State and Effect Logics

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Introduction

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- Idea (Birkhoff, von Neumann 1936): change the laws of logic itself.
  - Keep double negation.
  - Drop distributivity: $P \wedge (Q \vee R) \leftrightarrow (P \wedge Q) \vee (P \wedge R)$.

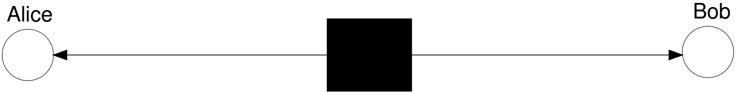
Motivation
The Hardy Box
State-and-Effect Triangles

Type Theory
Affine Type Theory
Logic over an Affine Type Theory
Probabilistic Triangles
Classical Triangles
Measurement

The Hardy Box Revisited
Related Work
For the Future

The Hardy Experiment

Described in [Hardy, 1993].
A box shoots out pairs electrons in two directions.
Alice and Bob can each choose to measure their spin vertically (V), or at angle 76.35^o (T). They get the answers $+$ or $-$.



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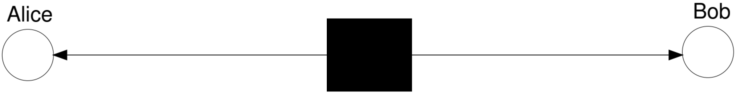

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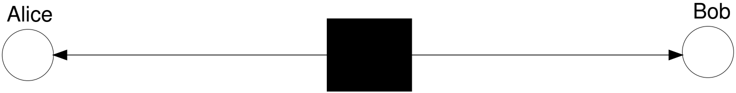

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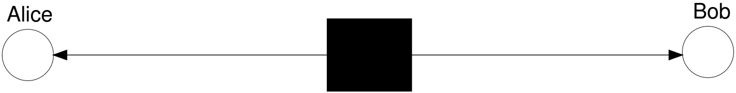

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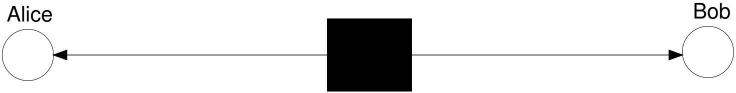

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A Natural Assumption

Alice	Bob	++	+−	−+	−−
V	V	1			
V	T	0			
T	V	0			
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The box sends out electrons in different *states*. The *state* of each electron determines the probability of the outcome $+$ or a $-$ for each measurement.

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Alice: $V+$ $T+$ Bob: $T-$ $V-$

Alice's choice of measurement affects Bob's outcome.

This 'influence' travels faster than light. [Yin et al., 2013]

It cannot be used to send information from Alice to Bob (*no-signalling*).

Unusual Features

- The observer effect
  - Observing the system changes its state.
  - *Propositions have side-effects*
- The uncertainty principle
  - Alice cannot use $V+$ and $T+$ in the same proposition.
  - *Propositional connectives become partial operations*

A Setup for Quantum Theory

$$\mathbf{CStar}_{CPU}{}^{\mathrm{op}}$$

Associated with every physical system is a C^**-algebra* A.
The computations from A to B are the CPU-maps $B \to A$.

A Setup for Quantum Theory

$$\mathbf{CStar}_{CPU}{}^{\mathrm{op}}$$

The *states* of the system correspond to the CPU-maps $A \to \mathbb{C}$.
A computation can be read as a *state transformer*.

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$$\mathbf{CStar}_{CPU}^{\mathrm{op}} \xrightarrow{\mathrm{Stat}} \mathbf{Conv}_{[0,1]}$$

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These form a *convex set* over $[0,1]$.

Definition (Convex Set)

A convex set over $[0,1]$ consists of a set X and an operation: given $r_1, \ldots, r_n \in [0,1]$ with $r_1 + \cdots + r_n = 1$ and $x_1, \ldots, x_n \in X$, returns an element

$$r_1 x_1 + \cdots + r_n x_n \in X$$

such that certain equations hold.

A Setup for Quantum Theory

$$\mathbf{CStar}_{CPU}{}^{\mathrm{op}} \xrightarrow{\mathrm{Stat}} \mathbf{Conv}_{[0,1]}$$

The *effects* of A correspond to the elements $x \in A$ such that $0 \leq x \leq 1$.
A computation can be read as a *predicate transformer* (weakest precondition semantics).

