Categorical Semantics for Logic-Enriched Type Theories

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The Curry-Howard Isomorphism

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Trivial Fact

It is possible to write a linear syntax for natural deduction proofs, and then write $\Gamma \vdash P : \phi$ for 'P is a proof of ϕ (that depends on the free variables and hypotheses Γ)'

The Curry-Howard Isomorphism

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Non-trivial Fact

When we do so:

- the rules for conjunction are the rules for product type;
- the rules for impliciation are the rules for non-dependent function type;
- the rules for universal quantification are (almost) the rules for dependent function type;
- the rules for classical logic are the rules for control operators (usually);
- the rules for modal logic are the rules for metavariables;
- etc.

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In this talk, 'Curry-Howard' shall mean the second.

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5. We should instead turn Curry-Howard into a mathematical object.

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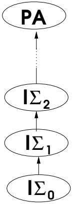
We have:

- systems of first-order arithmetic



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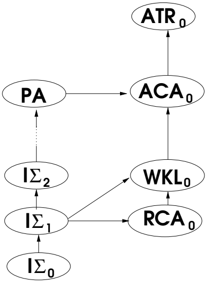

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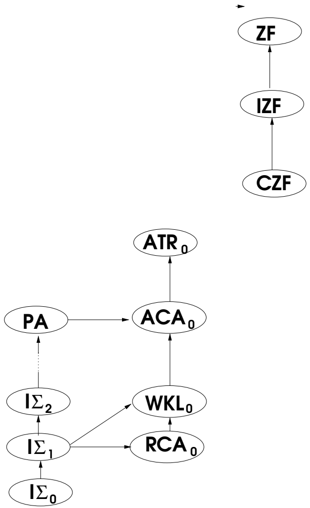

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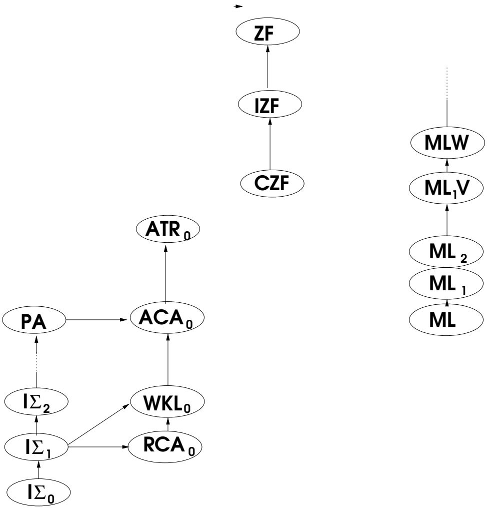

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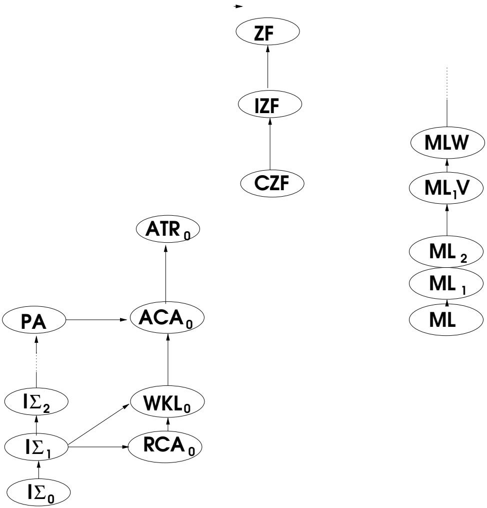

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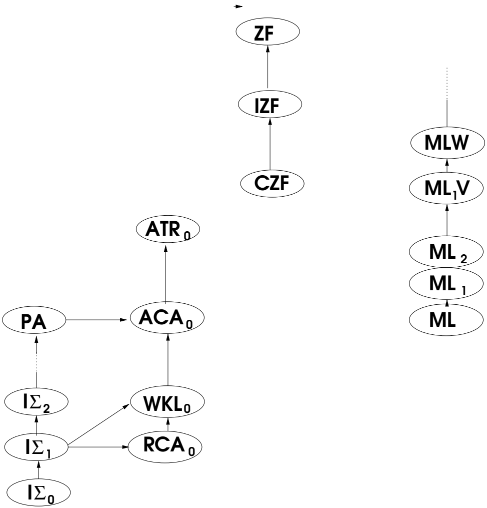

