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Infinite Omniscient Sets in Constructive Mathematics

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Abstract

We reprove some results concerning infinite sets that satisfy a certain omniscience principle that were first put forward by Escardó in [11]. We accomplish this by first introducing constructive mathematics and establishing a simple Bishop-esque constructive framework. Within this framework, we prove that $\mathbb{N}_\infty$ the subset of all descending sequences in $2^{\mathbb{N}}$, satisfies this omniscience principle and then we show how this can be generalised to many other subsets of $2^{\mathbb{N}}$. From the omniscience of $\mathbb{N}_\infty$ we then prove some properties concerning strongly extensional mappings and compare these results with Ishihara's First Trick. Finally we generalise these properties to other omniscient sets with mixed results.

Contents

Chapter 1

Introduction

A distinctive feature of constructive mathematics is that propositions are not either true or false *a priori*. Whereas in classical mathematics any proposition has a truth value (sometimes we just do not know which one yet), in a constructive setting we only consider a proposition true once we have supplied a proof of the statement or false once we have shown that the statement leads to a contradiction. Because we need any proof to be a finite routine, it follows that, given an infinite set X and an arbitrary predicate P on that set, we cannot in general decide whether P holds for every element in X or not. Thus, a natural question to ask would be: "For which sets are we able to decide this?" That is: can we prove for arbitrary P: $\forall x \in X\,[\,P(x)\,] \vee \exists x \in X\,[\,\neg P(x)\,]$.

One way to answer this question is by introducing the *omniscience* property, where a set is called an omniscient set if for any predicate on that set we can decide if the predicate is true for all elements in the set or it is false for at least one such element[11]. When we succeed in constructing a function that finds the element for which it is false (if it exists), we call the set *searchable*. In "Infinite sets that satisfy the principle of omniscience in all varieties of constructive mathematics"[11] Escardó shows that not only finite sets satisfy this property. He first and foremost shows that $\mathbb{N}_\infty$, the set of infinite descending binary sequences, is omniscient and searchable. It is important to note that he does this in a quite minimalistic constructive setting, akin to Bishop mathematics. The paper continues by proving interesting properties of $\mathbb{N}_\infty$ using the omniscience property and it concludes with an algorithm for constructing new omniscient sets.

In the present paper, we follow in the footsteps of Escardó's work in [11]. Our main purpose is exploring the work of Escardó in [11] by establishing a proper constructive framework in which to do so and that will assume no prior knowledge of constructive mathematics. We will reprove the omniscience of both $\mathbb{N}_\infty$ and of the *squashed sum*, a construction that creates a new set from existing omniscient sets. Furthermore, we look at properties of

$\mathbb{N}_\infty$ related to strong extensionality that are derived from the omniscience of $\mathbb{N}_\infty$. Our contribution here is an inspection into how far these properties generalise to other omniscient sets using the case of $\mathbb{N}_2\infty$, an instantation of the squashed sum construct.

Other research focuses more on the computational aspects of searchable sets, such as [13] and [12], that discuss how to perform exhaustive search on searchable sets from type-theoretic and category-theoretic points of view. This theory can be implemented quite easily in functional programming lanuages, as shown by [14] and its follow-up [15]. We will purely concentrate on the 'bare' mathematical aspects of omniscient sets. An example application in this direction can be found in constructive reverse mathematics[9]. The omniscience of $\mathbb{N}_\infty$ has recently been used to prove that the Cantor-Schröder-Bernstein theorem constructively implies the law of excluded middle[17].

In the next chapter, we will introduce constructive mathematics to those unfamiliar with it and elaborate on certain notions that we will need in later chapters. Chapter 3 formally introduces the concepts of omniscient and searchable sets and illustrates this using our infinite set $\mathbb{N}_\infty$. In Chapter 4 a way is shown to generalise the result of Chapter 3 to many other sets. Finally, Chapter 5 discusses a variant of Ishihara's First Trick[8] resulting from the omniscience of $\mathbb{N}_\infty$ and we try to slightly generalise these results to other omniscient sets.

Chapter 2

Preliminaries

This paper assumes a basic knowledge of classical first-order logic and set theory. The following sections contain some basic notions, mostly related to constructive mathematics, that the reader needs to be familiar with to understand the chapter(s) following this one.

We start with some conventions regarding certain sets and sequences.

- In this paper, $\mathbb{N}$ starts with 0.
- For the set $\{0, 1\}$ we will use the symbol 2.
- For any two sets X and Y, we use X^Y to mean the set of all functions from Y to X: $\{ f \mid f : Y \to X \}$.
  - In particular, any predicate p over a set X is an element of 2^X: it assigns a truth value to each element of the set X.
- Sets of infinite sequences of elements of a set X are written as $X^{\mathbb{N}}$: we think of them as functions that map each 'position' in the sequence to an element of X.
  - With the Cantor space we mean the 'canonical' Cantor space of infinite binary sequences and we will write it accordingly as $2^{\mathbb{N}}$.
- We will write the n-th element of a sequence α as $\alpha(n)$.
- $\overline{\alpha}n$ denotes the *initial segment* of length n of a sequence α: it is the finite sequence of length n such that $\overline{\alpha}n(m) = \alpha(m)$ for all $0 \leq m \leq n - 1$.
  - We sometimes also use the notation $\beta \sqsubseteq \alpha$ to mean that β is an initial segment of α for a sequence α and finite sequence β.

2.1 Constructive Mathematics

For those unfamiliar with constructive mathematics, we will try to give a small introduction to its foundations.

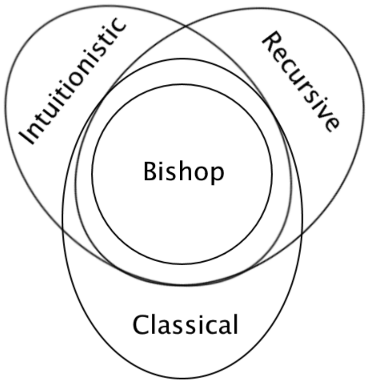
Compared to classical mathematics, practitioners of constructive mathematics wield a stricter notion of what counts as a proof. to consider a problem solved we must be able to explicitly produce a solution to said problem. Consequently, a proof to a problem should contain a method to explicitly compute a solution[1]. This method or *algorithm* should be a finite and deterministic procedure[5]. In practice, this strictness mainly affects the validity of classical 'existence proofs': often the existence of a mathematical object is asserted by showing that assuming its non-existence leads to a contradiction. These types of proofs do generally not include any method to find the object whose existence is asserted. In constructive mathematics, where 'there exists' is equivalent with 'we can compute/construct', this is unacceptable[4].

Within the field of constructive mathematics, we can distinguish multiple schools of thought. We will briefly mention some of the more established ones. These are, in chronological order of their first appearance: Brouwer's intuitionism, Markov's recursive constructive mathematics, and Bishop's constructive mathematics. What follows is a thoroughly simplified view that forgoes their respective philosophical underpinnings, but which will suffice for the purpose of this investigation. We can consider both intuitionism and recursive mathematics, as well as classical mathematics, as extensions of Bishop's constructive mathematics (see Figure 2.1) in the sense that every correct proof of a proposition in Bishop's mathematics is also valid in any of these other schools of mathematics[1]. These 'extensions' are mutually incompatible: each has theorems that can only be proven in that particular variety and not in the other two (or Bishop's mathematics).

For instance, the Cantor-Schröder-Bernstein theorem fails to be true in any of the constructive varieties[17], even though it is a long-established fact in classical mathematics. On the other hand, we can prove both within intuitionism and recursive mathematics that every mapping from [0,1] to $\mathbb{R}$ is pointwise continuous. This is clearly false classically and cannot be proven or disproven in Bishop's mathematics. Brouwer's intuitionism includes original principles that allow us to go even further and prove that every mapping from [0,1] to $\mathbb{R}$ is *uniformly* continuous. This is however false in recursive mathematics. On the other hand, Markov's recursive mathematics adheres to a strong constructive variant of Church's Thesis which is linked to its belief that constructive can be identified with recursive. This is incompatible with intuitionism and it is not accepted in Bishop's mathematics. Readers

[1]Even then, we note that such propositions can only be understood as statements that each school interprets in its own way.

Figure 2.1: Sketch of the relationship between the varieties of mathematics



are advised to consult [5] if they are interested in learning more about the differences between these constructive varieties and how they contrast with classical mathematics. For this paper we adopt a Bishop-esque setting, without intuitionistic or recursive principles. We reiterate that everything proven here will still be valid in those contexts.

2.2 Intuitionistic Logic

The first thing to note is that intuitionistic logic, despite its name, serves as a basis for logic in all varieties of constructive mathematics; i.e., its rules are valid in Bishop's mathematics as well.

The second thing to note when discussing intuitionistic logic is that logic has a different place in constructive mathematics than it does in classical mathematics. In classical mathematics, logic comes, in a certain sense, 'before' mathematics and acts as a foundation for it. In a constructive setting however, logic is just a part of mathematics. It is only descriptive: the rules and interpretation of logical symbols are formulated to reflect what we consider to be true based on our own (mathematical) intuition[5]. Nevertheless, we will see that intuitionistic logic can be compared to standard logic quite well from a proof-theoretic point of view.

A proposition is any *meaningful* statement. Here, meaningful means that we have an idea of what a proof of the statement would look like and would recognise a proof of the statement if we were presented with one[1]. Only when we have (dis)proven a proposition, we will consider it true or false[5]. Just like in standard logic, we can use logical connectives and quantifiers to form new propositions and predicates out of old ones. The 'standard' way to interpret or explain these symbols within constructive mathematics is the so-called *BHK-interpretation* (after *Brouwer*, *Heyting*, and *Kolmogorov*)[18]. Following the remarks at the beginning of Section 2.1, it should not come as a surprise that some connectives are interpreted in a stricter sense.

Let P and Q be propositions, then:

- Having a proof of $P \vee Q$ means we either have a proof of P or a proof of Q. Unlike in standard logic, it is not enough to prove that P and Q cannot both be false [$\neg(\neg P \wedge \neg Q)$] because this does not give us a method to construct a proof for either one of them.

- Having a proof of $P \wedge Q$ means we have a proof of P and also a proof of Q.

- Having a proof of $P \implies Q$ means that we have a routine that, when applied to a proof of P, produces a proof of Q.

- Having a proof of $\perp$ is impossible; $\perp$ (absurdity / contradiction / falsum) has no proof. The concept of contradiction is considered a primitive notion[16].

- The proposition $\neg P$ is equivalent with $P \implies \perp$. That is, having a proof of $\neg P$ means we can show that any (hypothetical) proof of P leads to a contradiction.

- Having a proof of $\exists x\, P(x)$ means we have a method to explicitly find an x such that $P(x)$ holds.

- Have a proof of $\forall x\, P(x)$ we have a method to construct a proof of $P(x)$ for any x.

2.2.1 Principles of Omniscience

In essence, intuitionistic logic can be described as standard logic without the *Law of Excluded Middle* or *Principle of Omniscience* (**PO**)

Definition 2.1 (**PO**).

For any proposition P, we can assert $P \vee \neg P$.

