Inductive Types in Pure Type Systems

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1 Inductive Types as Π-types

It is possible to define some inductive datatypes using only Π-types and λ-abstractions. The most well-known example is the definition of the type of natural numbers in System F as the type of Church numerals:

$$\forall X.X \to (X \to X) \to X$$

It is possible to do likewise for a large class of inductive definitions. We here give a general scheme for defining these inductive types within PTSs.

We use the scheme for inductive definitions of types given in [**?**]. These results seem to apply only to what Dybjer calls ordinary induction; that is, where the premises of each introduction rule are objects of the inductive type, as opposed to generalised induction, where the premises may also be functions with codomain the inductive type. The method also does not seem to extend to simultaneous induction. The one frill we have been able to incorporate is the use of indices.

Definition 1 ((Ordinary) Induction Scheme) *An* (ordinary) induction scheme *consists of:*

- *A sequence of declarations* $A_1 : \sigma_1, \ldots, A_l : \sigma_l$*, the* parameters.
- *A sequence of declarations* $a_1 : \alpha_1, \ldots, a_m : \alpha_m$*, the* indices.
- *A sort* s*, of which the inductive types shall be objects.*
- *A natural number* n*, the number of* introduction rules.
- *For* $i = 1, \ldots, n$*, a sequence of declarations* $b_{i1} : \beta_{i1}, \ldots, b_{is_i} : \beta_{is_i}$*, the* side conditions *to the* i*th introduction rule.*
- *For* $i = 1, \ldots, n$*, a sequnce of variables* $u_{i1}, \ldots, u_{it_i}$*, each with a corresponding sequence of terms* $p_{ij1}[\vec{A}, \vec{b_i}], \ldots, p_{ijm}[\vec{A}, \vec{b_i}]$ *of length* m*. (Informally, the* i*th introduction rule has* t_i *premises. The* j*th premise is*

  $$u_{ij} : \mathcal{I}(\vec{A}, p_{ij1}, \ldots, p_{ijm})$$

  where $\mathcal{I}(\vec{A}, a_1, \ldots, a_m)$ *is the family of inductive types being defined.*

- *For $i = 1, \ldots, n$, a sequence of terms $q_{i1}[\vec{A}, \vec{b_i}]$, ..., $q_{im}[\vec{A}, \vec{b_i}]$ of length m. (Informally, the ith introduction rule constructs a term of type $\mathcal{I}(\vec{A}, q_{i1}, \ldots, q_{im})$.*

- *A set E of sorts, the sorts over which one may eliminate.*

such that:

$$\vec{A} : \vec{\sigma}, \vec{a} : \vec{\alpha} \text{ valid}$$
$$\vec{A} : \vec{\sigma}, \vec{b_i} : \vec{\beta_i} \vdash \vec{p_i j} : \vec{\alpha} \qquad (j = 1, \ldots, t_i)$$
$$\vec{A} : \vec{\sigma}, \vec{b_i} : \vec{\beta_i} \vdash \vec{q_i} : \vec{\alpha}$$

The scheme $\mathcal{S}$ given above represents the family of inductive types $\mathcal{I}_{\mathcal{S}}$ defined by the formation rule

$$\frac{\vec{A} : \vec{\sigma} \quad \vec{a} : \vec{\alpha}}{\mathcal{I}_{\mathcal{S}}(\vec{A}, \vec{a}) : s}$$

the n introduction rules

$$\frac{\vec{A} : \vec{\sigma} \quad \vec{b_i} : \vec{\beta_i} \quad u_{i1} : \mathcal{I}_{\mathcal{S}}(\vec{A}, \vec{p_{i1}}) \quad \cdots \quad u_{it_i} : \mathcal{I}_{\mathcal{S}}(\vec{A}, \vec{p_{it_i}})}{\kappa_i(\vec{A}, \vec{b_i}, \vec{u_i}) : \mathcal{I}_{\mathcal{S}}(\vec{A}, \vec{q_i})} (i = 1, \ldots, n)$$

and the elimination rules

$$\frac{\begin{array}{c} \vec{A} : \vec{\sigma} \quad C : \Pi \vec{a} : \vec{\alpha}.t \\ e_1 : \Pi \vec{b_1} : \vec{\beta_1}.\Pi v_{11} : C\vec{p_{11}}. \cdots .\Pi v_{1t_1} : C\vec{p_{1t_1}}.C\vec{q_1} \\ \vdots \\ e_n : \Pi \vec{b_n} : \vec{\beta_n}.\Pi v_{n1} : C\vec{p_{n1}}. \cdots .\Pi v_{nt_n} : C\vec{p_{nt_n}}.C\vec{q_n} \\ \vec{a} : \vec{\alpha} \quad c : \mathcal{I}_{\mathcal{S}}(\vec{A}, \vec{a}) \end{array}}{\epsilon(\vec{A}, C, \vec{e}, \vec{a}, c) : C\vec{a}} (t \in E)$$

with

$$\epsilon(\vec{A}, C, \vec{e}, \vec{q_i}, c_i(\vec{A}, \vec{b_i}, \vec{u_i})) = e_i \vec{b_i} \epsilon(\vec{A}, C, \vec{e}, \vec{p_{i1}}, u_{i1}) \cdots \epsilon(\vec{A}, C, \vec{e}, \vec{p_{it_i}}, u_{it_i})$$

