The Consistency of Various Theories of Inductive Types

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1 PAL plus

A *signature* in PAL$^+$is a sequence of *constant declarations* of the form

$$c : K$$

and *equations* of the form

$$(\Delta)(t = t' : T)$$

where t, t' and T have arity 0.

Definition 1 *A PAL$^+$signature Σ is* consistent *iff there is no term M such that*

$$X : \mathbf{Type} \vdash_\Sigma M : \mathrm{El}\,(X) \quad .$$

Our aim in this note is to establish the consistency of four PAL$^+$signatures:

- MLTT — Martin-Löf Type Theory
- MLTTC — Martin-Löf Type Theory with classical logic
- UTT — Universal Type Theory
- UTTC — Universal Type Theory with classical logic

We define:

- A kind or context is *tiny* iff it does not contain the symbol **Type**.
- The *small* kinds are those of the form $(\Delta)K$, where Δ is tiny. (It will itself be tiny iff K is of the form $\mathrm{El}\,(M)$, and not if $K \equiv \mathbf{Type}$.)
- The *small* contexts are those such that:
  - For each declaration $x : K$, the kind K is small.
  - For each definition $v[\Delta] = t : T$, the context Δ is tiny.

- The *bijou* kinds are those of the form $(\Delta)K$, where Δ is small.

A signature is *bijou* iff only bijou kinds occur in it.

All the signatures we work with shall be bijou. We shall find that, while large (i.e. non-small) contexts are not needed establishing judgements involving bijou signatures. Furthermore, when finding a set-theoretic interpretation for our type theories, it shall be very convenient to only have to interpret the small contexts. To this end, we prove the following result.

Theorem 1 *Let Γ be a small context, Σ a bijou context, and K a bijou kind. For any term M, if*

$$\Gamma \vdash_\Sigma M : K$$

in PAL$^+$, then there is a term N such that $M \twoheadrightarrow N$, and a derivation of

$$\Gamma \vdash_\Sigma N : K$$

in which no large context appears.

2 Inductive Types

We have adapted the following definitions from [1]. Note that we are using terms in a different manner: our "inductive schema" would there be a sequence of inductive schemata, for example.

An *inductive schema* $\mathcal{S}$ has the form

$$\begin{aligned} \text{Inductive} I &: (\Delta)\mathbf{Type} \\ \text{Constructors} \kappa_1 &: (\Delta, \Theta_1, (\Phi_{11})I[\Delta], \ldots, (\Phi_{1r_1})I[\Delta])I[\Delta], \\ &\cdots \\ \kappa_n &: (\Delta, \Theta_n, (\Phi_{n1})I[\Delta], \ldots, (\Phi_{nr_n})I[\Delta])I[\Delta] \end{aligned}$$

where I and each κ_i is a constant, Δ is a small context, and each Θ_i and Φ_{ij} is a tiny context. I should have arity $|\Delta|$, and κ_i have arity $|\Delta| + |\Theta_i| + r_i$.

We associate with each inductive schema a chosen set of variables

$$x_{ij}, y_{ij} \qquad \text{for } 1 \leq i \leq n, 1 \leq j \leq r_i \enspace .$$

where the arity of each x_{ij} and y_{ij} is the same as the length of Φ_{ij}.

We make the following definitions:

- For $i = 1, \ldots, n$, Φ_i is the tiny context

$$x_{i1} : (\Phi_{i1})I[\Delta], \ldots, x_{ir_i} : (\Phi_{ir_i})I[\Delta] \enspace .$$

- For $i = 1, \ldots, n$, if M is a 0-ary object and z a 0-ary variable, then $\Phi_i^o[[z]M]$ is the tiny context

$$y_{i1} : (\Phi_{i1})\text{El}\left([x_{i1}[\Phi_{i1}]/z]M\right), \ldots, y_{ir_i} : (\Phi_{ir_i})\text{El}\left([x_{ir_i}[\Phi_{ir_i}]/z]M\right) \enspace .$$

- For $i = 1, \ldots, n$, if M and N are 0-ary objects and z a 0-ary variable then $\Phi_i^\natural[[z]M, [z]N]$ is the following sequence of terms:

$$\begin{array}{c}\langle \text{let } v[\Phi_{i1}] = [x_{i1}[\Phi_{i1}]/z]M : \mathrm{El}\,([x_{i1}[\Phi_{i1}]/z]N) \text{ in } v, \\ \ldots, \\ \text{let } v[\Phi_{ir_i}] = [x_{ir_i}[\Phi_{ir_i}]/z]M : \mathrm{El}\,([x_{ir_i}[\Phi_{ir_i}]/z]N) \text{ in } v\rangle\end{array}$$

- $\Psi_{\mathcal{S}}$ is the small context

$$\begin{array}{rcl} c & : & (I[\Delta])\mathbf{Type}, \\ f_1 & : & (\Theta_1, \Phi_1, \Phi_1^o[[z]c[z]])c[\kappa_1[\Delta, \Theta_1, \Phi_1]], \\ & \vdots & \\ f_n & : & (\Theta_n, \Phi_n, \Phi_n^o[[z]c[z]])c[\kappa_n[\Delta, \Theta_n, \Phi_n]] \end{array}$$

The inductive schema $\mathcal{S}$ is *valid* under the signature Σ iff I and each κ_i are not declared in Σ, and the following judgements are all derivable:

$$\begin{array}{rcll} \Delta & \vdash_\Sigma & \text{valid}, & \\ \Delta, \Theta_i & \vdash_\Sigma & \text{valid} & (1 \le i \le n), \\ \Delta, \Theta_i, \Phi_{ij} & \vdash_\Sigma & \text{valid} & (1 \le i \le n, 1 \le j \le r_i) \end{array}$$