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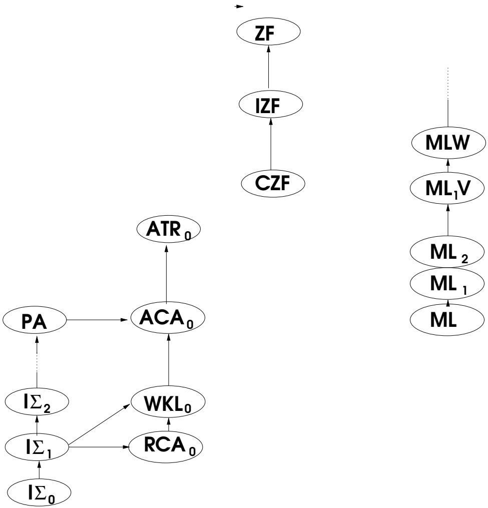

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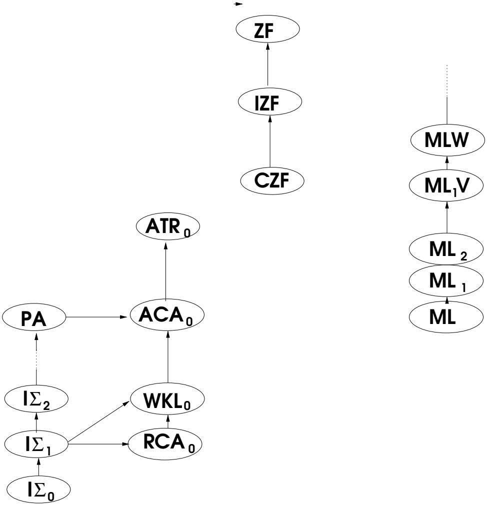
  - If propositions really were types, it should be easy.



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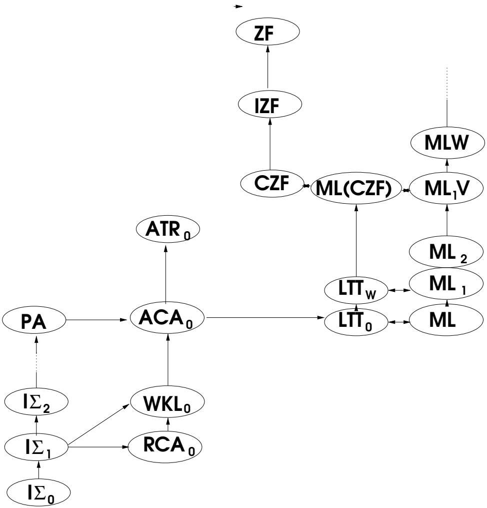

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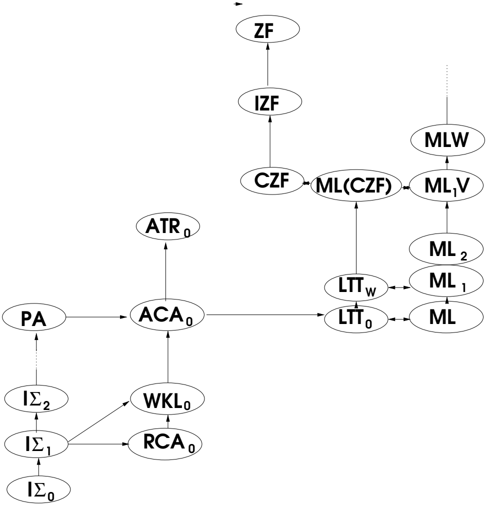

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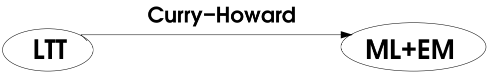

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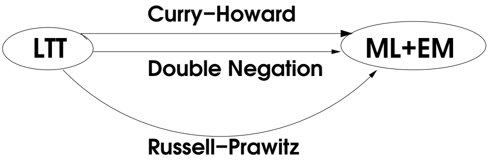
- Syntactic translations are possible.
- Curry-Howard becomes just one of a family.