Definition 2 *Let $\mathcal{S}$ be an induction scheme as given above. Let $I[\vec{A}, \vec{a}]$, $\kappa_i[\vec{A}, \vec{b_i}, \vec{u_i}]$ $(i = 1, \ldots, n)$, and $\epsilon_t[\vec{A}, C, \vec{e}, \vec{a}, c]$ $(t \in E)$ be terms with free variables as shown ($\vec{e}$ a sequence of variables of length n). We say that $(I, \vec{\kappa}, \{\epsilon_t \mid t \in E\})$* satisfies *the scheme $\mathcal{S}$, and I is a family of inductive types of scheme $\mathcal{S}$, iff the following* *are derivable rules of deduction in* $\lambda\mathbf{S}$*:*

$$\frac{\Gamma \vdash \vec{B} : \vec{\sigma} \quad \Gamma \vdash \vec{b} : [\vec{B}/\vec{A}]\alpha}{\Gamma \vdash I[\vec{B}, \vec{b}] : s}$$

$$\frac{\Gamma \vdash \vec{B} : \vec{\sigma} \quad \Gamma \vdash \vec{c_i} : [\vec{B}/\vec{A}]\vec{\beta_i} \quad \Gamma \vdash v_{i1} : I[\vec{B}, \vec{p_{i1}}[\vec{B}, \vec{c_i}]] \quad \cdots \quad \Gamma \vdash v_{it_i} : I[\vec{B}, \vec{p_{it_i}}[\vec{B}, \vec{c_i}]]}{\Gamma \vdash \kappa_i[\vec{B}, \vec{c_i}, \vec{v_i}] : I[\vec{B}, \vec{q_i}[\vec{B}, \vec{c_i}]]} \, (i = 1, \ldots, n)$$

$$\frac{\begin{array}{c} \Gamma \vdash \vec{B} : \vec{\sigma} \quad \Gamma \vdash D : \Pi\vec{a} : [\vec{B}/\vec{A}]\vec{\alpha}.t \\ \Gamma \vdash f_1 : \Pi\vec{b_1} : \vec{\beta_1}.\Pi x_{11} : D\vec{p_{11}}[\vec{B}, \vec{b_1}]. \cdots .\Pi x_{1t_1} : D\vec{p_{it_i}}[\vec{B}, \vec{b_1}].D\vec{q_1}[\vec{B}, \vec{b_1}] \\ \vdots \\ \Gamma \vdash f_n : \Pi\vec{b_n} : \vec{\beta_n}.\Pi x_{n1} : D\vec{p_{n1}}[\vec{B}, \vec{b_n}]. \cdots .\Pi x_{nt_n} : D\vec{p_{nt_n}}[\vec{B}, \vec{b_n}].D\vec{q_n}[\vec{B}, \vec{b_n}] \\ \Gamma \vdash \vec{b} : [\vec{B}/\vec{A}]\vec{\alpha} \quad \Gamma \vdash d : I[\vec{B}, \vec{b}] \end{array}}{\Gamma \vdash \epsilon_t[\vec{B}, D, \vec{f}, \vec{b}, d] : D\vec{b}} \, (t \in E)$$

and

$$\epsilon_t[\vec{A}, C, \vec{e}, \vec{q_i}[\vec{A}, \vec{b_i}], \kappa_i[\vec{A}, \vec{b_i}, \vec{u_i}]] =_\beta e_i\vec{b_i}\epsilon_t[\vec{A}, C, \vec{e}, \vec{p_{i1}}, u_{i1}] \cdots \epsilon_t[\vec{A}, C, \vec{e}, \vec{p_{it_i}}, u_{it_i}]$$

Theorem 3 *Let $\mathcal{S}$ be an inductive scheme as given above with $E = \{t\}$ a singleton. Let*

$$\begin{array}{rcl} I[\vec{A}, \vec{a}] & \equiv & \Pi X : \Pi\vec{a} : \vec{\alpha}.t. \\ & & (\Pi\vec{b_1} : \vec{\beta_1}.\Pi u_{11} : X\vec{p_{11}}. \cdots .\Pi u_{1t_1} : X\vec{p_{1t_1}}.X\vec{q_1}) \to \\ & & \cdots \to \\ & & (\Pi\vec{b_n} : \vec{\beta_n}.\Pi u_{n1} : X\vec{p_{n1}}. \cdots .\Pi u_{nt_n} : X\vec{p_{nt_n}}.X\vec{q_n}) \to \\ & & X\vec{a} \\ \kappa_i[\vec{A}, \vec{b_i}, \vec{u_i}] & \equiv & \lambda X : \Pi\vec{a} : \vec{\alpha}.t. \\ & & \lambda f_1 : \Pi\vec{b_1} : \vec{\beta_1}.\Pi u_{11} : X\vec{p_{11}}. \cdots .\Pi u_{1t_1} : X\vec{p_{1t_1}}.X\vec{q_1}. \\ & & \cdots . \\ & & \lambda f_n : \Pi\vec{b_n} : \vec{\beta_n}.\Pi u_{n1} : X\vec{p_{n1}}. \cdots .\Pi u_{nt_n} : X\vec{p_{nt_n}}.X\vec{q_n}. \\ & & f_i\vec{b_i}(u_{i1}X\vec{f}) \cdots (u_{it_i}X\vec{f}) \\ \epsilon_t[\vec{A}, C, \vec{e}, \vec{a}, c] & \equiv & cC\vec{e} \end{array}$$

If

$$\vec{A} : \vec{\sigma}, \vec{a} : \vec{\alpha} \vdash I[\vec{A}, \vec{a}] : s$$

then $(I, \vec{\kappa}, \epsilon_t)$ satisfies $\mathcal{S}$.