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$$\mathbf{EMod}_{[0,1]}{}^{\mathrm{op}} \xleftarrow{\ \mathrm{Pred}\ } \mathbf{CStar}_{CPU}{}^{\mathrm{op}} \xrightarrow{\ \mathrm{Stat}\ } \mathbf{Conv}_{[0,1]}$$

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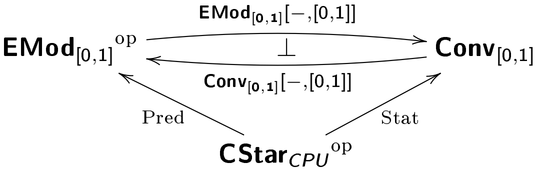
These form an *effect module* over $[0, 1]$.

Definition (Effect Module)

An *effect module* over the effect monoid M is a set A with a *partial* operation $\ovee : A \times A \to A$, and (total) operations $(-)^{\perp} : A \to A$, $\cdot : [0, 1] \times A \to A$ such that certain axioms hold

A Setup for Quantum Theory



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A computation can be read as a *predicate transformer* (weakest precondition semantics).
These form an *effect module* over $[0, 1]$.
There is an adjunction between these categories, in both cases 'homming into $[0, 1]$'.

A Setup for Probabilistic Computation

$$\mathrm{Kl}\,(\mathcal{D})$$

Let $\mathcal{D}B$ be the set of all probability distributions over B.
A computation from A to B is a function $A \to \mathcal{D}B$.

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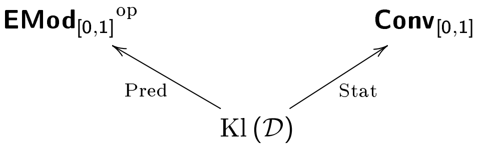

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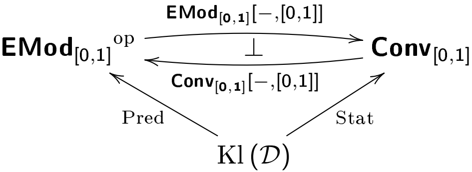

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Affine Categories

An *affine* category is a symmetric monoidal category $\mathcal{C}$ with finite coproducts such that:

- the tensor unit I is a terminal object
- the tensor distributes over $+$

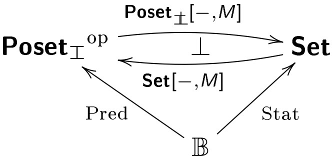
Examples

- Any Cartesian closed category is affine.
- $\mathrm{Kl}(\mathcal{D})$, $\mathrm{Kl}(\mathcal{P}_*)$, $\mathbf{CStar}_{CPU}{}^{\mathrm{op}}$ are all affine.

State-and-Effect Triangles

Let $\mathbf{Poset}_{\top}$ be the category of bounded posets (posets with $\top$ and $\bot$).
A *state-and-effect triangle* consists of:

- an affine category $\mathbb{B}$ of *types* and *computations*;
- a bounded poset M of *scalars* (propositions or truth values);
- a symmetric comonoidal functor $\mathrm{Pred} : \mathbb{B} \to \mathbf{Poset}_{\top}{}^{\mathrm{op}}$ that preserves coproducts;
- a symmetric monoidal functor $\mathrm{Stat} : \mathbb{B} \to \mathbf{Set}$ that preserves the terminal object;
- a natural transformation $\models : \mathrm{Stat} \to \mathbf{Poset}[\mathrm{Pred}-, M]$.

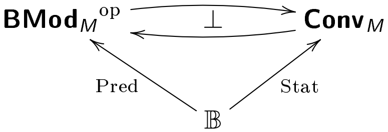


Example: $\mathrm{Kl}(\mathcal{P}_*)$.

Classical State-and-Effect Triangles

A *classical state-and-effect triangle* consists of:

- an affine category $\mathbb{B}$ of *types* and *computations*;
- an Boolean algebra M of *scalars* (propositions or truth values);
- a symmetric comonoidal functor $\mathrm{Pred} : \mathbb{B} \to \mathbf{BMod}_M{}^{\mathrm{op}}$ that preserves the terminal object and coproducts;
- a symmetric monoidal functor $\mathrm{Stat} : \mathbb{B} \to \mathbf{Conv}_M$ that preserves the terminal object;
- a natural transformation $\models : \mathrm{Stat} \to \mathbf{BMod}_M[\mathrm{Pred}-, M]$.