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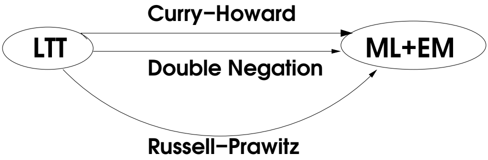

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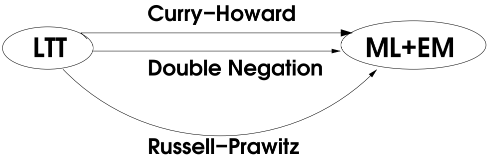
- We need a semantics for LTTs.



1. Logic-Enriched Type Theories
   - Syntax

2. Categorical Semantics
   - Introduction to Categorical Semantics
   - Categorical Semantics for Logic-Enriched Type Theories
   - Soundness and Completeness Theorems

3. Applications
   - Conservativity of ACA_0 over PA
   - Bounded Quantification

Syntax of an LTT

LTT_0 is a system with:
Judgement forms:

$$\Gamma \vdash A \ \mathrm{Type} \quad \Gamma \vdash M : A$$
$$\Gamma \vdash \phi \ \mathrm{Prop} \quad \Gamma \vdash P : \phi$$

and associated equality judgements.

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LTT_0 is a system with:

- arrow types $A \to B$
  with objects $\lambda x : A.M$

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  with objects $\hat{\mathbb{N}}$ and $M \hat{\times} M$

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  with propositions $M =_A M$, $\neg\phi$, $\phi \wedge \psi$, $\forall x : A.\phi$, ...

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- a *propositional universe* $\mathrm{prop}\ \mathrm{Type}$

We write $\mathrm{Set}\,(A)$ for $A \to \mathrm{prop}$.

The Propositional Universe

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- We can only eliminate $\mathbb{N}$ over small types.
- We can only use proof by induction with small propositions.
- Adding a new type or connective is conservative. Adding it to the universes is not.

Categorical Semantics

We can give semantics to a type theory in a variety of ways:

Map types to sets, ω-sets, PERs, sheaves, domains, ...

To save repeating work, we:

- define the properties a category must have for us to build a semantics from its objects;
- give semantics to the theory in an *arbitrary* category with those properties.

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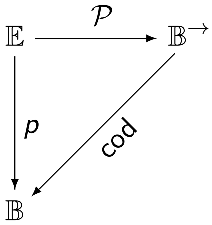

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such that

- $p = \mathrm{cod} \circ \mathcal{P}$ is a fibration
- $\mathbb{B}$ has a terminal object

$$\begin{array}{ccc} \mathbb{E} & \xrightarrow{\mathcal{P}} & \mathbb{B}^{\to} \\ {\scriptstyle p}\downarrow & \swarrow_{\mathrm{cod}} & \\ \mathbb{B} & & \end{array}$$

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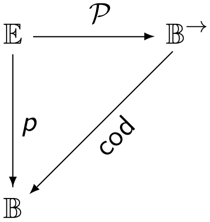

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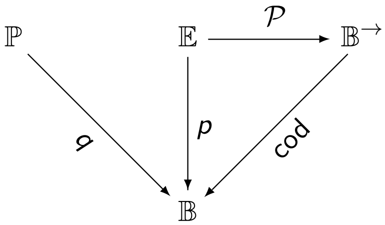

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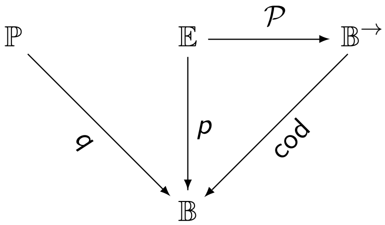

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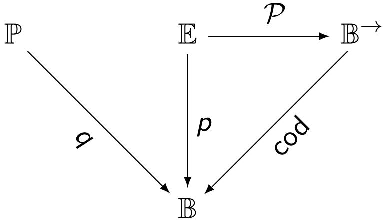
- for every object $\Gamma \vdash A\ \mathrm{Type}$ in $\mathbb{E}$, a right adjoint $\pi^* \dashv \forall$ and a left adjoint $\exists \dashv \pi^*$
- such that $\mathbb{P}$ is a locally Cartesian closed category.



