Bachelor thesis
Computing Science

Radboud University

Fusible Numbers and the Provability of Computability

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January 2021

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Abstract

A riddle about measuring time by lighting a bunch of fuses gives rise to computable functions that grow too fast for there to be a simple proof of computability. By formalising the concept of the riddle we define the set of *fusible numbers* $\mathcal{F}$. Erickson et al. [7] have shown some interesting statements that are derived from this set. We aim to bring part of these results to a larger audience by elaborating on the proofs and definitions needed. We show that $\mathcal{F}$ is well-ordered with order type at least ε_0, and that their density grows very fast. A recursively defined subset of $\mathcal{F}$, the *tame fusible numbers* leads to some of the facts about $\mathcal{F}$ just mentioned, and algorithm M. Algorithm M is incredibly easy to implement, and is computable, but by the construction of the tame fusible numbers we can show that Peano arithmetic cannot prove it to be computable.

Chapter 1

Introduction

A fundamental part of computer science is concerned with the computability of functions. We consider a function computable if some Turing machine could be constructed that models the function and terminates on each given input. If we want to prove a function to be computable, Peano Arithmetic (PA) is an obvious first choice to use as a proof system due to its simplicity and Turing completeness: Any computable function f can be expressed in PA. The question then arises: Can PA also prove the computability of f?

It turns out: often, but not always. Despite its simplicity, PA is surprisingly strong [2, 3]. Take for example the Ackermann function: It was designed as a total, computable, yet not primitive recursive function [5] and is the classical example of how a simple looking function can still have a ridiculously large complexity. And yet PA has no problem proving it to be computable. However, we know that there must be statements that can be expressed but not proven by PA, thanks to Gödel [10]. And plenty of such statements have been found already [8, 12, 14, 15, 16]. Though most examples can already be expressed quite simply, the one found by Erickson et al. [7] is of an incredibly simple form. They found that PA can express this function, but not prove it to be computable:

$$M(x) : \textit{if } x < 0 \textit{ return } -x, \textit{ else return } M(x - M(x-1))/2$$

This function M is derived from the construction of the set of so-called *Fusible Numbers* $\mathcal{F}$, based on a riddle where we measure time by lighting fuses. The set is formally introduced by Erickson et al. [7], specifically because some true statements can be derived from it that can be expressed in but not proven to be true by PA.We will be discussing their results about M and those leading up to them to make them available to a larger audience.

Following are the main results that we will discuss, and an overview of how this thesis is structured.

1.1 Results

The main point of this thesis is to provide a more intuitive explanation of the results about algorithm M found by Erickson et al. [7]. We elaborate on the existing proofs and definitions, add lemmas and examples where necessary, and provide some necessary background knowledge.

The result about M is described in theorem 1.3, and theorems 1.1 and 1.2 are necessary for its proof.

Theorem 1.1 (Part of Erickson et al. [7] Theorem 1.1). *The set $\mathcal{F}$ of fusible numbers, when ordered by the usual order "$<$" on $\mathbb{R}$, is well-ordered, with order type at least ε_0.*

This shows us how the fusible numbers are related to ordinals, which will help us with the result about PA. Erickson et al. [7] also show that the order type is *exactly* ε_0. Though that is interesting for other results, it is not strictly necessary to get to the result about M. The fact that $\mathcal{F}$ is well-ordered means each of its elements has a successor that is also in $\mathcal{F}$. This also holds for every subset of $\mathcal{F}$.

Theorem 1.2 (Erickson et al. [7] Theorem 1.2). *The density of the fusible numbers along the real line grows very fast: Let $g(n)$ be the largest gap between consecutive fusible numbers in $\mathcal{F} \cap [n, \infty)$. Then $g(n)^{-1} \geq F_{\varepsilon_0}(n-7)$ for all $n \geq 8$, where F_α denotes the fast-growing hierarchy.*

This shows a property of the fusible numbers that will help us find the final result.

Theorem 1.3 (Erickson et al. [7] Theorem 1.3.3). *Consider the algorithm "$M(x)$: if $x < 0$ return $-x$, else return $M(x - M(x-1))/2$." Then M terminates on real inputs, although PA cannot prove the statement "M terminates on all natural inputs."*

Theorem 1.3 is the final result about M. It confirms that even a computable function as seemingly simple as this can not necessarily be proven computable by PA.

1.2 Overview

- Chapter 2 contains the necessary background on ordinals and fast-growing functions.
- Chapter 3 contains the definition of the set of fusible numbers $\mathcal{F}$ and a proof of its well-orderedness, proving the first part of theorem 1.1.

- Chapter 4 introduces the tame fusible numbers $\mathcal{F}'$. We derive M from them and find the lower bound of the order type of $\mathcal{F}$, which proves the second part of 1.1 Here we also outline and clarify the proof of Theorem 1.2 by Erickson et al. [7].
- Chapter 5 describes how Erickson et al. [7] find a fast-growing function for Theorem 1.2, and clarifies how that leads to a proof.
- Chapter 6 gives some information on proving in PA and includes the results about M as in Theorem 1.3.

