Adding Logic to a PTS

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1 Introduction

The Curry-Howard correspondence [?] [?] is the well-known fact that many type theories contain an embedded system of logic; that is, there is a translation from some system of logic into the type theory that takes propositions to types, and proofs of a proposition to objects of the corresponding type. Many logically interesting operations correspond to type-theoretically interesting ones; for example, conjunction to Cartesian product, implication to function space, cut elimination to reduction of lambda terms. It is now common practice to use the internal logic of a type system to reason about the objects of the non-logical types without further thought.

For all its familiarity, it is surprisingly difficult to make the correspondence precise. I have never seen a presentation of the correspondence for a general class of type systems. Even for individual systems, it is rarely trivial to formally establish the isomorphism; the more expressive the logic, the more work that is involved. For type systems in which an object can depend on a proof, the correspondence is difficult to state, let alone prove.

In this paper, we shall attempt to remedy the situation somewhat. We work with Pure Type Systems (PTSs); first presented in [?], these form a general theory encompassing a large class of type systems. Our aim is as follows: we begin with a PTS $\mathbf{S}$, which will be the collection of objects about which we reason. We then choose a set X of 'powers' that we wish our system of logic to have; over what types one can quantify, what artities the atomic predicates can have, and which connectives exist. Given these data, we construct a PTS $\mathbf{S} + L_X(I)$, which we will use as a type theory with embedded logic. It has the following desiderata:

1. The structure of the non-logical types is precisely that of $\mathbf{S}$ (that is, $\mathbf{S} + L_X(I)$ is a conservative extension of $\mathbf{S}$).
2. There is a correspondence between $\text{Log}_X(\mathbf{S})$, the 'natrual' way to define a system of logic with powers X, and the logical types of $\mathbf{S} + L_X(I)$.

After a summary of the theory of PTSs in Section 2, we give a description of the various logical systems that are available to us in Section 3. A logic is specified by choosing any or all of five 'powers' that determine over which types predicates can be formed, and which connectives exist. As the only type forming operation available in a PTS is Π, so only implication and universal quantification are available in our logical systems.

In Section 4, we define $\mathbf{S}+L_X(I)$, and give the translations in both directions. The translation from $\lambda(\mathbf{S}+L_X(I))$ operates by splitting an $\mathbf{S}+L_X(I)$-context Γ into an $\mathbf{S}$-context Γ^-, a set of atomic predicates Γ_{pred}, and a set of hypotheses (propositions) Γ_{proof}. Any judgement with context Γ then corresponds either to an $\mathbf{S}$-judgement with context Γ^-, a definition of a predicate from atomic predicates Γ_{pred} in $\text{Log}_X(\mathbf{S})$, or a proof of a proposition in $\text{Log}_X(\mathbf{S})$ from hypotheses Γ_{proof}. The translation in the other direction is much simpler.

Finally, in Section 5, we show how this result can be generalised to give a condition for one PTS to be a conservative extension of another.

2 Pure Type Systems

For ease of reference, and to establish our notation, we present here a summary of the basic theory of PTSs.

A *specification of a PTS* is a triple (S, A, R), where S is a set of *sorts*, A a set of pairs of sorts, the *axioms*, and R a set of triples of sorts, the *rules*. We shall often write (s_1, s_2) for the rule (s_1, s_2, s_2).

Given a specification $S = (\mathcal{S}, \mathcal{A}, \mathcal{R})$, the PTS λS is as follows:

The *terms* of λS are defined by the grammar

$$Term\ M ::= x \mid s \mid \Pi x : M.M \mid \lambda x : M.M \mid MM$$

where x is a variable and s a sort. x is bound in B but not A within $\Pi x : A.B$ and $\lambda x : A.B$. We identify terms up to α-conversion. We denote by $[M/x]N$ the result of substituting M for each free occurrence of x in N, relabelling bound variables to avoid variable capture. We write $A \to B$ for $\Pi x : A.B$ when $x \notin FV(B)$.

We define β-reduction and conversion by the contraction rule:

$$(\lambda x : A.M)N \rightsquigarrow_\beta [N/x]M$$

A *context* is a finite sequence of pairs $\Gamma \equiv x_1 : A_1, \ldots, x_n : A_n$, where x_1, $\ldots$, x_n are distinct variables, and A_1, $\ldots$, A_n are terms. Its *domain*, $\text{dom}\,\Gamma$, is defined to be the set $\{x_1, \ldots, x_n\}$. A *judgement* is a triple $\Gamma \vdash M : A$, where Γ is a context, and M and A are terms.

The *derivable* judgements, or *theorems* of λS, are given by the following rules:

$$\text{(axioms)} \quad \frac{}{\vdash c : s}\ ((c, s) \in \mathcal{A})$$

$$\text{(start)} \quad \frac{\Gamma \vdash A : s}{\Gamma, x : A \vdash x : A}\ (x \notin \text{dom}\,\Gamma)$$

$$\text{(weakening)}\quad \frac{\Gamma \vdash A : B \quad \Gamma \vdash C : s}{\Gamma, x : C \vdash A : B}\ (x \notin \operatorname{dom}\Gamma)$$

$$\text{(product)}\quad \frac{\Gamma \vdash A : s_1 \quad \Gamma, x : A \vdash B : s_2}{\Gamma \vdash \Pi x : A.B : s_3}\ ((s_1, s_2, s_3) \in \mathcal{R})$$

$$\text{(application)}\quad \frac{\Gamma \vdash F : \Pi x : A.B \quad \Gamma \vdash a : A}{\Gamma \vdash Fa : [a/x]B}$$

$$\text{(abstraction)}\quad \frac{\Gamma, x : A \vdash b : B \quad \Gamma \vdash \Pi x : A.B : s}{\Gamma \vdash \lambda x : A.b : \Pi x : A.B}$$

$$\text{(conversion)}\quad \frac{\Gamma \vdash A : B \quad \Gamma \vdash B' : s}{\Gamma \vdash A : B'}\ (B =_\beta B')$$

A context Γ is *valid* iff $\Gamma \vdash M : A$ for some M, A.

We shall freely use the following results about PTSs. For proofs of the first eight, see [**?**]; for the Condensing Lemma, see [**?**].

Theorem 1 (Church-Rosser) *If $A =_\beta B$, then there exists C such that $A \rhd_\beta C$ and $B \rhd_\beta C$.*

Lemma 2 (Substitution Lemma) *If $\Gamma, x : A, \Delta \vdash B : C$ and $\Gamma \vdash a : A$ then $\Gamma, [a/x]\Delta \vdash [a/x]B : [a/x]C$.*

Lemma 3 (Generation Lemma for PTSs)

1. *If $\Gamma \vdash s : C$, where s is a sort, then, for some sort s', $C =_\beta s'$ and $(s, s') \in \mathcal{A}$.*
2. *If $\Gamma \vdash x : C$, where x is a variable, then, for some B, $C =_\beta B$ and $x : B \in \Gamma$.*
3. *If $\Gamma \vdash \Pi x : A.B : C$, then, for some rule (s_1, s_2, s_3), $\Gamma \vdash A : s_1$, $\Gamma, x : A \vdash B : s_2$, and $C =_\beta s_3$.*
4. *If $\Gamma \vdash \lambda x : A.b : C$, then, for some B and sort s, $\Gamma \vdash \Pi x : A.B : s$, $\Gamma, x : A \vdash b : B$, and $C =_\beta \Pi x : A.B$.*
5. *If $\Gamma \vdash Fa : C$, then, for some A, B, $\Gamma \vdash F : \Pi x : A.B$, $\Gamma \vdash a : A$, and $C =_\beta [a/x]B$.*

Lemma 4 (Type Validity) *If $\Gamma \vdash M : A$, then either A is a sort or $\Gamma \vdash A : s$ for some sort s.*

Lemma 5 (Thinning) *If $\Gamma \vdash M : A$, $\Gamma \subseteq \Delta$ and Δ is valid, then $\Delta \vdash M : A$.*

Lemma 6 (Subject Reduction) *If $\Gamma \vdash M : A$ and $M \rhd_\beta N$, then $\Gamma \vdash N : A$.*

Lemma 7 (Predicate Reduction) *If $\Gamma \vdash M : A$ and $A \rhd_\beta B$, then $\Gamma \vdash M : B$.*

Lemma 8 (Context Conversion) *If $\Gamma \vdash M : A$, Γ' is a legal context, and $\Gamma =_\beta \Gamma'$, then $\Gamma' \vdash M : A$.*

Lemma 9 (Condensing) *If $\Gamma, x : A, \Delta \vdash B : C$ and $x \notin FV(\Delta) \cup FV(B) \cup FV(C)$, then $\Gamma, \Delta \vdash B : C$.*

3 Adding Logic to a PTS Primitively

We describe here what we view as the natural method of adding intuitionistic predicate logic to a PTS, to form the formal system $\mathrm{Log}_X(\mathbf{S})$. This will be our 'control sample'; when we attempt to add logic using the PTS mechanism alone, we shall judge how well we have succeeded by how close we come to $\mathrm{Log}_X(\mathbf{S})$

Let $\mathbf{S}$ be a specification of a PTS, and I a set of sorts of $\mathbf{S}$. (In $\mathrm{Log}_X(\mathbf{S})$, one will be able to quantify over a type A iff $A : s$ for some $s \in I$.)

