Proof and Definition in Logic and Type Theory

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Logic turns philosophical disagreements into formal questions.

Definition and Proof

Definition and proof are interleaved in a body of mathematics.

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is total?

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Induction For which predicates $\phi[x]$ can we prove $\forall x \phi[x]$ by induction?

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Predicate logic emphasises proof — no primitive mechanism for definition.

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Let $GCD(a, b, c)$ be the formula

$$c \mid a \wedge c \mid b \wedge \forall x (x \mid a \rightarrow x \mid b \rightarrow x \mid c)$$

Theorem of PA: $\forall x \forall y \exists! z GCD(x, y, z)$.
We may safely add a function symbol gcd and the axiom $GCD(a, b, gcd(a, b))$.

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Methods of Proof affect Methods of Definition

RCA_0 is a system of *second-order arithmetic*. Its language deals with:

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If we add Σ_k-induction, we obtain bounded Σ_k-abstraction:

$$\{x \leq n \mid \phi[x]\}$$

can be proved to exist for $\phi[x]$ a Σ_k-formula.

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Language deals with judgements

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- A proof of $A \wedge B$ consists of a proof of A and a proof of B.
- A proof of $A \vee B$ is either a proof of A or a proof of B.
- A proof of $A \to B$ is a function that takes a proof of A and returns a proof of B.

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We cannot change the logic without changing the objects, or *vice versa*.

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  It thus has two 'worlds' — the logical world and the type theory world.
  These two worlds interact but can be modified separately.

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- Divide categories into basic and ideal categories.
  - Natural numbers form a basic category.
  - For any category A, the sets of As form an ideal category.
- When defining a set, we may only quantify over the basic categories.

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  Restricted induction $\phi[0] \rightarrow \forall x(\phi[x] \rightarrow \phi[x']) \rightarrow \forall x \phi[x]$ for $\phi[x]$ arithmetic

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Shortcomings: Weyl uses

- sets of sets
- definition of sets by recursion
- full induction

$\mathrm{LTT}_{\mathrm{W}}$

Weyl's Foundation as a Logic-Enriched Type Theory

Type Theory World

Universe U (objects are the basic categories)

$$\mathbb{N}$$

$$\times$$

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$$Set(A)$$

Logical World

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All results from *Das Kontinuum* have been formalised in $\mathrm{LTT_W}$ using proof assistant Plastic.

ACA_0 is Embeddable in $\mathrm{LTT}_{\mathrm{W}}$

Define a mapping

Formulas of ACA_0 $\longrightarrow$ Formulas of $\mathrm{LTT}_{\mathrm{W}}$

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Formulas of ACA_0	$\longrightarrow$	Formulas of $\mathrm{LTT}_{\mathrm{W}}$
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$\forall x$		$\forall x : \mathbb{N}$
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But $\mathrm{LTT}_{\mathrm{W}}$ goes further:

- has types $\mathit{Set}(\mathit{Set}(\mathbb{N}))$, ...
- allows definition by recursion in $\mathit{Set}(A)$
- allows full induction

ACA and ACA_0^+

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ACA_0^+ is ACA_0 extended with iteration over ω:

$$\exists X \forall j \forall n(\langle n, j \rangle \in X \leftrightarrow \phi[n, (X)^j])$$

for $\phi[x, X]$ arithmetic, where

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Equivalent to ACA_0+ $TJ^\omega(\emptyset)$ exists.

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- $\mathrm{LTT}^1_{\mathrm{W}}-$: restrict $E_{\mathbb{N}}$ to U and induction to *prop*.
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				Proof Theoretic Ordinal
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ACA	$\hookrightarrow$	$\mathrm{LTT_W}^1$		ϵ_{ϵ_0}
$\mathrm{ACA_0}^+$	$\overset{?}{\hookrightarrow}$	$\mathrm{LTT_W}^2-$		$\phi_2(0) = \epsilon_{\epsilon_\epsilon}$
				$\ddots$
		$\mathrm{LTT_W}$	$\dashrightarrow$ ML_1	$\phi_{\epsilon_0}(0)$

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- Which functions are definable in $\mathrm{LTT}_{\mathrm{W}}$?
- Develop a hierarchy of LTTs, by strengthening and weakening universes and eliminators, similar to Reverse Mathematics.

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set definable by $S : \mathrm{Set}(\mathbb{N} \times \mathbb{N})$	$\langle \overline{n}, \overline{f(n)} \rangle \in S$ is provable
	$\neg \langle \overline{n}, \overline{m} \rangle \in S$ is provable for other m
	$\forall x \exists! y \langle x, y \rangle \in S$ is provable
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