Proof

- If $\Gamma \vdash \vec{B} : \vec{\sigma}$ and $\Gamma \vdash \vec{b} : [\vec{B}/\vec{A}]\vec{\alpha}$, then

  $$\Gamma \vdash I[\vec{B}, \vec{b}] : s$$

  by Substitution

- Suppose $\Gamma \vdash \vec{B} : \vec{\sigma}$, $\Gamma \vdash \vec{c_i} : [\vec{B}/\vec{A}]\vec{\beta_i}$, and $\Gamma \vdash v_{ij} : I[\vec{B}, \vec{p_{ij}}[\vec{B}, \vec{c_i}]]$ for $j = 1, \ldots, t_i$. Now, as $I[\vec{A}, \vec{a}]$ is well-typed,

$$\begin{aligned}\Gamma_1 \equiv\ & \Gamma, X : \Pi\vec{a} : [\vec{B}/\vec{A}]\vec{\alpha}.t, \\ & f_1 : \Pi\vec{b_1} : [\vec{B}/\vec{A}]\vec{\beta_1}.\Pi u_{11} : X\vec{p_{11}}[\vec{B}, \vec{b_1}]. \cdots .\Pi u_{1t_1} : X\vec{p_{1t_1}}[\vec{B}, \vec{b_1}].X\vec{q_1}[\vec{B}, \vec{b_1}], \\ & \ldots, \\ & f_n : \Pi\vec{b_n} : [\vec{B}/\vec{A}]\vec{\beta_n}.\Pi u_{n1} : X\vec{p_{n1}}[\vec{B}, \vec{b_n}]. \cdots .\Pi u_{nt_n} : X\vec{p_{nt_n}}[\vec{B}, \vec{b_n}].X\vec{q_n}[\vec{B}, \vec{b_n}] \text{ valid}\end{aligned}$$

Now,

$$\Gamma_1 \vdash v_{ij} : I[\vec{B}, \vec{p_{ij}}[\vec{B}, \vec{c_i}]]$$

i.e.

$$\begin{aligned}\Gamma_1 \ \vdash\ & v_{ij} : \Pi X : \Pi\vec{a} : [\vec{B}/\vec{A}]\vec{\alpha}.t. \\ & (\Pi\vec{b_1} : [\vec{B}/\vec{A}]\vec{\beta_1}.\Pi u_{11} : X[\vec{B}/\vec{A}]\vec{p_{11}}. \cdots .\Pi u_{1t_1} : X[\vec{B}/\vec{A}]\vec{p_{1t_1}}.X[\vec{B}/\vec{A}]\vec{q_1}) \to \\ & \cdots \to \\ & (\Pi\vec{b_n} : [\vec{B}/\vec{A}]\vec{\beta_n}.\Pi u_{n1} : X[\vec{B}/\vec{A}]\vec{p_{n1}}. \cdots .\Pi u_{nt_n} : X[\vec{B}/\vec{A}]\vec{p_{nt_n}}.X[\vec{B}/\vec{A}]\vec{q_n}) \to \\ & X\vec{p_{ij}}[\vec{B}, \vec{c_i}] \\ \therefore \Gamma_1 \ \vdash\ & v_{ij}X\vec{f} : X\vec{p_{ij}}[\vec{B}, \vec{c_i}] \\ \therefore \Gamma_1 \ \vdash\ & f_i\vec{c_i}(v_{i1}X\vec{f}) \cdots (v_{it_i}X\vec{f}) : X\vec{q_i}[\vec{B}, \vec{c_i}] \\ \therefore \Gamma \ \vdash\ & \kappa_i[\vec{B}, \vec{c_i}, \vec{v_i}] : \Pi X : \Pi\vec{a} : [\vec{B}/\vec{A}]\vec{\alpha}.t. \\ & \quad (\Pi\vec{b_1} : [\vec{B}/\vec{A}]\vec{\beta_1}.\Pi u_{11} : X\vec{p_{11}}[\vec{B}, \vec{b_1}]. \cdots .\Pi u_{1t_1} : X\vec{p_{1t_1}}[\vec{B}, \vec{b_1}].X\vec{q_1}[\vec{B}, \vec{b_1}]) \to \\ & \quad \cdots \to \\ & \quad (\Pi\vec{b_n} : [\vec{B}/\vec{A}]\vec{\beta_n}.\Pi u_{n1} : X\vec{p_{n1}}[\vec{B}, \vec{b_n}]. \cdots .\Pi u_{nt_n} : X\vec{p_{nt_n}}[\vec{B}, \vec{b_n}].X\vec{q_n}[\vec{B}, \vec{b_n}]) \to \\ & \quad X\vec{q_i}[\vec{B}, \vec{c_i}]\end{aligned}$$

i.e.

$$\Gamma \vdash \kappa_i[\vec{B}, \vec{c_i}, \vec{v_i}] : I[\vec{B}, \vec{c_i}]$$