We must also make a choice as to what powers the logical system will have. The features that can be mimicked by PTSs are:

- First-order predicates
- Higher-order predicates
- Implication
- First-order quantification
- Higher-order quantification

To this end, let $X \subseteq \{1OP, HOP, \rightarrow, \forall, \forall_H\}$ be the set of powers we choose to include.

We define a *predicate type* in the legal context Γ inductively as follows:

- **Prop** is a predicate type in any legal context.
- If $1OP \in X$, there exists $s \in I$ such that $\Gamma \vdash A : s$, and T is a predicate type in $\Gamma, x : A$, then $\Pi x : A.T$ is a predicate type in Γ.
- If $HOP \in X$, and T and U are predicate types in Γ, then $T \rightarrow U$ is a predicate type in Γ.

A *predicate signature* in the context Γ is a sequence of pairs

$$y_1 : F_1, \ldots, y_m : F_m$$

where $y_1, \ldots, y_m$ are distinct variables not in $\mathrm{dom}\,\Gamma$, and $F_1, \ldots, F_m$ are predicate types in the context Γ.

The class of *predicates* in the context Γ is defined by the following grammar. Which clauses are included in the grammar depends on the contents of X as shown.

$$\begin{array}{llll} \text{Predicate } \phi ::= & & x & \\ & [& \mid \lambda x : M.\phi \mid \phi M & \text{if } 1OP \in X &] \\ & [& \mid \lambda x : T.\phi \mid \phi\phi & \text{if } HOP \in X &] \\ & [& \mid \phi \rightarrow \phi & \text{if } \rightarrow \in X &] \\ & [& \mid \forall x : M.\phi & \text{if } \forall \in X &] \\ & [& \mid \forall x : T.\phi & \text{if } \forall_H \in X &] \end{array}$$

where x is a variable, M a term of $\lambda\mathbf{S}$, and T a Γ-predicate type. (We count formulas as special cases of predicates — predicates of type **Prop**. We do not distinguish between predicates and formulas at the raw grammar.)

The class of *proof terms* in the context Γ is defined by the following grammar, that again depends on X.

$$\begin{array}{llll} \text{Proof term } p ::= & & x & \\ & [& \mid \lambda x : \phi.p \mid pp & \text{if } \to \in X \quad] \\ & [& \mid \lambda x : M.p \mid pM & \text{if } \forall \in X \quad] \\ & [& \mid \lambda x : T.p \mid p\phi & \text{if } \forall_H \in X \quad] \end{array}$$

where x is a variable, ϕ a predicate, M a term of $\lambda\mathbf{S}$, and T a Γ-predicate type.

A *logical judgement* is an expression of the form

$$\Gamma \vdash_\Delta P : T$$

or

$$\Gamma \vdash_\Delta z_1 : \phi_1, \ldots, z_n : \phi_n \Rightarrow p : \psi$$

where Γ is a context of $\mathbf{S}$, Δ is a predicate signature in Γ, T is a Γ-predicate type, z_1, ..., z_n are variables, p is a Γ-proof term, and P, ϕ, ϕ_1, ..., ϕ_n and ψ are Γ-predicates.

We give the rules for deriving logical judgements that are available to us in Figure 3. (atom), (hyp) and (conv) are always rules of $\mathrm{Log}_X(S)$; the others are included or not depending on the contents of X as shown.

An *hypothesis* in the context Γ and predicate context Δ is a pair $z : \phi$, where z is a proof variable and $\Gamma \vdash_\Delta \phi : \mathbf{Prop}$. An easy induction shows that, if $\Gamma \vdash_\Delta z_1 : \phi_1, \ldots, z_n : \phi_n \Rightarrow p : \psi$, then $z_1 : \phi_1$, ..., $z_n : \phi_n$ is a list of hypotheses. We shall sometimes use Φ to denote an arbitrary list of hypotheses.

Lemma 10 *If* $\Gamma, x : A, \Gamma' \vdash_\Delta P : T$ *and*

$$\Gamma \vdash a : A$$

then

$$\Gamma, [a/x]\Gamma' \vdash_\Delta [a/x]P : [a/x]T$$

Proof Induction on $\Gamma, x : A, \Gamma' \vdash_\Delta P : T$. **QED**

Lemma 11 (Thinning) *Suppose* $\Gamma \subseteq \Gamma'$ *are* S*-contexts,* $\Delta \subseteq \Delta'$ *are* Γ*-predicate contexts, and* $\Phi \subseteq \Phi'$ *are lists of* Γ, Δ*-hypotheses. Then* Δ *and* Δ' *are* Γ'*-predicate contexts,* Φ *and* Φ' *are lists of* Γ', Δ'*-hypotheses, and:*

- *If* $\Gamma \vdash_\Delta P : T$*, then* $\Gamma' \vdash_{\Delta'} P : T$.
- *If* $\Gamma \vdash_\Delta \Phi \Rightarrow \psi$*, then* $\Gamma' \vdash_{\Delta'} \Phi' \Rightarrow \psi$.

Lemma 12 (Context Conversion) *Suppose* Γ *and* Γ' *are legal* S*-contexts, and* $\Gamma =_\beta \Gamma'$.

1. *If* $\Gamma \vdash_\Delta P : T$*, then* $\Gamma' \vdash_\Delta P : T$.
2. *If* $\Gamma \vdash_\Delta \Phi \Rightarrow p : \psi$*, then* $\Gamma' \vdash_\Delta \Phi \Rightarrow p : \psi$.

- **Common**

$$(\text{atom})\ \frac{}{\Gamma \vdash_\Delta H : T}\ (H : T \in \Delta) \qquad (\text{hyp})\ \frac{\Gamma \vdash_\Delta \phi_1 : \mathbf{Prop} \quad \cdots \quad \Gamma \vdash_\Delta \phi_n : \mathbf{Prop}}{\Gamma \vdash_\Delta z_1 : \phi_1, \ldots, z_n : \phi_n \Rightarrow z_i : \phi_i}$$

$$(\text{pred\_conv})\ \frac{\Gamma \vdash_\Delta P : T}{\Gamma \vdash_\Delta P : U}\ (T =_\beta U) \qquad (\text{prop\_conv})\ \frac{\Gamma \vdash_\Delta \Phi \Rightarrow p : \psi \quad \Gamma \vdash_\Delta \psi' : \mathbf{Prop}}{\Gamma \vdash_\Delta \Phi \Rightarrow p : \psi'}\ (\psi =_\beta \psi')$$

- 1OP

$$(\text{1abs})\ \frac{\Gamma, x : A \vdash_\Delta P : T \quad \Gamma \vdash A : s}{\Gamma \vdash_\Delta \lambda x : A.P : \Pi x : A.T}\ (s \in I) \qquad (\text{1app})\ \frac{\Gamma \vdash_\Delta P : \Pi x : A.T \quad \Gamma \vdash a : A}{\Gamma \vdash_\Delta Pa : [a/x]T}$$

- HOP

$$(\text{Habs})\ \frac{\Gamma \vdash_{\Delta, H:T} P : U}{\Gamma \vdash_\Delta \lambda H : T.P : T \to U} \qquad (\text{Happ})\ \frac{\Gamma \vdash_\Delta P : T \to U \quad \Gamma \vdash_\Delta Q : T}{\Gamma \vdash_\Delta PQ : U}$$

- $\to$

$$(\to)\ \frac{\Gamma \vdash_\Delta \phi : \mathbf{Prop} \quad \Gamma \vdash_\Delta \psi : \mathbf{Prop}}{\Gamma \vdash_\Delta \phi \to \psi : \mathbf{Prop}}$$

$$(\to I)\ \frac{\Gamma \vdash_\Delta \Phi, z : \phi \Rightarrow p : \psi}{\Gamma \vdash_\Delta \Phi \Rightarrow \lambda z : \phi.p : \phi \to \psi}$$

$$(\to E)\ \frac{\Gamma \vdash_\Delta \Phi \Rightarrow p : \phi \to \psi \quad \Gamma \vdash_\Delta \Phi \Rightarrow q : \phi}{\Gamma \vdash_\Delta \Phi \Rightarrow pq : \psi}$$

- $\forall$

$$(\forall)\ \frac{\Gamma, x : A \vdash_\Delta \phi : \mathbf{Prop} \quad \Gamma \vdash A : s}{\Gamma \vdash_\Delta \forall x : A.\phi : \mathbf{Prop}}\ (s \in I)$$

$$(\forall I)\ \frac{\Gamma, x : A \vdash_\Delta \Phi \Rightarrow p : \phi \quad \Gamma \vdash A : s}{\Gamma \vdash_\Delta \Phi \Rightarrow \lambda x : A.p : \forall x : A.\phi}\ (s \in I, x \notin FV(\Phi))$$

$$(\forall E)\ \frac{\Gamma \vdash_\Delta \Phi \Rightarrow p : \forall x : A.\phi \quad \Gamma \vdash a : A}{\Gamma \vdash_\Delta \Phi \Rightarrow pa : [a/x]\phi}$$

