A Type Theory for Formalising Category Theory

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Notation is Important

Euclid IX.36

If as many numbers as we please beginning from an unit be set out continuously in double proportion, until the sum of all becomes prime, and if the sum multiplied into the last make some number, the product will be perfect.

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If $1 + 2 + \cdots + 2^k = p$ is prime, then $p2^k$ is perfect.

From a Category Theory Textbook

(We have defined $y : Nat(D(r, -), K) \simeq Kr$.)
The Yoneda map y [...] is natural in K and r. To state this fact formally, we must consider K as an object in the functor category $\mathbf{Set}^D$, regard both domain and codomain of the map y as functors of the pair $\langle K, r \rangle$, and consider this pair as an object in the category $\mathbf{Set}^D \times D$. The codomain for y is then the evaluation functor E, which maps each pair $\langle K, r \rangle$ to the value Kr of the functor K at the object r; the domain is the functor N which maps the object $\langle K, r \rangle$ to the set $Nat(D(r, -), K)$ of all natural transformations and which maps a pair of arrows $F : K \to K'$, $f : r \to r'$ to $Nat(D(f, -), F)$.

In my Notation

Let $N = \lambda K.\lambda r.Nat(\lambda x.D(r, x), K)$ and $E = \lambda K.\lambda r.Kr$. There is a natural isomorphism between N and E, namely

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Problem: How can we guarantee that a lambda-expression of the appropriate type defines a functor or a natural transformation?

Categories

A *category* $\mathcal{A}$ consists of:

- a class $|\mathcal{A}|$ of *objects*;
- given any two objects A, B, a set $\mathcal{A}[A, B]$ of *arrows* from A to B. We write $f : A \to B$ iff f is an arrow from A to B;
- for any object A, an arrow $1_A : A \to A$, the *identity* on A;
- given arrows $f : A \to B$, $g : B \to C$, an arrow $g \circ f : A \to C$, the *composite* of g with f

such that, for all $f : A \to B$, $g : B \to C$, $h : C \to D$,

$$\begin{aligned} 1_B \circ f &= f \\ f \circ 1_A &= f \\ h \circ (g \circ f) &= (h \circ g) \circ f \end{aligned}$$

Functors

Given categories $\mathcal{A}$, $\mathcal{B}$, a *functor* $F : \mathcal{A} \to \mathcal{B}$ consists of:

- for every object A on $\mathcal{A}$, an object FA of $\mathcal{B}$;
- for every arrow $f : A \to A'$ in $\mathcal{A}$, an arrow $Ff : FA \to FA'$

such that

$$\begin{aligned} F1_A &= 1_{FA} \\ F(g \circ f) &= Fg \circ Ff \end{aligned}$$

Natural Transformations

Given functors $F, G : \mathcal{A} \to \mathcal{B}$, a *natural transformation* $\tau : F \to G$ consists of:

- for every object A of $\mathcal{A}$, an arrow $\tau_A : FA \to GA$ in $\mathcal{B}$

such that

- for every arrow $f : A \to B$ in $\mathcal{A}$,

$$Gf \circ \tau_A = \tau_B \circ Ff$$

The Setup

We have six syntactic categories:

- *categories* $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$, ...
- *objects* A, B, C, ..., X, Y, Z
- *sets* S, T, ...
- *elements* a, b, c, ..., x, y, z
- *propositions* ϕ, ψ, ...
- *proofs* δ, ..., p, q

A *context* consists of declarations of the form $X : \mathcal{A}$, $x : S$ and $p : \phi$.

The Setup

We have six judgement forms:

$$\begin{array}{rcl} \Gamma & \vdash & \mathcal{A}\ \mathrm{cat} \\ \Gamma & \vdash & A : \mathcal{A} \\ \Gamma & \vdash & S\ \mathrm{set} \\ \Gamma & \vdash & a : S \\ \Gamma & \vdash & \phi\ \mathrm{prop} \\ \Gamma & \vdash & \delta : \phi \end{array}$$

Reduction

We are going to introduce a reduction relation. We define *convertibility* $\simeq$ as usual, and have the rules

$$\frac{\Gamma \vdash A : \mathcal{A} \qquad \Gamma \vdash \mathcal{B} \text{ cat}}{\Gamma \vdash A : \mathcal{B}} \; (\mathcal{A} \simeq \mathcal{B})$$

$$\frac{\Gamma \vdash a : S \quad \Gamma \vdash T \text{ set}}{\Gamma \vdash a : T} \; (S \simeq T)$$

$$\frac{\Gamma \vdash \delta : \phi \qquad \Gamma \vdash \psi \text{ prop}}{\Gamma \vdash \delta : \psi} \; (\phi \simeq \psi)$$

Propositions

Let there be *propositions*.

Propositions

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Let propositions have *proofs*.

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Let propositions have *proofs*.
There is no notion of equality between proofs.