The signature $\Sigma + \mathcal{S}$, the result of adding to Σ the inductive type defined by $\mathcal{S}$, then consists of Σ followed by the following declarations:

$$\begin{array}{rcl} I & : & (\Delta)\mathbf{Type}, \\ \kappa_1 & : & (\Delta, \Theta_1, \Phi_1)I[\Delta], \\ & \vdots & \\ \kappa_n & : & (\Delta, \Theta_n, \Phi_n)I[\Delta], \\ E_I & : & (\Delta, \Psi_{\mathcal{S}}, z : I[\Delta])c[z], \\ (\Delta, \Psi_{\mathcal{S}}, \Theta_1, \Phi_1)(E_I[\Delta, \Psi_{\mathcal{S}}, \kappa_1[\Delta, \Theta_1, \Phi_1]] & = & f_1[\Theta_1, \Phi_1, \Phi_1^\natural[[z]E_I[\Delta, \Psi_{\mathcal{S}}, z], [z]c[z]]] : c[\kappa_1[\Delta, \Theta_1, \Phi_1]]) \\ & \vdots & \\ (\Delta, \Psi_{\mathcal{S}}, \Theta_n, \Phi_n)(E_I[\Delta, \Psi_{\mathcal{S}}, \kappa_n[\Delta, \Theta_n, \Phi_n]] & = & f_n[\Theta_n, \Phi_n, \Phi_n^\natural[[z]E_I[\Delta, \Psi_{\mathcal{S}}, z], [z]c[z]]] : c[\kappa_n[\Delta, \Theta_n, \Phi_n]]) \end{array}$$

Examples

1. We spell out the details of one example. The schema $\mathcal{S}_{\text{List}}$ of *lists* is the schema

$$\begin{array}{rcl} \text{Inductive List} & : & (A : \mathbf{Type})\mathbf{Type} \\ \text{Constructors emp} & : & (A : \mathbf{Type})\mathrm{List}[A], \\ \text{cons} & : & (A : \mathbf{Type}, a : A, \mathrm{List}[A])\mathrm{List}[A] \end{array}$$

In this case,

$$\begin{array}{rcl} n = 2, i_1 & = & 0, i_2 = 1, \\ \Delta & \equiv & A : \mathbf{Type}, \\ \Theta_1 & \equiv & \langle\rangle, \\ \Theta_2 & \equiv & a : A, \\ \Phi_{21} & \equiv & \langle\rangle \end{array}$$

Adding this schema to a signature consists of adding the following declarations and equations:

$$(A : \mathbf{Type}, C : (\text{List}[A])\mathbf{Type}, f_1 : C[\text{emp}[A]], f_2 : (a : A, l : \text{List}[A], C[l])C[\text{cons}[A$$

$$(A : \mathbf{Type}, C : (\text{List}[A])\mathbf{Type}, f_1 : C[\text{emp}[A]], f_2 : (a : A, l : \text{List}[A], C[l])C[\text{cons}[A, a, l], a : A, l : \text{List}[A$$

2. The *schema for disjoint unions*, $\mathcal{S}_+$, is as follows:

$$\begin{array}{rcl} \text{Inductive}+ & : & (\mathbf{Type}, \mathbf{Type})\mathbf{Type}, \\ \text{Constructors}\iota_1 & : & (A : \mathbf{Type}, B : \mathbf{Type}, A) + [A, B], \\ \iota_2 & : & (A : \mathbf{Type}, B : \mathbf{Type}, B) + [A, B] \end{array}$$

3. The *schema for products*, $\mathcal{S}_\Pi$, is as follows:

$$\begin{array}{rcl} \text{Inductive}\Pi & : & (A : \mathbf{Type}, (A)\mathbf{Type})\mathbf{Type}, \\ \text{Constructors}\lambda & : & (A : \mathbf{Type}, B : (A)\mathbf{Type}, b : (a : A)B[a])\Pi[A, B] \end{array}$$

   We write $A \to B$ for $\Pi[A, \text{let } v[x : A] = B \text{ in } v]$ where $x \notin FV(B)$.

4. The *schema for the empty type*, $\mathcal{S}_\emptyset$, is;

$$\begin{array}{rcl} \text{Inductive}\emptyset & : & \mathbf{Type}, \\ \text{Constructors} & & \end{array}$$

   In a signature containing $\mathcal{S}_\Pi$ and $\mathcal{S}_\emptyset$, we write $\neg A$ for $A \to \emptyset$.

3 Some Type Theories

- *Martin-Löf Type Theory*, MLTT, is the result of adding every possible inductive schema to the empty signature.

- *Martin-Löf Type Theory with Classical Logic*, MLTTC, is the result of adding
$$EM : (A : \mathbf{Type})A + \neg A$$
to MLTT.

- UTT is the result of adding every possible inductive schema to the following signature:

$$(A : \mathbf{Type}, P : (A)\mathbf{Prop}, R : (\mathrm{Prf}[\forall[A, P]])\mathbf{Prop}, f : (g : (a : A)\mathrm{Prf}[P[a]])\mathrm{Prf}[R[\Lambda[A, P, g]]], g : (a : A)$$

  The above shall be referred to as the *basic signature* of UTT.