- $\forall_H$

$$(\forall_H)\ \frac{\Gamma \vdash_{\Delta, H:T} \phi : \mathbf{Prop}}{\Gamma \vdash_\Delta \forall H : T.\phi : \mathbf{Prop}}$$

$$(\forall_H I)\ \frac{\Gamma \vdash_{\Delta, H:T} \Phi \Rightarrow p : \phi}{\Gamma \vdash_\Delta \Phi \Rightarrow \lambda H : T.p : \forall H : T.\phi}\ (H \notin FV(\Phi))$$

$$(\forall_H E)\ \frac{\Gamma \vdash_\Delta \Phi \Rightarrow p : \forall H : T.\phi \quad \Gamma \vdash_\Delta P : T}{\Gamma \vdash_\Delta \Phi \Rightarrow pP : [P/H]\phi}$$

Figure 1: Rules of deduction for $\mathrm{Log}_X(S)$

Proof Easy induction on the derivation of the premise in each case, using Context Conversion for λS. **QED**

Lemma 13 (Proposition Reduction) *Suppose* $\phi \rhd \psi$.

1. *If* $\Gamma \vdash_\Delta \phi : T$, *then* $\Gamma \vdash_\Delta \psi : T$.
2. *If* $\Gamma \vdash_\Delta \Phi \Rightarrow p : \phi$, *then* $\Gamma \vdash_\Delta \Phi \Rightarrow p : \psi$.

Proof

1. Easy induction on derivation of the premise.
2. Using 1 and (prop_conv).

QED

4 Adding Logic with the PTS Mechanism

We can mimic $\mathrm{Log}_X(\mathbf{S})$ without leaving the PTS mechanism.

Again, let $\mathbf{S}$ be a PTS specification, I a set of sorts of $\mathbf{S}$, and $X \subseteq \{1OP, HOP, \rightarrow, \forall, \forall_H\}$. The specification $\mathbf{S} + L_X(I)$ is then as follows:

The sorts of $\mathbf{S} + L_X(I)$ are the sorts of $\mathbf{S}$ together with new sorts **Prop**, **Pred**.

The axioms of $\mathbf{S} + L_X(I)$ are the axioms of $\mathbf{S}$ together with $(\mathbf{Prop}, \mathbf{Pred})$.

The rules of $\mathbf{S} + L_X(I)$ are the rules of $\mathbf{S}$ together with:

- If $1OP \in X$, $(s, \mathbf{Pred})$ is a rule for all $s \in I$.
- If $HOP \in X$, $(\mathbf{Pred}, \mathbf{Pred})$ is a rule.
- If $\rightarrow \in X$, $(\mathbf{Prop}, \mathbf{Prop})$ is a rule.
- If $\forall \in X$, $(s, \mathbf{Prop})$ is a rule for all $s \in I$.
- If $\forall_H \in X$, $(\mathbf{Pred}, \mathbf{Prop})$ is a rule.

(There is a slight discrepancy in the naming of the new sorts. The types of sort **Prop** are (or are interpreted as) propositions. However, the types of sort **Pred** correspond to predicate arities; it is the objects of these types that are the predicates.)

Example 14 *Let E be the empty specification (with no sorts, no axioms and no rules), and $\rightarrow$ the specification with two sorts $*$, $\Box$, one axiom $(*, \Box)$, and one rule $(*, *)$, the weakest specification in the λ-cube. Recall the specifications* *of the L-cube. (See [**?**] among other places for a discussion of the two cubes). Then*

$$
\begin{array}{rcl}
PROP & = & E + L_{\{\rightarrow\}}(\emptyset) \\
PROP2 & = & E + L_{\{\rightarrow,\forall_H\}}(\emptyset) \\
PROP\underline{\omega} & = & E + L_{\{HOP,\rightarrow\}}(\emptyset) \\
PROP\omega & = & E + L_{\{HOP,\rightarrow,\forall_H\}}(\emptyset) \\
PRED & = & \rightarrow + L_{\{1OP,\rightarrow,\forall\}}(\{*\}) \\
PRED2 & = & \rightarrow + L_{\{1OP,\rightarrow,\forall,\forall_H\}}(\{*\}) \\
PRED\underline{\omega} & = & \rightarrow + L_{\{1OP,HOP,\rightarrow,\forall\}}(\{*\}) \\
PRED\omega & = & \rightarrow + L_{\{1OP,HOP,\rightarrow,\forall,\forall_H\}}(\{*\})
\end{array}
$$

with $\mathbf{Prop} \equiv *^p$, $\mathbf{Pred} \equiv \Box^p$ *in each case.*

$\mathbf{S} + L_X(I)$ is equivalent to the system described in Section 3. To make this statement precise, we first define, for each valid context Γ of $\mathbf{S} + L_X(I)$, the context Γ^- of $\mathbf{S}$, the Γ^--predicate signature Γ_{pred}, and the list of $\Gamma^-, \Gamma_{\text{pred}}$-hypotheses Γ_{proof}. Informally, Γ^- consists of the entries $x : A$ of Γ for which A is a type of $\mathbf{S}$, Γ_{pred} those for which A is a predicate type, and Γ_{proof} those for which A is a proposition. Before we can give the formal definition, we need the following lemmata:

Lemma 15 *If* $\Gamma \vdash A : \mathbf{Pred}$*, then* $A \equiv \Pi x_1 : A_1. \cdots .\Pi x_n : A_n.\mathbf{Prop}$ *for some* $A_1, \ldots, A_n$.

Proof Induction on A. The most difficult case is the vacuous application case:
If $\Gamma \vdash Fa : \mathbf{Pred}$, then, by Generation, there exist A, B such that

$$
\begin{array}{rcl}
\Gamma & \vdash & F : \Pi x : A.B \\
\Gamma & \vdash & a : A \\
{}[a/x]B & \triangleright & \mathbf{Pred}
\end{array}
$$

By Type Validity and Generation, there is a rule (s_1, s_2, s_3) such that

$$
\begin{array}{rcll}
\Gamma & \vdash & A : s_1 & \\
\Gamma, x : A & \vdash & B : s_2 & \\
\therefore \Gamma & \vdash & [a/x]B : s_2 & \text{(Substitution)} \\
\therefore \Gamma & \vdash & \mathbf{Pred} : s_2 & \text{(Subject Reduction)}
\end{array}
$$

and this is impossible. **QED**

Lemma 16 *If* $\Gamma \vdash \Pi x_1 : A_1. \cdots .\Pi x_n : A_n.\mathbf{Prop} : s$*, then* $s \equiv \mathbf{Pred}$.

Proof Easy induction on n. **QED**

Lemma 17 (Uniqueness of Types for Predicates) *If* $\Gamma \vdash M : \Pi x_1 : A_1. \cdots .\Pi x_n : A_n.\mathbf{Prop}$ *and* $\Gamma \vdash M : X$*, then* $X =_\beta \Pi x_1 : A_1. \cdots .\Pi x_n : A_n.\mathbf{Prop}$.

Proof Induction on M.

If M is a variable, then $\Pi x_1 : A_1. \cdots .\Pi x_n : A_n.\mathbf{Prop} =_\beta X$ by Generation.

If $M \equiv \lambda x_1 : B.C$, then, by Generation, $n > 0$ and

$$A_1 =_\beta B, \Pi x_2 : A_2. \cdots .\Pi x_n : A_n.\mathbf{Prop} =_\beta D, \quad \Gamma, x_1 : B \vdash C : D$$

Also, $X =_\beta \Pi x_1 : B.D'$, where

$$\Gamma, x_1 : B \vdash C : D'$$

We have

$$\begin{array}{rcll} \Gamma, x_1 : A_1 & \vdash & C : D & \text{(Context Conversion)} \\ \therefore \Gamma, x_1 : A_1 & \vdash & C : \Pi x_2 : A_2. \cdots .\Pi x_n : A_n.\mathbf{Prop} & \text{(conversion)} \\ \therefore D' & =_\beta & \Pi x_2 : A_2. \cdots .\Pi x_n : A_n.\mathbf{Prop} & \text{(i.h.)} \\ \therefore X & =_\beta & \Pi x_1 : B.\Pi x_2 : A_2. \cdots .\Pi x_n : A_n.\mathbf{Prop} & \\ & =_\beta & \Pi x_1 : A_1.\Pi x_2 : A_2. \cdots .\Pi x_n : A_n.\mathbf{Prop} & \end{array}$$