Chapter 2

Ordinals

Ordinals, or *ordinal numbers*, say something about the size of a well-ordered set. Say we have two ordered sets X and Y. We could simply compare them by size, as we do with *cardinality*, but that does not always give us enough information. For example, $\mathbb{N}$ and $\mathbb{Q}$ have the same cardinality ($\aleph_0$), but there is no order-preserving bijection between them so they have different order types.

If X and Y are well-ordered, however, they must also be of the same order type. This is where ordinals come in. Each well-ordered set has the same order type as exactly one ordinal. As such, we use the ordinal as a name for that order type.

This enables us to differentiate between different types of infinities, in particular different countable ones.

2.1 Formal Definition

We follow Jech [13] for the formal definitions. Ordinal numbers are defined as sets containing all ordinals smaller than them, and they are well-ordered by $\in$. It holds for ordinal α and β

$$\alpha < \beta \text{ if and only if } \alpha \in \beta \text{ and } \alpha = \{\beta : \beta < \alpha\} \tag{2.1}$$

The finite ordinals are related to the natural numbers $\mathbb{N}$.

For example the ordinal 0 is defined by the empty set $\emptyset$ (we write $0 = \emptyset$). And 1= $\{0\} = \{\{\}\}(= \{\emptyset\})$, so by (1) 0<1. Another example: $3 = \{0, 1, 2\}$ so 1<3

We differentiate between two types of ordinals: the *successor* and *limit* ordinals.

Definition 2.1 (successor ordinal)**.** An ordinal α is a *successor ordinal* if it is the successor of another ordinal. In other words, $\alpha = \beta + 1$, where β can be any ordinal.

Definition 2.2 (limit ordinal)**.** An ordinal α is a *limit ordinal* if it is not a successor ordinal.

To illustrate this difference, look at the natural numbers. All nonzero natural numbers are successor ordinals, as for each nonzero $n \in \mathbb{N}$ we can find an $m \in \mathbb{N}$ such that $m + 1 = n$.

As implied, 0 is not a successor ordinal, so it is a limit ordinal. This follows the definition, as the ordinals are limited by zero: there is no ordinal smaller than 0, so it cannot be the successor of any ordinal.

A more interesting limit ordinal is ω:

Definition 2.3 (ω)**.** The least nonzero limit ordinal is ω. This is defined by the set of all natural numbers.

To be clear, following this definition, $\omega + 1$ is a successor limit.

Definition 2.4 (τ_n)**.** Given $n \geq 0$, τ_n is defined by

$$\begin{aligned} \tau_0 &= 1 \\ \tau_{n+1} &= \omega^{\tau_n} \end{aligned}$$

Definition 2.5 (ε_0)**.** $\varepsilon_0 = \lim_{n \to \infty} \tau_n$.

2.2 Basic Arithmetic

Again we follow Jech [13]. In the previous section we have made use of the addition of ordinals. However it is important to note that this and other arithmetical functions differ from those in regular arithmetic. Following are a few definitions, and a brief discussion of the implied differences after that:

Definition 2.6 (Addition)**.** For all ordinal numbers α

(i) $\alpha + 0 = \alpha$,
(ii) $\alpha + (\beta + 1) = (\alpha + \beta) + 1$, for all β,
(iii) $\alpha + \beta = \lim_{\xi \to \beta}(\alpha + \xi)$ for all limits $\beta > 0$

Definition 2.7 (Multiplication)**.** For all ordinal numbers α

(i) $\alpha \cdot 0 = 0$,
(ii) $\alpha \cdot (\beta + 1) = \alpha \cdot \beta + \alpha$, for all β,
(iii) $\alpha \cdot \beta = \lim_{\xi \to \beta}(\alpha \cdot \xi)$ for all limits $\beta > 0$

Definition 2.8 (Exponentiation)**.** For all ordinal numbers α

(i) $\alpha^0 = 1$,
(ii) $\alpha^{\beta+1} = \alpha^\beta \cdot \alpha$, for all β,
(iii) $\alpha + \beta = \lim_{\xi \to \beta}(\alpha + \xi)$ for all limits $\beta > 0$

In regular arithmetic, addition and multiplication are commutative. Note that here they are not. If the second argument is a limit larger than the first argument, then by (iii), we drop the first argument. Take for example $1 + \omega = \lim_{\xi \to \omega}(1 + \xi) = \omega \neq \omega + 1$. If the first argument is also a limit, but still smaller than the second, the same happens: $\omega + \omega^2 = \omega^2$, but if the first argument is larger than the first then addition and multiplication work as normal. Note for another example that $(\omega^2 + 1) + \omega = \omega^2 + \omega + 1 \neq \omega^2 + 1 + \ \omega = \omega^2 + \omega$.

2.3 Cantor Normal Form and ε_0

Every ordinal α can be written uniquely in the so-called *Cantor Normal Form* as follows

$$\alpha = \omega^{\alpha_1} \cdot n_1 + \cdots + \omega^{\alpha_k} n_k$$

where $k \geq 0$, $n_1, \ldots, n_k$ nonzero natural numbers, and $\alpha_1 \ldots \alpha_k$ ordinals.