- *UTT with Classical Logic*, UTTC, is the result of adding
$$EM : (P : \mathbf{Prop})\mathrm{Prf}[P \vee \neg P]$$
to UTT, where we define in UTT:
$$\begin{aligned} P \to Q &\equiv \forall[\mathrm{Prf}[P], \mathrm{let}\ v[x : \mathrm{Prf}[P]] = Q \text{ in } v] \\ P \vee Q &\equiv \forall[\mathbf{Prop}, \mathrm{let}\ v[X : \mathbf{Prop}] = (P \to X) \to (Q \to X) \to X \text{ in } v] \\ \bot &\equiv \forall[\mathbf{Prop}, \mathrm{let}\ v[X : \mathbf{Prop}] = X \text{ in } v] \\ \neg P &\equiv P \to \bot \end{aligned}$$

4 Set Theoretic Interpretations

4.1 Preliminaries

A set theory T is some set of axioms in the language whose sole non-logical symbol is $\in$, such that the axioms of BS are theorems of T.

When interpreting a PAL$^+$theory in a set theory T, we are going to refer to entities, such as classes, superclasses and class terms, that are not objects of T. Such talk can always be decoded into statements about formulas of T. The following notes indicate how. For the full details, see the appendix.

- A *class* is specified by a formula $\phi[x]$ with a distinguished free variable x. We write this class as
$$\{x \mid \phi[x]\} \ .$$
If X is the class $\{x \mid \phi[x]\}$, then $a \in X$ is defined to be the formula
$$\phi[a] \ .$$

We adopt other turns of phrase, too: e.g. "X is a class of ordered pairs" is the formula

$$\forall x(\phi[x] \to \exists y \exists z\, x = \langle y, z \rangle)$$

We identify every set a with the class $\{x \mid x \in a\}$.

- A *class term* is a class F of ordered pairs such that

  $$\forall x, y, z(\langle x, y \rangle \in F \to \langle x, z \rangle \in F \to y = z) \ .$$

  If $\exists y \langle x, y \rangle \in F$, we say that F is *defined* at x, and denote the unique y such that $\langle x, y \rangle \in F$ by $F(x)$.

  The *domain* of a class term F is the class of all objects x at which F is defined. An *n-ary class term* is one whose domain consists solely of n-tuples.

  Given classes A and B, we write $F : A \to B$ for: "F is a class term with domain A such that

  $$(\forall x \in A) F(x) \in B \ .''$$

- A *superclass* is specified by a formula $\phi[X]$ in the language formed by extending the language of T with the unary predicate symbol X. We write this superclass as

  $$\{X \mid \phi[X]\} \ .$$

  If $\mathbf{A}$ is the superclass $\{X \mid \phi[X]\}$, and C is the class $\{x \mid \psi[x]\}$, then

  $$C \in \mathbf{A}$$

  is the formula formed by replacing each occurrence of Xt in $\phi[X]$ with $\psi[t]$, where t is any term of T.

  We adopt other turns of phrase, too: e.g. "$\mathbf{A}$ is a superclass of class terms" is the statement "For every class C, if $C \in \mathbf{A}$ then C is a class term". This is a statement about T; it cannot be represented as a formula of T.

4.2 Interpretation of a Signature

Interpretation An *interpretation* of a bijou PAL$^+$-signature Σ is a mapping $\mathcal{I}$ assigning an n-ary class term $\mathcal{I}(c)$ to each n-ary constant c.

An *valuation* is a function ρ assigning an n-ary function $\rho(x)$ to each n-ary variable x. We shall denote by e the valuation mapping every variable to the empty set, $\emptyset$.

Given an interpretation $\mathcal{I}$ and valuation ρ, we shall define:

- For some n-ary objects k, an n-ary class term $[\![k]\!]_\rho$.
- For some n-ary tiny kinds K, a set $(\!|K|\!)_\rho$ of n-ary functions.

- For some n-ary small kinds K, a class $\llparenthesis K \rrparenthesis_\rho$ of n-ary functions.
- For some n-ary bijou kinds K, a superclass $\llparenthesis K \rrparenthesis_\rho$ of n-ary class constructors.
- For some n-ary tiny contexts Δ, a set $\llparenthesis \Delta \rrparenthesis_\rho$ of n-tuples.
- For some n-ary small contexts Δ, a class $\llparenthesis \Delta \rrparenthesis_\rho$ of n-tuples.

In this definition and the sequel, we write "$E_1 \simeq E_2$" for "E_1 is defined if and only if E_2 is defined, in which case they have the same value".

$$
\begin{array}{rcl}
\llbracket c \rrbracket_\rho & = & \mathcal{I}(c) \\
\llbracket x \rrbracket_\rho & = & \rho(x) \\
\llbracket f(k_1, \ldots, k_n) \rrbracket_\rho & \simeq & \llbracket f \rrbracket_\rho \left(\llbracket k_1 \rrbracket_\rho, \ldots, \llbracket k_n \rrbracket_\rho \right) \\
\llbracket \text{let } v[\Delta] = t : T \text{ in } k \rrbracket_\rho & \simeq & \llbracket k \rrbracket_{\rho[v := (\lambda \vec{a} \in \llbracket \Delta \rrbracket_\rho) \llbracket t \rrbracket_{\rho[\Delta := \vec{a}]}]}
\end{array}
$$