If $M \equiv Fa$, then, for some C, D, C', D',

$$\begin{array}{rccc} \Gamma \vdash & F : \Pi y : C.D \Gamma & \vdash F : \Pi y : C'.D' \\ \Gamma \vdash & a : C \Gamma & \vdash a : C' \\ [a/y]D =_\beta & \Pi x_1 : A_1. \cdots .\Pi x_n : A_n.\mathbf{Prop}[a/y]D' & =_\beta X \\ \therefore [a/y]D \rhd & \Pi x_1 : A_1^*. \cdots .\Pi x_n : A_n^*.\mathbf{Prop} & \end{array}$$

where $A_1 \rhd A_1^*$, ..., $A_n \rhd A_n^*$. By Type Validity and Generation, there exists a rule (s_1, s_2, s_3) such that

$$\begin{array}{rcll} \Gamma & \vdash & C : s_1 & \\ \Gamma, y : C & \vdash & D : s_2 & \\ \therefore \Gamma & \vdash & [a/y]D : s_2 & \text{(Conversion)} \\ \therefore \Gamma & \vdash & \Pi x_1 : A_1^*. \cdots .\Pi x_n : A_n^*.\mathbf{Prop} : s_2 & \text{(Subject Reduction)} \\ \therefore s_2 & \equiv & \mathbf{Pred} & \text{(Lemma 16)} \\ \therefore D & \equiv & \Pi x_1 : D_1. \cdots .\Pi x_n : D_n.\mathbf{Prop} & \text{(Lemma 15)} \end{array}$$

where $[a/y]D_1 =_\rhd A_1^*$, ..., $[a/y]D_n \rhd A_n^*$.

So we have

$$\Gamma \vdash F : \Pi y : C.\Pi x_1 : D_1. \cdots .\Pi x_n : D_n.\mathbf{Prop}$$

Therefore, the i.h. gives

$$\begin{array}{rcl} \Pi y : C'.D' & =_\beta & \Pi y : C.\Pi x_1 : D_1. \cdots .\Pi x_n : D_n.\mathbf{Prop} \\ \therefore X & =_\beta & [a/y]D' \\ & =_\beta & \Pi x_1 : [a/y]D_1. \cdots .\Pi x_n : [a/y]D_n.\mathbf{Prop} \\ & =_\beta & \Pi x_1 : A_1. \cdots .\Pi x_n : A_n.\mathbf{Prop} \end{array}$$

QED

Lemma 18 *At most one of*

- $\Gamma \vdash A : \mathbf{Prop}$
- $\Gamma \vdash A : \mathbf{Pred}$
- $\Gamma \vdash A : s$ *for some sort* s *of* S

holds for a given Γ, A.

Proof By Lemmata 15 and 16, we cannot have the first and second, nor the second and third, alternatives holding together. By Lemma 17, we cannot have the first and third alternatives holding together. **QED**

Definition 19 *Define, for each legal* $S + L_X(I)$*-context* Γ*, the lists* Γ^-, Γ_{pred} *and* Γ_{proof} *as follows:*

- $\langle\rangle^- \equiv \langle\rangle_{\text{pred}} \equiv \langle\rangle_{\text{proof}} \equiv \langle\rangle$
-

$$
\begin{aligned}
(\Gamma, x : A)^- &\equiv \begin{cases} \Gamma^-, x : A & \textit{if } \Gamma \vdash A : s, \textit{where } s \textit{ is a sort of } S \\ \Gamma^- & \textit{if } \Gamma \vdash A : \mathbf{Prop} \textit{ or } \Gamma \vdash A : \mathbf{Pred} \end{cases} \\
(\Gamma, x : A)_{\text{pred}} &\equiv \begin{cases} \Gamma_{\text{pred}}, x : A & \textit{if } \Gamma \vdash A : \mathbf{Pred} \\ \Gamma_{\text{pred}} & \textit{if } \Gamma \vdash A : \mathbf{Prop} \textit{ or } \Gamma \vdash A : s \textit{ for some sort } s \textit{ of } S \end{cases} \\
(\Gamma, x : A)_{\text{proof}} &\equiv \begin{cases} \Gamma_{\text{proof}}, x : A & \textit{if } \Gamma \vdash A : \mathbf{Prop} \\ \Gamma_{\text{proof}} & \textit{if } \Gamma \vdash A : \mathbf{Pred} \textit{ or } \Gamma \vdash A : s \textit{ for some sort } s \textit{ of } S \end{cases}
\end{aligned}
$$

(By the Generation Lemma, one of each pair of cases holds; by Lemma 18, only one holds.)

We can now give translations between $\lambda(\mathbf{S} + L_X(I))$ and $\text{Log}_X(\mathbf{S})$. The translation one way is almost trivial:

Theorem 20 *1. If* $\Gamma \vdash_\Delta P : T$*, then* $\Gamma, \Delta \vdash_{\mathbf{S}+L_X(I)} P : T$.

2. If $\Gamma \vdash_\Delta x_1 : \phi_1, \ldots, x_n : \phi_n \Rightarrow p : \psi$*, then* $\Gamma, \Delta, x_1 : \phi_1, \ldots, x_n : \phi_n \vdash_{\mathbf{S}+L_X(I)} p : \psi$.

Proof

1. First show, for each Γ-predicate context Δ, that Γ, Δ is a legal $\mathbf{S} + L_X(I)$-context, by induction on Δ. Then prove the result by induction on $\Gamma \vdash_{\Gamma_{\text{pred}}} \phi : \mathbf{Prop}$.
2. Induction on $\Gamma \vdash_\Delta x_1 : \phi_1, \ldots, x_n : \phi_n \Rightarrow p : \psi$.

QED

The main work comes in giving the translation in the opposite direction:

Theorem 21 *Suppose $\Gamma \vdash_{\mathbf{S}+L_X(I)} M : A$. Then:*

1. *Γ^- is a legal* **S***-context, Γ_{pred} is a Γ^--predicate context, and Γ_{proof} is a list of $\Gamma^-, \Gamma_{\mathrm{pred}}$-hypotheses.*

2. *Exactly one of the following holds:*

   (a) $\Gamma^- \vdash_{\mathbf{S}} M : A$.

   (b) $\Gamma \vdash A : \mathbf{Prop}$, *and* $\Gamma^- \vdash_{\Gamma_{\mathrm{pred}}} \Gamma_{\mathrm{proof}} \Rightarrow M : A$.

   (c) $\Gamma \vdash A : \mathbf{Pred}$, *and* $\Gamma^- \vdash_{\Gamma_{\mathrm{pred}}} M : A$.

   (d) $A \equiv \mathbf{Pred}$, *and M is a Γ^--predicate type.*

Proof Note first that, by Lemma 18 and the fact that $\Gamma \nvdash s : \mathbf{Prop}$ and $\Gamma \nvdash s : \mathbf{Pred}$ for any sort s of S (by Generation), at most one of 2a, 2b, 2c and 2d can hold for a given Γ, M, A.

We prove 1 and that at least one of 2a, 2b, 2c, 2d holds, by induction on $\Gamma \vdash M : A$.

(axioms)

$$\frac{}{\vdash c : s}\ (c : s \in A)$$

$\langle\rangle$ is a legal **S**-context, $\langle\rangle$ is an $\langle\rangle$-predicate context, and $\langle\rangle$ is a list of $\langle\rangle, \langle\rangle$-hypotheses.

If $c : s$ is an axiom of **S**, then 2a holds.

If $c \equiv \mathbf{Prop}$, $s \equiv \mathbf{Pred}$, then 2d holds.

(start)

$$\frac{\Gamma \vdash A : s}{\Gamma, x : A \vdash x : A}\ (x \notin \mathrm{dom}\,\Gamma)$$

- If s is a sort of **S**, then:

  $$\begin{array}{rcll} \Gamma^- & \vdash & A : s & \text{(i.h.)} \\ \therefore \Gamma^-, x : a & \vdash & A : s & \text{(weakening)} \end{array}$$

  Thus, $(\Gamma, x : A)^- \equiv \Gamma^-, x : A$ is a legal **S**-context, and 2a holds.
  $(\Gamma, x : A)_{\mathrm{pred}} \equiv \Gamma_{\mathrm{pred}}$ is a $\Gamma^-, x : A$-predicate context (by (weakening)), and $(\Gamma, x : A)_{\mathrm{proof}} \equiv \Gamma_{\mathrm{proof}}$ is a list of $\Gamma^-, x : A, \Gamma_{\mathrm{pred}}$-hypotheses.

- If $s \equiv \mathbf{Prop}$, then:
  $(\Gamma, x : A)^- \equiv \Gamma^-$ is a legal **S**-context.
  $(\Gamma, x : A)_{\mathrm{pred}} \equiv \Gamma_{\mathrm{pred}}$ is a Γ^--predicate context.
  2c holds of the premise (by i.h.), so $\Gamma^- \vdash_{\Gamma_{\mathrm{pred}}} A : \mathbf{Prop}$. $\therefore$ $(\Gamma, x : A)_{\mathrm{proof}} \equiv \Gamma_{\mathrm{proof}}, x : A$ is a list of $\Gamma^-, \Gamma_{\mathrm{pred}}$-hypotheses. By (hyp),

  $$\Gamma^- \vdash_{\Gamma_{\mathrm{pred}}} \Gamma_{\mathrm{proof}}, x : A \Rightarrow x : A$$

  and so 2b holds of the conclusion.