Example: **Set**, any Boolean topos.

Syntax of QPEL

$$\text{Type } A ::= A \otimes A \mid I \mid A + B$$

- Terms $s, t, \ldots$ intended to represent quantum programs.
- Effects $\phi, \psi, \ldots$ intended to represent predicates on quantum states.

Judgement forms:

- $\Gamma \vdash t : A$
- $\Gamma \vdash s = t : A$
- $\Gamma \vdash \phi \text{ eff}$
- $\Gamma \vdash \phi \leq \psi$

Properties of QPEL

This is a *linear* type system – no Contraction:

$$\frac{\Gamma, x : A, y : A \vdash t[x, y] : B}{\Gamma, x : A \vdash t[x, x] : B}$$

Allowing Contraction would violate the no-cloning theorem

Typing System

Structural Rules

$$\frac{\Gamma, x : A, y : B, \Delta \vdash \mathcal{J}}{\Gamma, y : B, x : A, \Delta \vdash \mathcal{J}} \qquad \frac{}{\Gamma \vdash x : A} (x : A \in \Gamma)$$

Tensor Products

$$\frac{\Gamma \vdash M : A \qquad \Delta \vdash N : B}{\Gamma, \Delta \vdash M \otimes N : A \otimes B}$$

$$\frac{\Gamma \vdash M : A \otimes B \qquad \Delta, x : A, y : B \vdash N : C}{\Gamma, \Delta \vdash \mathrm{let}\ x \otimes y = M\ \mathrm{in}\ N : C}$$

$$\mathrm{let}\ x \otimes y = M \otimes N\ \mathrm{in}\ P = [M/x, N/y]P$$

$$M = \mathrm{let}\ x \otimes y = M\ \mathrm{in}\ x \otimes y$$

Semantics

Define:

- an object $[\![A]\!] \in \mathcal{V}$ for each type A
- an object $[\![\Gamma]\!] \in \mathcal{V}$ for each context Γ
- an arrow $[\![M]\!] : [\![\Gamma]\!] \to [\![A]\!]$ for each term $\Gamma \vdash M : A$

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Example: If $\Gamma \vdash M : A \otimes B$ and $\Delta, x : A, y : B \vdash N : C$, then

$$[\![\Gamma, \Delta \vdash \mathrm{let}\ x \otimes y = M\ \mathrm{in}\ N : C]\!]$$

is

$$[\![\Gamma]\!] \otimes [\![\Delta]\!] \xrightarrow{[\![M]\!] \otimes \mathrm{id}} [\![A]\!] \otimes [\![B]\!] \otimes [\![\Delta]\!] \cong [\![\Delta]\!] \otimes [\![A]\!] \otimes [\![B]\!] \xrightarrow{[\![N]\!]} [\![C]\!]$$

The models are exactly the affine categories.

Effects

$$(\top)\ \frac{}{\Gamma \vdash \top\ \mathrm{eff}} \qquad (\bot)\ \frac{}{\Gamma \vdash \bot\ \mathrm{eff}}$$

$$(\text{case-prop})\ \frac{\Gamma \vdash M : A + B \qquad \Delta, x : A \vdash \phi\ \mathrm{eff} \qquad \Delta, y : B \vdash \psi\ \mathrm{eff}}{\Gamma, \Delta \vdash \mathrm{case}\ M \mathrm{of}\ \mathrm{inl}\,(x) \mapsto \phi \mid \mathrm{inr}\,(y) \mapsto \psi\ \mathrm{eff}}$$

$$(\text{let-prop})\ \frac{\Gamma \vdash M : A \otimes B \qquad \Delta, x : A, y : B \vdash \phi\ \mathrm{eff}}{\Gamma, \Delta \vdash \mathrm{let}\ x \otimes y = M \mathrm{in}\ \phi\ \mathrm{eff}}$$

$$(\leq\text{-ref})\ \frac{\Gamma \vdash \phi\ \mathrm{eff}}{\Gamma \vdash \phi \leq \phi} \qquad (\leq\text{-trans})\ \frac{\Gamma \vdash \phi \leq \psi \qquad \Gamma \vdash \psi \leq \chi}{\Gamma \vdash \phi \leq \chi}$$

$$(\leq\text{-}\top)\ \frac{\Gamma \vdash \phi\ \mathrm{eff}}{\Gamma \vdash \phi \leq \top} \qquad (\bot\text{-}\leq)\ \frac{\Gamma \vdash \phi\ \mathrm{eff}}{\Gamma \vdash \bot \leq \phi}$$