$$
\begin{array}{rcl}
\llparenthesis \mathbf{Type} \rrparenthesis_\rho & = & \mathbf{V} \\
\llparenthesis \mathrm{El}\,(M) \rrparenthesis_\rho & \simeq & \llbracket M \rrbracket_\rho \\
\llparenthesis (\Delta) K \rrparenthesis_\rho & \simeq & \{ F \mid F \text{ is a class term with domain } \llbracket \Delta \rrbracket_\rho \text{ and } (\forall \vec{a} \in \llbracket \Delta \rrbracket_\rho) F(\vec{a}) \in \llparenthesis K \rrparenthesis_{\rho[\Delta := \vec{a}]}
\end{array}
$$

$$
\begin{array}{rcl}
\llbracket \langle\rangle \rrbracket_\rho & = & \{ \langle\rangle \} \\
\llbracket \Delta, x : K \rrbracket_\rho & \simeq & \{ \vec{a} :: b \mid \vec{a} \in \llbracket \Delta \rrbracket_\rho, b \in \llparenthesis K \rrparenthesis_{\rho[\Delta := \vec{a}]} \} \\
\llbracket \Delta, v[\Delta'] = t : T \rrbracket_\rho & \simeq & \{ \vec{a} :: b \mid \vec{a} \in \llbracket \Delta \rrbracket_\rho, b = (\lambda \vec{c} \in \llbracket \Delta' \rrbracket_{\rho[\Delta := \vec{a}]}) \llbracket t \rrbracket_{\rho[\Delta := \vec{a}, \Delta' := \vec{c}]} \}
\end{array}
$$

Lemma 1 *For any expression E (which may be an object, kind or context), if*

$$(\forall x \in FV(E)) \rho(x) = \sigma(x)$$

then

$$\llbracket E \rrbracket_\rho \simeq \llbracket E \rrbracket_\sigma \quad .$$

Proof A simple induction on E. **QED**

Lemma 2 *For any expression E, n-ary object k and n-ary variable x,*

$$\llbracket [k/x] E \rrbracket_\rho \mathrel{<\!\!\!\!\simeq} \llbracket E \rrbracket_{\rho[x := \llbracket k \rrbracket_\rho]} \quad .$$

We have $\llbracket [k/x] E \rrbracket_\rho \simeq \llbracket E \rrbracket_{\rho[x := \llbracket k \rrbracket_\rho]}$ provided $x \in FV(E)$.

Proof A simple induction on E.

$$
\begin{array}{rcl}
[\![[k/x]c]\!]_\rho & \simeq & [\![c]\!]_\rho \\
& = & \mathcal{I}(c) \\
& \mathrel{<\!\!\!\simeq} & [\![c]\!]_{\rho[x:=[\![k]\!]_\rho]} \\
[\![[k/x]x]\!]_\rho & \simeq & [\![k]\!]_\rho \\
& \simeq & [\![x]\!]_{\rho[x:=[\![k]\!]_\rho]}
\end{array}
$$

For y a variable other than x:

$$
\begin{array}{rcl}
[\![[k/x]y]\!]_\rho & \simeq & [\![y]\!]_\rho \\
& = & \rho(y) \\
& \mathrel{<\!\!\!\simeq} & \rho[x := [\![k]\!]_\rho](y) \\
& \simeq & [\![y]\!]_{\rho[x:=[\![k]\!]_\rho]}
\end{array}
$$

The other cases are straightforward. **QED**

A valuation ρ *satisfies* a small context Γ iff:

- For every declaration $x : K$ in Γ, $\rho(x) \in (\!|K|\!)_\rho$.
- For every definition $v[\Delta] = t : T$ in Γ,

$$\rho(v) = (\lambda \vec{a} \in [\![\Delta]\!]_\rho)\, [\![t]\!]_{\rho[\Delta:=\vec{a}]} \quad .$$

An interpretation is *sound* iff:

- For every declaration $c : K$ in Σ, $\rho(c) \in (\!|K|\!)_e$.
- For every equation $(\Delta)(k = k' : K)$ in Σ, and every valuation ρ that satisfies Δ,

$$[\![k]\!]_\rho = [\![k']\!]_\rho \quad .$$

Theorem 2 (Soundness Theorem) *Let $\mathcal{I}$ be a sound interpretation of the valid bijou signature Σ, and ρ a valuation that satisfies Γ.*

1. *If* $\Gamma, \Delta \vdash_\Sigma \mathrm{valid}$, *then* $[\![\Delta]\!]_\rho \downarrow$.
2. *If* $\Gamma \vdash_\Sigma K$, *then* $[\![K]\!]_\rho \downarrow$.
3. *If* $\Gamma \vdash_\Sigma k : K$, *then* $[\![k]\!]_\rho \in [\![K]\!]_\rho$.
4. *If* $\Gamma \vdash_\Sigma K = K'$, *then* $[\![K]\!]_\rho = [\![K']\!]_\rho$.
5. *If* $\Gamma \vdash_\Sigma k = k' : K$, *then* $[\![k]\!]_\rho = [\![k']\!]_\rho \in [\![K']\!]_\rho$.

Proof The proof is by induction on the derivation, which we assume involves no large contexts. The non-trivial cases are as follows.