- If $s \equiv \mathbf{Pred}$, then: $(\Gamma, x : A)^- \equiv \Gamma^-$ is a legal $\mathbf{S}$-context.

  2d holds of the premise (by i.h.), so A is a Γ^--predicate type. Therefore, $(\Gamma, x : A)_{\text{pred}} \equiv \Gamma_{\text{pred}}, x : A$ is a Γ^--predicate context.

  $(\Gamma, x : A)_{\text{proof}} \equiv \Gamma_{\text{proof}}$ is a list of $\Gamma^-, \Gamma_{\text{pred}}$-hypotheses, which are $\Gamma^-, \Gamma_{\text{pred}}, x : A$-hypotheses.

  $\Gamma^- \vdash_{\Gamma_{\text{pred}}, x:A} x : A$ by (atom).

(weakening)

$$\frac{\Gamma \vdash M : A \quad \Gamma \vdash B : s}{\Gamma, x : B \vdash M : A} \; (x \notin \operatorname{dom} \Gamma)$$

- If s is a sort of $\mathbf{S}$, then:

$$\begin{array}{rcll} \Gamma^- & \vdash_{\mathbf{S}} & B : s & \text{(i.h.)} \\ \therefore \Gamma^-, x : B & \vdash_{\mathbf{S}} & x : B & \text{(start)} \end{array}$$

  Thus, $(\Gamma, x : B)^- \equiv \Gamma^-, x : B$ is a legal $\mathbf{S}$-context,

  $(\Gamma, x : B)_{\text{pred}} \equiv \Gamma_{\text{pred}}$ is a Γ^--predicate context, hence a $\Gamma^-, x : B$-predicate context.

  $(\Gamma, x : B)_{\text{proof}} \equiv \Gamma_{\text{proof}}$ is a list of $\Gamma^-, \Gamma_{\text{pred}}$-hypotheses, hence a list of $\Gamma^-, x : B, \Gamma_{\text{pred}}$-hypotheses.

  By Thinning, whichever of 2a, 2b, 2c, 2d holds of $\Gamma \vdash M : A$ holds of the conclusion.

- If $s \equiv \mathbf{Prop}$, then, by i.h.,

$$\Gamma^- \vdash_{\Gamma_{\text{pred}}} B : \mathbf{Prop}$$

  $(\Gamma, x : B)^- \equiv \Gamma^-$ is a legal $\mathbf{S}$-context.

  $(\Gamma, x : B)_{\text{pred}} \equiv \Gamma_{\text{pred}}$ is a Γ^--predicate context.

  $(\Gamma, x : B)_{\text{proof}} \equiv \Gamma_{\text{proof}}, x : B$ is a list of $\Gamma^-, \Gamma_{\text{pred}}$-hypotheses.

  By Thinning, whichever of 2a, 2b, 2c, 2d holds of $\Gamma \vdash M : A$ holds of the conclusion.

- If $s \equiv \mathbf{Pred}$, then, by i.h., B is a Γ^--predicate type. Therefore, $(\Gamma, x : B)^- \equiv \Gamma^-$ is a legal $\mathbf{S}$-context.

  $(\Gamma, x : B)_{\text{pred}} \equiv \Gamma_{\text{pred}}, x : B$ is a Γ^--predicate context.

  $(\Gamma, x : B)_{\text{proof}} \equiv \Gamma_{\text{proof}}$ is a list of $\Gamma^-, \Gamma_{\text{pred}}$-hypotheses, hence of $\Gamma^-, \Gamma_{\text{pred}}, x : B$-hypotheses.

  By Thinning, whichever of 2a, 2b, 2c, 2d holds of $\Gamma \vdash M : A$ holds of the conclusion.

(product)

$$\frac{\Gamma \vdash A : s_1 \quad \Gamma, x : A \vdash B : s_2}{\Gamma \vdash \Pi x : A.B : s_3} \; ((s_1, s_2, s_3) \in R)$$

1 holds by i.h.

- If (s_1, s_2, s_3) is a rule of **S**:

$$\begin{array}{rll} \Gamma^- & \vdash_{\mathbf{S}} & A : s_1 \qquad \text{(i.h.)} \\ \Gamma^-, x : A & \vdash_{\mathbf{S}} & B : s_2 \qquad \text{(i.h.)} \\ \therefore \Gamma^- & \vdash_{\mathbf{S}} & \Pi x : A.B : s_3 \qquad \text{(product)} \end{array}$$

  Thus, 2a holds.

- If $1OP \in X$ and $(s_1, s_2, s_3) \equiv (s_1, \mathbf{Pred}, \mathbf{Pred})$, where $s_1 \in I$, then, by i.h., B is a $\Gamma^-, x : A$-predicate type, and

$$\Gamma^- \vdash_{\mathbf{S}} A : s_1$$

  Therefore, $\forall x : A.B$ is a Γ^--predicate type. Thus, 2d holds.

- If $HOP \in X$ and $(s_1, s_2, s_3) \equiv (\mathbf{Pred}, \mathbf{Pred}, \mathbf{Pred})$, then, by i.h., A and B are Γ^--predicate types. Therefore, $x \notin FV(B)$, so $\Pi x : A.B \equiv A \to B$ is a Γ^--predicate type. Hence, 2d holds.

- If $\to \in X$ and $(s_1, s_2, s_3) \equiv (\mathbf{Prop}, \mathbf{Prop}, \mathbf{Prop})$, then, by i.h.,

$$\begin{array}{rll} \Gamma^- & \vdash_{\Gamma_{\text{pred}}} & A : \mathbf{Prop} \\ \Gamma^- & \vdash_{\Gamma_{\text{pred}}} & B : \mathbf{Prop} \\ \therefore \Gamma^- & \vdash_{\Gamma_{\text{pred}}} & A \to B : \mathbf{Prop} \qquad (\to) \end{array}$$

  Also, $\Pi x : A.B \equiv A \to B$. Thus, 2c holds.

- If $\forall \in X$ and $(s_1, s_2, s_3) \equiv (s_1, \mathbf{Prop}, \mathbf{Prop})$, where $s_1 \in I$, then, by i.h.,

$$\begin{array}{rll} \Gamma^- & \vdash_{\mathbf{S}} & A : s_1 \\ \Gamma^-, x : A & \vdash_{\Gamma_{\text{pred}}} & B : \mathbf{Prop} \\ \therefore \Gamma^- & \vdash_{\Gamma_{\text{pred}}} & \forall x : A.B : \mathbf{Prop} \qquad (\forall) \end{array}$$

  Thus, 2c holds.

- If $\forall_H \in X$ and $(s_1, s_2, s_3) \equiv (\mathbf{Pred}, \mathbf{Prop}, \mathbf{Prop})$, then, by i.h.,

$$\begin{array}{rll} \Gamma^- & \vdash_{\Gamma_{\text{pred}}, x:A} & B : \mathbf{Prop} \\ \therefore \Gamma^- & \vdash_{\Gamma_{\text{pred}}} & \forall x : A.B : \mathbf{Prop} \qquad (\forall_H) \end{array}$$

  Thus, 2c holds.

(application)

$$\frac{\Gamma \vdash F : \Pi x : A.B \quad \Gamma \vdash a : A}{\Gamma \vdash Fa : [a/x]B}$$

1 holds by i.h.

- If 2a holds of $\Gamma \vdash F : \Pi x : A.B$, then we have

$$\Gamma^- \vdash_{\mathbf{S}} F : \Pi x : A.B$$

By Type Validity and Generation,

$$\Gamma^- \vdash_{\mathbf{S}} A : s$$

for some sort s of $\mathbf{S}$, and so 2a holds of $\Gamma \vdash a : A$. Therefore,

$$\begin{array}{rcll} \Gamma^- & \vdash & a : A & \text{(i.h.)} \\ \therefore \Gamma^- & \vdash & Fa : [a/x]B & \text{(application)} \end{array}$$

Thus, 2a holds of the conclusion.

- If 2b holds of $\Gamma \vdash F : \Pi x : A.B$, then

$$\begin{array}{rcl} \Gamma^- & \vdash_{\Gamma_{\text{pred}}} & \Pi x : A.B : \mathbf{Prop} \\ \Gamma^- & \vdash_{\Gamma_{\text{pred}}} & \Gamma'_{\text{proof}} \Rightarrow F : \Pi x : A.B \end{array}$$

Inverting the first judgement:

Case One

$$\begin{array}{rcl} \rightarrow & \in & X \\ \Pi x : A.B & \equiv & A \rightarrow B \\ \Gamma^- & \vdash_{\Gamma_{\text{pred}}} & A : \mathbf{Prop} \\ \Gamma^- & \vdash_{\Gamma_{\text{pred}}} & B : \mathbf{Prop} \end{array}$$

Therefore, 2b holds of $\Gamma \vdash a : A$. We have

$$\begin{array}{rcll} \Gamma^- & \vdash_{\Gamma_{\text{pred}}} & \Gamma_{\text{proof}} \Rightarrow a : A & \\ \therefore \Gamma^- & \vdash_{\Gamma_{\text{pred}}} & \Gamma_{\text{proof}} \Rightarrow Fa : B & (\rightarrow E) \end{array}$$

and so 2b holds of the conclusion.