Sets

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$$\frac{\Gamma \vdash a : S}{\Gamma \vdash \mathrm{ref}\,(a) : a = a : S}$$

$$\frac{\Gamma, x : S \vdash \phi[x] \ \mathrm{prop} \qquad \Gamma \vdash \delta : a = b : S \qquad \Gamma \vdash \delta' : \phi[a]}{\Gamma \vdash \mathrm{sub}\,([x : S]\phi[x], \delta, \delta') : \phi[b]}$$

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$$\frac{\Gamma \vdash A : \mathcal{C}}{\Gamma \vdash 1_A : A \to A : \mathcal{C}}$$

$$\frac{\Gamma \vdash f : A \to B : \mathcal{C} \qquad \Gamma \vdash g : B \to C : \mathcal{C}}{\Gamma \vdash g \circ f : A \to C : \mathcal{C}}$$

$$\frac{\Gamma \vdash f : A \to B : \mathcal{C}}{\Gamma \vdash \mathrm{unitl}\,(f) : 1_B \circ f = f : A \to B : \mathcal{C}}$$

$$\frac{\Gamma \vdash f : A \to B : \mathcal{C}}{\Gamma \vdash \mathrm{unitr}\,(f) : f \circ 1_A = f : A \to B : \mathcal{C}}$$

$$\frac{\Gamma \vdash f : A \to B : \mathcal{C} \qquad \Gamma \vdash g : B \to C : \mathcal{C} \qquad \Gamma \vdash h : C \to D : \mathcal{C}}{\Gamma \vdash \mathrm{assoc}\,(f, g, h) : h \circ (g \circ f) = (h \circ g) \circ f : A \to D : \mathcal{C}}$$

Implication

Given propositions ϕ and ψ, let there be a proposition $\phi \to \psi$.

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$$\frac{\Gamma, p : \phi \vdash \delta : \psi}{\Gamma \vdash \mathrm{impI}([p : \phi]\delta) : \phi \to \psi}$$

$$\frac{\Gamma \vdash \delta : \phi \to \psi \qquad \Gamma \vdash \delta' : \phi}{\Gamma \vdash \mathrm{impE}(\delta, \delta') : \psi}$$

Functions

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$$\frac{\Gamma, x : S \vdash b : T}{\Gamma \vdash \lambda x : S.b : S \to T}$$

$$\mathrm{S}\ \frac{\Gamma \vdash f : S \to T \qquad \Gamma \vdash a : S}{\Gamma \vdash f(a) : T}$$

$$\frac{\Gamma, x : S \vdash b : T \qquad \Gamma \vdash a : S}{\Gamma \vdash \beta([x : S]b, a) : (\lambda x : S.b)(a) = [a/x]b : T}$$

$$\frac{\Gamma, x : S \vdash \delta : f(x) = g(x) : T}{\Gamma \vdash \mathrm{ext}\,([x : S]\delta) : f = g : S \to T}$$

$$\Gamma \vdash F : \mathcal{A} \to \mathcal{B} \quad \Gamma \vdash A : \mathcal{A} \Gamma \vdash FA : \mathcal{B}$$

$$\Gamma \vdash F : \mathcal{A} \to \mathcal{B} \quad \Gamma \vdash f : A \to A' \Gamma \vdash Ff : FA \to FA' : \mathcal{B}$$

$$\Gamma \vdash \tau : F \to G : \mathcal{A} \to \mathcal{B} \quad \Gamma \vdash A : \mathcal{A} \Gamma \vdash \tau_A : FA \to GA : \mathcal{B}$$

Given categories $\mathcal{A}$ and $\mathcal{B}$, let there be a category $\mathcal{A} \to \mathcal{B}$, whose objects are called *functors* and whose arrows are called *natural transformations*.

$$\Gamma, X : \mathcal{A} \vdash B : \mathcal{B} \Gamma \vdash \Lambda X : \mathcal{A}.B : \mathcal{A} \to \mathcal{B}$$

$$\Gamma, X : \mathcal{A} \vdash f : FX \to GX : \mathcal{B} \Gamma \vdash \lambda X : \mathcal{A}.f : F \to G : \mathcal{A} \to \mathcal{B}$$

$$\Gamma \vdash \tau : F \to G : \mathcal{A} \to \mathcal{B} \quad \Gamma \vdash f : A \to B : \mathcal{A} \Gamma \vdash \mathrm{nat}\,(\tau, f) : Gf \circ \tau_A = \tau_B \circ Ff : FA \to GB : \mathcal{B}$$

We introduce rules to guarantee the desired properties of functors and natural transformations:

$$\Gamma \vdash F : \mathcal{A} \to \mathcal{B} \quad \Gamma \vdash A : \mathcal{A} \Gamma \vdash \mathrm{funcid}\,(F, A) : F1_A = 1_{FA} : FA \to FA : \mathcal{B}$$

$$\Gamma \vdash F : \mathcal{A} \to \mathcal{B} \quad \Gamma \vdash f : A \to B : \mathcal{A} \quad \Gamma \vdash g : B \to C : \mathcal{A} \Gamma \vdash \mathrm{funcomp}\,(F, f, g) : F(g \circ f) = Fg \circ Ff : FA \to FC : \mathcal{B}$$