Contexts and assumptions

- $$\frac{\Gamma, \Delta \vdash_\Sigma K \text{ kind}}{\Gamma, \Delta, x : K \vdash_\Sigma \text{valid}}$$

  By i.h., $[\![\Delta]\!]_\rho \downarrow$. Further, for any $\vec{a} \in [\![\Delta]\!]_\rho$, we have $\rho[\Delta := \vec{a}]$ satisfies Γ, Δ (formally, this needs to be proven by induction on Δ), and so $(\!|K|\!)_{\rho[\Delta:=\vec{a}]} \downarrow$. Therefore, $[\![\Delta, x : K]\!]_\rho \downarrow$, as this set is

  $$\{\vec{a} :: b \mid \vec{a} \in [\![\Delta]\!]_\rho , b \in (\!|K|\!)_{\rho[\Delta:=\vec{a}]}\} \ .$$

- $$\frac{\Gamma \vdash_\Sigma \text{valid}}{\Gamma \vdash_\Sigma c : K} (c : K \in \Sigma)$$

  $$\begin{array}{rcll} [\![c]\!]_\rho & = & \mathcal{I}(c) & \\ & \in & [\![K]\!]_e & (\mathcal{I} \text{ is sound}) \\ & \simeq & [\![K]\!]_\rho & (\text{Lemma 1}) \end{array}$$

- $$\frac{\Gamma \vdash_\Sigma \vec{k} : \Delta}{\Gamma \vdash_\Sigma [\vec{k}/\Delta]k = [\vec{k}/\Delta]k' : [\vec{k}/\Delta]K} (\Delta)(k = k' : K) \in \Sigma$$

  Let

  $$\begin{array}{rcl} \Sigma & \equiv & \Sigma_0, (\Delta)(k = k' : K), \Sigma_1 \\ \Delta & \equiv & x_1 : K_1, \ldots, x_n : K_n \\ \vec{k} & \equiv & k_1, \ldots, k_n \end{array}$$

  We claim that $\rho[\Delta := \left[\!\left[\vec{k}\right]\!\right]_\rho]$ satisfies Δ. First note that the i.h. gives $[\![k_i]\!]_\rho \downarrow$ for each i, so the valuation is well defined. We must show that, for $i = 1, \ldots, n$,

  $$[\![k_i]\!]_\rho \in [\![K_i]\!]_{\rho[\Delta:=[\![\vec{k}]\!]_\rho]}$$

  The proof shall be by induction on i.

  Assume that, for $j < i$, we have

  $$[\![k_j]\!]_{\rho[\Delta:=[\![\vec{k}]\!]_\rho]} \in [\![K_j]\!]_{\rho[\Delta:=[\![\vec{k}]\!]_\rho]}$$

  It follows that $\rho[\Delta := \left[\!\left[\vec{k}\right]\!\right]_\rho]$ satisfies $x_1 : K_1, \ldots, x_{i-1} : K_{i-1}$. Now, by Context Validity, the premises must contain a subderivation of

  $$x_1 : K_1, \ldots, x_{i-1} : K_{i-1} \vdash_{\Sigma_0} K_i \text{ kind} \ .$$

The i.h. therefore gives

$$\llbracket K_i \rrbracket_{\rho[\Delta := \llbracket \vec{k} \rrbracket_\rho]} \downarrow$$

We therefore have

$$\begin{aligned} \llbracket k_i \rrbracket_\rho &\in \llbracket [k_{i-i}/x_{i-1}] \cdots [k_1/x_1] K_i \rrbracket_\rho && \text{(i.h.)} \\ &= \llbracket K_i \rrbracket_{\rho[x_1 := \llbracket k_1 \rrbracket_\rho, \ldots, x_{i-1} := \llbracket k_{i-1} \rrbracket_\rho]} && \text{(Lemma 2)} \\ &= \llbracket K_i \rrbracket_{\rho[\Delta := \llbracket \vec{k} \rrbracket_\rho]]} && \text{(Lemma 1)} \end{aligned}$$

- $$\frac{\Gamma \vdash_\Sigma \text{valid}}{\Gamma \vdash_\Sigma x : K} \, (x : K \in \Gamma)$$

  From the assumption that ρ satisfies Γ, we have $\rho(x) \in \llbracket K \rrbracket_\rho$.

  `INCOMPLETE`

We have shown by induction that $\rho[\Delta := \left\llbracket \vec{k} \right\rrbracket_\rho]$ satisfies Δ. Hence,

$$\begin{aligned} \left\llbracket [\vec{k}/\Delta] k \right\rrbracket_\rho &\simeq \llbracket k \rrbracket_{\rho[\Delta := \llbracket \vec{k} \rrbracket_\rho]} && \text{(Lemma 2)} \\ &= \llbracket k' \rrbracket_{\rho[\Delta := \llbracket \vec{k} \rrbracket_\rho]} && (\mathcal{I} \text{ is sound}) \\ &\simeq \left\llbracket [\vec{k}/\Delta] k' \right\rrbracket_\rho && \text{(Lemma 2)} \end{aligned}$$

Formation rules for parametric kinds

- $$\frac{\Gamma, \Delta \vdash_\Sigma T \text{ kind}}{\Gamma \vdash_\Sigma (\Delta) T \text{ kind}}$$

  By Context Validity, the premise contains a subderivation of $\Gamma, \Delta \vdash_\Sigma$ valid, and so $\llbracket \Delta \rrbracket_\rho \downarrow$. Further, for $\vec{a} \in \llbracket \Delta \rrbracket_\rho$, we have that $\rho[\Delta := \vec{a}]$ satisfies Γ, Δ, and so

  $$(\!| T |\!)_{\rho[\Delta := \vec{a}]} \downarrow \quad .$$

  Therefore, $(\!| (\Delta) T |\!)_\rho \downarrow$.