While **PO** is a cornerstone and at the same time almost trivial fact of classical logic, it is not accepted in constructive mathematics. Using the interpretations of intuitionistic logic, one can see why - **PO** makes a not-so-trivial statement: for any proposition P, we either can explicitly produce a proof of P or we can show that P leads to a contradiction.

Note that constructive mathematics does not *refute* **PO**. It is just not *universally* valid but remains so in a lot of cases. For example:

Theorem 2.2. $\forall n \in \mathbb{N}\, [\, n = 0 \vee n \neq 0\,]$

Without the validity of **PO**, this needs a proof.

Proof. By mathematical induction: in the case that $n = 0$, then $n = 0$ (so $n = 0 \vee n \neq 0$). Now assume that for an arbitrary $n \in \mathbb{N}$, we know whether $n = 0$ or $n \neq 0$. In both cases $n+1 \neq 0$. Thus, $\forall n \in \mathbb{N}\,[\, n = 0 \vee n \neq 0\,]$. $\square$

For this proof to make sense, mathematical induction needs to be constructively valid. Fortunately, the principle of induction is perfectly acceptable within Bishop's constructive mathematics (and any variant that admits (potentially) infinite sets)[2]. This principle has been formalised within *Heyting Arithmetic* (**HA**)[18]. **HA** has adopted the axioms of *Peano Arithmetic* but uses intuitionistic logic instead of classical logic underneath it. Within this formalization, Theorem 2.2 trivially follows from the inductively defined natural numbers. An informal justification for the principle goes as follows: assume that for a certain proposition P we have a proof of $P(0)$ and a proof of $P(x) \implies P(x+1)$, where the latter means that we have a finite routine that produces a proof of $P(x+1)$ when applied to a proof of $P(x)$. Then we can prove $P(n)$ for any natural number n by starting with our proof of $P(0)$ and our routine, followed by iteratively applying *modus ponens* a finite number of times. For a more fundamental philosophical justification, we refer to [6] and [10].

One of the consequences of not accepting **PO**, is that we can no longer use the so-called 'proof by contradiction': proving P by assuming $\neg P$ and showing this leads to a contradiction. Constructively, this does not produce a method to actually find P. It only allows us to conclude $\neg P \implies \bot$ or, equivalently, $\neg\neg P$. This also immediately shows that the 'double negation elimination' rule is invalid in constructive mathematics as well; for if $\neg\neg P \implies P$ were valid, it would then imply **PO** by *modus ponens* applied to the valid $\neg\neg(P \vee \neg P)$. More generally in constructive mathematics, we see that negative statements are oftentimes weaker than their positive (classically equivalent) counterparts. For example: $\exists x\, P(x) \implies \neg\forall x \neg P(x)$ is valid, because when we can explicitly produce an x such that $P(x)$ holds, this contradicts the assumption that $\forall x \neg P(x)$; hence $\neg\forall x \neg P(x)$. The reverse implication does not hold however: just because we can show that $\forall x \neg P(x)$ is contradictory does not give us a method to find an x so $P(x)$ does hold.

Continuing the last example, we easily see that the reverse implication *does* hold when x ranges over a finite domain. Because there are only finitely many x to check, this constitutes to a finite routine for finding the element we need to produce. In general, classically valid principles and proofs also hold constructively when we restrict ourselves to finite mathematical objects. Therefore, research into constructive mathematics has been concentrated on infinite objects, usually through infinite sequences. In particular, we will focus on the Cantor space $2^{\mathbb{N}}$.

Aside from 'full' **PO**, constructive mathematics recognises other principles of omniscience. We will use three of these, the *Lesser Principle of Om-*

niscience (**LPO**), the *Weaker Lesser Principle of Omniscience* (**WLPO**), and *Markov's Principle* (**MP**).

Definition 2.3 (**LPO**)**.**

$$\forall \alpha \in 2^{\mathbb{N}} \, [\, \forall i \in \mathbb{N} \, [\, \alpha(i) = 1 \,] \, \vee \, \exists i \in \mathbb{N} \, [\, \alpha(i) = 0 \,] \,]$$

Definition 2.4 (**WLPO**)**.**

$$\forall \alpha \in 2^{\mathbb{N}} \, [\, \forall i \in \mathbb{N} \, [\, \alpha(i) = 1 \,] \, \vee \, \neg \forall i \in \mathbb{N} \, [\, \alpha(i) = 1 \,] \,]$$

Definition 2.5 (**MP**)**.**

$$\forall \alpha \in 2^{\mathbb{N}} \, [\, \neg \, (\forall i \in \mathbb{N} \, [\, \alpha(i) = 1 \,] \,) \implies \exists i \in \mathbb{N} \, [\, \alpha(i) = 0 \,] \,]$$

As should be obvious from the name, **PO** implies **LPO**, which in turn implies **WLPO**. Furthermore, **LPO** implies **MP**, while **WLPO** and **MP** together imply **LPO**:

$$\mathbf{PO} \implies \mathbf{LPO} \iff \mathbf{WLPO} \wedge \mathbf{MP}$$

Both **LPO** and **WLPO** are invalid in all types of constructive mathematics (and can even proven to be false within both intuitionistic and recursive mathematics). **MP** on the other hand is a valid principle within recursive mathematics, but invalid in most other schools of constructive mathematics[5]. Proponents argue that the hypothesis of **MP** guarantees we will find one. However, we have no prior bound on how long it will take to find the element we need and therefore represents a type of unbounded search. For proofs of the aforementioned implications, as well as an extensive overview on the subject, we refer to [9]. Later on, we will prove the invalidity of other statements by showing they are equivalent to one of these principles.

2.3 Constructive Set Theory

In a similar vein as the rules for logic, constructive set theory contains some stricter notions compared to its classical counterpart. Most of these should not come as a surprise anymore after the previous sections.

In order to define a set, we must both give a method for constructing members of this set and tell how we can show that two members of this set are equal[5][1]. In other words, for each set we can explicitly produce members of this set and each set comes with its own equality, which should be an equivalence relation. This latter part is where constructive sets really deviate from classical ones, as from a constructive viewpoint it is not clear how equality statements ("$x = y$") could be proven for two objects from arbitrary sets[1]. Furthermore, to prevent constructions such as "the set of all sets”, a set X can only contain objects that have been (or could have been) defined before defining X[16][5]. As usual, we use $x \in X$ to denote that x is a member of the set X and $x \notin X$ that x is not.

As in classical set theory, we can construct new sets out of existing ones. The key difference is that such constructions are only meaningful if performed on two sets that have a common 'superset' (and hence a common equivalence relation). Otherwise, we have no way of comparing elements of these sets. We define set inclusion of two sets X and Y as follows:

$$X \subseteq Y \iff \forall \alpha \in X \, \exists \beta \in Y \, [\, \alpha = \beta \,]$$

where $=$ is the equality relation that was defined on Y. Using set inclusion, we can define equality on the set level in the familiar way:

$$X = Y \iff X \subseteq Y \wedge Y \subseteq X$$

Further, we can construct the intersection ($\cap$) and union ($\cup$) of two sets:

$$z \in X \cap Y \iff z \in X \wedge z \in Y$$

$$z \in X \cup Y \iff z \in X \vee z \in Y$$

As the above definition makes clear, stating that an element is in a union also means we can know in which one of the sets that makes up the union it is. For relative set complements, let $Y \subseteq X$ (as defined above). Then we define the complement of Y in X, $X \setminus Y$, as follows:

$$X \setminus Y = \{\, \alpha \in X \mid \forall \beta \in Y \, [\, \alpha \neq \beta \,] \,\}$$

2.3.1 Decidable equality

Even though every set comes equipped with its own equality (Section 2.3), we do not demand that we must be able to prove the equality of two members of said set. To wit, we must know what must be done to show equality between two members, but we do not need to have a finite method to actually do it for arbitrary members of the set. Sets that do have such a finite routine, have what we call *decidable* equality.

Definition 2.6 (**decidable equality**)**.** A set X has *decidable equality* when $\forall x, y \in X \, [\, x = y \vee x \neq y \,]$.

Examples of such sets are finite sets[2] and $\mathbb{N}$. For the natural numbers this follows with some extra work from Theorem 2.2.

Many other sets do not have this property however, and we will illustrate this for $2^{\mathbb{N}}$. First, we define equality between elements of $2^{\mathbb{N}}$. A natural way

[2]Here, as in [2], we consider a set finite if there exists a bijection between that set and a set $\mathbb{N}_{<n} = \{\, 0, 1, ..., n-1 \,\}$ for a certain $n \in \mathbb{N}$.

to do this is check for equality pointwise. So, for any two elements α and β in $2^{\mathbb{N}}$, we define

$$\alpha = \beta \iff \forall i \in \mathbb{N}\,[\,\alpha(i) = \beta(i)\,]$$

A quick check confirms that this is indeed an equivalence relation. Now we will show that this equality is not decidable.

Proposition 2.7. $2^{\mathbb{N}}$ *does not have decidable equality.*

Proof. Suppose $2^{\mathbb{N}}$ does have decidable equality. Let $\underline{1} \in 2^{\mathbb{N}}$ be the sequence such that $\forall i \in \mathbb{N}\,[\,\underline{1}(i) = 1\,]$. Because we have decidable equality, we should be able to decide whether $\alpha = \underline{1}$ for any $\alpha \in 2^{\mathbb{N}}$. Using our definition of equality on $2^{\mathbb{N}}$, this means we can decide for all $\alpha \in 2^{\mathbb{N}}$ if $\forall i \in \mathbb{N}\,[\,\alpha(i) = 1\,]$ or not. But this is **WLPO**, therefore we cannot accept that $2^{\mathbb{N}}$ has decidable equality. □

2.3.2 Apartness

So far, we have discussed the idea of equality on a set and we have casually used the symbol for inequality ($\neq$) in Theorem 2.2 and in our definition of set complements. One might guess correctly that we used inequality as the negation of equality:

$$x \neq y \iff \neg\,(x = y)$$

As we have mentioned before, in constructive mathematics negative statements are oftentimes weaker than positive ones. We would therefore prefer to have a positively formulated notion of two elements being distinct. Constructive mathematicians have found this stronger notion of inequality in the *apartness relation*[19] [3]. We use # as the symbol for this relation (as opposed to Bishop, who used $\neq$ to mean apartness[2]) and read $x \# y$ as '*x is apart from y*'. More precisely:

Definition 2.8. (apartness relation) a relation # on a set X is an apartness relation if it has the following three properties for $x, y, z \in X$:

- $\neg\,(x \# x)$ (irreflexive)
- $x \# y \implies y \# x$ (symmetric)
- $x \# y \implies (z \# x) \vee (z \# y)$ (co-transitive)

The complement of an apartness relation is an equivalence relation, so if two elements are not apart, they are equivalent. This makes it immediately clear why this is a suitable candidate for a positive version of inequality. We call the apartness relation a *tight* apartness if the negation of apartness not just means equivalence but actually implies equality:

- $\neg\,(x \# y) \implies x = y$ (tightness)

For sets that can be regarded as metric spaces, we have a standard way to define apartness. First of all, a *metric* on a set is defined the same constructively as it is classically[2]:

Definition 2.9 (metric). A metric $d : X \times X \to \mathbb{R}_{\geq 0}$ is a map on a previously defined set X, such that for $x, y, z \in X$ the following holds:

- $d(x, y) = 0 \iff x = y$
- $d(x, y) = d(y, x)$ (symmetric)
- $d(x, y) \leq d(x, z) + d(z, y)$ (triangle inequality)

Given a metric d on a set X and x and y elements from this set, we then define $x \# y \iff d(x, y) > 0$.