Case Two

$$\begin{array}{rcl} \forall & \in & X \\ \Gamma^-, x : A & \vdash_{\Gamma_{\text{pred}}} & B : \mathbf{Prop} \end{array}$$

Then 2a holds of $\Gamma \vdash a : A$; i.e.

$$\begin{array}{rcll} \Gamma^- & \vdash_{\mathbf{S}} & a : A & \\ \therefore \Gamma^- & \vdash_{\Gamma_{\text{pred}}} & Fa : [a/x]B & (\forall E) \end{array}$$

and so 2b holds of the conclusion.

Case Three

$$\begin{array}{lll} \forall_H & \in & X \\ \Gamma^- & \vdash_{\Gamma_{\text{pred}}, x:A} & B : \mathbf{Prop} \end{array}$$

Then 2c holds of $\Gamma \vdash a : A$, i.e.

$$\begin{array}{llll} \Gamma^- & \vdash_{\Gamma_{\text{pred}}} & a : A & \\ \therefore \Gamma^- & \vdash_{\Gamma_{\text{pred}}} & Fa : [a/x]B & (\forall_H E) \end{array}$$

- If 2c holds of $\Gamma \vdash F : \Pi x : A.B$, then

$$\Gamma^- \vdash_{\Gamma_{\text{pred}}} F : \Pi x : A.B$$

There are two possibilities:

Case One

$$\begin{array}{llll} 1OP & \in & X & \\ \Gamma^- & \vdash_{\mathbf{S}} & A : s & (s \in I) \end{array}$$

and B is a $\Gamma^-, x : A$-predicate type. Then 2a holds of $\Gamma \vdash a : A$, i.e.

$$\begin{array}{llll} \Gamma^- & \vdash_{\mathbf{S}} & a : A & \\ \therefore \Gamma^- & \vdash_{\Gamma_{\text{pred}}} & Fa : [a/x]B & (1\text{app}) \end{array}$$

Thus, 2c holds of the conclusion.

Case Two

$$\begin{array}{lll} HOP & \in & X \\ \Pi x : A.B & \equiv & A \to B \end{array}$$

and A and B are Γ^--predicate types. Then 2c holds of $\Gamma \vdash a : A$, and so

$$\begin{array}{llll} \Gamma^- & \vdash_{\Gamma_{\text{pred}}} & a : A & \\ \therefore \Gamma^- & \vdash_{\Gamma_{\text{pred}}} & Fa : B & (\text{Happ}) \end{array}$$

Thus, 2c holds of the conclusion.

(abstraction)

$$\frac{\Gamma, x : A \vdash b : B \quad \Gamma \vdash \Pi x : A.B : s}{\Gamma \vdash \lambda x : A.b : \Pi x : A.B}$$

1 holds by i.h.

- If 2a holds of $\Gamma \vdash \Pi x : A.B : s$, then

$$\Gamma^{-} \vdash_{\mathbf{S}} \Pi x : A.B : s$$

Generation easily shows that 2a holds of $\Gamma, x : A \vdash b : B$; i.e.

$$\begin{array}{rll} \Gamma^{-}, x : A & \vdash_{\mathbf{S}} & b : B \\ \therefore \Gamma^{-} & \vdash_{\mathbf{S}} & \lambda x : A.b : \Pi x : A.B \qquad \text{(abstraction)} \end{array}$$

- If 2c holds of $\Gamma \vdash \Pi x : A.B : s$, then $s \equiv \mathbf{Prop}$, and

$$\Gamma^{-} \vdash_{\Gamma_{\text{pred}}} \Pi x : A.B : \mathbf{Prop}$$

Case One

$$\begin{array}{rcl} \rightarrow & \in & X \\ \Pi x : A.B & \equiv & A \rightarrow B \\ \Gamma^{-} & \vdash_{\Gamma_{\text{pred}}} & A : \mathbf{Prop} \\ \Gamma^{-} & \vdash_{\Gamma_{\text{pred}}} & B : \mathbf{Prop} \end{array}$$

By i.h., 2b holds of $\Gamma, x : A \vdash b : B$; so

$$\Gamma^{-} \vdash_{\Gamma_{\text{pred}}} \Gamma_{\text{proof}}, x : A \Rightarrow b : B$$

where $b \rhd b$, $B \rhd B$. So

$$\Gamma^{-} \vdash_{\Gamma_{\text{pred}}} \Gamma_{\text{proof}} \Rightarrow \lambda x : A.b : A \rightarrow B \qquad (\rightarrow I)$$

Thus, 2b holds of the conclusion.

Case Two

$$\begin{array}{rcl} \forall & \in & X \\ \Gamma^{-}, x : A & \vdash_{\Gamma_{\text{pred}}} & B : \mathbf{Prop} \end{array}$$

By i.h., 2b holds of $\Gamma, x : A \vdash b : B$, so

$$\begin{array}{rll} \Gamma^{-}, x : A & \vdash_{\Gamma_{\text{pred}}} & \Gamma_{\text{proof}} \Rightarrow b : B \\ \therefore \Gamma^{-} & \vdash_{\Gamma_{\text{pred}}} & \Gamma_{\text{proof}} \Rightarrow \lambda x : A.b : \forall x : A.B \qquad (\forall I) \end{array}$$

Thus, 2b holds of the conclusion.

Case Three

$$\begin{array}{rcl} \forall_{H} & \in & X \\ \Gamma^{-} & \vdash_{\Gamma_{\text{pred}}, x:A} & B : \mathbf{Prop} \end{array}$$

By i.h., 2b holds of $\Gamma, x : A \vdash b : B$, so

$$\begin{array}{rll} \Gamma^{-} & \vdash_{\Gamma_{\text{pred}}, x:A} & \Gamma_{\text{proof}} \Rightarrow b : B \\ \therefore \Gamma^{-} & \vdash_{\Gamma_{\text{pred}}} & \Gamma_{\text{proof}} \Rightarrow \lambda x : A.b : \forall x : A.B \end{array}$$

Thus, 2b holds of the conclusion.

- If 2d holds of $\Gamma \vdash \Pi x : A.B : s$, then 2c holds of $\Gamma, x : A \vdash b : B$.

  Case One

$$\begin{array}{rll} 1OP & \in & X \\ \Gamma^-, x : A & \vdash_{\Gamma_{\text{pred}}} & b : B \\ \therefore \Gamma^- & \vdash_{\Gamma_{\text{pred}}} & \lambda x : A.b : \Pi x : A.B \qquad \text{(1abs)} \end{array}$$

  Case Two

$$\begin{array}{rll} HOP & \in & X \\ \Gamma^- & \vdash_{\Gamma_{\text{pred}}, x:A} & b : B \\ \therefore \Gamma^- & \vdash_{\Gamma_{\text{pred}}} & \lambda x : A.b : A \to B \qquad \text{(Habs)} \end{array}$$

  In either case, 2c holds of the conclusion.

(conversion)

$$\frac{\Gamma \vdash A : B \quad \Gamma \vdash B' : s}{\Gamma \vdash A : B'} \; (B =_\beta B')$$

1 holds by i.h.

By Church-Rosser, there exists X such that $B \rhd X$, $B' \rhd X$.

- Suppose 2a is true of $\Gamma \vdash B' : s$. Then s is a sort of **S** and

$$\Gamma \vdash X : s$$

  by Subject Reduction; hence, $\Gamma \nvdash B : \mathbf{Prop}$, $\Gamma \nvdash B : \mathbf{Pred}$, $B \not\equiv \mathbf{Pred}$ lest we have

$$\Gamma \vdash X : \mathbf{Prop}, \quad \Gamma \vdash X : \mathbf{Pred}, \text{ or } \Gamma \vdash \mathbf{Pred} : s$$

  by Subject Reduction. The first two of these would contradict Lemma 18, the third is impossible by Generation.
  Therefore, 2a is true of $\Gamma \vdash A : B$. Hence,

$$\begin{array}{rll} \Gamma^- & \vdash_{\mathbf{S}} & A : B \\ \Gamma^- & \vdash_{\mathbf{S}} & B' : s \\ \therefore \Gamma^- & \vdash_{\mathbf{S}} & A : B' \qquad \text{(conversion)} \end{array}$$

  and so 2a holds of the conclusion.