- Suppose $\Gamma \vdash \vec{B} : \vec{\sigma}$; $\Gamma \vdash D : \Pi\vec{a} : \vec{\alpha}.t$; for $i = 1, \ldots, n$, $\Gamma \vdash f_i : \Pi\vec{b_i} : \vec{\beta_i}.\Pi x_{i1} : D\vec{p_{i1}}[\vec{B}, \vec{b_i}]. \cdots .\Pi x_{it_i} : D\vec{p_{it_i}}[\vec{B}, \vec{b_i}].D\vec{q_i}[\vec{B}, \vec{b_i}]$; $\Gamma \vdash \vec{b} : [\vec{B}/\vec{A}]\vec{\alpha}$; and $\Gamma \vdash d : I[\vec{B}, \vec{b}]$. Then

$$\begin{aligned}\Gamma \ \vdash\ & d : \Pi X : \Pi\vec{a} : [\vec{B}/\vec{A}]\vec{\alpha}.t. \\ & \quad (\Pi\vec{b_1} : [\vec{B}/\vec{A}]\vec{\beta_1}.\Pi u_{11} : X\vec{p_{11}}[\vec{B}, \vec{b_1}]. \cdots .\Pi u_{1t_1} : X\vec{p_{1t_1}}[\vec{B}, \vec{b_1}].X\vec{q_1}[\vec{B}, \vec{b_1}]) \to \\ & \quad \cdots \to \\ & \quad (\Pi\vec{b_n} : [\vec{B}/\vec{A}]\vec{\beta_n}.\Pi u_{n1} : X\vec{p_{n1}}[\vec{B}, \vec{b_n}]. \cdots .\Pi u_{nt_n} : X\vec{p_{nt_n}}[\vec{B}, \vec{b_n}].X\vec{q_n}[\vec{B}, \vec{b_n}]) \to \\ & \quad X\vec{b} \\ \therefore \Gamma \ \vdash\ & dD\vec{f} : D\vec{b}\end{aligned}$$

- $$\begin{aligned}
\epsilon_t[\vec{A}, C, \vec{e}, \vec{q_i}[\vec{A}, \vec{b_i}], \kappa_t[\vec{A}, \vec{b_i}, \vec{u_i}]] &\equiv \kappa_i[\vec{A}, \vec{b_i}, \vec{u_i}]C\vec{e} \\
&\equiv \{\lambda X : \Pi\vec{a} : \vec{\alpha}.t. \\
&\quad \lambda f_1 : \Pi\vec{b_1} : \vec{\beta_1}.\Pi u_{11} : X\vec{p_{11}}.\cdots.\Pi u_{1t_1} : X\vec{p_{1t_1}}.X\vec{q_1}. \\
&\quad \cdots . \\
&\quad \lambda f_n : \Pi\vec{b_n} : \vec{\beta_n}.\Pi u_{n1} : X\vec{p_{n1}}.\cdots.\Pi u_{nt_n} : X\vec{p_{nt_n}}.X\vec{q_n}. \\
&\quad f_i\vec{b_i}(u_{i1}X\vec{f})\cdots(u_{it_i}X\vec{f})\}C\vec{e} \\
&\twoheadrightarrow_\beta e_i\vec{b_i}(u_{i1}C\vec{e})\cdots(u_{it_i}C\vec{e}) \\
&\equiv e_i\vec{b_i}\epsilon_t[\vec{A}, C, \vec{e}, \vec{p_{i1}}[\vec{A}, \vec{b_i}], u_{i1}]\cdots\epsilon_t[\vec{A}, C, \vec{e}, \vec{p_{it_i}}[\vec{A}, \vec{b_i}], u_{it_i}]
\end{aligned}$$

QED

Thus, the only condition required for the inductive type to exist is that the PTS specification allows the formation of the type $I[\vec{A}, \vec{a}]$. The weakest possible condition that ensures this is given in the following corollary:

Corollary 4 *Suppose there exist sorts*

- t^+
- τ_{ij} *for* $i = 1, \ldots, n,\ j = 1, \ldots, t_i$*;*
- κ_{ij} *for* $i = 1, \ldots, n,\ j = 1, \ldots, s_i$*;*
- β^+_{ij} *for* $i = 1, \ldots, n,\ j = 1, \ldots, s_i$*;*
- ρ_i *for* $i = 1, \ldots, n$*;*
- α^+_i *for* $i = 1, \ldots, l$*;*
- μ_i *for* $i = 1, \ldots, l$*;*
- ν_i *for* $i = 1, \ldots, n$

such that

1. $t : t^+ \in \mathcal{A}$.
2. $\vec{A} : \vec{\sigma}, a_1 : \alpha_1, \ldots, a_{i-1} : \alpha_{i-1} \vdash \alpha_i : \alpha^+_i \qquad (i = 1, \ldots, l)$
3. $\vec{A} : \vec{\sigma}, b_{i1} : \beta_{i1}, \ldots, b_{i,j-1} : \beta_{i,j-1} \vdash \beta_{ij} : \beta^+_{ij} \qquad (i = 1, \ldots, n, j = 1, \ldots, s_i)$
4. $(\alpha^+_l, t^+, \mu_l) \in \mathcal{R}$
5. $(\alpha^+_i, \mu_{i+1}, \mu_i) \in \mathcal{R} \qquad (i = 1, \ldots, l-1)$
6. $(t, t, \tau_{it_i}) \in \mathcal{R} \qquad (i = 1, \ldots, n)$

7. $(t, \tau_{i,j+1}, \tau_{ij}) \in \mathcal{R} \qquad (i = 1, \ldots, n, j = 1, \ldots, t_i - 1)$

8. $(\beta^+_{i,s_i}, \tau_{i1}, \kappa_{is_i}) \in \mathcal{R} \qquad (i = 1, \ldots, n)$

9. $(\beta^+_{ij}, \kappa_{i,j+1}, \kappa_{ij}) \in \mathcal{R} \qquad (i = 1, \ldots, n, j = 1, \ldots, s_i - 1)$

10. $(\kappa_{n1}, t, \nu_n) \in \mathcal{R}$

11. $(\kappa_{i1}, \nu_{i+1}, \nu_i) \in \mathcal{R} \qquad (i = 1, \ldots, n)$

12. $(\mu_1, \nu_1, s) \in \mathcal{R}$

Then there exists an inductive type of scheme $\mathcal{S}$ with $E = \{t\}$.