As an example, we will consider apartness on $2^{\mathbb{N}}$ in this way. Let $d : 2^{\mathbb{N}} \times 2^{\mathbb{N}} \to \mathbb{Q}_{\geq 0}$ be defined by

$$(\alpha, \beta) \mapsto \begin{cases} 0 & \text{if } \alpha = \beta \\ 2^{-\min\{\, i \mid \alpha(i) \neq \beta(i) \,\}} & \text{else} \end{cases}$$

First, we check that d indeed is a metric on $2^{\mathbb{N}}$ and then we show this induces an apartness relation as defined above.

Proposition 2.10. *d is a metric on $2^{\mathbb{N}}$.*

Proof. The property that $d(\alpha, \beta) = 0 \iff \alpha = \beta$ follows immediately from our construction of d. Symmetry of d comes from the symmetric property of equality (and, by extension, negation of equality) on any set. When we keep in mind that the codomain of d has no negative numbers, the triangle equality holds almost trivially in the case that α, β and γ are not all distinct. If they are, we need to do a little more work.

The argument relies on the fact that $\mathbb{N}$ and $\mathbb{Q}$ have decidable equality. For $\mathbb{N}$ we have seen this before and equality of elements $\frac{a}{b}, \frac{c}{d} \in \mathbb{Q}$ is defined by $\frac{a}{b} = \frac{c}{d} \iff ab = cd$. That this is decidable follows from decidability on $\mathbb{N}$.

Let $j := \min\{\, i \mid \alpha(i) \neq \beta(i) \,\}$, $k := \min\{\, i \mid \alpha(i) \neq \gamma(i) \,\}$, and $l := \min\{\, i \mid \gamma(i) \neq \beta(i) \,\}$. If $k \leq j$ or $l \leq j$, then $d(\alpha, \beta) = 2^{-j} \leq 2^{-k} = d(\alpha, \gamma)$ or $d(\alpha, \beta) = 2^{-j} \leq 2^{-l} = d(\gamma, \beta)$ respectively. Particularly, in both cases we have that $d(\alpha, \beta) \leq d(\alpha, \gamma) + d(\gamma, \beta)$. Finally, if $j < k$ and $j < l$, then $\alpha(j) = \gamma(j)$ and $\gamma(j) = \beta(j)$. Thus, by the transitivity property of equality, also $\alpha(j) = \beta(j)$. But this contradicts j, so this situation cannot happen. Therefore, the triangle inequality holds for d and we conclude that d is a metric on $2^{\mathbb{N}}$. □

Proposition 2.11. *The relation defined by $\alpha \# \beta \iff d(\alpha, \beta) > 0$ is an apartness relation on $2^{\mathbb{N}}$.*

Proof. # is irreflexive, because $d(\alpha,\alpha) = 0$ by definition of d. Therefore, $\neg\,(d(\alpha,\alpha) > 0)$ and thus $\neg\,(\alpha \# \alpha)$. # is symmetric because d is symmetric. Finally, if $\alpha \# \beta$, then $d(\alpha,\beta) > 0$. By the triangle equality we know for any γ that $0 < d(\alpha,\beta) \leq d(\alpha,\gamma) + d(\gamma,\beta)$. This allows us to conclude that $d(\alpha,\gamma) > 0 \vee d(\gamma,\beta) > 0$ (and we can check which one because $d(\alpha,\gamma)$ and $d(\gamma,\beta)$ are in $\mathbb{Q}$, which has decidable equality). This is equivalent to the statement that $\alpha \# \gamma \ \vee \ \gamma \# \beta$. So # is co-transitive and therefore # is an apartness relation on $2^{\mathbb{N}}$. □

Proposition 2.12. *The apartness relation on* $2^{\mathbb{N}}$ *is a tight apartness.*

Proof. First, we show that $\alpha \# \beta \iff \exists i \in \mathbb{N}\,[\,\neg\,(\alpha(i) = \beta(i))\,]$. If $\alpha \# \beta$, then $d(\alpha,\beta) > 0$. In fact, $d(\alpha,\beta) = 2^{-i_0}$ for some $i_0 \in \mathbb{N}$. For this i_0, $\alpha(i_0) \neq \beta(i_0)$ (because that is how we constructed d). Conversely, assume that $\exists i \in \mathbb{N}\,[\,\neg\,(\alpha(i) = \beta(i))\,]$. Then in particular: $\neg\,(\forall i \in \mathbb{N}\,[\,\alpha(i) = \beta(i)\,])$. But this is equivalent to saying that $\alpha \neq \beta$ and therefore $d(\alpha,\beta) = 2^{-i} > 0$ for a certain $i \in \mathbb{N}$. Hence, $\alpha \# \beta$.

Now we can prove tight apartness as follows: $\neg\,(\alpha \# \beta) \implies \neg\,(\exists i \in \mathbb{N}\,[\,\neg\,(\alpha(i) = \beta(i))\,])$. This in turn means that $\forall i \in \mathbb{N}\,[\,\neg\neg\,(\alpha(i) = \beta(i))\,]$. But $\alpha(i)$ and $\beta(i)$ are elements of 2. This is a set with decidable equality, and therefore we are allowed to say that $\neg\neg\,(\alpha(i) = \beta(i)) \implies \alpha(i) = \beta(i)$. Thus, $\forall i \in \mathbb{N}\,[\,\alpha(i) = \beta(i)\,]$. In other words: $\alpha = \beta$. □

Chapter 3

Omniscience & Searchability

We will first introduce the concepts of *omniscience* and *searchability*, certain properties of sets, as introduced by Escardó[11].

Definition 3.1 (**omniscience**)**.** A set X is called an *omniscient set* if for any predicate on X we can decide whether the predicate holds for all elements or fails to hold for some elements. More precisely, X is omniscient if

$$\forall (p : X \to 2)\,[\,\forall x \in X\,[\,p(x) = 1\,] \;\vee\; \exists x \in X\,[\,p(x) = 0\,]\,].$$

For $X = \mathbb{N}$, predicates are just elements of $2^{\mathbb{N}}$ and we see that this statement is equivalent to **LPO**, so $\mathbb{N}$ is not omniscient. Every finite set however is of course omniscient: for every predicate we can just check whether it holds for each element. And classically every set is trivially omniscient because of **PO**.

Definition 3.2 (**selection function**)**.** A *selection function* for X is an $\varepsilon : (X \to 2) \to X$ satisfying

$$\forall p : X \to 2\,[\,p(\varepsilon(p)) = 1 \implies \forall x \in X\,[\,p(x) = 1\,]\,].$$

So ε finds, for each predicate p, an element $x \in X$ for which the predicate does not hold ($p(x) = 0$), if it exists. A set X is *searchable* if there is a selection function for X.

As with omniscience, every set is searchable classically by using the axiom of choice. Intuitively, sets are searchable constructively if we can construct such a choice function that we get for free classically. Since the axiom of choice implies **PO** (Diaconescu's theorem), it makes sense that every searchable set is an omniscient set. It easily follows from how the selection function is constructed:

Lemma 3.3. *A searchable set is an omniscient set.*

Proof. Let X be a searchable set with selection function ε. For an arbitrary predicate p on X, $p(\varepsilon(p))$ equals 0 or 1; we know which one because 2 has decidable equality. If $p(\varepsilon(p)) = 1$, then $\forall x \in X\, [\, p(x) = 1\,]$ by definition of ε. If $p(\varepsilon(p)) = 0$, then $\exists x \in X\, [\, p(x) = 0\,]$: just take $x = \varepsilon(p)$. $\square$

3.1 Omniscience of $\mathbb{N}_\infty$

The set $\mathbb{N}_\infty$ is the subset of the Cantor space containing all descending sequences; more precisely: $\mathbb{N}_\infty = \{\, \alpha \in 2^{\mathbb{N}} \mid \forall i \in \mathbb{N}\, [\, \alpha(i+1) \leq \alpha(i)\,]\, \}$ and it has elements $\underline{n} = 1^n 0^\omega$ and $\infty = 1^\omega$. The main purpose of this section will be showing that $\mathbb{N}_\infty$ is omniscient by proving the following theorem:

Theorem 3.4 ($\mathbb{N}_\infty$ **is searchable**). *The set* $\mathbb{N}_\infty$ *is searchable with the functional* $\varepsilon : (\mathbb{N}_\infty \to 2) \to \mathbb{N}_\infty$ *defined by*

$$\varepsilon(p)(i) = min\{\, p(\underline{n}) \mid n \leq i\, \}$$

Before we can do that however, some work must be done first. Problems arise that are related to the structure of $\mathbb{N}_\infty$. While similar to $\mathbb{N}$, we will soon see that we cannot just identify it with $\mathbb{N}$ plus an extra point added at infinity. To immediately show that specifically the element ∞ will be troublesome for us, we start with the following simple statement.

Proposition 3.5. *If* $\forall \alpha \in \mathbb{N}_\infty\, [\, \alpha \neq \infty \implies \exists n \in \mathbb{N}\, [\, \alpha = \underline{n}\,]\,]$*, then* ***MP****.*

Proof. First of all: $\alpha \neq \infty$ just means that $\neg\, (\forall i \in \mathbb{N}\, [\, \alpha(i) = 1\,])$. If $\exists n \in \mathbb{N}\, [\, \alpha = \underline{n}\,]$, then $\exists m \in \mathbb{N}\, [\, \alpha(m) = 0\,]$: take $m = n$. Thus, if we were to assert that $\alpha \neq \infty \implies \exists n \in \mathbb{N}\, [\, \alpha = \underline{n}\,]$, then $\neg\, (\forall i \in \mathbb{N}\, [\, \alpha(i) = 1\,] \implies \exists i \in \mathbb{N}\, [\, \alpha(i) = 0\,]$. In other words, it would imply **MP**. $\square$

The problem is that $\mathbb{N}_\infty$ does not have decidable equality: we cannot in general decide if an arbitrary element in $\mathbb{N}_\infty$ is equal to ∞. Escardó[11] illustrates this through a trio of statements that relate it (as a set) to the set $\underline{\mathbb{N}} = \{\, \underline{n} \mid n \in \mathbb{N}\, \}$, which we will restate and then give proof of below (Proposition 3.6). This then motivates the construction further in this section, which will allow us to make a step in the proof of Theorem 3.4 for which we would otherwise need decidable equality. For now, recall the meaning of **LPO** (Definition 2.3), **WLPO** (Definition 2.4), and **MP** (Definition 2.5).