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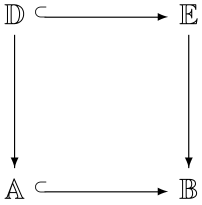
- a substructure (intended to represent the small types and small contexts)



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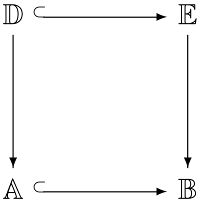

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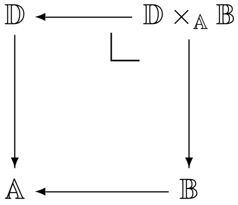

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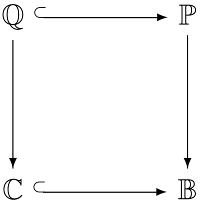

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$$\begin{array}{ccc} \mathbb{Q} & \hookrightarrow & \mathbb{P} \\ \downarrow & & \downarrow \\ \mathbb{C} & \hookrightarrow & \mathbb{B} \end{array}$$

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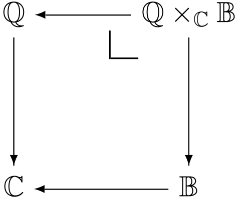

Categorical Semantics for Universes

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We require $\top \to \langle x : \mathbb{N} \rangle \to \langle x : \mathbb{N} \rangle$ to be a weak fibred natural number object in both of these right-hand-sides.

Interpretation

Given an $\mathrm{LTT_W}$-category $\mathcal{C}$, define:

- for every valid context Γ, an object $[\![\Gamma]\!]$ of $\mathbb{B}$;
- for every type A such that $\Gamma \vdash A \ \mathrm{Type}$, an object $[\![\Gamma \vdash A]\!]$ of $\mathbb{E}$ such that $p\,[\![\Gamma \vdash A]\!] = [\![\Gamma]\!]$
- for every term M such that $\Gamma \vdash M : A$, an arrow $[\![\Gamma \vdash M]\!] : [\![\Gamma]\!] \to \mathrm{dom}\,\mathcal{P}\,[\![\Gamma \vdash A]\!]$
- for every proposition ϕ such that $\Gamma \vdash \phi \mathrm{prop}$, an object $[\![\Gamma \vdash \phi]\!]$ of $\mathbb{P}$ over $[\![\Gamma]\!]$

Soundness Theorem

Theorem

Every judgement is true in any $\mathrm{LTT_W}$*-category. That is:*

1. *If* $\Gamma \vdash A = B$ *then* $[\![\Gamma \vdash A]\!] = [\![\Gamma \vdash B]\!]$
2. *If* $\Gamma \vdash M = N : A$ *then* $[\![\Gamma \vdash M]\!] = [\![\Gamma \vdash N]\!]$
3. *If* $\Gamma \vdash \phi = \psi$ *then* $[\![\Gamma \vdash \phi]\!] = [\![\Gamma \vdash \psi]\!]$
4. *If there is a proof* $\Gamma \vdash P : \phi$ *then there is a vertical arrow* $\top \to [\![\Gamma \vdash \phi]\!]$ *in the fibre* $\mathbb{P}/ [\![\Gamma]\!]$.

Proof.

Induction on derivations. □

Completeness Theorem

Theorem

If a judgement is true in every category $\mathcal{C}$, then it is derivable in T.

Proof.