Proof By 1,

$$\vec{A} : \vec{\sigma}, \vec{a} : \vec{\alpha} \vdash t : t^+$$

By 2,

$$\vec{A} : \vec{\sigma}, a_1 : \alpha_1, \ldots, a_{l-1} : \alpha_{l-1} \vdash \alpha_l : \alpha_l^+$$

$\therefore$ By 4,

$$\vec{A} : \vec{\sigma}, a_1 : \alpha_1, \ldots, a_{l-1} : \alpha_{l-1} \vdash \Pi a_l : \alpha_l.t : \mu_l$$

Suppose

$$\vec{A} : \vec{\sigma}, a_1 : \alpha_1, \ldots, a_i : \alpha_i \vdash \Pi a_{i+1} : \alpha_{i+1}. \cdots .\Pi a_l : \alpha_l.t : \mu_{i+1}$$

$(i = 1, \ldots, l - i)$. By 2,

$$\vec{A} : \vec{\sigma}, a_1 : \alpha_1, \ldots, a_{i-1} : \alpha_{i-1} \vdash \alpha_i : \alpha_i^+$$

Therefore, by 5,

$$\vec{A} : \vec{\sigma}, a_1 : \alpha_1, \ldots, a_{i-1} : \alpha_{i-1} \vdash \Pi a_i : \alpha_i.\Pi a_{i+1} : \alpha_{i+1}. \cdots .\Pi a_l : \alpha_l.t : \mu_i$$

Thus, by backward induction,

$$\vec{A} : \vec{\sigma} \vdash \Pi\vec{a} : \vec{\alpha}.t : \mu_1$$

For $i = 1, \ldots, n$,

$$\begin{aligned} \vec{A} : \vec{\sigma}, X : \Pi\vec{A} : \vec{\alpha}.t, \vec{b_i} : \vec{\beta}_i, u_{i1} : X\vec{p_{i1}}, \ldots, u_{it_i} : X\vec{p_{it_i}} &\vdash X\vec{q_i} : t \\ \vec{A} : \vec{\sigma}, X : \Pi\vec{A} : \vec{\alpha}.t, \vec{b_i} : \vec{\beta}_i, u_{i1} : X\vec{p_{i1}}, \ldots, u_{i,t_i-1} : X\vec{p_{i,t_i-1}} &\vdash X\vec{p_{it_i}} : t \end{aligned}$$

Therefore, by 6,

$$\vec{A} : \vec{\sigma}, X : \Pi\vec{A} : \vec{\alpha}.t, \vec{b_i} : \vec{\beta}_i, u_{i1} : X\vec{p_{i1}}, \ldots, u_{i,t_i-1} : X\vec{p_{i,t_i-1}} \vdash \Pi u_{it_i} : X\vec{p_{it_i}}.X\vec{q_i} : \tau_{it_i}$$

$$\vec{A} : \vec{\sigma}, X : \Pi\vec{A} : \vec{\alpha}.t, \vec{b_i} : \vec{\beta}_i, u_{i1} : X\vec{p_{i1}}, \ldots, u_{ij} : X\vec{p_{ij}} \vdash X\vec{p_{i,j+1}} : t \qquad (j = 1, \ldots, t_i - 1)$$

Therefore, using 7 and backward induction,

$$\vec{A} : \vec{\sigma}, X : \Pi\vec{A} : \vec{\alpha}.t, \vec{b_i} : \vec{\beta}_i \vdash \Pi u_{i1} : X\vec{p_{i1}}. \cdots .\Pi u_{it_i} : X\vec{p_{it_i}}.X\vec{q_i} : \tau_{i1}$$

A similar backward induction using 8 and 9 (and 3) gives

$$\vec{A} : \vec{\sigma}, X : \Pi\vec{A} : \vec{\alpha}.t \vdash \Pi\vec{b_i} : \vec{\beta_i}.\Pi u_{i1} : X\vec{p_{i1}}.\cdots.\Pi u_{it_i} : X\vec{p_{it_i}}.X\vec{q_i} : \kappa_{i1}$$

And yet another using 10 and 11 gives

$$\begin{aligned}\vec{A} : \vec{\sigma}, \vec{a} : \vec{\alpha}, X : \Pi\vec{A} : \vec{\alpha}.t \quad \vdash \quad & (\Pi\vec{b_1} : \vec{\beta_1}.\Pi u_{11} : X\vec{p_{11}}.\cdots.\Pi u_{1t_1} : X\vec{p_{1t_1}}.X\vec{q_1}) \to \\ & \cdots \to \\ & (\Pi\vec{b_n} : \vec{\beta_n}.\Pi u_{n1} : X\vec{p_{n1}}.\cdots.\Pi u_{nt_n} : X\vec{p_{nt_n}}.X\vec{q_n}) \to \\ & X\vec{a} : \nu_1\end{aligned}$$

Finally, 12 gives

$$\vec{A} : \vec{\sigma}, \vec{a} : \vec{\alpha} \vdash I[\vec{A}, \vec{a}] : s$$

as required. **QED**

Of course, in most applications, many of these types will be identified, and the conditions will reduce to a few. Here are two examples, the predicative and impredicative cases:

