3) equals that of the righthandside of (3.3). Hence our only concern is the first component. Put $\hat{a}(X_2,\ldots,X_n) := a(X_2,\ldots,X_n) - \lambda X_2^{i_2}\ldots X_n^{i_n}$. On the lefthandside we have:

$$sF\varphi_{|1} = sX_1 + s\mu X_2^{i_2}\ldots X_n^{i_n} + s\lambda X_2^{i_2}\ldots X_n^{i_n} + s\hat{a}(X_2,\ldots,X_n) + s\ell_1(X_2,\ldots,X_n) \tag{3.4}$$

and on the righthandside:

$$\varphi(s \begin{pmatrix} X_1 + \tilde{a}(X_2,\ldots,X_n) + \ell_1(X_2,\ldots,X_n) \\ X_2 + \ell_2(X_3,\ldots,X_n) \\ \vdots \\ X_{n-1} + \ell_{n-1}(X_n) \\ X_n \end{pmatrix})_{|1} \tag{3.5}$$

$$= \quad sX_1 + s\tilde{a}(X_2,\ldots,X_n) + s\ell_1(X_2,\ldots,X_n) + \mu \prod_{j=2}^{n}(sX_j + s\ell_j(X_{j+1},\ldots,X_n))^{i_j}$$

By subtracting equation (3.5) from equation (3.4) under the assumption that equation (3.3) holds, we get:

$$s(\mu+\lambda)X_2^{i_2}\ldots X_n^{i_n} + s\hat{\tilde{a}}(X_2,\ldots,X_n) = \mu \prod_{j=2}^{n}(sX_j + s\ell_j(X_{j+1},\ldots,X_n))^{i_j} \tag{3.6}$$

where $\hat{\tilde{a}} = \hat{a} - \tilde{a}$. Now we have to derive a relation for μ to achieve that equation (3.3) indeed holds. We can do this by restricting equation (3.6) to the coefficients of $X_2^{i_2}\ldots X_n^{i_n}$. With lemma 3.5 we see that the restriction of the righthandside of (3.6) to $X_2^{i_2}\ldots X_n^{i_n}$ gives $\mu s^{i_2+\ldots+i_n}$, so we get:

$$s\mu + s\lambda = s^{i_2+\ldots+i_n}\mu$$

and from this equation we can compute μ:

$$\mu = \frac{\lambda}{s^{i_2+\ldots+i_n-1} - 1}$$

Note that we have assumed that $i_2 + \ldots + i_n \geq 2$ so $s^{i_2+\ldots+i_n-1} - 1 \neq 0$, hence μ is well defined. □

Now we are able to give the proof of theorem 3.2.

Proof. By theorem 1.6 we may assume that $F = (F_1, \dots, F_n)$ is on triangular form. We use induction on n. If $n = 1$ F degenerates to the identical map X_1 and the theorem follows immediately.

If $n = 2$ we can write

$$F = \begin{pmatrix} X_1 + a(X_2) + \ell_1(X_2) \\ X_2 \end{pmatrix}$$

where $a = \sum_{i=2}^m a_i X_2^i$ and $\ell_1 = aX_2$, the linear part. In particular we have that the leading monomial of a is $a_m X_2^m$. So with lemma 3.6 we know that there exists a map φ_m on triangular form such that

$$\varphi_m^{-1} sF\varphi_m = \begin{pmatrix} sX_1 + \tilde{a}(X_2) + s\ell_1(X_2) \\ sX_2 \end{pmatrix}.$$

where $\deg(\tilde{a}) < m$. By applying the same lemma m times (if necessary we can use φ_j is the identity) we find a sequence $\varphi_1, \dots, \varphi_m$ such that

$$\varphi_1^{-1} \dots \varphi_m^{-1} sF\varphi_m \dots \varphi_1 = s \begin{pmatrix} X_1 + \ell_1(X_2) \\ X_2 \end{pmatrix}$$

So $\varphi_s := \varphi_m \circ \dots \circ \varphi_1$ is as desired. Now consider $F = (F_1, F_2, \dots, F_n)$. Put $\tilde{F} := (F_2, \dots, F_n)$ and $\tilde{X} := (X_2, \dots, X_n)$. Then by the induction hypothesis we know that there exists an invertible polynomial map $\tilde{\varphi}_s$ such that

$$\tilde{\varphi}_s^{-1} s\tilde{F}\tilde{\varphi}_s = sJ_{\tilde{X}}\tilde{F}(0).$$

So with $\chi = (X_1, \tilde{\varphi}_s)$ and with the notation

$$F = (X_1 + a(X_2, \dots, X_n) + \ell_1(X_2, \dots, X_n), \tilde{F})$$

we get

$$\chi^{-1} sF\chi = s \begin{pmatrix} X_1 + \tilde{a}(X_2, \dots, X_n) + \ell_1(X_2, \dots, X_n) \\ X_2 + \ell_2(X_3, \dots, X_n) \\ \vdots \\ X_{n-1} + \ell_{n-1}(X_n) \\ X_n \end{pmatrix}$$

Now we only have to make the first component linear. Let r be the rank of the leading monomial in $\tilde{a}(X_2, \dots, X_n)$. With lemma 3.6 we know that there exists a φ_r such that

$$\varphi_r^{-1}\chi^{-1} sF\chi\varphi_r = s \begin{pmatrix} X_1 + \tilde{a}_r(X_2, \dots, X_n) + \ell_1(X_2, \dots, X_n) \\ X_2 + \ell_2(X_3, \dots, X_n) \\ \vdots \\ X_{n-1} + \ell_{n-1}(X_n) \\ X_n \end{pmatrix}$$

where the rank of the leading monomial of $\tilde{a}_r(X_2, \dots, X_n) < r$. So after r applications of lemma 3.6 we have obtained a sequence $\varphi_1, \dots, \varphi_r$ such that

$$\varphi_1^{-1} \dots \varphi_r^{-1} \chi s F \chi \varphi_r \dots \varphi_1 = s \begin{pmatrix} X_1 + \ell_1(X_2, \dots, X_n) \\ X_2 + \ell_2(X_3, \dots, X_n) \\ \vdots \\ X_{n-1} + \ell_{n-1}(X_n) \\ X_n \end{pmatrix}$$

which proves the theorem. □

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