We define $[\![\Gamma \vdash \phi\ \mathrm{eff}]\!] \in \mathrm{Pred}\,[\![\Gamma]\!]$ for every effect in context Γ.
We define $(\![\vdash \phi\ \mathrm{eff}]\!) \in M$ for every effect in the empty context

Tensor Product of Effects

$$(\text{mult})\ \frac{\Gamma \vdash \phi\ \text{eff} \qquad \Delta \vdash \psi\ \text{eff}}{\Gamma, \Delta \vdash \phi \otimes \psi\ \text{eff}}$$

$$(\otimes\text{-monoL})\ \frac{\Gamma \vdash \phi \leq \phi' \qquad \Delta \vdash \psi\ \text{eff}}{\Gamma, \Delta \vdash \phi \otimes \psi \leq \phi' \otimes \psi}$$

$$(\otimes\text{-monoR})\ \frac{\Gamma \vdash \phi\ \text{eff} \qquad \Delta \vdash \psi \leq \psi'}{\Gamma, \Delta \vdash \phi \otimes \psi \leq \phi \otimes \psi'}$$

$$(\text{unit}_L)\ \frac{\Gamma \vdash \phi\ \text{eff}}{\Gamma \vdash \top \otimes \phi \Leftrightarrow \phi} \qquad (\text{unit}_R)\ \frac{\Gamma \vdash \phi\ \text{eff}}{\Gamma \vdash \phi \otimes \top \Leftrightarrow \phi}$$

$$(\text{assoc})\ \frac{\Gamma \vdash \phi\ \text{eff} \qquad \Delta \vdash \psi\ \text{eff} \qquad \Theta \vdash \chi\ \text{eff}}{\Gamma, \Delta, \Theta \vdash \phi \otimes (\psi \otimes \chi) \Leftrightarrow (\phi \otimes \psi) \otimes \chi}$$

$$(\text{comm})\ \frac{\vdash \phi\ \text{eff} \qquad \vdash \psi\ \text{eff}}{\vdash \phi \otimes \psi \Leftrightarrow \psi \otimes \phi}$$

This makes the functor $\mathbb{B} \to \mathbf{Poset}_{\perp}{}^{\mathrm{op}}$ into a symmetric comonoidal functor.f

Probabilistic Triangles

For probabilistic triangles, we add these rules:

$$\frac{\Gamma \vdash \phi \text{ eff}}{\Gamma \vdash \phi^{\perp} \text{ eff}} \qquad \frac{\Gamma \vdash \phi \leq \psi^{\perp}}{\Gamma \vdash \phi \ovee \psi \text{ eff}}$$

$$\frac{\Gamma \vdash \phi \leq \psi}{\Gamma \vdash \psi^{\perp} \leq \phi^{\perp}} \qquad \frac{\Gamma \vdash \phi \text{ eff}}{\Gamma \vdash \phi \leq \phi^{\perp\perp}} \qquad \frac{\Gamma \vdash \phi \text{ eff}}{\Gamma \vdash \phi^{\perp\perp} \leq \phi}$$

$$(\mathsf{dist}_L)\ \frac{\Gamma \vdash \phi \perp \psi \qquad \Delta \vdash \chi \text{ eff}}{\begin{array}{c}\Gamma, \Delta \vdash \phi \otimes \chi \perp \psi \otimes \chi \\ \Gamma, \Delta \vdash (\phi \ovee \psi) \otimes \chi \Leftrightarrow \phi \otimes \chi \ovee \psi \otimes \chi\end{array}}$$

$$(\mathsf{dist}_R)\ \frac{\Gamma \vdash \phi \text{ eff} \qquad \Delta \vdash \psi \perp \chi}{\begin{array}{c}\Gamma, \Delta \vdash \phi \otimes \psi \perp \phi \otimes \chi \\ \Gamma, \Delta \vdash \phi \otimes (\psi \ovee \chi) \Leftrightarrow \phi \otimes \psi \ovee \phi \otimes \chi\end{array}}$$

We have a soundness and completeness result for these: the derivable judgements are those true in every probabilistic state-and-effect triangle.