Corollary 5 *Suppose $t : s \in \mathcal{A}$ and $(s, s), (t, t), (t, s), (s, t, s) \in \mathcal{R}$. Let $\mathcal{S}$ be an induction scheme with $E = \{t\}$ such that, for $i = 1, \ldots, l$, either*

$$\vec{A} : \vec{\sigma}, a_1 : \alpha_1, \ldots, a_{i-1} : \alpha_{i-1} \vdash \alpha_i : s$$

or

$$\vec{A} : \vec{\sigma}, a_1 : \alpha_1, \ldots, a_{i-1} : \alpha_{i-1} \vdash \alpha_i : t;$$

and, for $i = 1, \ldots, n$, $j = 1, \ldots, s_i$,

$$\vec{A} : \vec{\sigma}, b_{i1} : \beta_{i1}, \ldots, b_{i,j-1} : \beta_{i,j-1} \vdash \beta_{ij} : t$$

Then there exists an inductive type of scheme $\mathcal{S}$.

Proof Take

$$\begin{array}{rcll} t^+ & \equiv & s & \\ \tau_{ij} & \equiv & t & (\forall i, j) \\ \kappa_{ij} & \equiv & t & (\forall i, j) \\ \beta^+_{ij} & \equiv & t & (\forall i, j) \\ \mu_i & \equiv & s & (\forall i) \\ \nu_i & \equiv & t & (\forall i) \end{array}$$

and α_i^+ to be whichever of s and t ensures that

$$\vec{A} : \vec{\sigma}, a_1 : \alpha_1, \ldots, a_{i-1} : \alpha_{i-1} \vdash \alpha_i : \alpha_i^+$$

QED

Corollary 6 *Suppose $s : s^+ \in \mathcal{A}$. Let $\mathcal{S}$ be an induction scheme with $E = \{s\}$ such that, for $i = 1, \dots, l$,*

$$\vec{A} : \vec{\sigma}, a_1 : \alpha_1, \dots, a_{i-1} : \alpha_{i-1} \vdash \alpha_i : \alpha_i^+$$

and, for $i = 1, \dots, n$ and $j = 1, \dots, s_i$,

$$\vec{A} : \vec{\sigma}, b_{i1} : \beta_{i1}, \dots, b_{i,j-1} : \beta_{i,j-1} \vdash \beta_{ij} : \beta_{ij}^+$$

where each α_i^+ and β_{ij}^+ is a sort. Suppose

$$(s, s), (s^+, s), (\alpha_i^+, s), (\beta_{ij}^+, s) \in \mathcal{R}$$

Then there is an inductive type of scheme $\mathcal{S}$.

Proof Take every sort in the hypothesis of Corollary 4 other than α_i^+ and β_{ij}^+ to be s. **QED**

Remark 7 *Let* **Prop** *be a sort whose types are intended to be propositions. Assume there is a sort* $\mathbf{Prop}^+$ *such that*

$$\begin{array}{rcl} \mathbf{Prop} : \mathbf{Prop}^+ & \in & \mathcal{A} \\ (\mathbf{Prop}, \mathbf{Prop}) & \in & \mathcal{R} \\ (\mathbf{Prop}^+, \mathbf{Prop}) & \in & \mathcal{R} \end{array}$$

The logical connectives can be introduced as inductive definitions of types of sort **Prop** *in the familiar manner. If one applies the above method to these inductive definitions (with elimination sort* **Prop***), one obtains the impredicative definitions of the connectives. For example, $\wedge$ can be introduced with the inductive definition:*

- *Parameters ϕ :* **Prop***, ψ :* **Prop***.*
- *No indices.*
- *Sort* **Prop***.*
- *One introduction rule.*
- *Side conditions $p : \phi$, $q : \psi$.*
- *No premises.*
- *Sole elimination sort* **Prop***.*

Applying the above method gives the definitions

$$\phi \wedge \psi \equiv \lambda X : \mathbf{Prop}.(\phi \to \psi \to X) \to X : \mathbf{Prop}$$

which is the usual impredicative definition of $\wedge$.

1. $0 \in \mathbb{N}$
2. $(\forall n \in \mathbb{N}) n' \in \mathbb{N}$
3. $(\forall n \in \mathbb{N}) n' \neq 0$
4. $(\forall m, n \in \mathbb{N})(m' = n' \Rightarrow m = n)$
5. $\phi(0) \Rightarrow (\forall n \in \mathbb{N})(\phi(n) \Rightarrow \phi(n')) \Rightarrow (\forall n \in \mathbb{N})\phi(n)$

Figure 1: Peano's Axioms for the Natural Numbers

2 Peano's Third Axiom

Assume we have a sort **Prop** of propositions, and, when $\Gamma \vdash a : A$ and $\Gamma \vdash b : A$, a proposition $\Gamma \vdash a =_A b : \mathbf{Prop}$.

The equivalent of Peano's axioms for an arbitrary inductive type of scheme S are as follows. For simplicity's sake, we assume S has no indices, and the side conditions of an introduction rule do not depend on each other.

