A polynomial counterexample to the Markus-Yamabe Conjecture

Anna Cima[*], Arno van den Essen[‡], Armengol Gasull[§], Engelbert Hubbers[‡], Francesc Mañosas[§]

Abstract

We give a polynomial counterexample to both the Markus-Yamabe Conjecture and the discrete Markus-Yamabe problem for all dimensions ≥ 3.

Introduction

The following conjecture was explicitly stated by Markus and Yamabe in [13]: let $F : \mathbb{R}^n \to \mathbb{R}^n$ be a $\mathcal{C}^1$-class vector field with $F(0) = 0$ and such that for all $x \in \mathbb{R}^n$ all eigenvalues of $JF(x)$ have negative real part, then 0 is a global attractor of the autonomous system $\dot{x} = F(x)$ i.e. every solution tends to the origin as t tends to infinity.

A particular case of the above is the so-called Kalman Conjecture, see [1]. The Markus-Yamabe conjecture has been studied by many authors and several partial results are obtained (for a summary of results we refer to the papers [3], [8], [10] and [14]).

The conjecture was solved affirmatively in the case $n = 2$ for polynomial vector fields by Meisters and Olech in [15], 1988. In the same year Barabanov published a paper [1] containing ideas to construct a $\mathcal{C}^1$-counterexample to the Kalman Conjecture for all $n \geq 4$ and so to the Markus-Yamabe Conjecture. In fact in [2], 1994, such a counterexample, even analytic, was constructed by Bernat and Llibre. Finally in 1993 the Markus-Yamabe Conjecture was completely solved affirmatively for $n = 2$ independently by Feßler

[*]Universitat Politècnica de Catalunya, Barcelona, Spain
[‡]University of Nijmegen, The Netherlands
[§]Universitat Autònoma de Barcelona, Spain

in [9] and Gutierrez in [12]. In 1994 another proof of the same fact was given by Glutsuk in [11]. However the Markus-Yamabe Conjecture remained open for $n = 3$ and for polynomial vector fields for all $n \geq 3$.

We would also like to mention that it was pointed out in [5] and [16] that an affirmative answer to the polynomial Markus-Yamabe Conjecture would imply a positive answer to the famous Jacobian Conjecture. Furthermore recently a discrete version of the Markus-Yamabe Conjecture was proposed by Cima, Gasull and Mañosas in [4] i.e. let $F : \mathbb{R}^n \to \mathbb{R}^n$ be a polynomial map with $F(0) = 0$ and such that for all $x \in \mathbb{R}^n$ all eigenvalues of $JF(x)$ have absolute value smaller than one, does it follow that for each $x \in \mathbb{R}^n$ $F^k(x)$ tends to zero if k tends to infinity? It was shown in [4] that the answer is affirmative if $n = 2$. However, in [7], 1995, van den Essen and Hubbers gave a family of counterexamples to this question for all $n \geq 4$.

In this paper we show that for each $n \geq 3$ there is a polynomial map, similar to the family in [7], which provides a counterexample to the polynomial Markus-Yamabe Conjecture! In contrast with the $\mathcal{C}^1$-class counterexamples given in [1] and [2] which have a periodic orbit, our example has orbits which escape to infinity if t tends to infinity. Finally we also give a counterexample to the discrete Markus-Yamabe problem for all $n \geq 3$.

1 The counterexample

Theorem 1.1 *Let $n \geq 3$ and $F : \mathbb{R}^n \to \mathbb{R}^n$ be given by*

$$F(x_1, \ldots, x_n) = (-x_1 + x_3 d(x)^2, -x_2 - d(x)^2, -x_3, \ldots, -x_n)$$

where $d(x) = x_1 + x_3 x_2$. Then F is a counterexample to the Markus-Yamabe Conjecture. More precisely there exists a solution of $\dot{x} = F(x)$ which tends to infinity if t tends to infinity.

Proof. One easily verifies that for all $x \in \mathbb{R}^n$ all eigenvalues of $JF(x)$ are equal to -1. Finally one checks that

$$\begin{aligned} x_1(t) &= 18e^t \\ x_2(t) &= -12e^{2t} \\ x_3(t) &= e^{-t} \\ &\vdots \\ x_n(t) &= e^{-t} \end{aligned}$$

is a solution of $\dot{x} = F(x)$ which obviously tends to infinity as t tends to infinity. $\square$

To conclude this section we also give a new counterexample to the discrete Markus-Yamabe problem for all $n \geq 3$.

Theorem 1.2 *Let $n \geq 3$ and $F : \mathbb{R}^n \to \mathbb{R}^n$ be given by*

$$F(x_1, \ldots, x_n) = (\tfrac{1}{2}x_1 + x_3 d(x)^2, \tfrac{1}{2}x_2 - d(x)^2, \tfrac{1}{2}x_3, \ldots, \tfrac{1}{2}x_n)$$

where $d(x) = x_1 + x_3 x_2$. Then F is a counterexample to the discrete Markus-Yamabe question. More precisely, there exists an initial condition $x^{(0)}$ such that the sequence $x^{(n+1)} = F(x^{(n)})$, tends to infinity when n tends to infinity.

