Control Operators and Natural Numbers are Degenerate

Robin Adams

September 20, 2010

We work in a type theory that has both μ-operators, and a type of natural numbers. Suppose we have a term $iszero$ such that

$$\begin{array}{rcl} iszero 0 & \to & 1 \\ iszero(Sn) & \to & 0 \end{array}$$

In addition to the above, our type theory has the following reduction rules:

$$S(\mu\alpha.n) \to \mu\alpha.[S[]/\alpha]n \quad (1)$$
$$iszero(\mu\alpha.n) \to \mu\alpha.[iszero[]/\alpha]n \quad (2)$$
$$[\alpha]\mu\beta.n \to [\alpha/\beta]n \quad (3)$$

where, for $C[]$ a context, the term

$$[C[]/\alpha]n$$

is defined to be the result of replacing every subterm of the form $[\alpha]M$ with $[\alpha]C[M]$.

(There may be other reduction rules, too; we do not care.)

Consider the term

$$M \equiv \mu\alpha.[\alpha]S(\mu\beta.[\alpha]0)$$

Theorem 1 *The term $iszero M$ is not confluent.*

Proof Here is one reduction sequence for $iszero M$:

$$\begin{array}{c} iszero M \\ \mu\alpha.[\alpha]iszero(S(\mu\beta.[\alpha]iszero(0))) \\ \mu\alpha.[\alpha]0 \end{array}$$

Here is another

$$\begin{gathered} iszero M \\ \mu\alpha.[\alpha] iszero(S(\mu\beta.[\alpha] iszero(0))) \\ \mu\alpha.[\alpha] iszero(\mu\beta.[\alpha] iszero(0)) \\ \mu\alpha.[\alpha]\mu\beta.[\alpha] iszero(0) \\ \mu\alpha.[\alpha] iszero(0) \\ \mu\alpha.[\alpha]1 \end{gathered}$$

Corollary 1 *If we add to the type theory the following rule (sometimes called the η-rule for control operators):*

$$\mu\alpha.[\alpha]M \to M$$

then we have $0 \simeq 1$*.*

Note that, in the proof above, the first reduction sequence corresponds to a lazy or call-by-name strategy, while the second corresponds to a call-by-value strategy.

Thus, if we want to have both control operators and inductive types, we must choose what to give up:

1. We may choose to restrict reduction to call-by-value only, or to call-by-name only.

2. We may choose to remove rule (1) or (2): control operators only reduce under constructors, or only under eliminators, or neither. (The option 'neither' was explored by $\lambda\mu$PRL; the other two have not been explored, to the best of my knowledge.)