Classical Triangles

We can add to the logic *conjunction*:

$$\mathsf{Effect}\phi ::= \cdots \mid \phi \wedge \phi$$

with rules which ensure

$$\phi \wedge \psi \leq \phi \qquad \phi \wedge \psi \leq \psi$$

$$\phi \wedge \phi^{\perp} \Leftrightarrow 0$$

$$\frac{\Gamma \vdash \phi \leq \psi \qquad \Gamma \vdash \phi \leq \chi}{\Gamma \vdash \phi \leq \psi \wedge \chi}$$

$$\frac{\Gamma \vdash \phi \perp \psi \qquad \Gamma \vdash \phi \perp \chi}{\Gamma \vdash (\phi \ovee \psi) \wedge (\phi \ovee \chi) \Leftrightarrow \phi \ovee (\psi \wedge \chi)}$$

Theorem
The models of this logic are exactly the classical state-and-effect triangles.

Measurement with Side Effects

How do we give propositions side-effects?

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For now:

$$\text{(measure)}\ \frac{\Gamma \vdash 1 \leq \bigovee_{i=1}^{n} \phi_i \qquad \Delta \vdash M_i : A \qquad (1 \leq i \leq n)}{\Gamma, \Delta \vdash \mathrm{measure}\ {}_{i=1}^{n}\phi_i \mapsto M_i : A}$$

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$$
\begin{aligned}
(\mathrm{measure}\ {}_{i=1}^{n}\phi_i \mapsto M_i) &= (\mathrm{measure}\ {}_{i=1}^{n}\phi_{p(i)} \mapsto M_{p(i)}) \\
(\mathrm{measure}\ \phi_1 \mapsto M_1 \mid \cdots \mid \phi_n \mapsto M_n \mid 0 \mapsto M_{n+1}) & \\
&= \mathrm{measure}\ \phi_1 \mapsto M_1 \mid \cdots \mid \phi_n \mapsto M_n \\
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\end{aligned}
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(\text{measure } 1 \mapsto M) &= M
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. . .
We want a more disciplined approach.

Chains of Adjunctions

Define the 'lift' monad $\mathcal{L} : \mathbb{B} \to B$ by: $\mathcal{L}A = A + I$.

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Define the functor $\Box : \mathrm{Kl}(\mathcal{L}) \to \mathbf{Poset}^{\mathrm{op}}$ as follows

$$\Box A = \mathrm{Pred} A$$
$$\Box(f : A \to B + I)(p) = F(p, \top)$$

where F is the arrow

$$PredB \times PredI \longrightarrow \mathrm{Pred}(B + I) \xrightarrow{\mathrm{Pred} f} \mathrm{Pred} A$$

Intuitively,

$$\Box f(p) = \{a \in A : f(a) \text{ is defined and satisfies } p\}$$

Chains of Adjunctions

Define the 'lift' monad $\mathcal{L} : \mathbb{B} \to B$ by: $\mathcal{L}A = A + I$.
Define the functor $\Box : \mathrm{Kl}(\mathcal{L}) \to \mathbf{Poset}^{\mathrm{op}}$ as follows

$$\Box A = \mathrm{Pred} A$$
$$\Box(f : A \to B + I)(p) = F(p, \top)$$

where F is the arrow

$$PredB \times PredI \longrightarrow \mathrm{Pred}(B + I) \xrightarrow{\mathrm{Pred} f} \mathrm{Pred} A$$

Let $\int \Box$ be the *Grothendieck completion* with

- objects all pairs (A, p) with $p \in \mathrm{Pred} A$
- arrows $f : (A, p) \to (B, q)$ all arrows $f : A \to B + I$ such that $p \leq \Box f(q)$

Chains of Adjunctions

$$\begin{array}{c} \int \square \\ \big\downarrow \\ \mathbb{B} \end{array}$$

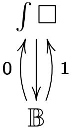
Forgetful functor

Chains of Adjunctions

$$\begin{array}{c} \int \Box \\ {\scriptstyle 0}\,\uparrow\;\downarrow \\ \mathbb{B} \end{array}$$