Proposition 3.6. *We have the following equivalencies:*

1. ***LPO*** $\iff \mathbb{N}_\infty = \underline{\mathbb{N}} \cup \{\, \infty\, \}$
2. ***WLPO*** $\iff \mathbb{N}_\infty = \mathbb{N}_\infty \setminus \{\, \infty\, \} \cup \{\, \infty\, \}$
3. ***MP*** $\iff \mathbb{N}_\infty \setminus \{\, \infty\, \} = \underline{\mathbb{N}}$

Proof of 3.6.1. (⇒) Let $\alpha \in \mathbb{N}_\infty$. Assuming **LPO**, we know that either $\exists i \in \mathbb{N}\,[\,\alpha(i) = 0\,]$ or $\forall i \in \mathbb{N}\,[\,\alpha(i) = 1\,]$. In the first case, $\alpha = \underline{n}$, for some $n \leq i$. Therefore, $\alpha \in \underline{\mathbb{N}} \subseteq \underline{\mathbb{N}} \cup \{\,\infty\,\}$. In the second case, $\alpha = \infty$. Thus, $\alpha \in \{\,\infty\,\} \subseteq \underline{\mathbb{N}} \cup \{\,\infty\,\}$. Conversely, let $\alpha \in \underline{\mathbb{N}} \cup \{\,\infty\,\}$. In both cases, $\alpha \in \mathbb{N}_\infty$.

(⇐) Let $\alpha \in \mathbb{N}_\infty$. Because $\mathbb{N}_\infty = \underline{\mathbb{N}} \cup \{\,\infty\,\}$, we know that $\exists \beta \in \underline{\mathbb{N}} \cup \{\,\infty\,\}\,[\,\alpha = \beta\,]$. Hence $\exists \beta \in \underline{\mathbb{N}}\,[\,\alpha = \beta\,]$ or $\exists \beta \in \{\,\infty\,\}\,[\,\alpha = \beta\,]$. In the former case, $\beta = \underline{n}$, for some $n \in \mathbb{N}$, and so $\exists i \in \mathbb{N}\,[\,\beta(i) = 0\,]$. In the latter case, $\beta = \infty$ and thus $\forall i \in \mathbb{N}\,[\,\beta(i) = 1\,]$. But this is exactly **LPO**. □

Proof of 3.6.2. (⇒) Let $\alpha \in \mathbb{N}_\infty$. Assuming **WLPO**, we know that either $\forall i \in \mathbb{N}\,[\,\alpha(i) = 1\,]$ or $\neg\forall i \in \mathbb{N}\,[\,\alpha(i) = 1\,]$. In the first case, $\alpha = \infty$ and so $\alpha \in \{\,\infty\,\} \subseteq \mathbb{N}_\infty \setminus \{\,\infty\,\} \cup \{\,\infty\,\}$. In the second case, we know that $\{\,\infty\,\}$ only contains one element, ∞, which is exactly the element such that $\forall i \in \mathbb{N}\,[\,\infty(i) = 1\,]$ and therefore it is immediately obvious that $\alpha \neq \infty$ and thus $\alpha \in \mathbb{N}_\infty \setminus \{\,\infty\,\} \subseteq \mathbb{N}_\infty \setminus \{\,\infty\,\} \cup \{\,\infty\,\}$. Conversely, let $\alpha \in \mathbb{N}_\infty \setminus \{\,\infty\,\} \cup \{\,\infty\,\}$. In both cases, $\alpha \in \mathbb{N}_\infty$.

(⇐) Let $\alpha \in \mathbb{N}_\infty$. Because $\mathbb{N}_\infty = \mathbb{N}_\infty \setminus \{\,\infty\,\} \cup \{\,\infty\,\}$, we know that $\exists \beta \in \mathbb{N}_\infty \setminus \{\,\infty\,\} \cup \{\,\infty\,\}\,[\,\alpha = \beta\,]$. Hence $\exists \beta \in \mathbb{N}_\infty \setminus \{\,\infty\,\}\,[\,\alpha = \beta\,]$ or $\exists \beta \in \{\,\infty\,\}\,[\,\alpha = \beta\,]$. In the former case, $\beta \neq \infty$, which is equivalent to stating that $\neg\forall i \in \mathbb{N}\,[\,\beta(i) = 1\,]$. In the latter case, $\beta = \infty$ and so $\forall i \in \mathbb{N}\,[\,\beta(i) = 1\,]$. But this is exactly **WLPO**. □

Proof of 3.6.3. (⇒) Let $\alpha \in \mathbb{N}_\infty \setminus \{\,\infty\,\}$. Then $\alpha \neq \infty$, so $\neg\forall i \in \mathbb{N}\,[\,\alpha(i) = 1\,]$. By **MP**, this means that $\exists i \in \mathbb{N}\,[\,\alpha(i) = 0\,]$. This in turn means that $\alpha = \underline{n}$, for some $n \leq i$. Thus $\alpha \in \underline{\mathbb{N}}$. Conversely, let $\alpha \in \underline{\mathbb{N}}$. Then $\alpha = \underline{n}$ for some $n \in \mathbb{N}$, and it immediately follows that $\alpha \in \mathbb{N}_\infty \setminus \{\,\infty\,\}$.

(⇐) Let $\alpha \in \mathbb{N}_\infty \setminus \{\,\infty\,\}$. Then $\alpha \neq \infty$. The result then follows from Proposition 3.5. □

We recognise that $\mathbb{N}_\infty$ and $\underline{\mathbb{N}} \cup \{\,\infty\,\}$ are not identical, but they are still remarkably similar: it is 'just' an issue of decidability. To see exactly how similar (and to give it a name), Escardó[11] introduces the following concept:

Definition 3.7 (full subset). For any set X, a subset Y of X is called *full* if the complement of Y in X is empty.

This is equivalent to a differently formulated concept originally introduced by Brouwer[7]:

Definition 3.8 (congruent sets). Two sets are *congruent* if either set cannot have an element that the other does not have.

This concept deals exactly with these issues where sets are not identical but cannot be distinguished either. Heyting[16] proves the following:

Proposition 3.9. *Let X be a set and Y a subset of X. Then X and $X' = Y \cup (X \setminus Y)$ are congruent.*

Proof. Since $X' \subseteq X$, X' does not have an element that X does not have. Now suppose x is an element of X but not of X', then x is in particular not an element of Y. Therefore, x must be an element of $X \setminus Y$. But that means that x is an element of X', contradicting our assumption. So there does not exists such element x. $\square$

By taking $X = \mathbb{N}_\infty$ and $X' = \underline{\mathbb{N}} \cup \{ \infty \}$, we now easily see that these are congruent sets. Similarly, Escardó proves that $\underline{\mathbb{N}} \cup \{ \infty \}$ is a full subset of $\mathbb{N}_\infty$. Even though this already follows from the above proposition we shall restate it here because its argument relies more on the structure of $\mathbb{N}_\infty$, which is what has so far caused our problems in the first place. The lemma will also be useful on its own later.

Lemma 3.10. $\forall \alpha \in \mathbb{N}_\infty \, [\forall n \in \mathbb{N} \, [\, \alpha \neq \underline{n} \,] \implies \alpha = \infty \,]$.

Proof. Let $\alpha \in \mathbb{N}_\infty$ and assume the antecedent is true. Now suppose $\alpha(i) = 0$ for some $i \in \mathbb{N}$. Then (by definition) $\alpha = \underline{n}$ for some $n \leq i$, contradicting our assumption. So $\alpha(i)$ must be equal to 1. This is true for any $i \in \mathbb{N}$, so $\forall i \in \mathbb{N} \, [\alpha(i) = 1) \,]$. In other words: $\alpha = \infty$. $\square$

The following lemma about full subsets is what is going to allow us to circumvent the need for decidable equality on $\mathbb{N}_\infty$.

Lemma 3.11. *Let C be a full subset of some set X, and let Y be a set with decidable equality. If a function $f : X \to Y$ is constant on C, it is constant on X.*

Proof. Let $f(c) = y$ for all $c \in C$, and suppose that $f(x) \neq y$ for some $x \in X$. Because functions are extensional, this means that $x \neq c$ for all $c \in C$. This contradicts that C is a full subset of X. Because Y has decidable equality, we can conclude that $f(x) = y$ for every $x \in X$. $\square$

We are now ready to prove the omniscience of $\mathbb{N}_\infty$.

Proof of Theorem 3.4. $\varepsilon(p)(i)$ is point-wise defined as $\min\{ p(\underline{n}) \mid n \leq i \}$. It is clear then that $\varepsilon(p)$ is once again an element of $\mathbb{N}_\infty$. Now we just need to confirm that $p(\varepsilon(p)) = 1 \implies \forall \alpha \in \mathbb{N}_\infty \, [\, p(\alpha) = 1 \,]$ for all predicates on $\mathbb{N}_\infty$. So, let p be an arbitrary predicate such that $p(\varepsilon(p)) = 1$.

Suppose $\varepsilon(p) = \underline{n}$ for some $n \in \mathbb{N}$. Then $p(\underline{n}) = p(\varepsilon(p)) = 1$ by assumption. $\varepsilon(p) = \underline{n}$ also means that $\varepsilon(p)(n) = 0$ and $\varepsilon(p)(i) = 1$ ($\forall i \leq n$), by definition of ε, which means that $p(\underline{n}) = 0$. We have arrived at a contradiction and must conclude that $\varepsilon(p) \neq \underline{n}$. Applying Lemma 3.10, this means that $\varepsilon(p) = \infty$.

$\varepsilon(p) = \infty$ implies that $p(\infty) = p(\varepsilon(p)) = 1$ by our assumption. Furthermore, it also means that $\varepsilon(p)(i) = 1$ for every $i \in \mathbb{N}$. Then by our definition there does not exist an i for which there is an $n \leq i$ such that $p(\underline{n}) = 0$. In other words, $\forall n \in \mathbb{N} [\, p(\underline{n}) = 1 \,]$. Now we know that $\forall \alpha \in \underline{\mathbb{N}} [\, p(\alpha) = 1 \,]$ and we know that $\forall \alpha \in \{ \infty \} [\, p(\alpha) = 1 \,]$.

Taken together, we know that $\forall \alpha \in \underline{\mathbb{N}} \cup \{ \infty \} [\, p(\alpha) = 1 \,]$. However, by lack of decidable equality we do not know in which 'case' we are for a general element of $\mathbb{N}_\infty$. Luckily, p has a constant value on $\underline{\mathbb{N}} \cup \{ \infty \}$, a full subset of $\mathbb{N}_\infty$. And because the set 2 is decidable, we can apply Lemma 3.11 to conclude that $p(\alpha) = 1$ for all $\alpha \in \mathbb{N}_\infty$. $\square$

Chapter 4

Squashed sums

Suppose we have a certain number of searchable subsets of the Cantor space. Escardó shows that their *squashed sum* is once again searchable. For this construction we will rescale each subset and we must first confirm that this will not affect searchability.

Lemma 4.1. *Let X and Y be sets, $f : X \to Y$ some function and suppose X is searchable with selection function ε. Then $f(X)$ is searchable.*

Proof. We construct the selection function ε_f as follows:

$$\varepsilon_f(p) = f(\varepsilon(p \circ f)).$$

$p \circ f$ is again a predicate on X and therefore can be searched. The resulting $x \in X$, found by ε, is then associated with the corresponding element in Y by f. □

Definition 4.2 (**squashed sum**)**.** Let X_i be searchable subsets of $2^{\mathbb{N}}$ (for $i \in \mathbb{N}$). We then construct their *squashed sum* $\overline{\sum\limits_{i \in \mathbb{N}}} X_i$ as follows: first we prefix every member of every set X_i with $1^i 0$, creating new disjoint searchable sets $1^i 0 X_i$. Then we define membership of the squashed sum as follows:

$$\alpha \in \overline{\sum_{i \in \mathbb{N}}} X_i \iff \forall n \in \mathbb{N}\, \exists \beta \in \bigcup_{i \in \mathbb{N}} 1^i 0 X_i \,[\, \overline{\alpha} n = \overline{\beta} n \,]$$

So, a sequence α is in the squashed sum if every initial segment of α is also the initial segment of some element in the union of X_i.