Introduction rule for global definitions

Typing and equality rules for global definitions

$$\frac{\Gamma \vdash_\Sigma \text{valid}}{\Gamma \vdash_\Sigma v = \text{let } v[\Delta] = t : T \text{ in } v : (\Delta) T} \, (v[\Delta] = t : T \in \Gamma)$$

Let $\Gamma \equiv \Gamma_0, v[\Delta] = t : T, \Gamma_1$. From the hypothesis that ρ satisfies Γ, we have

$$\rho(v) = (\lambda \vec{a} \in \llbracket \Delta \rrbracket_\rho) \, \llbracket t \rrbracket_{\rho[\Delta := \vec{a}]}$$

Now,

$$
\begin{aligned}
[\![\text{let } v[\Delta] = t : T \text{ in } v]\!]_\rho &\simeq [\![v]\!]_{\rho[v:=(\lambda\vec{a}\in[\![\Delta]\!]_\rho)[\![t]\!]_{\rho[\Delta:=\vec{a}]}]} \\
&\simeq (\lambda\vec{a} \in [\![\Delta]\!]_\rho)\,[\![t]\!]_{\rho[\Delta:=\vec{a}]} \\
&= \rho(v) \\
&= [\![v]\!]_\rho
\end{aligned}
$$

Further, by Context Validity, the premise contains a subderivation of

$$\Gamma, \Delta \vdash_\Sigma t : T \ \ .$$

The i.h. therefore gives that, for $\vec{a} \in [\![\Delta]\!]_\rho$,

$$
\begin{aligned}
[\![t]\!]_{\rho[\Delta:=\vec{a}]} &\in (\!|T|\!)_{\rho[\Delta:=\vec{a}]} \\
\therefore (\lambda\vec{a} \in [\![\Delta]\!]_\rho)\,[\![t]\!]_{\rho[\Delta:=\vec{a}]} &\in \{F \mid F \text{ is a class term with domain } [\![\Delta]\!]_\rho \text{ and } (\forall\vec{a} \in [\![\Delta]\!]_\rho) F(\vec{a}) \in (\!|T|\!)_{\rho[\Delta:=\vec{a}]}\} \\
&= (\!|(\Delta)T|\!)_\rho
\end{aligned}
$$

let **-introduction rules**

- $$\frac{\Gamma, v[\Delta] = t : T \vdash_\Sigma K \text{ kind}}{\Gamma \vdash_\Sigma \text{let } v[\Delta] = t : T \text{ in } K \text{ kind}}$$

  $$[\![\text{let } v[\Delta] = t : T \text{ in } K]\!]_\rho \simeq [\![K]\!]_{\rho[v:=(\lambda\vec{a}\in[\![\Delta]\!])[\![t]\!]_{\rho[\Delta:=\vec{a}]}]}$$

  By i.h., it is therefore sufficient to show that $\rho[v := (\lambda\vec{a} \in [\![\Delta]\!])\,[\![t]\!]_{\rho[\Delta:=\vec{a}]}]$ satisfies $\Gamma, v[\Delta] = t : T$. By the definition of satisfaction, all we have to do is show that $\rho[v := (\lambda\vec{a} \in [\![\Delta]\!])\,[\![t]\!]_{\rho[\Delta:=\vec{a}]}]$ exists. But this follows from the fact that $[\![v[\Delta] = t : T]\!]_\rho \downarrow$, given to us by applying the i.h. to the subderivation of

  $$\Gamma, v[\Delta] = t : T \vdash_\Sigma \text{valid}$$

  that must exist by Context Validity.

Equality rules for let **-expressions** We give here only the relevant calculations:

$$
\begin{aligned}
\left[\!\!\left[(\text{let } v[\Delta] = t : T \text{ in } v)[\vec{k}]\right]\!\!\right]_\rho &\simeq [\![\text{let } v[\Delta] = t : T \text{ in } v]\!]_\rho \left(\left[\!\!\left[\vec{k}\right]\!\!\right]_\rho\right) \\
&\simeq [\![v]\!]_{\rho[v:=(\lambda\vec{a}\in[\![\Delta]\!]_\rho)[\![t]\!]_{\rho[\Delta:=\vec{a}]}]} \left(\left[\!\!\left[\vec{k}\right]\!\!\right]_\rho\right) \\
&\simeq ((\lambda\vec{a} \in [\![\Delta]\!]_\rho)\, [\![t]\!]_{\rho[\Delta:=\vec{a}]})\left(\left[\!\!\left[\vec{k}\right]\!\!\right]_\rho\right) \\
&\overset{\sim}{\Rightarrow} [\![t]\!]_{\rho[\Delta:=[\![\vec{k}]\!]_\rho]} \\
&\overset{\sim}{\Rightarrow} \left[\!\!\left[[\vec{k}/\Delta]t\right]\!\!\right]_\rho \\
[\![\text{let } v[\Delta] = g[\Delta] \text{ in } k]\!]_\rho &\simeq [\![k]\!]_{\rho[v:=(\lambda\vec{a}\in[\![\Delta]\!]_\rho)[\![g[\Delta]]\!]_{\rho[\Delta:=\vec{a}]}]} \\
&\simeq [\![k]\!]_{\rho[v:=(\lambda\vec{a}\in[\![\Delta]\!]_\rho)[\![g]\!]_{\rho[\Delta:=\vec{a}]}([\![\Delta]\!]_{\rho[\Delta:=\vec{a}]})]} \\
&\simeq [\![k]\!]_{\rho[v:=(\lambda\vec{a}\in[\![\Delta]\!]_\rho)[\![g]\!]_\rho(\vec{a})]} \qquad \text{(Lemma 1)} \\
&\overset{\sim}{\Leftarrow} [\![k]\!]_{\rho[v:=[\![g]\!]_\rho]} \qquad ([\![g]\!]_\rho \in (\!|(\Delta)T|\!)_\rho) \\
&\overset{\sim}{\Rightarrow} [\![[g/v]k]\!]_\rho \qquad \text{(Lemma 2)} \\
[\![[\text{let } v[\Delta] = t : T \text{ in } v/v]k]\!]_\rho &\overset{\sim}{\Leftarrow} [\![k]\!]_{\rho[v:=[\![\text{let } v[\Delta]=t:T \text{ in } v]\!]_\rho]} \\
&\simeq [\![k]\!]_{\rho[v:=[\![v]\!]_{\rho[v:=(\lambda\vec{a}\in[\![\Delta]\!]_\rho)[\![t]\!]_{\rho[\Delta:=\vec{a}]}]}]} \\
&\simeq [\![k]\!]_{\rho[v:=(\lambda\vec{a}\in[\![\Delta]\!]_\rho)[\![t]\!]_{\rho[\Delta:=\vec{a}]}]} \\
&\simeq [\![\text{let } v[\Delta] = t : T \text{ in } k]\!]_\rho
\end{aligned}
$$

