The EfProb Library for Probabilistic Calculations

https://efprob.cs.ru.nl/

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CALCO Tools
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EfProb

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  - $\mathcal{K}\ell(\mathcal{D})$, the Kleisli category for the distribution monad
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➡ EfProb: a Python library for discrete/continuous/quantum probability calculations

Talk plan

1. Language: States, Predicates, and Channels

2. Implementation and Demo

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Discrete probability, states & predicates

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A **state** on a set X is a distribution $\sigma \in \mathcal{D}(X)$, i.e. $\sigma\colon X \to [0,1]$ with $\sum_x \sigma(x) = 1$. Often denoted as a formal convex sum, e.g. $\frac{1}{3}|a\rangle + \frac{1}{2}|b\rangle + \frac{1}{6}|c\rangle$, for $a, b, c \in X$

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Operations for states $\sigma \in \mathcal{D}(X)$, $\tau \in \mathcal{D}(Y)$, $\omega \in \mathcal{D}(X \times Y)$

- Joint state $\sigma \otimes \tau \in \mathcal{D}(X \times Y)$ by $(\sigma \otimes \tau)(x,y) = \sigma(x) \cdot \tau(y)$
- Marginal state $\omega_1 \in \mathcal{D}(X)$ by $\omega_1(x) = \sum_y \omega(x,y)$

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- Truth $\mathbf{1}_X$, falsity $\mathbf{0}_X$, and negation $({\sim}p)(x) = 1 - p(x)$
- Sequential conjunction $p \,\&\, q$ by $(p \,\&\, q)(x) = p(x) \cdot q(x)$

Discrete prob., validity & conditioning

For a state $\sigma \in \mathcal{D}(X)$ and a predicate $p \in [0,1]^X$, the validity

$$\sigma \models p \quad := \quad \sum_x \sigma(x) \cdot p(x) \quad \in [0,1]$$

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When the validity $\sigma \models p$ is non-zero, the conditional state $\sigma/p \in \mathcal{D}(X)$ ('σ given p') defined by

$$(\sigma/p)(x) \quad := \quad \frac{\sigma(x) \cdot p(x)}{\sigma \models p}$$

- *Updated state after we observe that p holds*

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Comparing to traditional notation, for events $A, B \subseteq X$

- $\mathrm{P}(A) \quad = \quad \sigma \models \mathbf{1}_A$
- $\mathrm{P}(A \mid B) \quad = \quad \sigma/\mathbf{1}_B \models \mathbf{1}_A$

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State/predicate transformation by a channel $c\colon X \to Y$

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Forward and backward learning are described as
$c \gg (\sigma/p)$ and $\sigma/(c \ll q)$ [Jacobs & Zanasi, MFPS 2016]

Uniform language: continuous / quantum

For **continuous probability**:

States are probability density functions $\sigma\colon X \to \mathbb{R}_{\geq 0}$ such that $\int \sigma(x)\,\mathrm{d}x = 1$ (where $X \subseteq \mathbb{R}^n$)

Predicates are (measurable) functions $p\colon X \to [0,1]$

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States are positive trace-one matrices

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For **quantum probability**:

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- Operations $\sigma \models p$ etc. exist in quantum theory.
  Note: $p \,\&\, q \neq q \,\&\, p$

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EfProb: Implementation in Python

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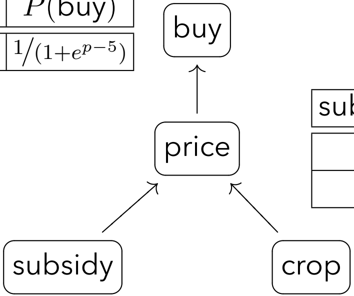
- Works well for moderate-size problems; but not meant for large-scale computation
- Can be helpful eg. for research and teaching

Demo: Hybrid Bayesian Network example

From [Cobb & Shenoy, 2006]

price	$P(\text{buy})$
p	$1/(1+e^{p-5})$



subsidy	crop	$P(\text{price})$
t	c	$\mathcal{N}(20-c,1)$
f	c	$\mathcal{N}(10-c,1)$

$P(\text{subsidy})$
0.7

$P(\text{crop})$
$\mathcal{N}(5,1)$

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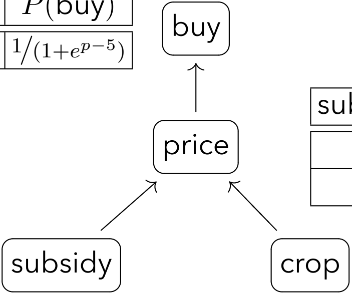

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- ‘subsidy’ and ‘buy’ have discrete domain $\{t, f\}$
- ‘crop’ and ‘price’ have continuous domain $\mathbb{R}$
- $\mathcal{N}(\mu,\sigma)$ normal (Gaussian) distribution

Conclusions

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