This squashed sum is again a subset of $2^{\mathbb{N}}$. Next, we will see that this set is equivalent to the closure of $\bigcup\limits_{i \in \mathbb{N}} 1^i 0 X_i$ in the *metric space* $2^{\mathbb{N}}$.

Definition 4.3 (**open ball**[18])**.** For a set X with a metric d (recall Definition 2.9), $x \in X$ and $r \in \mathbb{R}$, we call

$$U(r, x) := \{\, y \in X \mid d(x, y) < r \,\}$$

the *open ball* of x with radius r.

Definition 4.4 (**closure**[18])**.** For a set X with a metric and $Y \subseteq X$, we define the *closure* $\overline{Y}$ of Y as follows:

$$\overline{Y} := \{x \in X \mid \forall r > 0 \, \exists y \in Y \, [\, y \in U(r, x) \,] \, \}$$

where $U(r, x)$ is the open ball as defined in Definition 4.3.

Proposition 4.5. *The squashed sum* $\overline{\sum_{i \in \mathbb{N}}} X_i$ *coincides with* $\overline{\bigcup_{i \in \mathbb{N}} 1^i 0 X_i}$*, the closure of* $\bigcup_{i \in \mathbb{N}} 1^i 0 X_i$ *in* $2^{\mathbb{N}}$*, using the metric as in Proposition 2.10.*

Proof. $\beta \in U(r, \alpha) \iff d(\alpha, \beta) < r \iff 2^{-\min\{\, i \,|\, \alpha(i) \neq \beta(i) \,\}} < r \iff \min\{\, i \mid \alpha(i) \neq \beta(i) \,\} > \log_2 \frac{1}{r} \iff \min\{\, i \mid \alpha(i) \neq \beta(i) \,\} \geq \lceil \log_2 \frac{1}{r} \rceil$, where $\lceil \log_2 \frac{1}{r} \rceil$ means $\log_2 \frac{1}{r}$ rounded up to the nearest integer. Define $r_* := \lceil \log_2 \frac{1}{r} \rceil$. Continuing, $\min\{\, i \mid \alpha(i) \neq \beta(i) \,\} \geq r_* \iff \overline{\beta} r_* = \overline{\alpha} r_*$.
So, using Definition 4.4, $\overline{\bigcup_{i \in \mathbb{N}} 1^i 0 X_i} = \{\alpha \in 2^{\mathbb{N}} \mid \forall r > 0 \, \exists \beta \in \bigcup_{i \in \mathbb{N}} 1^i 0 X_i \, [\, \beta \in U(r, \alpha) \,] \, \} = \{\alpha \in 2^{\mathbb{N}} \mid \forall r > 0 \, \exists \beta \in \bigcup_{i \in \mathbb{N}} 1^i 0 X_i \, [\, \overline{\beta} r_* = \overline{\alpha} r_* \,] \, \}$. That this is equivalent to $\overline{\sum_{i \in \mathbb{N}}} X_i$ follows from the fact that $r_* \to \infty$ as $r \to 0$. □

An alternative definition of the squashed sum is used in [11], where it is claimed that this is also equivalent to the closure of $\bigcup_{i \in \mathbb{N}} 1^i 0 X_i$ in $2^{\mathbb{N}}$. Hence, we get the following result.

Corollary 4.5.1. $\forall \alpha \in \overline{\sum_{i \in \mathbb{N}}} X_i \, [\, \forall n \in \mathbb{N} \, [\, 1^n 0 \sqsubseteq \alpha \implies \alpha \in 1^n 0 X_n \,] \,]$.

It would however again be **LPO** to assert that $\overline{\sum_{i \in \mathbb{N}}} X_i = \bigcup_{i \in \mathbb{N}} 1^i 0 X_i \cup \{\, \infty \,\}$ because of problems with decidability. And in a similar vein as with $\mathbb{N}_\infty$, this will cause issues in proving searchability. Luckily, we can apply the same trick here as we did then. First a small result:

Lemma 4.6. $\forall \alpha \in 2^{\mathbb{N}} \, [\, \forall n \in \mathbb{N} \, [\, 1^n 0 \not\sqsubseteq \alpha \,] \implies \alpha = \infty \,]$.

Proof. For arbitrary $i \in \mathbb{N}$, if $\alpha(i) = 0$ then $1^n 0 \sqsubseteq \alpha$ for some $n \leq i$. This contradicts our hypothesis, so $\alpha(i) = 1$ for all $i \in \mathbb{N}$ and thus $\alpha = \infty$. □

Now we will use the above result to show that $\bigcup_{i \in \mathbb{N}} 1^i 0 X_i \cup \{\, \infty \,\}$ is a full subset of $\overline{\sum_{i \in \mathbb{N}}} X_i$.

Lemma 4.7. $\forall \alpha \in \overline{\sum_{i \in \mathbb{N}}} X_i \, [\, \forall i \in \mathbb{N} \, [\, \alpha \notin 1^i 0 X_i \,] \implies \alpha = \infty \,]$.

Proof. By definition of the squashed sum, if $\alpha \in \overline{\sum_{i\in\mathbb{N}}} X_i$ and $1^n0 \sqsubseteq \alpha$ for some n, then $\exists \beta \in \bigcup_{i\in\mathbb{N}} 1^i0X_i\, [\, \overline{\alpha}(n+1) = \overline{\beta}(n+1)\,]$. Then by Corollary 4.5.1, $\alpha \in 1^i0X_i$. But we assumed this is not the case, so we know that $1^i0 \not\sqsubseteq \alpha$ for any $i \in \mathbb{N}$. It follows from Lemma 4.6 that $\alpha = \infty$. □

The main idea for our selection function for the squashed sum is that we combine the existing selection functions of our known searchable sets into a single construction. We formalise this as follows.

Definition 4.8 (filter function). Let $X_i \subseteq 2^{\mathbb{N}}$ (with $i \in \mathbb{N}$) be searchable sets, each with their own selection function ε_{X_i}. Furthermore, let $\varepsilon_{1^i0X_i}$ be the selection function for the prefixed set 1^i0X_i, constructed from ε_{X_i} as in Lemma 4.1. Let Y be the squashed sum of the X_i (as in Definition 4.2). We define its *filter function* $\zeta : (Y \to 2) \to 2^{\mathbb{N}}$ by $\zeta(p)(i) = p(\varepsilon_{1^i0X_i}(p))$

Lemma 4.9. *Let ζ be as in Definition 4.8. Then for any predicate p we have*

$$1^n0 \sqsubseteq \zeta(p) \iff p(\varepsilon_{1^n0X_n}(p)) = 0 \,\wedge\, \forall i < n\, [\, p(\varepsilon_{1^i0X_i}(p)) = 1\,].$$

Proof. $1^n0 \sqsubseteq \zeta(p)$ if and only if $\overline{\underline{n}}(n+1) = \overline{\zeta(p)}(n+1)$. This is the same as saying that $\zeta(p)(n) = 0$ and $\zeta(p)(i) = 1$ for all $0 \leq i < n$. The conclusion then follows by the definition of ζ. □

Now we have everything we need to prove that our construction of the squashed sum is once again a searchable set.

Theorem 4.10. *Let $X_i \subseteq 2^{\mathbb{N}}$ (with $i \in \mathbb{N}$) be searchable sets, each with their own selection function ε_{X_i}. Let Y be their squashed sum (as in Definition 4.2) and ζ its filter function (Definition 4.8). Then Y is searchable with the functional $\varepsilon : (Y \to 2) \to Y$ defined by*

$$\varepsilon(p)(i) = 0 \iff \exists n \leq i\, [\, 1^n0 \sqsubseteq \zeta(p) \,\wedge\, \varepsilon_{1^n0X_n}(p)(i) = 0\,]$$

Proof. Let $p : \mathbb{N}_\infty \to 2$ be such that $p(\varepsilon(p)) = 1$. Suppose $1^n0 \sqsubseteq \varepsilon(p)$ for some $n \in \mathbb{N}$. For this n we have that $1^n0 \sqsubseteq \zeta(p)$ and thus $\varepsilon(p)(i) = 0 \iff \varepsilon_{1^n0X_n}(p)(i) = 0$ because that is how ε is defined. Therefore $\varepsilon(p) = \varepsilon_{1^n0X_n}(p)$. By assumption this means that $p(\varepsilon_{1^n0X_n}(p)) = 1$. But $1^n0 \sqsubseteq \zeta(p)$ also means that $p(\varepsilon_{1^n0X_n}(p)) = 0$ by Lemma 4.9, so we have a contradiction and must conclude that there is no n such that $1^n0 \sqsubseteq \varepsilon(p)$.

Lemma 4.6 now gives us that $\varepsilon(p) = \infty$ and therefore $p(\infty) = p(\varepsilon(p) = 1$ by assumption. That $\varepsilon(p) = \infty$ also means that $\zeta(p) = \infty$ and it follows from Lemma 4.9 that $p(\alpha) = 1$ for every $\alpha \in \bigcup_{i\in\mathbb{N}} 1^i0X_i$.

In general, we can now conclude that $\forall \alpha \in \bigcup_{i\in\mathbb{N}} 1^i0X_i \cup \{\, \infty \,\}\, [\, p(\alpha) = 1\,]$. Because $\bigcup_{i\in\mathbb{N}} 1^i0X_i \cup \{\, \infty \,\}$ is a full subset of Y (Lemma 4.7), we can once again invoke Lemma 3.11 to conclude that $p(\alpha) = 1$ for any $\alpha \in Y$. We conclude that our constructed ε is a selection function for Y and therefore Y, the squashed sum of the X_i, is searchable. □

Now that we have proven that squashed sums of searchable sets are once again searchable, we can look at certain concrete instantiations of these sums.

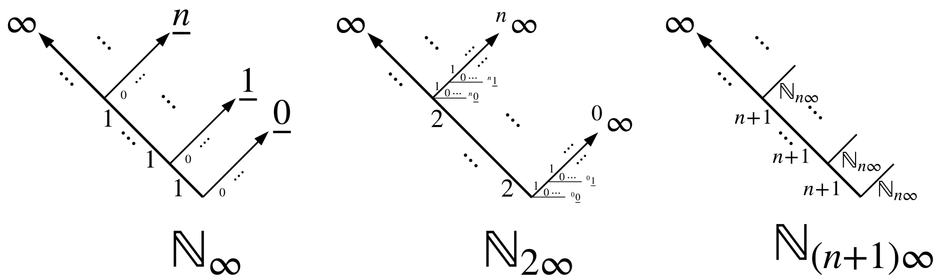
Firstly, when we take $X_i = \{ \underline{0} \}$ for all $i \in \mathbb{N}$, we get $\overline{\sum_{i \in \mathbb{N}}} X_i = \mathbb{N}_\infty$. Next, we introduce larger sets of descending sequences.