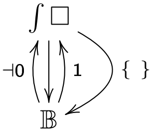
$$A \mapsto (A, \bot)$$

Chains of Adjunctions



$$A \mapsto (A, \top)$$

Chains of Adjunctions



$$A \mapsto (A, \top)$$

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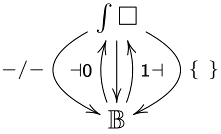
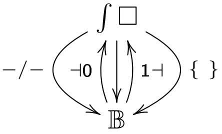


$$A \mapsto (A, \top)$$

A *state-and-effect triangle with comprehension and quotients* is one in which this chain of adjunctions exists and for all p

$$\{X|p\} \cong X/p^{\perp}$$

Chains of Adjunctions



$$A \mapsto (A, \top)$$

A *state-and-effect triangle with comprehension and quotients* is one in which this chain of adjunctions exists and for all p

$$\{X|p\} \cong X/p^{\perp}$$

Idea

$$A \longrightarrow \{A \mid \phi\} \cong A/\phi^{\perp} \longrightarrow A$$

is the action (side-effect) of measuring ϕ on A.

Comprehension

We give ourselves a new judgement form,

$$\Gamma \mid \phi \vdash \mathcal{J}$$

A term $\Gamma \mid \phi \vdash M : A$ is interpreted as an arrow $(\Gamma, \phi) \to (A, \top)$.
A predicate $\Gamma \mid \phi \vdash \psi \text{ eff}$ is interpreted as an element of $\mathrm{Pred}\{\Gamma \mid \phi\}$.

$$(\text{comp-weak})\ \frac{\Gamma \vdash \phi \text{ eff} \qquad \Gamma \vdash \mathcal{J}}{\Gamma \mid \phi \vdash \mathcal{J}} \qquad (\text{comp-id})\ \frac{\Gamma \vdash \phi \text{ eff}}{\Gamma \mid \phi \vdash \top \leq \phi}$$

$$(\text{comp-¡})\ \frac{}{\Gamma \mid \bot \vdash \text{¡} : A} \qquad (\text{comp-false})\ \frac{\Gamma \vdash M : A \qquad \Gamma \vdash N : A}{\Gamma \mid \bot \vdash M = N : A}$$

Comprehension

$$(\text{comp})\ \frac{x : A \vdash \phi \ \mathrm{eff}}{\{x : A \mid \phi\} \ \ \mathrm{type}}$$

$$(\text{compl})\ \frac{x : A \vdash \phi \ \mathrm{eff} \qquad \Gamma \vdash M : A \qquad \Gamma \vdash \top \leq [M/x]\phi}{\Gamma \vdash M|_{x.\phi} : \{x : A \mid \phi\}}$$

$$(\text{comp-let})\ \frac{\Gamma \vdash M : \{x : A \mid \phi\} \qquad \Delta, x : A \mid \phi \vdash N : B}{\Gamma, \Delta \vdash \mathrm{let} \ \ x|_\phi = M \ \mathrm{in} \ N : B}$$

$$\begin{aligned} \top &\leq [(\mathrm{let} \ \ x|_\phi = M \ \mathrm{in} \ x)/x]\phi \\ (\mathrm{let} \ \ x|_\phi = M|_{x.\phi} \ \mathrm{in} \ N) &= N[M/x] \\ M &= (\mathrm{let} \ \ x|_\phi = M \ \mathrm{in} \ x|_{x.\phi} : \{x : A \mid \phi\}) \end{aligned}$$

Quotients

$$(\text{quotient})\ \frac{x : A \vdash \phi \ \mathrm{eff}}{\vdash (x : A/\phi)\ \ \mathrm{type}}$$

$$(\text{quot-in})\ \frac{\Gamma \vdash M : A \qquad x : A \vdash \phi \ \mathrm{eff}}{\Gamma \vdash \xi_\phi(M) : (x : A/\phi) + 1}$$

$$(\text{quot-in-inr})\ \frac{x : A \vdash \phi \ \mathrm{eff}}{x : A \vdash \phi \Leftrightarrow \mathsf{inr?}\,(\xi_\phi(x))}$$

$$(\text{quot-out})\ \frac{\Gamma \vdash M : (x : A/\phi) \qquad \Delta, x : A \vdash N : B + 1 \qquad \Delta, x : A \vdash \phi \leq \mathsf{inr?}\,(N)}{\Gamma, \Delta \vdash \mathrm{let}\ \xi_\phi(x) = M \mathrm{in}\ \ N : B + 1}$$