- If 2c is true of $\Gamma \vdash B' : s$, then $s \equiv \mathbf{Prop}$. Now, $\Gamma \vdash X : \mathbf{Prop}$ by Subject Reduction, so $\Gamma \vdash B : \mathbf{Prop}$ by considering cases as above. Therefore, 2b is true of $\Gamma \vdash A : B$. So we have,

$$\begin{array}{rll} \Gamma^- & \vdash_{\Gamma_{\text{pred}}} & \Gamma_{\text{proof}} \Rightarrow A : B \\ \Gamma^- & \vdash_{\Gamma_{\text{pred}}} & B' : \mathbf{Prop} \\ \therefore \Gamma^- & \vdash_{\Gamma_{\text{pred}}} & \Gamma_{\text{proof}} \Rightarrow A : B' \qquad \text{(conv)} \end{array}$$

  Thus, 2b is true of the conclusion.

- If 2d is true of $\Gamma \vdash B' : s$, then

  $$\Gamma \vdash X : \mathbf{Pred}$$

  Considering cases as above, we find that

  $$\Gamma \vdash B : \mathbf{Pred}$$

  and so 2c is true of $\Gamma \vdash A : B$. Therefore,

  $$\begin{array}{lll} \Gamma^- & \vdash_{\Gamma_{\text{pred}}} & A : B \\ \therefore \Gamma^- & \vdash_{\Gamma_{\text{pred}}} & A : B' \qquad \text{(pred\_conv)} \end{array}$$

  and so 2c is true of the conclusion.

QED

Corollary 22 *$\lambda(\mathbf{S} + NL(I))$ is a conservative extension of $\lambda\mathbf{S}$.*

Proof Suppose $\Gamma \vdash M : A$, Γ is an **S**-context, and M and A are **S**-terms. By Theorem 21,

$$\Gamma^- \vdash_{\mathbf{S}} M : A$$

(as A cannot reduce to a $\Gamma^-, \Gamma_{\text{pred}}$-proposition, nor be **Pred** nor a Γ^--predicate type). It is therefore sufficient to show that $\Gamma^- \equiv \Gamma$ for all **S**-contexts Γ legal in $\lambda(S + NL(I))$. The proof is by induction on Γ.

$$\langle\rangle^- \equiv \langle\rangle$$

Suppose $\Gamma^- \equiv \Gamma$. Consider the valid **S**-context $\Gamma, x : A$. By Theorem 21, if $\Gamma \vdash A : \mathbf{Prop}$ or $\Gamma \vdash A : \mathbf{Pred}$, then A is not a Γ^--term in **S**, hence not a Γ-term in **S**. This is a contradiction. Therefore, $\Gamma \vdash A : s$ for some sort s of **S**; and so

$$(\Gamma, x : A)^- \equiv \Gamma^-, x : A \equiv \Gamma, x : A$$

QED

The two translations are exact inverses of one another.

5 Conservative Extensions of PTSs

Let $\mathbf{S}_1 = (\mathcal{S}, \mathcal{A}, \mathcal{R})$ be a PTS specification, and $\mathbf{S}_2 = (\mathcal{S} \cup \mathcal{S}', \mathcal{A} \cup \mathcal{A}', \mathcal{R} \cup \mathcal{R}')$ be an extension of $\mathbf{S}_1$ such that:

- For each new axiom $c : s \in \mathcal{A}'$, $c, s \in \mathcal{S}'$.
- For each new rule $(s_1, s_2, s_3) \in \mathcal{R}'$, $s_2, s_3 \in \mathcal{S}'$.

Lemma 23 *It is never the case that $\Gamma \vdash A : s$ and $\Gamma \vdash A : s'$, where $s \in \mathcal{S}$ and $s' \in \mathcal{S}'$.*

Proof As in Definition 5.1 of [?], we divide the terms of any PTS into *sort-like* and *variable-like* terms as follows:

- Every sort is sort-like.
- Every variable is variable-like.
- $\lambda x : A.B$ and BA are the same kind as B.
- $\Pi x : A.B$ is sort-like.

We write $\mathcal{T}_S$ for the set of all sort-like terms, and $\mathcal{T}_V$ for the set of all variable-like terms.

Claim 1 If A is variable-like, $\Gamma \vdash A : B$ and $\Gamma \vdash A : C$, then $B =_\beta C$.

Proof See [?], Lemma 4.3

Define $\Sigma_\Gamma(A)$ for each context Γ and sort-like term A as follows:

$$\begin{array}{rcl} \Sigma_\Gamma(s) & = & \{t \in \mathcal{S} \mid s : t \in \mathcal{A}\} \\ \Sigma_\Gamma(\lambda x : A.B) & = & \Sigma_{\Gamma, x:A}(B) \\ \Sigma_\Gamma(BA) & = & \Sigma_\Gamma(B) \\ \Sigma_\Gamma(\Pi x : A.B) & = & \{s_3 \in \mathcal{S} \mid (\exists s_1, s_2 \in \mathcal{S})((s_1, s_2, s_3) \in \mathcal{R} \wedge \\ & & \quad ((A \in \mathcal{T}_S \wedge s_1 \in \Sigma_\Gamma(A)) \vee (A \in \mathcal{T}_V \wedge \Gamma \vdash A : s_1)) \wedge \\ & & \quad ((B \in \mathcal{T}_S \wedge s_2 \in \Sigma_{\Gamma, x:A}(B)) \vee (B \in \mathcal{T}_V \wedge \Gamma, x : A \vdash B : s_2)))\} \end{array}$$

Claim 2 If A is sort-like and $\Gamma \vdash A : s$, then $s \in \Sigma_\Gamma(A)$.

Proof See [?] Corollary 5.4

Thus, to prove this lemma, it is sufficient to prove that, for $s \in \mathcal{S}$ and $s' \in \mathcal{S}'$, it is never the case that $s, s' \in \Sigma_\Gamma(A)$ for any context Γ and $A \in \mathcal{T}_S$. The proof is by induction on A.

$c \in \mathcal{S} \cup \mathcal{S}'$

Suppose $s, s' \in \Sigma_\Gamma(c)$. Then $c : s, c : s' \in \mathcal{A} \cup \mathcal{A}'$. As $s \in \mathcal{S}$, $c : s \in \mathcal{A}$, hence $c \in \mathcal{S}$. As $s' \in \mathcal{S}'$, $c : s' \in \mathcal{A}'$, hence $c \in \mathcal{S}'$. This is a contradiction.

$\lambda x : A.B$

Immediate from i.h.

BA

Immediate from i.h.

$\Pi x : A.B$

Suppose $s_3, s'_3 \in \Sigma_\Gamma(\Pi x : A.B)$, where $s_3 \in \mathcal{S}$, $s'_3 \in \mathcal{S}'$. Let $(s_1, s_2, s_3) \in \mathcal{R}$, $(s'_1, s'_2, s'_3) \in \mathcal{R}'$ verify the membership condition of $\Sigma_\Gamma(\Pi x : A.B)$. Then $s_2 \in \mathcal{S}$ and $s'_2 \in \mathcal{S}'$.

If B is sort-like, then $s_2, s'_2 \in \Sigma_{\Gamma,x:A}(B)$, which contradicts i.h.

If B is variable-like, then

$$\begin{array}{rcll} \Gamma, x : A & \vdash & B : s_2 & \\ \Gamma, x : A & \vdash & B : s'_2 & \\ \therefore s_2 & \equiv & s'_2 & \text{(Claim 1)} \end{array}$$

This is again a contradiction.

QED

Definition 24 *For each legal* $\mathbf{S}_2$*-context* Γ*, we define the* $\mathbf{S}_1$*-context* Γ^- *thus:*

- $\langle\rangle^- \equiv \langle\rangle$
- $(\Gamma, x : A)^- \equiv \begin{cases} \Gamma^-, x : A & \textit{if } \Gamma \vdash A : s,\ s \in \mathcal{S} \\ \Gamma^-, x : A & \textit{if } \Gamma \vdash A : s,\ s \in \mathcal{S}' \end{cases}$

Theorem 25 *Suppose* $\Gamma \vdash_{\mathbf{S}_2} M : A$*. Then* Γ^- *is a legal* $\mathbf{S}_1$*-context, and either*

1. $\Gamma^- \vdash_{\mathbf{S}_1} M : A$*, or*
2. $\Gamma \vdash_{\mathbf{S}_2} A : s$ *or* $A \equiv s$ *for some* $s \in \mathcal{S}'$*; and some* $t \in \mathcal{S}'$ *appears in* Γ *or* A*.*

Proof Induction on $\Gamma \vdash M : A$.

(axioms)

$$\frac{}{\vdash c : s}\ (c : s \in \mathcal{A} \cup \mathcal{A}')$$

$\langle\rangle$ is a legal $\mathbf{S}_1$-context. If $c : s \in \mathcal{A}$, then 1 holds; if $c : s \in \mathcal{A}'$, 2 holds.