Definition 4.11 (sets of descending sequences). For any $n \in \mathbb{N}$, we define

$$\mathbb{N}_{n\infty} = \{ \alpha \in \{ 0, 1, ..., n \}^{\mathbb{N}} \mid \forall i \in \mathbb{N} [\alpha(i+1) \leq \alpha(i)] \},$$

the set of all descending sequences in $n^{\mathbb{N}}$.

Figure 4.1: Visualisation of the sets $\mathbb{N}_\infty$, $\mathbb{N}_{2\infty}$ and more generally $\mathbb{N}_{(n+1)\infty}$. The main branch of $\mathbb{N}_{(n+1)\infty}$ has countably many copies of $\mathbb{N}_{n\infty}$.



Proposition 4.12. *If $X_i = \mathbb{N}_\infty$ for every i, the squashed sum $\overline{\sum_{i \in \mathbb{N}}} X_i$ is equivalent to $\mathbb{N}_{2\infty}$.*

Proof. We can either prefix each set X_i with 2^i instead of 1^i0 in our construction of the squashed sum, or we can construct a function $f : \overline{\sum_{i \in \mathbb{N}}} X_i \to 3^{\mathbb{N}}$ by

$$f(\alpha)(i) = \begin{cases} 2 & \text{if } \forall n \leq i [\alpha(n) = 1] \\ \alpha(i+1) & \text{else} \end{cases}.$$ □

We can repeat this construction and take $X_i = \mathbb{N}_{2\infty}$ to create searchable set $\mathbb{N}_{3\infty}$ and more generally we see that the set $\mathbb{N}_{(n+1)\infty}$ is searchable as the squashed sum of $X_i = \mathbb{N}_{n\infty}$ for all i. This aligns with the idea that we can think of these sets of descending sequences as highly structured branching trees. When we start producing a sequence in $\mathbb{N}_{n\infty}$ we can at each step produce an n or branch off into a copy of $\mathbb{N}_{(n-1)\infty}$; see Figure 4.1.

Chapter 5

Strong Extensionality in Omniscient Sets

A function is extensional if it preserves equality: if a is equal to b then $f(a)$ is equal to $f(b)$ for any extensional function f and any a and b in the domain of f. Constructively, equality has a counterpart in apartness (see subsection 2.3.2) and consequently there is a useful property that describes that a function preserves apartness.

Definition 5.1 (**strong extensionality**)**.** A function f is *strongly extensional* if $f(x) \# f(y)$ implies that $x \# y$ for all x and y.

5.1 Strong extensionality in $\mathbb{N}_\infty$

We start with a simple result that gives an alternative characterisation of strong extensionality for predicates on $\mathbb{N}_\infty$.

Lemma 5.2. *For* $p : \mathbb{N}_\infty \to 2$*,* p *is strongly extensional* $\iff \forall \alpha \in \mathbb{N}_\infty\, [\, p(\alpha) \neq p(\infty) \implies \alpha \# \infty\,]$*.*

Proof. Remember that for decidable sets (such as 2) inequality and apartness coincide. It follows that if p is strongly extensional, the condition certainly holds. Conversely, let $p(\alpha) \neq p(\beta)$. Then also $p(\alpha) \neq p(\infty)$ or $p(\beta) \neq p(\infty)$ because of the co-transitivity of apartness on 2. Suppose $p(\alpha) \neq p(\infty)$ (the case $p(\beta) \neq p(\infty)$ is exactly similar). Then $\alpha \# \infty$ and therefore $\alpha \# \beta$ or $\infty \# \beta$, this time by co-transitivity of apartness on $\mathbb{N}_\infty$. In the first case we are done and in the second case we have that both α and β are of the form $\underline{n}$ for some $n \in \mathbb{N}$, say $\alpha = \underline{a}$ and $\beta = \underline{b}$, because they are both apart from ∞. We cannot have that $a = b$ because then $\alpha = \beta$ and therefore $p(\alpha) = p(\beta)$ (because all functions are extensional), which contradicts our assumption. So, $a \neq b$ and consequently $\alpha \# \beta$. So, $p(\alpha) \neq p(\beta)$ implies that $\alpha \# \beta$ for all $\alpha, \beta \in \mathbb{N}_\infty$. $\square$

Escardó[11] introduces the following lemma that is a consequence of the omniscience of $\mathbb{N}_\infty$:

Lemma 5.3 (Ishihara's First Trick for $\mathbb{N}_\infty$). *For all strongly extensional predicates s on $\mathbb{N}_\infty$,*

$$\forall n \in \mathbb{N}\,[\,s(\underline{n}) = s(\infty)\,] \vee \exists n \in \mathbb{N}\,[\,s(\underline{n}) \neq s(\infty)\,]$$

Proof. Let s be an arbitrary strongly extensional predicate. The omniscience of $\mathbb{N}_\infty$ states that $\forall p : \mathbb{N}_\infty \to 2\,[\forall \alpha \in \mathbb{N}_\infty\,[p(\alpha) = 1\,] \vee \exists \alpha \in \mathbb{N}_\infty\,[p(\alpha) = 0\,]\,]$. We can then plug in the predicate $p_0 := \alpha \mapsto s(\alpha) = s(\infty)$ for p. We then get that $\forall \alpha \in \mathbb{N}_\infty\,[\,s(\alpha) = s(\infty)\,] \vee \exists \alpha \in \mathbb{N}_\infty\,[\,s(\alpha) \neq s(\infty)\,]$. If for every element α of $\mathbb{N}_\infty$ the equality $s(\alpha) = s(\infty)$ holds, then it holds in particular for all α of the form $\underline{n}$, which are elements of $\mathbb{N}_\infty$ for each $n \in \mathbb{N}$. If on the other hand we have an element α such that $s(\alpha) \neq s(\infty)$, then $\alpha \# \infty$ because s is strongly extensional (and inequality and apartness coincide on 2). Then there is an $i \in \mathbb{N}$ such that $\alpha(i) = 0$ and that means $\alpha = \underline{n}$ for some $n \leq i$. Taking both cases together, we can conclude that $\forall n \in \mathbb{N}\,[\,s(\underline{n}) = s(\infty)\,] \vee \exists n \in \mathbb{N}\,[\,s(\underline{n}) \neq s(\infty)\,]$. □

Escardó[11] then claims that Lemma 5.3 is the essence of *Ishihara's First Trick*, which is the following statement[8]:

Lemma 5.4 (Ishihara's First Trick). *Let (X, d_X) be a complete metric space, (Y, d_Y) a metric space, $f : X \to Y$ strongly extensional and $(x_i)_{i=0}^{\infty}$ a sequence in X converging to x. Then we have for all $b > a > 0$ that:*

$$\forall i\,[\,d_Y(f(x_i), f(x)) < b\,] \vee \exists i\,[\,d_Y(f(x_i), f(x)) > a\,]$$

We can motivate this as follows: we know that for metric spaces being apart is equivalent to having a strictly positive distance, and strong extensionality gives that if $d_Y(f(x_i), f(x)) > a$ (so $f(x_i) \# f(x)$), then $d_X(x_i, x) > c$ for some $c > 0$. For $\alpha, \beta \in \mathbb{N}_\infty$, this is the same as there being an index i such that $\alpha(i) \neq \beta(i)$. For apartness from ∞, this even simplifies to there being an i such that $\alpha(i) = 0$. Picking a and b can then be seen as choosing how much of the sequences we are willing to inspect (say $\overline{\alpha} n_a$) to try and find this i. Then either we find an $i < n_a$ and can conclude that α and β are apart or we do not and cannot conclude apartness *so far*. It is true for arbitrary a and b, which means that we can decide whether α is apart from ∞.

In [8] this trick then gets generalised to the following statement, from which the original trick can be derived.

Proposition 5.5 (Ishihara's generalised First Trick). *Let (X, d_X) be a complete metric space, P and Q subsets of X such that $P \cup Q = X$ and $x \in X$ such that $\forall y \in X\,[\,x \neq y \vee y \notin Q\,]$. Then for each sequence $(x_i)_{i=0}^{\infty}$ in X converging to x:*

$$\forall i\,[\,x_i \in P\,] \vee \exists i\,[\,x_i \in Q\,]$$

Like the original trick, Escardó translates this proposition to a version that is applicable to $\mathbb{N}_\infty$. Instead of converging sequences he uses strongly extensional mappings. The following lemma shows why this works.

Lemma 5.6. *Let X be a complete metric space and let $(x_i)_{i=0}^{\infty}$ be a sequence in X converging to l. Then the sequence can be extended to a strongly extensional map $f : \mathbb{N}_\infty \to X$ with $f(\infty) = l$.*

Proof. For every $\alpha \in \mathbb{N}_\infty$ we construct a converging sequence $y_\alpha = (y_i)_{i=0}^{\infty}$ in X inductively as follows:

$$y_0 = x_0 \text{ and } y_{i+1} = \begin{cases} x_{i+1} & \text{if } \alpha(i) = 1 \\ y_i & \text{if } \alpha(i) = 0 \end{cases}.$$

For an example of this construction, see Figure 5.1. It follows that $y_\infty = (x_i)_{i=0}^{\infty}$. Each of these y_α converges to a certain l_α; for $\alpha = \underline{n}$, $l_\alpha = x_n$.

Now we define our map f as follows: $f(\alpha) = l_\alpha$. We see that $f(\infty) = l_\infty = l$, as required. Furthermore, it is an extension of our original sequence by identifying $\underline{n} \in \mathbb{N}_\infty$ with $n \in \mathbb{N}$. □

Figure 5.1: Example of sequence construction for $\alpha = \underline{n}$

i	0	1	2	⋯	$n-2$	$n-1$	n	$n+1$	$n+2$	⋯
$(x_i)_{i=0}^{\infty}$	x_0	x_1	x_2	⋯	x_{n-2}	x_{n-1}	x_n	x_{n+1}	x_{n+2}	⋯
$\alpha = \underline{n}$	1	1	1	⋯	1	1	0	0	0	⋯
$y_\alpha = (y_i)_{i=0}^{\infty}$	x_0	x_1	x_2	⋯	x_{n-2}	x_{n-1}	x_n	x_n	x_n	⋯

Furthermore, we do not require X to be a complete metric space anymore because we are not considering sequences and it suffices to require that X has tight apartness. Finally, its subsets P and Q are now assumed to be disjoint. This restriction is necessary to avoid the need for the axiom of choice[11].