Introduce the reduction rule $(\Lambda X : \mathcal{A}.B)A \rightsquigarrow [A/X]B$.

$$\Gamma, X : \mathcal{A} \vdash f : FX \to GX : \mathcal{B} \quad \Gamma \vdash A : \mathcal{A} \Gamma \vdash \beta_{\mathrm{nat}}([X : \mathcal{A}]f, A) : (\lambda X : \mathcal{A}.f)_A = [A/X]f : FA \to GA : \mathcal{B}$$

What should $(\Lambda X : \mathcal{A}.B)f$ mean?

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The definition is:

$$\begin{array}{rcl}
\{f/X\}X & \equiv & f \\
\{f/X\}Y & \equiv & 1_Y \\
\{f/X\}(\Lambda Y:\mathcal{C}.D) & \equiv & \lambda Y:\mathcal{C}.\{f/X\}D \\
\{f:A\to A'/X\}(FB) & \equiv & [A'/X]F\{f/X\}B \circ (\{f/X\}F)_{[A/X]B}
\end{array}$$

Define a new notion of substitution.

$$\{f:A\to A'/X\}B$$

(f an element, X an object variable, A an object, result an element)
We want the following rule to be admissible:

$$\Gamma, X:\mathcal{A} \vdash B:\mathcal{B} \quad \Gamma \vdash f:A\to A':\mathcal{A}\Gamma \vdash \{f:A\to A'/X\}B : [A/X]B \to [A'/X]B : \mathcal{B}$$

Now introduce the rule:

$$\Gamma, X : \mathcal{A} \vdash B : \mathcal{B} \quad \Gamma \vdash f : A \to A' : \mathcal{A}\Gamma \vdash \beta_{\mathrm{func}}([X : \mathcal{A}]B, f) : (\Lambda X : \mathcal{A}.B)f = \{f/X\}B : [A/X]B \to [A'/X]B : \mathcal{B}$$

$$\Gamma \vdash \sigma : F \to G : \mathcal{A} \to \mathcal{B}$$
$$\Gamma \vdash \tau : G \to H : \mathcal{A} \to \mathcal{B}$$
$$\Gamma \vdash A : \mathcal{A}$$
$$\Gamma \vdash \mathrm{nat}_{\mathrm{comp}}(\sigma, \tau, A) : (\sigma \circ \tau)_A = \sigma_A \circ \tau_A : FA \to HA : \mathcal{B}$$

Our rules for categories automatically give us identity natural transformations, and
We introduce rules specifying how these behave under application.

$$\Gamma \vdash F : \mathcal{A} \to \mathcal{B} \quad \Gamma \vdash A : \mathcal{A} \Gamma \vdash \mathrm{nat}_{\mathrm{id}}(F, A) : (1_F)_A = 1_{FA} : FA \to FA : \mathcal{B}$$

Special Functors

We can define

$$\begin{aligned} 1_{\mathcal{A}} &\equiv \Lambda X : \mathcal{A}.X \\ G \circ F &\equiv \Lambda X : \mathcal{A}.G(FX) \end{aligned}$$

The following equalities then hold:

$$\begin{aligned} 1_{\mathcal{B}} \circ F &\simeq F \\ F \circ 1_{\mathcal{A}} &\simeq F \\ H \circ (G \circ F) &\simeq (H \circ G) \circ F \end{aligned}$$

And So On . . .

We can go on to introduce:

- negation, conjunction, disjunction, quantification over sets, quantification over the objects of a category

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- sets $\{x : S \mid \phi[x]\}$ and categories $\Sigma X : \mathcal{A}.S[X]$

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- sets $\{x : S \mid \phi[x]\}$ and categories $\Sigma X : \mathcal{A}.S[X]$
- comma categories, universals, limits, adjunctions, . . .

Technical Comments

We have:

- an *intensional* notion of equality between sets, categories and objects;
- an *extensional* notion of equality between elements;
- *no* notion of equality between proofs.

(We are also free to define notions, e.g. equivalence of propositions, isomorphism of objects.)
It is thus halfway between good and evil.

Categories may not depend on object variables.

Sets, objects and categories may not depend on element variables.

- If an element variable x may occur in an expression $E[x]$, you must have an answer to the question 'If $x = y$, what is the relation between $E[x]$ and $E[y]$?'
- If an object variable X may occur in an expression $E[X]$, you must have an answer to the question 'Given an arrow $f : X \to Y$, what is the relation between $E[X]$ and $E[Y]$?'

To Do:

Justify system by

- proving metatheoretic properties
- giving semantics in terms of category theory
- formalising loads of examples

Extend to 2-categories.

Implement this system?

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I want to formalise fibred category theory in a non-evil way

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- and Conor hasn’t written a paper with that title yet.