Define the category $\mathrm{Cl}(T)$, the *classifying category* of T, thus:

- the objects of $\mathbb{B}$ are the valid contexts;
- the objects of $\mathbb{E}$ are the pairs (Γ, A) such that $\Gamma \vdash A\ \mathrm{Type}$, quotiented by equality;
- ...

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In fact, $\mathrm{Cl}(T)$ is an initial object in the metacategory of $\mathrm{LTT_W}$-categories. The interpretation given earlier is the unique functor $\mathrm{Cl}(T) \to \mathbb{C}$.

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In fact, $\mathrm{Cl}(T)$ is an initial object in the metacategory of $\mathrm{LTT_W}$-categories. The interpretation given earlier is the unique functor $\mathrm{Cl}(T) \to \mathbb{C}$. This is the sort of thing that gets category theorists excited.

Conservativity of LTT_0 over PA

I have previously given *syntactic* proofs that LTT_0 is conservative over PA. We can now give a *semantic* proof of the same result.

Theorem

LTT_0 *is conservative over* PA.

Proof.

From any model $\mathcal{M}$ of PA, we construct a model of LTT_0.

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Define the *higher-order recursive* (hor) functions to be those built up from $0^{\mathcal{M}}$ and $S^{\mathcal{M}}$ by composition, primitive recursion, pairing, projection, lambda-abstraction and application.

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Define the *arithmetic* predicates to be those built up from equality by Boolean operations and quantification over $|\mathcal{M}|$.

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where $A \rightarrow B$ is the set of hor functions from A to B,
and PA is the set of arithmetic subsets of A.

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The objects of $\mathbb{P}$ over $b \in \mathbb{B}$ are all subsets of b. □

Note that $\mathbb{E}$ and $\mathbb{P}$ are radically different.

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We can similarly prove LTT_0 conservative over ACA_0.

Corollary

ACA_0 *is conservative over* PA.

Bounded Quantification

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We can define bounded quantification by elimination $\mathbb{N}$ over prop:

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Conversely, any formula in prop in the new LTT corresponds to a Σ_0-formula in $I\Sigma_0(exp)$.
(Show that the functions in $\mathbb{N} \to \mathbb{N}$ are all defined by a Σ_0-formula in $I\Sigma_0(exp)$. Use the fact that the Σ_0-definable functions are closed under primitive recursion.)

Conclusion

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- What is the proof-theoretic ordinal of this LTT?
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Please bring me some more.

Syntax of an LTT

LTT_0 is a system with:

Judgement forms:

$$\begin{array}{ll} \Gamma \vdash A \text{ type} & \Gamma \vdash M : A \\ \Gamma \vdash \phi \text{ Prop} & \Gamma \vdash P : \phi \end{array}$$

and associated equality judgements.

Type	A	$::=$	
Term	M	$::=$	x
Proposition	ϕ	$::=$	
Proof	P	$::=$	

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- arrow types

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\begin{array}{lrcl}
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\\
\text{Proposition} & \phi & ::= & \\
\\
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Syntax of an LTT

LTT_0 is a system with:

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\end{array}
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 & & & \forall x : A.\phi \mid \exists x : A.\phi \\
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\end{array}$$

We can give a semantic proof of this result:

A function $f : \mathbb{N}^n \to \mathbb{N}$ is *definable* in *PA* iff there is a formula $\phi[x_1, \ldots, x_n, y]$ such that:

1. for all a_1, …, a_n, $\mathrm{PA} \vdash \phi[\overline{a_1}, \ldots, \overline{a_n}, \overline{f(a_1, \ldots, a_n)}]$;
2. $\mathrm{PA} \vdash \forall x_1 \cdots \forall x_n \exists! y \phi[x_1, \ldots, x_n, y]$

Theorem

The functions definable in PA *are exactly the* ϵ_0*-recursive functions.*

Proof.

Construct a model of LTT_0 in which the arrows are the ϵ_0-recursive functions. Then apply conservativity. □

History of LTTs

2002 Aczel and Gambino [?] define translations between Constructive ZF (CZF) and the type theory $\mathrm{ML}_1\mathrm{V}$.

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From this work, I take the message:

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