Proposition 5.7 (Ishihara's generalised First Trick for $\mathbb{N}_\infty$). *Let X be a tight set, P and Q disjoint subsets of X such that $P \cup Q = X$ and $f : \mathbb{N}_\infty \to X$ a strongly extensional map such that $\forall y \in X\,[\, y \# f(\infty) \lor y \notin Q\,]$. Then:*

$$\forall n \in \mathbb{N}\,[\, f(\underline{n}) \in P\,] \lor \exists n \in \mathbb{N}\,[\, f(\underline{n}) \in Q\,]$$

Proof. Define p as the predicate p on X with $p(y) = 0 \iff y \in Q$. Because $f(\infty)$ is by definition not apart from itself, this means that $p(f(\infty)) = 1$ and thus $\forall y \in X\,[\, y \# f(\infty) \lor y \notin Q\,] \iff \forall y \in X\,[\, y \# f(\infty) \lor p(y) = p(f(\infty))\,] \iff \forall y \in X\,[\, p(y) \neq p(f(\infty)) \implies y \# f(\infty))\,]$. This is particularly true for all elements of X that can be written as $f(\alpha)$ for some $\alpha \in \mathbb{N}_\infty$. Now we get $\forall \alpha \in \mathbb{N}_\infty\,[\, p(f(\alpha)) \neq p(f(\infty)) \implies y \# f(\infty)\,]$ and consequently the predicate $s : \mathbb{N}_\infty \to 2$ defined by $s = p \circ f$ is strongly extensional by Lemma 5.2. Lemma 5.3 then gives that $\forall n \in \mathbb{N}\,[\,s(\underline{n}) = s(\infty)\,] \vee \exists n \in \mathbb{N}\,[\,s(\underline{n}) \neq s(\infty)\,]$. In the first case $s(\underline{n}) = 1$ for all $n \in \mathbb{N}$ and therefore $f(\underline{n})$ is not in Q and by assumption it is then in P. In the second case there is some $f(\underline{n})$ such that $p(f(\underline{n})) = s(\underline{n}) = 0$ so $f(\underline{n}) \in Q$. □

5.2 Strong extensionality in $\mathbb{N}_{2\infty}$

We will now try to see if we can prove Ishihara's trick from the omniscience of $\mathbb{N}_{2\infty}$ instead of $\mathbb{N}_\infty$. Recall that $\mathbb{N}_{2\infty}$ is the subset of all descending sequences of $3^{\mathbb{N}}$, and therefore has the following elements:

$$\infty = 2^\omega,\ {}^n\infty = 2^n 1^\omega, \text{ and } {}^n\underline{m} = 2^n 1^m 0^\omega.$$

Furthermore, we use $\mathbb{N}_{2\infty}(n)$ to mean the subset $\{\,\alpha \in \mathbb{N}_{2\infty} \mid \alpha(n) \neq 2 \wedge \forall i < n\,[\,\alpha(i) = 2\,]\,\}$.

An immediate difficulty compared to $\mathbb{N}_\infty$ is that where $\mathbb{N}_\infty$ only had one 'infinite element', $\mathbb{N}_{2\infty}$ has infinitely many of those: ∞ and ${}^n\infty$ for every $n \in \mathbb{N}$. We therefore cannot just replace $\mathbb{N}_\infty$ with $\mathbb{N}_{2\infty}$ in the formulation of Lemma 5.3 and achieve the same result. We can however prove the following derived lemma, which has a somewhat weaker conclusion.

Lemma 5.8 (Ishihara's First Trick for $\mathbb{N}_{2\infty}$)**.** *For all strongly extensional predicates s on $\mathbb{N}_{2\infty}$,*

$$\begin{aligned}\forall \alpha \in \mathbb{N}_{2\infty}\,[\,s(\alpha) = s(\infty)\,] \vee (\exists n \in \mathbb{N}\,[\,\exists \alpha \in \mathbb{N}_{2\infty}(n)\,[\,s(\alpha) \neq s(\infty)\,] \wedge \\ (\forall \beta \in \mathbb{N}_\infty\,[\,s(2^n : \beta) = s({}^n\infty)\,] \vee \exists m \in \mathbb{N}\,[\,s({}^n\underline{m}) \neq s(\infty)\,]]))\end{aligned}$$

Proof. Omniscience of $\mathbb{N}_{2\infty}$ gives that $\forall \alpha \in \mathbb{N}_{2\infty}\,[\,s(x) = s(\infty)\,] \vee \exists \alpha \in \mathbb{N}_{2\infty}\,[\,s(\alpha) \neq s(\infty)\,]$. The first case is as we want it, and for the second case we have $\alpha \mathrel{\#} \infty$ by the strong extensionality of s. So, $x \in \mathbb{N}_{2\infty}(n)$ for some $n \in \mathbb{N}$. Now we have $\forall \alpha \in \mathbb{N}_{2\infty}\,[\,s(\alpha) = s(\infty)\,] \vee$
$(\exists n \in \mathbb{N}\,[\,\exists \alpha \in \mathbb{N}_{2\infty}(n)\,[\,p(\alpha) \neq p(\infty)\,]$.

We can further distinguish within the second case. Let $s_n : \mathbb{N}_\infty \to 2$ be defined by $s_n(\beta) = s(2^n{:}\beta)$; these are again strongly extensional. By omniscience of $\mathbb{N}_\infty$, we know that $\forall \beta \in \mathbb{N}_\infty\,[\,s_n(\beta) = s_n(\infty)\,] \vee \exists \beta \in \mathbb{N}_\infty\,[\,s_n(\beta) \neq s_n(\infty)\,]$. In the latter case strong extensionality gives us that $\beta \mathrel{\#} \infty$ so $\beta = \underline{m}$ for some $m \in \mathbb{N}$. Putting it all together yields the desired result. □

For the generalised version, we will need two things: an equivalent of Lemma 5.2 and checking whether we can extend Cauchy sequences to strong extensional maps on $\mathbb{N}_{2\infty}$. For the latter, we can use the same construction as in Lemma 5.6 but instead define $y_{i+1} = x_{i+1}$ if $\alpha(i) = 2$ and $y_{i+1} = y_i$ else. In order to do more justice to the two-dimensional structure of the set, we can also extend double sequences to strong extensional maps on $\mathbb{N}_{2\infty}$.

Lemma 5.9. *Let X be a complete metric space and let $(x_{i,j})_{i,j=0}^{\infty}$ be a double sequence in X such that for all $n \in \mathbb{N}(x_{n,j})_{j=0}^{\infty}$ converges to l_n and $(x_{i,0})_{i=0}^{\infty}$ converges to l. Then the sequence can be extended to a strongly extensional map $f : \mathbb{N}_{2\infty} \to X$ with $f(\infty) = l$ and $\forall n \in \mathbb{N}\,[\, f({}^{n}\infty) = l_n \,]$.*

Proof. For every $\alpha \in \mathbb{N}_{2\infty}$ we construct a converging sequence $y_\alpha = (y_i)_{i=0}^{\infty}$ in X inductively as follows:

$$y_0 = x_{0,0} \text{ and } y_{i+1} = \begin{cases} x_{i+1,0} & \text{if } \alpha(i) = 2 \\ x_{n,i-n} & \text{if } \alpha(i) = 1 \text{ and } \alpha \in \mathbb{N}_{2\infty}(n) \\ y_i & \text{if } \alpha(i) = 0 \end{cases} \cdot$$

Note that we know that if $\alpha(i) = 1$ for some $i \in \mathbb{N}$ then $\exists n \leq i\,[\,\alpha \in \mathbb{N}_{2\infty}(n)\,]$. For an example of this construction, see Figure MISSING. It follows that $y_\infty = (x_{i,0})_{i=0}^{\infty}$ and $y_{{}^{n}\infty} = (x_{n,j})_{j=0}^{\infty}$. Each of these y_α converges to a certain l_α; for $\alpha = {}^{n}\underline{m}$, $l_\alpha = x_{n,m}$.

Now we define our map f as follows: $f(\alpha) = l_\alpha$. We see that $f(\infty) = l_\infty = l$ and $f({}^{n}\infty) = l_{{}^{n}\infty} = l_n$, as required. Furthermore, it is an extension of our original double sequence by identifying ${}^{n}\underline{m} \in \mathbb{N}_{2\infty}$ with $(n, m) \in \mathbb{N} \times \mathbb{N}$. $\square$

Figure 5.2: Example of y_α construction from $(x_{i,j})_{i,j=0}^{\infty}$ for $\alpha = {}^{n}\underline{m}$

	$\vdots$	$\vdots$		$\vdots$	$\vdots$	$\vdots$	$\iddots$			
	$x_{0,m+1}$	$x_{1,m+1}$	$\cdots$	$x_{n-1,m+1}$	$x_{n,m+1}$	$x_{n+1,m+1}$	$\cdots$			
	$x_{0,m}$	$x_{1,m}$	$\cdots$	$x_{n-1,m}$	$x_{n,m}$	$x_{n+1,m}$	$\cdots$			
$(x_{i,j})_{i,j=0}^{\infty}$	$x_{0,m-1}$	$x_{1,m-1}$	$\cdots$	$x_{n-1,m-1}$	$x_{n,m-1}$	$x_{n+1,m-1}$	$\cdots$			
	$\vdots$	$\vdots$		$\vdots$	$\vdots$	$\vdots$	$\cdots$			
	$x_{0,1}$	$x_{1,1}$	$\cdots$	$x_{n-1,1}$	$x_{n,1}$	$x_{n+1,1}$	$\cdots$			
	$x_{0,0}$	$x_{1,0}$	$\cdots$	$x_{n-1,0}$	$x_{n,0}$	$x_{n+1,0}$	$\cdots$			
$\alpha = {}^{n}\underline{m}$	2	2	$\cdots$	2	1	1	$\cdots$	1	0	0	$\cdots$
$y_\alpha = (y_i)_{i=0}^{\infty}$	$x_{0,0}$	$x_{1,0}$	$\cdots$	$x_{n-1,0}$	$x_{n,0}$	$x_{n,1}$	$\cdots$	$x_{n,m-1}$	$x_{n,m}$	$x_{n,m}$	$\cdots$
i	0	1	$\cdots$	$n-1$	n	$n+1$	$\cdots$	$n+m-1$	$n+m$	$n+m+1$	$\cdots$

Finding an equivalent of Lemma 5.2 is more problematic, as can be seen in the following lemma.

Lemma 5.10 (Naive translation of Lemma 5.2). *Let* $p : \mathbb{N}_{2\infty} \to 2$*. If* $\forall \alpha \in \mathbb{N}_{2\infty} [\, p(\alpha) \neq p(\infty) \implies \alpha \# \infty \,] \implies p$ *is strongly extensional, then* ***LPO****.*

Proof. Let p be a predicate for which the condition holds, and take $\alpha, \beta \in \mathbb{N}_{2\infty}$ such that $p(\alpha) \neq p(\beta)$. We try to show that $\alpha \# \beta$. By co-transitivity, we know that $p(\alpha) \neq p(\infty) \vee p(\beta) \neq p(\infty)$. Without loss of generality, assume $p(\alpha) \neq p(\infty)$. Then by assumption we conclude that $\alpha \# \infty$. By co-transitivity, we know that $\alpha \# \beta \vee \infty \# \beta$. In the first case we are done, so assume the second.

Since both α and β are apart from ∞, we know that $\alpha \in \mathbb{N}_{2\infty}(n)$ and $\beta \in \mathbb{N}_{2\infty}(m)$ for some $n, m \in \mathbb{N}$. If $n \neq m$, $\alpha \# \beta$ and we are done. If $n = m$ we have problem: we cannot check if $\alpha = {}^n\infty$ or $\beta = {}^n\infty$, this is **LPO**. □

Lemma 5.2 works because we can decide equality of every two elements in $\mathbb{N}$ that are apart from ∞. In the case of $\mathbb{N}_{2\infty}$ we still have all the elements ${}^n\infty$ to contend with. This leads us to the following alternative lemma with a stronger hypothesis.