1,2 If $\Gamma \vdash \vec{B} : \vec{\sigma}$, $\Gamma \vdash \vec{c_i} : [\vec{B}/\vec{A}]\vec{\beta_i}$, and $\Gamma \vdash u_{ij} : \mathcal{I}_S(\vec{B})$ for $j = 1, \ldots, t_i$, then

$$\Gamma \vdash \kappa_i(\vec{B}, \vec{c_i}, \vec{u_i}) : \mathcal{I}_S(\vec{B})$$

3 Let $i \neq j$. If

$$\begin{array}{lcl} \Gamma & \vdash & \vec{B} : \vec{\sigma} \\ \Gamma & \vdash & \vec{c_i} : [\vec{B}/\vec{A}]\vec{\beta_i} \\ \Gamma & \vdash & \vec{c_j} : [\vec{B}/\vec{A}]\vec{\beta_j} \\ \Gamma & \vdash & u_{ik} : \mathcal{I}_S(\vec{B}) \qquad (k = 1, \ldots, t_i) \\ \Gamma & \vdash & u_{jk} : \mathcal{I}_S(\vec{B}) \qquad (k = 1, \ldots, t_j) \\ \Gamma & \vdash & p : \kappa_i(\vec{B}, \vec{c_i}, \vec{u_i}) =_{\mathcal{I}_S(\vec{B})} \kappa_j(\vec{B}, \vec{c_j}, \vec{u_j}) \\ \Gamma & \vdash & \phi : \mathbf{Prop} \end{array}$$

then there is a proof q such that

$$\Gamma \vdash q : \phi$$

4 If

$$\begin{array}{lcl} \Gamma & \vdash & \vec{B} : \vec{\sigma} \\ \Gamma & \vdash & \vec{c_i} : [\vec{B}/\vec{A}]\vec{\beta_i} \\ \Gamma & \vdash & \vec{d_i} : [\vec{B}/\vec{A}]\vec{\beta_i} \\ \Gamma & \vdash & u_{ij} : \mathcal{I}_S(\vec{B}) \qquad (j = 1, \ldots, t_i) \\ \Gamma & \vdash & v_{ij} : \mathcal{I}_S(\vec{B}) \qquad (j = 1, \ldots, t_i) \\ \Gamma & \vdash & p : \kappa_i(\vec{B}, \vec{c_i}, \vec{u_i}) =_{\mathcal{I}_S(\vec{B})} \kappa_i(\vec{B}, \vec{d_i}, \vec{v_i}) \end{array}$$

then there are proofs q_j $(j = 1, \ldots, s_i)$, r_j $(j = 1, \ldots, t_i)$ such that

$$\begin{array}{rcl} \Gamma & \vdash & q_j : c_{ij} =_{[\vec{B}/\vec{A}]\beta_{ij}} d_{ij} \\ \Gamma & \vdash & r_j : u_{ij} =_{\mathcal{I}_S(\vec{B})} v_{ij} \end{array}$$

5 If $\Gamma \vdash \vec{B} : \vec{\sigma}$, $\Gamma, x : \mathcal{I}_S(\vec{B}) \vdash \phi : \mathbf{Prop}$, and

$$\Gamma, \vec{b_i} : [\vec{B}/\vec{A}]\vec{\beta_i}, \vec{u_i} : \mathcal{I}_S(\vec{B}) \vdash p : [\kappa_i(\vec{B}, \vec{b_i}, \vec{u_i})/x]\phi$$

then there is a proof q such that

$$\Gamma, x : \mathcal{I}_S(\vec{B}) \vdash q : \phi$$

Suppose we have a type $\mathcal{I}_S$ of scheme S in some type system. The equivalent of the first two Peano axioms are satisfied by definition. It is easy to show that the fourth and fifth axioms are satisfied provided **Prop** is an elimination sort.

The third axiom is more interesting. The traditional method of satisfying it can be summarised as follows:

> Suppose we have an axiom $\mathbf{Prop} : \mathbf{Prop}^+$, and $\mathbf{Prop}^+$ is an elimination sort. Suppose also there exist $\top$, $\bot$, $*$ such that, for any context Γ,
>
> $$\begin{array}{rcl} \Gamma & \vdash & \top : \mathbf{Prop} \\ \Gamma & \vdash & \bot : \mathbf{Prop} \\ \Gamma & \vdash & * : \top \end{array}$$
>
> and if $\Gamma \vdash p : \bot$ and $\Gamma \vdash \phi : \mathbf{Prop}$, there exists a term q such that $\Gamma \vdash q : \phi$. Assume also that, if $\Gamma, x : A \vdash \phi : \mathbf{Prop}$, $\Gamma \vdash p : a =_A b$, and $\Gamma \vdash q : [a/x]\phi$, then there is a term r such that $\Gamma \vdash r : [b/x]\phi$.
>
> Now, assume
>
> $$\begin{array}{rcll} \Gamma & \vdash & \vec{B} : \vec{\sigma} & \\ \Gamma & \vdash & \vec{c_i} : [\vec{B}/\vec{A}]\vec{\beta_i} & \\ \Gamma & \vdash & \vec{c_j} : [\vec{B}/\vec{A}]\vec{\beta_j} & \\ \Gamma & \vdash & u_{ik} : \mathcal{I}_S(\vec{B}) & (k = 1, \ldots, t_i) \\ \Gamma & \vdash & u_{jk} : \mathcal{I}_S(\vec{B}) & (k = 1, \ldots, t_j) \\ \Gamma & \vdash & p : \kappa_i(\vec{B}, \vec{c_i}, \vec{u_i}) =_{\mathcal{I}_S(\vec{B})} \kappa_j(\vec{B}, \vec{c_j}, \vec{u_j}) & \end{array}$$