QED

Theorem 3 *If there exists a sound interpretation of the valid bijou PAL$^+$ signature Σ in some set theory T, then*

$$T \vdash \Sigma \textit{ is consistent} \ .$$

Proof In T, we reason as follows:

Suppose there were a term M such that

$$X : \textbf{Type} \vdash_\Sigma M : X \ .$$

The valuation e, that maps every variable to the empty set, satisfies $X : \textbf{Type}$, so, by the Soundness Theorem,

$$[\![M]\!]_e \in [\![X]\!]_e = \emptyset \ .$$

This is a contradiction. **QED**

5 An Interpretation for UTT

5.1 Interpretations for Inductive Types

Let Σ be a valid bijou PAL$^+$signature, and let $\mathcal{S}$ be the following valid inductive schema in Σ:

$$\begin{aligned} &\text{Inductive} I \\ &\text{Constructors} \kappa_1 : [\Delta_1, \Theta_1] I, \\ &\vdots \\ &\kappa_k : [\Delta_k, \Theta_k] I \end{aligned}$$

Let $\mathcal{I}$ be a sound interpretation for Σ. We can extend $\mathcal{I}$ to a sound interpretation for $\Sigma + \mathcal{S}$ as follows:

Define a sequence of sets V_n for $n \in \mathbb{N}$ as follows:

$$V_0 = \emptyset$$

Given V_n, we define

$$\begin{aligned} V_{n+1} = \{\langle i, \vec{a}, \vec{b}\rangle \quad | \quad & 1 \leq i \leq k, \\ & \vec{a} \in [\![\Delta_i]\!], \\ & \vec{b} \in [\![\Theta_i]\!]_{\Delta_i := \vec{a}}\} \end{aligned}$$

where the interpretation used to define $[\![\Theta_i]\!]$ is $\mathcal{I}$ with I mapped to V_n.

Lemma 3 *For $m \leq n$, we have $V_m \subseteq V_n$.*

Let

$$V_\omega = \bigcup_{n=0}^{\infty} V_n \quad .$$

We define, for each $n \in \mathbb{N}$, the $k+1$-ary class term E_n as follows:

$$E_0 = \emptyset$$

Given E_n, E_{n+1} is defined by

$$E_{n+1}(f_1, \ldots, f_k, \langle i, \vec{a}, \vec{b}\rangle) = f_i(\vec{a}, \vec{b}, \vec{b}^\natural)$$

where

$$\begin{aligned} \langle b_1, \ldots, b_r\rangle^\natural \quad = \quad & \langle (\lambda x \in \operatorname{dom} b_1) E_n(f_1, \ldots, f_k, b_1(x)), \\ & \ldots \\ & (\lambda \vec{x} \in \operatorname{dom} b_r) E_n(f_1, \ldots, f_k, b_r(x))\rangle \end{aligned}$$

Thus, for E_{n+1} to be defined at a given $k+1$-tuple, its $k+1$st argument must be a member of V_{n+1}.

Lemma 4 *For $m \leq n$, we have $E_m \subseteq E_n$. That is, the domain of E_m is a subclass of the domain on E_n, and the two class terms agree on* $\operatorname{dom} E_m$.

Proof The proof is by induction on m.

$$E_0 = \emptyset \subseteq E_n$$

QED

It follows that $\bigcup_{n=0}^{\infty} E_n$ is a class term. We denote this class term by E.

We now extend $\mathcal{I}$ as follows:

- I is interpreted as V_ω.
- κ_i is interpreted as the class term which, given a tuple $\vec{a}$ of the same length as Δ_i, and a tuple $\vec{b}$ of the same length as Θ_i, returns the triple

$$\langle i, \vec{a}, \vec{b} \rangle \ .$$

- E_I is interpreted as the class term E.

Theorem 4 *This extended interpretation of* $\Sigma + \mathcal{S}$ *is sound.*

Examples

1. The empty type $\emptyset$, whose schema is simply

$$\text{Inductive } \emptyset$$

   In this case, each set V_n is the empty set, so V_ω is the empty set, and E the nowhere-defined function.

2. The disjoint union of two types:

$$\begin{array}{l} (A, B : \mathit{Type})\text{Inductive}+ \\ \qquad \text{Constructors}\iota_1 : (A)+ \\ \qquad \qquad \iota_2 : (B)+ \end{array}$$

   In this case, for every $n > 0$,

$$V_n = \{\langle 1, a \rangle \mid a \in [\![A]\!]\} \cup \{\langle 2, b \rangle \mid b \in [\![B]\!]\}$$

   This is the set we shall write as $A \uplus B$.