Quotients

$$(\beta\text{-quotient})\ \frac{\Gamma \vdash M : A \qquad x : A \vdash \phi\ \mathrm{eff} \qquad \Delta, x : A \vdash N : B + 1 \qquad \Delta, x : A \vdash \phi \leq \mathsf{inr?}\,(N)}{\Gamma, \Delta \vdash \ \mathrm{case}\ \xi_\phi(M)\mathrm{of}\ \mathrm{inl}\,(y) \mapsto \mathrm{let}\ \xi_\phi(x) = y \mathrm{in}\ N \mid \mathrm{inr}\,(z) \mapsto \mathrm{inr}\,(\langle\rangle) = [M/x]N : B + 1}$$

$$(\eta\text{-quotient})\ \frac{\Gamma \vdash M : (x : A/\phi) \qquad \Delta, u : (x : A/\phi) + 1 \vdash N : B}{\Gamma, \Delta \vdash \ \mathrm{let}\ \xi_\phi(x) = M \mathrm{in}\ [\xi_\phi(x)/u]N = [\mathrm{inl}\,(M)\,/u]N : (x : A/\phi) + 1}$$

Measurement

Let us say ϕ is sharp if

$$\{x : A \mid \phi\} \cong (x : A/phi^{\perp})$$

Consider the arrow $assert_{\phi}$:

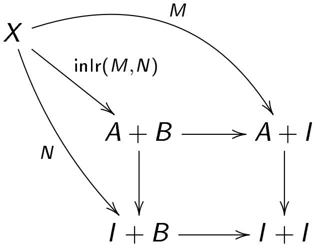
$$A \xrightarrow{\xi} (x : A/\phi^{\perp}) + 1 \cong \{x : A \mid \phi\} + 1 \longrightarrow A + 1$$

This represents this program:

> *Take input* $x : A$*. Measure* ϕ *on* x*. If the outcome is true, output* $\mathrm{inl}\,($*state after measurement*$)$*. If the outcome is false, output* $\mathrm{inr}\,(\langle\rangle)$.

Pullback Property

In all our categories $\mathbb{B}$, the following diagrams are pullbacks:



Let $\mathrm{inlr}\,(M, N)$ be the mediating arrow.

Then define

$$\text{measure } \phi(x) \mapsto M(x) \mid \phi(x)^{\perp} \mapsto N(x)$$

to be

$$\mathsf{case\ inlr}\ \big(\mathit{assert}_{\phi(x)}, \mathit{swap}(\mathit{assert}_{\phi(x)^{\perp}})\big)\ \mathsf{of}\ \mathrm{inl}\,(x) \mapsto M(x) \mid \mathrm{inr}\,(x) \mapsto N(x)$$

Theorem

In the category of von Neumann algebras, for every effect ϕ, there is a unique least sharp effect above ϕ.

For the future: build this into the syntax.

The theory *HARDY*

Type Constants

$$electron$$

Term Constants

$$initial : electron \otimes electron$$

Effect Constants $V+$, $T+$, both over *electron*. 1%, over I

Axioms

$$1\% \varovee 1\% \varovee \cdots \varovee 1\% \Leftrightarrow 1$$

$$1\% \leq V+(x) \otimes V+(y)$$

$$\text{let } x \otimes y = initial \text{ in } T+(x) \otimes V+(y) \Leftrightarrow 0$$

$$\text{let } x \otimes y = initial \text{ in } V+(x) \otimes T+(y) \Leftrightarrow 0$$

$$\text{let } x \otimes y = initial \text{ in } T+(x)^{\perp} \otimes T+(y)^{\perp} \Leftrightarrow 0$$

This theory is *consistent* in the probabilistic framework, and *inconsistent* in the classical framework.

Related Work

- Quantum programming languages without logic.
  - QPL, Quipper, quantum λ-calculus, ...
  - No syntax for logic (but can use matrices as predicates, see for example [d'Hondt and Panangaden, 2006]).
- LQP [Baltag et al., 2014]
  - Modal logic for reasoning about quantum states
  - Language for programs is an underspecification language.
- Equational reasoning
  - Measurement calculus, Staton's rules, ...
- Quantum protocols
  - CQP, QPAlg, qCCS, ...
  - Equational axioms and reduction rules for communicating quantum processes
  - No logic of effects over these processes

For the Future

- Find a syntactic charaterisation of the difference between probabilistic and quantum computation.
- Give separate syntactic characterisations of the 'weird' features of quantum theory:
  - non-locality/contextuality
  - non-determinism
  - no signalling
- Prove some quantum algorithms correct.
- Build a proof checker for quantum computing.

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