(start)

$$\frac{\Gamma \vdash A : s}{\Gamma, x : A \vdash x : A}\ (x \notin \operatorname{dom}\Gamma)$$

If $s \in \mathcal{S}$, then 2 cannot hold of the premise (by Generation, since $s : t \in \mathcal{A}' \Rightarrow s \in \mathcal{S}'$), so 1 does. Therefore,

$$\begin{array}{rcl} \Gamma^- & \vdash_{\mathbf{S}_1} & A : s \\ \therefore \Gamma^-, x : A & \vdash_{\mathbf{S}_1} & x : A \end{array}$$

and $(\Gamma, x : A)^- \equiv \Gamma^-, x : A$.

If $s \in \mathcal{S}'$, then 2 holds of the conclusion, and $(\Gamma, x : A)^- \equiv \Gamma^-$ is a legal $\mathbf{S}_1$-context.

(weakening)

$$\frac{\Gamma \vdash M : A \quad \Gamma \vdash B : s}{\Gamma, x : B \vdash M : A} \; (x \notin \mathrm{dom}\,\Gamma)$$

If $s \in \mathcal{S}$, then we have

$$\begin{array}{rcll} \Gamma^- & \vdash & B : s & \\ \therefore \Gamma^-, x : B & \vdash & x : B & \quad \text{(start)} \end{array}$$

and so $(\Gamma, x : B)^- \equiv \Gamma^-, x : B$ is a legal $\mathbf{S}_1$-context. If $s \in \mathcal{S}'$, then $(\Gamma, x : B)^- \equiv \Gamma^-$, which is $\mathbf{S}_1$-legal.

Whichever of 1, 2 holds of the first premise holds of the conclusion, by (weakening).

(product)

$$\frac{\Gamma \vdash A : s_1 \quad \Gamma, x : A \vdash B : s_2}{\Gamma \vdash \Pi x : A.B : s_3} \; ((s_1, s_2, s_3) \in \mathcal{R} \cup \mathcal{R}')$$

If $(s_1, s_2, s_3) \in \mathcal{R}$, then 1 holds of each premise. Therefore,

$$\begin{array}{rcll} \Gamma^- & \vdash_{\mathbf{S}_1} & A : s_1 & \\ \Gamma^-, x : A & \vdash_{\mathbf{S}_1} & B : s_2 & \\ \therefore \Gamma^- & \vdash_{\mathbf{S}_1} & \Pi x : A.B : s_3 & \quad \text{(product)} \end{array}$$

If $(s_1, s_2, s_3) \in \mathcal{R}'$, then $s_3 \in \mathcal{S}'$, so 2 holds of the conclusion. Some $t \in \mathcal{S}'$ appears in $\Gamma, x : A$ or B, hence in Γ or $\Pi x : A.B$.

(application)

$$\frac{\Gamma \vdash F : \Pi x : A.B \quad \Gamma \vdash a : A}{\Gamma \vdash Fa : [a/x]B}$$

If 1 holds of the first premise, then

$$\Gamma^- \vdash_{\mathbf{S}_1} \Pi x : A.B : s$$

for some $s \in \mathcal{S}$. Therefore, there is a rule $(s_1, s_2, s) \in \mathcal{R}$ such that

$$\begin{array}{rcl} \Gamma^- & \vdash_{\mathbf{S}_1} & A : s_1 \\ \Gamma^-, x : A & \vdash_{\mathbf{S}_1} & B : s_2 \end{array}$$

Thus, 1 holds of the second premise. We thus have

$$\begin{array}{rcl} \Gamma^- & \vdash_{\mathbf{S}_1} & F : \Pi x : A.B \\ \Gamma^- & \vdash_{\mathbf{S}_1} & a : A \\ \therefore \Gamma^- & \vdash_{\mathbf{S}_1} & Fa : [a/x]B \end{array}$$

If 2 holds of the second premise, then

$$\Gamma \vdash_{\mathbf{S}_2} \Pi x : A.B : s$$

for some $s \in \mathcal{S}'$. Therefore, there is a rule $(s_1, s_2, s) \in \mathcal{R}'$ such that

$$\begin{array}{rll} \Gamma & \vdash_{\mathbf{S}_2} & A : s_1 \\ \Gamma, x : A & \vdash_{\mathbf{S}_2} & B : s_2 \\ \therefore \Gamma & \vdash_{\mathbf{S}_2} & [a/x]B : s_2 \qquad \text{(Substitution)} \end{array}$$

$s_2 \in \mathcal{S}'$, so 2 holds of the conclusion. Some $t \in \mathcal{S}'$ occurs in Γ or a, hence in Γ or Fa.

(abstraction)

$$\frac{\Gamma, x : A \vdash b : B \quad \Gamma \vdash \Pi x : A.B : s}{\Gamma \vdash \lambda x : A.b : \Pi x : A.B}$$

If $s \in \mathcal{S}'$, then, by Generation, there is a rule $(s_1, s_2, s) \in \mathcal{R}'$ such that

$$\Gamma, x : A \vdash B : s_2$$

Now, $s_2 \in \mathcal{S}'$. Therefore, 2 holds of the first premise. Therefore, some $t \in \mathcal{S}'$ occurs in $\Gamma, x : A$ or b, hence in Γ or $\lambda x : A.b$. Thus, 2 holds of the conclusion.

Suppose $s \in \mathcal{S}$. Then there is a rule $(s_1, s_2, s) \in \mathcal{R}$ such that

$$\begin{array}{rll} \Gamma & \vdash & A : s_1 \\ \Gamma, x : A & \vdash & B : s_2 \end{array}$$

Therefore, 1 holds of the first premise. We therefore have

$$\begin{array}{rll} \Gamma^-, x : A & \vdash_{\mathbf{S}_1} & b : B \\ \Gamma^- & \vdash_{\mathbf{S}_1} & \Pi x : A.B : s \\ \therefore \Gamma^- & \vdash_{\mathbf{S}_1} & \lambda x : A.b : \Pi x : A.B \end{array}$$

and so 1 holds of the conclusion.

(conversion)

$$\frac{\Gamma \vdash A : B \quad \Gamma \vdash B' : s}{\Gamma \vdash A : B'} \; (B =_\beta B')$$

Suppose $s \in \mathcal{S}'$. Let X be such that $B, B' \rhd X$. If 1 held of the first premise, then, for some $t \in \mathcal{S}$,

$$\Gamma^- \vdash_{\mathbf{S}_1} B : t \vee B \equiv t$$

(by Type Validity for $\mathbf{S}_1$). Therefore,

$$\begin{array}{c} \Gamma^- \vdash_{\mathbf{S}_1} X : t \vee X \equiv t \qquad \text{(Subject Reduction)} \\ \therefore \Gamma \vdash_{\mathbf{S}_2} X : t \vee X \equiv t \end{array}$$

(by the fact that $\mathbf{S}_2$ extends $\mathbf{S}_1$ and Thinning). Also

$$\Gamma \vdash_{\mathbf{S}_2} X : s$$

by Subject Reduction. $\Gamma \vdash_{\mathbf{S}_2} X : t$ would contradict Lemma 23; $X \equiv t$ would contradict Generation.

Therefore, 2 holds of the first premise. Hence, some $t \in \mathcal{S}'$ occurs in Γ or A. Thus, 2 holds of the conclusion.

Suppose $s \in \mathcal{S}$. Let X be such that $B, B' \rhd X$. If 2 held of the first premise, then, for some $s' \in \mathcal{S}'$,

$$\Gamma \vdash B : s \vee B \equiv s'$$

$$\therefore \Gamma \vdash X : s' \vee X \equiv s'$$

We also have $\Gamma \vdash X : s$, which contradicts Lemma 23.

Therefore, 1 holds of both premises. Hence,

$$\begin{array}{rcl} \Gamma^- & \vdash & A : B \\ \Gamma^- & \vdash & B' : s \\ \therefore \Gamma^- & \vdash & A : B' \end{array}$$

and so 1 holds of the conclusion.

QED

Corollary 26 $\mathbf{S}_2$ *is a conservative extension of* $\mathbf{S}_1$*.*

Proof An easy induction using Theorem 25, if Γ is an $\mathbf{S}_1$-context legal in $\mathbf{S}_2$, then $\Gamma^- \equiv \Gamma$. Therefore, if $\Gamma \vdash_{\mathbf{S}_2} M : A$, Γ is an $\mathbf{S}_1$-context, and M and A are $\mathbf{S}_1$-terms, then

$$\begin{array}{rcll} \Gamma^- & \vdash_{\mathbf{S}_1} & M : A & \quad \text{(Theorem 25)} \\ \therefore \Gamma & \vdash_{\mathbf{S}_1} & M : A & \end{array}$$

QED

6 Conclusion

We have established in a precise form the Curry-Howard isomorphism for a wide variety of Pure Type Systems, that includes and goes beyond those of the L-cube. The uniformity of the correspondence is brought out by the fact that the proof was virtually identical for all the different systems of logic.

If we wish to extend the correspondence to include conjunction and existential quantification, then we need to have Σ-types in our type systems. If we also wish disjunction and negation, we shall need disjoint unions and an empty type. I fully intend to extend this result to an extension of PTSs that includes these features in the near future.