   The eliminator E is defined by

$$\begin{array}{rcll} E(f, g, \langle 1, a \rangle) & \simeq & f(a) & (a \in [\![A]\!]) \\ E(f, g, \langle 2, b \rangle) & \simeq & g(b) & (b \in [\![B]\!]) \end{array}$$

3. Non-dependent function space:

$$\begin{array}{l} (A, B : \mathit{Type})\text{Inductive} \rightarrow \\ \qquad \text{Constructors}\lambda : ((A)B) \rightarrow \end{array}$$

   In this case, for every $n > 0$,

$$V_n = \{\langle 1, f \rangle \mid f : [\![A]\!] \rightarrow [\![B]\!]\}$$

   The eliminator E is defined by

$$E(F, \langle 1, f \rangle) \simeq F(f) \qquad (f : [\![A]\!] \rightarrow [\![B]\!])$$

5.2 Consistent Theories of Inductive Types

Theorem 5 *Martin-Löf Type Theory is consistent.*

Proof The empty interpretation of the empty signature is trivially sound. The result now follows by Theorem 4. **QED**

Theorem 6 *There is a sound interpretation of Martin-Löf Type Theory with Classical Logic in $ZF + V = L$. Therefore, if ZF is consistent, then Martin-Löf Type Theory with Classical Logic is consistent.*

Proof It is sufficient to find a class term EM that, given a set a, returns a set

$$\begin{array}{rcl} EM(a) & \in & [\![X + (X \to \emptyset)]\!]_{X:=a} \\ & = & a + \{\langle 1, f \rangle \mid f : a \to \emptyset\} \end{array}$$

Let $<$ be a well-ordering of L. We define the class term by

$$EM(a) = \begin{cases} \langle 1, x \rangle & \text{if } a \neq \emptyset \\ \langle 2, \langle 1, \emptyset \rangle \rangle & \text{if } a = \emptyset \end{cases}$$

where x is the $<$-least member of a. **QED**

Theorem 7 *There is a sound interpretation of UTT with excluded middle.*

Proof We begin by interpreting the basic signature as follows:
Choose some object $*$.
Prop is interpreted as $\{0, 1\}$.
Prf is interpreted as the function PRF on $\{0, 1\}$, where

$$\begin{array}{rcl} PRF(0) & = & \emptyset \\ PRF(1) & = & \{*\} \end{array}$$

$\forall$ is interpreted as the class term ALL that, given a set a and a function $f : a \to \{0, 1\}$, returns:

$$\begin{array}{r} \text{if } (\forall x \in a) f(x) = 1 \\ 0 \text{if } (\exists x \in a) f(x) = 0 \end{array}$$

Λ and $E_\forall$ are each interpreted as the class term that maps every set to $*$.

The only difficult point in verifying that this interpretation is sound is checking the types of Λ and $E_\forall$. This is done as follows:

Suppose $\langle A, P, f \rangle$ satisfy the domain of Λ, that is:

- A is a set.
- $P : A \to \{0, 1\}$.

- $f \in \Pi_{a \in A} PRF(P(a))$

Then, for every $a \in A$, we have $PRF(P(a)) \neq \emptyset$ (as $f(a) \in PRF(P(a))$), hence $P(a) = 1$. Thus, $ALL(A, P) = 1$, and so

$$\Lambda(A, P, f) = * \in \{*\} = PRF(ALL(A, P))$$

Suppose $\langle A, P, R, f, z \rangle$ satisfy the domain of $E_\forall$; that is,

- A is a set.
- $P : A \to \{0, 1\}$
- $R : PRF(ALL(A, P)) \to \{0, 1\}$
- $f \in \Pi_{g \in \Pi_{a \in A} PRF(P(a))} PRF(R(*))$
- $z \in PRF(ALL(A, P))$

Then $PRF(ALL(A, P)) \neq \emptyset$, so $ALL(A, P) = 1$; therefore, $P(a) = 1$ for each $a \in A$. Thus, the function g defined on A by

$$g(a) = * \qquad (a \in A)$$

is a member of the set $\Pi_{a \in A} PRF(P(a))$. Hence,

$$f(g) \in PRF(R(*))$$

and so $PRF(R(*)) \neq \emptyset$; therefore, $R(*) = 1$.

We have $PRF(ALL(A, P)) = \{*\}$, so $z = *$. Therefore,

$$E_\forall(A, P, R, f, z) = * \in \{*\} = PRF(R(*)) = PRF(R(z))$$

as required.

The inductive types are interpreted as in Theorem 4.

Finally, EM is interpreted as the function from $\{0, 1\}$ to $\{0, 1\}$, that maps both 0 and 1 to 1. It is easily verified that $[\![P \vee \neg P]\!]_\rho = 1$ for all ρ satisfying $P : \mathbf{Prop}$ (i.e. such that $\rho(P) \in \{0, 1\}$). **QED**

Note that this interpretation is also sound if we add to UTTC the equation for *proof irrelevance*:

$$(P : \mathbf{Prop}, p : \mathrm{Prf}[P], q : \mathrm{Prf}[P])(p = q : \mathrm{Prf}[P])$$

References

[1] Zhaohui Luo. *Computation and Reasoning: A Type Theory for Computer Science*. Number 11 in International Series of Monographs on Computer Science. Oxford University Press, 1994.