Proof. One easily verifies that for all $x \in \mathbb{R}^n$ the eigenvalues of $JF(x)$ are equal to $\frac{1}{2}$. Taking $x^{(0)} = (\frac{147}{32}, \frac{-63}{32}, 1, 0, \ldots, 0)$ it is easy to verify by induction that

$$x^{(n)} = (\tfrac{147}{32} \cdot 2^n, \tfrac{-63}{32} \cdot 2^{2n}, (\tfrac{1}{2})^n, 0, \ldots, 0)$$

which obviously tends to infinity as n tends to infinity. $\square$

2 Conclusion

Polynomial Markus-Yamabe Conjecture (both continuous and discrete) have two main motivations: first their interest as knowledge of the global behaviour of a dynamical system and second for their relation to the Jacobian Conjecture.

As follows from the survey made in the introduction and from the results of this paper only the second interest remains, in fact weaker versions of both polynomial Markus-Yamabe Conjectures are still of importance for the study of the Jacobian Conjecture. So to conclude this paper let us formulate these weaker versions:

Conjecture 2.1 *(see [5]) Let $F : \mathbb{R}^n \to \mathbb{R}^n$ be a polynomial map such that for all $x \in \mathbb{R}^n$ all eigenvalues of $JF(x)$ have negative real part, then F is injective.*

Conjecture 2.2 *(see [4]) Let $F : \mathbb{R}^n \to \mathbb{R}^n$ be a polynomial map such that for all $x \in \mathbb{R}^n$ all eigenvalues of $JF(x)$ have absolute value less than one, then F has a unique fixed point.*

It is shown in [5] and [16] that the first one implies the Jacobian Conjecture. The second one is equivalent to the Jacobian Conjecture as is proved in [4].

References

[1] N.E. Barabanov, *On a problem of Kalman*, Siberian Mathematical Journal **29** (1988), no. 3, 333–341.

[2] J. Bernat and J. Llibre, *Counterexamples to Kalman and Markus-Yamabe conjectures in dimension larger than 3*, preprint 1994. To appear in Dynamics of Continuous, Discrete and Impulsive systems.

[3] A. Cima, A. Gasull, J. Llibre, and F. Mañosas, *Global injectivity of polynomial maps via vector fields*, In van den Essen [6], Proceedings of the conference 'Invertible Polynomial maps', pp. 105–124.

[4] A. Cima, A. Gasull, and F. Mañosas, *The discrete Markus-Yamabe problem*, Prepublicacions Núm. 26/1995, Universitat Autònoma de Barcelona, 1995.

[5] A.R.P. van den Essen, *Conjectures and problems surrounding the Jacobian Conjecture*, In Sabatini [17], Extended version as Report 9345, University of Nijmegen, Toernooiveld, 6525 ED Nijmegen, The Netherlands, 1993.

[6] A.R.P. van den Essen (ed.), *Automorphisms of Affine Spaces*, Curaçao, Caribbean Mathematics Foundation, Kluwer Academic Publishers, July 4–8 1994, 1995, Proceedings of the conference 'Invertible Polynomial maps'.

[7] A.R.P. van den Essen and E.-M.G.M. Hubbers, *Chaotic Polynomial Automorphisms; counterexamples to several conjectures*, Report 9549, University of Nijmegen, Toernooiveld, 6525 ED Nijmegen, The Netherlands, 1995, To appear in Advances of Applied Mathematics.

[8] R. Feßler, *On the Markus-Yamabe Conjecture*, In van den Essen [6], Proceedings of the conference 'Invertible Polynomial maps', pp. 127–136.

[9] R. Feßler, *A solution of the two dimensional Global Asymptotic Jacobian Stability Conjecture*, Annales Polonici Mathematici **62** (1995), 45–75.

[10] A. Gasull, J. Llibre, and J. Sotomayor, *Global Asymptotic Stability of Differential Equations in the Plane*, J. of Differential Equations **91** (1991), 327–335.

[11] A.A. Glutsuk, *The complete solution of the Jacobian problem for planar vector fields*, Uspeki Mat. Nauk. **3** (1994), In Russian.

[12] C. Gutierrez, *A solution to the bidimensional Global Asymptotic Stability Conjecture*, In Sabatini [17], Workshop, I-38050 POVO (TN) ITALY, September 14-17 1993. Dipartimento di Matematica, Italia.

[13] L. Markus and H. Yamabe, *Global stability criteria for differential systems*, Osaka Math. Journal **12** (1960), 305–317.

[14] G.H. Meisters, *Jacobian problems in differential equations and algebraic geometry*, Rocky Mountain J. Math. **12** (1982), 679–705.

[15] G.H. Meisters and C. Olech, *Solution of the Global Asymptotic Stability Jacobian Conjecture for the Polynomial Case*, Analyse Mathématique et Applications, Gauthier-Villars, Paris, 1988, pp. 373–381.

[16] G.H. Meisters and C. Olech, *Global Stability, Injectivity, and the Jacobian Conjecture*, Proceedings of the **First World Congress of Nonlinear Analysts** (Tampa, Florida) (Lakshmikantham, ed.), Walter de Gruyter & Co. Berlin, August 19–26 1992.

[17] M. Sabatini (ed.), *Recent Results on the Global Asymptotic Stability Jacobian Conjecture*, Matematica 429, Università di Trento, 1994, Workshop, I-38050 POVO (TN) ITALY, September 14-17 1993. Dipartimento di Matematica, Italia.