Lemma 5.11 (Lemma 5.2 for $\mathbb{N}_{2\infty}$**).** *For* $p : \mathbb{N}_{2\infty} \to 2$*,* p *is strongly extensional* $\iff \forall \alpha \in \mathbb{N}_{2\infty} [\, p(\alpha) \neq p(\infty) \implies \alpha \# \infty \,] \wedge \forall n \in \mathbb{N} [\, \forall \alpha \in \mathbb{N}_{2\infty}(n) [\, p(\alpha) \neq p({}^n\infty) \implies \alpha \# {}^n\infty \,]\,]$.

Proof. If p is strongly extensional, the condition is an automatic consequence of it. Conversely, let $\alpha, \beta \in \mathbb{N}_{2\infty}$ and assume $p(\alpha) \neq p(\beta)$. We have to show that $\alpha \# \beta$. By co-transitivity, we know that $p(\alpha) \neq p(\infty) \vee p(\beta) \neq p(\infty)$. Without loss of generality, assume $p(\alpha) \neq p(\infty)$. Then by assumption we conclude that $\alpha \# \infty$. By co-transitivity, we know that $\alpha \# \beta \vee \infty \# \beta$. In the first case we are done, so assume the second.

Since both α and β are apart from ∞, we know that $\alpha \in \mathbb{N}_{2\infty}(n)$ and $\beta \in \mathbb{N}_{2\infty}(m)$ for some $n, m \in \mathbb{N}$. If $n \neq m$, $\alpha \# \beta$ and we are done. If $n = m$, then knowing that $p(\alpha) \neq p(\beta)$ and using co-transitivity gives us that $p(\alpha) \neq p({}^n\infty) \vee p(\beta) \neq p({}^n\infty)$. Without loss of generality, assume $p(\alpha) \neq p({}^n\infty)$. Then by assumption $\alpha \# {}^n\infty$ and again by co-transitivity $\beta \# \alpha \vee \beta \# {}^n\infty$. In the first case we are again done and in the second case we know that $\alpha = {}^n\underline{p}$ and $\beta = {}^n\underline{q}$ for some $p, q \in \mathbb{N}$. If $p = q$ we have $\alpha = \beta$ and hence $p(\alpha) = p(\beta)$ by extensionality of p, which contradicts that $p(\alpha) \neq p(\beta)$. Thus, $p \neq q$ and accordingly $\alpha \# \beta$. □

The above lemmas then culminate in the following generalised version of Ishihara's First Trick. Compared to the Proposition for $\mathbb{N}_\infty$, we need a stronger hypothesis and have a weaker conclusion because of the changes made in Lemma 5.11 and Lemma 5.8, respectively.

Proposition 5.12 (Ishihara's generalised First Trick for $\mathbb{N}_{2\infty}$**).** *Let* X *be a tight set,* P *and* Q *disjoint subsets of* X *such that* $P \cup Q = X$ *and*

$f : \mathbb{N}_{2\infty} \to X$ *a strongly extensional map such that* $\forall y \in X\,[\,(y \# f(\infty) \wedge \forall n \in \mathbb{N}\,[\,y \# f({}^{n}\infty)\,]) \vee y \notin Q\,]$. *Then:*

$$\forall \alpha \in \mathbb{N}_{2\infty}\,[\,f(\alpha) \in P\,] \vee \exists n \in \mathbb{N}\,[\,\exists \alpha \in \mathbb{N}_{2\infty}(n)\,[\,f(\alpha) \in Q\,]\,]$$

Proof. Define p on X such that $p(y) = 0 \iff y \in Q$. Because $f(\infty)$ is by definition not apart from itself, this means that $f(\infty) \notin Q$ so $p(f(\infty)) = 1$. The condition on f reduces to $\forall y \in X\,[\,(y \# f(\infty) \wedge \forall n \in \mathbb{N}\,[\,y \# f({}^{n}\infty)\,]) \vee p(y) = p(f(\infty))\,]$, which is equivalent to $\forall y \in X\,[\,p(y) \neq p(f(\infty)) \implies (y \# f(\infty) \wedge \forall n \in \mathbb{N}\,[\,y \# f({}^{n}\infty)\,])\,]$. This is then in particular true for all $f(\alpha) \in X$. By Lemma 5.11, the predicate $s : \mathbb{N}_{2\infty} \to 2$ defined by $s = p \circ f$ is strongly extensional. Then Lemma 5.8 gives us that $\forall \alpha \in \mathbb{N}_{2\infty}\,[\,s(\alpha) = s(\infty)\,]$ or $\exists n \in \mathbb{N}\,[\,\exists \alpha \in \mathbb{N}_{2\infty}(n)\,[\,p(\alpha) \neq p(\infty)\,]$. In the first case $s(\alpha) = 1$ for all $\alpha \in \mathbb{N}_{2\infty}$ and therefore $f(\alpha)$ is not in Q and by assumption it is then in P. In the second case there is some $f(\alpha)$ (with $\alpha \in \mathbb{N}_{2\infty}(n)$ for some $n \in \mathbb{N}$) such that $p(f(\alpha)) = s(\alpha) = 0$ so $f(\alpha) \in Q$. □

Chapter 6

Discussion

We have seen that $\mathbb{N}_\infty$ is an omniscient set: for any predicate on $\mathbb{N}$ we can decide whether there is an element for which the predicate does not hold. We have shown this by constructing a selection function that finds one such element in $\mathbb{N}_\infty$ if it exists. The squashed sum, a way to create a new set out of countable (not necessarily distinct) omniscient sets, is again omniscient. Furthermore, the squashed sum produced by taking $\mathbb{N}_\infty$ for every omniscient set turns out to be equivalent to $\nvDash_{2\infty}$, the set of descending sequences in $\{0, 1, 2\}$, and we can iterate this procedure to generate omniscient sets equivalent to $\mathbb{N}_{n\infty}$ (descending sequences in $\{0, 1, ..., n\}$). From the omniscience of $\mathbb{N}_\infty$ we were able to prove interesting properties related to strongly extensional mappings. The most important of these is a proposition that captures the essence of Ishihara's First Trick[8]. We have then investigated whether we can generalise this to other infinite omniscient sets by finding a suitable version of this proposition for $\mathbb{N}_{2\infty}$.

Since omniscience of $\mathbb{N}$ is **LPO**, it is somewhat surprising that we can conclude it for the similar $\mathbb{N}_\infty$. It is even more surprising when we consider the roadblock caused by the subtle difference between $\mathbb{N}_\infty$ and $\underline{\mathbb{N}} \cup \{\infty\}$: asserting they are equal is **LPO**. In [11] this was solved in a profound way by introducing full subsets (in our case $\underline{\mathbb{N}} \cup \{\infty\}$) that allowed us to conclude something about the larger set ($\mathbb{N}_\infty$). This exact same technique is then again useful when proving that the squashed sum is searchable. We think that in the original article the necessity of this construction and more specifically what exactly it solves is somewhat underexposed and we have therefore tried to make this more explicit.

The squashed sum allows us to prove omniscience for larger subsets of $2^{\mathbb{N}}$ which can then be identified with other sets (such as $\mathbb{N}_{2\infty}$. However, the appealing properties of $\mathbb{N}_\infty$ discussed in Section 5.1 do not readily transfer to these larger sets. The variant of Ishihara's First Trick is derived from the omniscience of $\mathbb{N}_\infty$, but that is not the only reason why $\mathbb{N}$ behaves so well because we see in Section 5.2 that a lot more work has to be done to reach similar results for $\mathbb{N}_{2\infty}$. When comparing these sets, a glaring difference is that $\mathbb{N}$ only has one 'infinite element'. This makes the set constructively easier to handle because oftentimes we can use ∞ as a special case and all the other elements can be easily compared. $\mathbb{N}_{2\infty}$ on the other hand has infinitely many 'infinite elements', which causes us to violate different principles of omniscience more easily. This observation suggests that potential omniscient sets with a finite number of 'infinite elements' would more easily acquire the properties proven for $\mathbb{N}$, but this requires more research.

The difficulties in proving similar properties for $\mathbb{N}_{2\infty}$ as for $\mathbb{N}_\infty$ in Section 5.2 have mostly resulted in lemmas and propositions with stronger hypotheses and/or weaker conclusions then their counterparts. This makes them immediately less valuable and not as generally usable. This is most clearly visible in the analogue of Ishihara's generalised First Trick for $\mathbb{N}_{2\infty}$ (Proposition 5.12. Whereas in the version for $\mathbb{N}_\infty$ (Proposition 5.7) we require our strongly extensional map $f : \mathbb{N}_\infty \to X$ to be such that $\forall y \in X\,[\,y \# f(\infty) \vee y \notin Q\,]$, in Proposition 5.12 we need the map such that $\forall y \in X\,[\,(y \# f(\infty) \wedge \forall n \in \mathbb{N}\,[\,y \# f({}^{n}\infty)\,]) \vee y \notin Q\,]$. We suddenly require knowing whether elements are apart from (infinitely) many fixed elements instead of just one. This is a consequence of the aforementioned differences between these two sets. We do suspect however that we can slightly weaken this condition. The reason we need it in the first place is because halfway through the proof of Proposition 5.12 we want to invoke Lemma 5.11. The current condition allows us to do that, but it concludes something that is stronger than what we actually require for the lemma. Unfortunately we have so far not succeeded in finding an alternative weaker condition that is strong enough for our purposes.

At the same time, it is also not clear that our version of Ishihara's (generalised) First Trick for $\mathbb{N}_{2\infty}$ and the accompanying lemmas are the most appropriate ones. We used the version(s) for $\mathbb{N}_\infty$ as a starting point and modified it to create a statement about $\mathbb{N}_{2\infty}$. There might be a more natural extension of the original Ishihara's (generalised) First Trick to the set $\mathbb{N}_{2\infty}$ (and more generally, $\mathbb{N}_{n\infty}$) that captures the essence of the trick and also takes into account the structure of the particular set.

Furthermore, we used $\mathbb{N}_{2\infty}$ as an example to see whether the behaviour of $\mathbb{N}_\infty$ transfers well but we would like to see how this goes for more general $\mathbb{N}_{n\infty}$ or other squashed sums. The lemmas and propositions proposed for $\mathbb{N}_{2\infty}$ in Section 5.2 seem to be quite easily adaptable to $\mathbb{N}_{n\infty}$ but this needs further investigation. A further investigation into the behaviour of squashed sums compared to $\mathbb{N}_\infty$ should then also discuss Ishihara's Second Trick, which we did not touch upon here but is discussed with regards to $\mathbb{N}_\infty$ in [11].

Finally, in a broader sense more research is needed towards other infinite omniscient sets. We now know of $\mathbb{N}_\infty$ and we can generate (infinitely many) new ones using the squashed sum construction, but these are all (equivalent to) subsets of $2^{\mathbb{N}}$. Much less is known about other types of omniscient sets, with different internal structures. Getting more insight into these sets can reveal more common properties of omniscient sets.

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