Let

$$\begin{aligned}
\psi[x] \quad \equiv \quad & \epsilon_{\mathbf{Prop}^+}(\vec{B}, \lambda x : \mathcal{I}_S(\vec{B}).\mathbf{Prop}, \\
& \lambda \vec{b_1} : [\vec{B}/\vec{A}]\vec{\beta_1}.\lambda \vec{x_1} : \mathbf{Prop}.\top, \\
& \ldots, \\
& \lambda \vec{b_{j-1}} : [\vec{B}/\vec{A}]\vec{\beta_{j-1}}.\lambda \vec{x_{j-1}} : \mathbf{Prop}.\top, \\
& \lambda \vec{b_j} : [\vec{B}/\vec{A}]\vec{\beta_j}.\lambda \vec{x_j} : \mathbf{Prop}.\bot, \\
& \lambda \vec{b_{j+1}} : [\vec{B}/\vec{A}]\vec{\beta_{j+1}}.\lambda \vec{x_{j+1}} : \mathbf{Prop}.\bot, \\
& \ldots, \\
& \lambda \vec{b_n} : [\vec{B}/\vec{A}]\vec{\beta_n}.\lambda \vec{x_n} : \mathbf{Prop}.\bot, x)
\end{aligned}$$

Then

$$\begin{aligned}
\Gamma, x : \mathcal{I}_S(\vec{B}) \quad & \vdash \quad \psi[x] : \mathbf{Prop} \\
\psi[\kappa_i(\vec{B}, \vec{c_i}, \vec{u_i})] \quad & =_\beta \quad \top \\
\psi[\kappa_j(\vec{B}, \vec{c_j}, \vec{u_j})] \quad & =_\beta \quad \bot
\end{aligned}$$

Therefore,

$$\Gamma \vdash * : \psi[\kappa_i(\vec{B}, \vec{c_i}, \vec{u_i})]$$

and so there is a proof q of

$$\Gamma \vdash q : \psi[\kappa_j(\vec{B}, \vec{c_j}, \vec{u_j})]$$

Therefore, whenever $\Gamma \vdash \phi : \mathbf{Prop}$, there is a proof r of

$$\Gamma \vdash r : \phi$$

When using the method of defining inductive types given in Section 1, this result is of no help if $\mathbf{Prop}^+$ is not the elimination sort. Far more useful is the following result: if there is a type in the elimination sort with two provably distinct objects, then the third axiom can be proven.

Theorem 8 *Assume:*

1. *If $\Gamma \vdash a : A$, then there exists p such that $\Gamma \vdash p : a =_A a$.*
2. *If $\Gamma \vdash p : a =_A b$, $\Gamma, x : A \vdash \phi : \mathbf{Prop}$, and $\Gamma \vdash q : [a/x]\phi$, then there exists r such that $\Gamma \vdash r : [b/x]\phi$.*

Suppose S is an induction scheme, and t is an elimination sort of S. Suppose

$$\Gamma \vdash A : t, \quad \Gamma \vdash a : A, \quad \Gamma \vdash b : A$$

and, whenever $\Gamma, \Delta \vdash p : a =_A b$ and $\Gamma, \Delta \vdash \phi : \mathbf{Prop}$, then there is a term q s.t. $\Gamma, \Delta \vdash q : \phi$. Then condition 3 above holds.

Proof Assume the hypotheses of condition 3. Let

$$
\begin{aligned}
T[x] \quad \equiv \quad & \epsilon_t(\vec{B}, \lambda x : \mathcal{I}_S(\vec{B}).A, \lambda \vec{b_1} : [\vec{B}/\vec{A}]\vec{\beta_1}.\lambda \vec{u_1} : \mathcal{I}_S(\vec{B}).a, \ldots, \lambda \vec{b_{j-1}} : [\vec{B}/\vec{A}]\vec{\beta_{j-1}}.\lambda \vec{u_{j-1}} : \mathcal{I}_S(\vec{B}).a, \\
& \lambda \vec{b_j} : [\vec{B}/\vec{A}]\vec{\beta_j}.\lambda \vec{u_j} : \mathcal{I}_S(\vec{B}).b, \\
& \lambda \vec{b_{j+1}} : [\vec{B}/\vec{A}]\vec{\beta_{j+1}}.\lambda \vec{u_{j+1}} : \mathcal{I}_S(\vec{B}).a, \ldots, \lambda \vec{b_n} : [\vec{B}/\vec{A}]\vec{\beta_n}.\lambda \vec{u_n} : \mathcal{I}_S(\vec{B}).a, x)
\end{aligned}
$$

Then

$$
\begin{aligned}
T[\kappa_i(\vec{B}, \vec{c_i}, \vec{u_i})] &=_\beta a \\
T[\kappa_j(\vec{B}, \vec{c_j}, \vec{u_j})] &=_\beta b
\end{aligned}
$$

There exists p_1 such that

$$
\begin{aligned}
\Gamma &\vdash p_1 : a =_A a \\
\therefore \Gamma &\vdash p_1 : T[\kappa_i(\vec{B}, \vec{c_i}, \vec{u_i})] =_A T[\kappa_i(\vec{B}, \vec{c_i}, \vec{u_i})]
\end{aligned}
$$

Therefore, there exists p_2 such that

$$
\begin{aligned}
\Gamma &\vdash p_2 : T[\kappa_i(\vec{B}, \vec{c_i}, \vec{u_i})] =_A T[\kappa_j(\vec{B}, \vec{c_j}, \vec{u_j})] \\
\therefore \Gamma &\vdash p_2 : a =_A b
\end{aligned}
$$

Therefore, there exists p_3 such that

$$\Gamma \vdash p_3 : \phi$$

QED