There exist ordinals such that in this normal form $\alpha = \omega^\alpha$. The smallest such ordinal is $\varepsilon_0 = \omega^{\omega^{\omega^{\cdots}}}$.

2.4 Natural sum and product $\oplus, \otimes$

Addition and multiplication on ordinals is not commutative. See for example the ordinals $\alpha = \omega + 1$ and $\beta = \omega^2$. Then $\alpha + \beta = \omega + 1 + \omega^2 = \omega^2 \neq \omega^2 + \omega + 1 = \beta + \alpha$. In most cases this is wanted behavior, but it also gives the addition and multiplication of unknown ordinals a degree of unpredictability. Using the Cantor normal form we can construct operations $\oplus$ and $\otimes$ that are commutative and associative. We follow the definition from Erickson et al. [7]. Let $\alpha = \omega^{\alpha_1} + \cdots + \omega^{\alpha_n}$ and $\beta = \omega^{\beta_1} + \cdots + \omega^{\beta_m}$ be two ordinals in Cantor normal form with $m, n \geq 0$ and $\alpha_1 \geq \cdots \geq \alpha_n$ and $\beta_1 \geq \cdots \geq \beta_m$.

- Natural sum $\oplus$

$$\alpha \oplus \beta = \omega^{\gamma_1} + \cdots + \omega^{\gamma_{n+m}},$$

where $\gamma_1, \ldots, \gamma_{n+m}$ are $\alpha_1, \ldots, \alpha_n, \beta_1, \ldots, \beta_m$ sorted in nonincreasing order.

- Natural product $\otimes$

$$\alpha \otimes \beta = \bigoplus_{i,j} \omega^{\alpha_i \oplus \beta_j}$$

2.5 Canonical descent

For a limit ordinal $\alpha < \varepsilon_0$ the *canonical sequence* $[\alpha]_n$ is a sequence converging to α when n goes to infinity, and is defined as follows: For limit $\beta < \omega^{\alpha+1}$ and ordinal γ let

$$[\omega^\gamma + \beta]_n = \omega^\gamma + [\beta]_n \tag{2.2}$$

$$[\omega^{\gamma+1}]_n = \omega^\gamma n \tag{2.3}$$

$$[\omega^\gamma]_n = \omega^{[\gamma]_n} \text{(only if } \gamma \text{ is a limit)} \tag{2.4}$$

For successor ordinals we define $[\alpha + 1]_n = \alpha$.

With these we know how to write each element of a canonical sequence. Take for example

$$\begin{aligned}[\omega^{\omega+2}7]_5 &= [\omega^{\omega+2}6 + \omega^{\omega+2}]_5 \\ &\overset{(2.2)}{=} \omega^{\omega+2}6 + [\omega^{\omega+2}]_5 \\ &\overset{(2.3)}{=} \omega^{\omega+2}6 + \omega^{\omega+1}5\end{aligned}$$

(and $[\omega^{\omega+2}7]_n = \omega^{\omega+2}6 + \omega^{\omega+1}n$).

Definition 2.9 (Canonical descent)**.** A *canonical descent* is a descending sequence of canonical sequences: $\alpha_1 > \alpha_2 > \cdots > \alpha_k$ in which $\alpha_{i+1} = [\alpha_i]_{n_i}$. So each α_i after the first one is the n_i-th element of the canonical sequence of the previous one.

These n_i are the *descent parameters* of the sequence.

For each pair of ordinals $\beta < \alpha < \varepsilon_0$ we can find a (the shortest) canonical descent from α to β. If the descent parameter of such a descent are at most k then we can write $\alpha \to_k \beta$. Note that this k is just an upper limit, not necessarily the least upper limit.

Lemma 2.10 (Erickson et al. [7] Lemma 3.1)**.** *Let $\alpha, \beta < \varepsilon_0$ be limit ordinals satisying $[\alpha]_n < \beta < \alpha$. Then $[\beta]_1 \geq [\alpha]_n$ and $[\beta]_2 > [\alpha] + 1$.*

This means in particular that if $[\alpha]_n < \beta < \alpha$, then every canonical descent from β to 0 must contain $[\alpha]_n$.

2.6 Fast-growing hierarchies

A fast-growing hierarchy are functions F_α (where α is an ordinal) that grow very fast. It holds that for ordinals $\beta > \alpha$, F_β grows much faster than F_α. It grows so much faster, that no matter how many times in a row we apply F_α, F_β will at some point overtake it: For every k, $F_\beta(n)$ will at some point overtake $F_\alpha^{(k)}(n)$. It's defined as follows:

$$\begin{aligned} f_0(n) &= n + 1 \\ f_{\alpha+1}(n) &= f_\alpha^{(n)}(n) \\ f_\alpha(n) &= f_{[\alpha]_n}(n) \text{ if } \alpha \text{ is a limit.} \end{aligned}$$

The exact definition of a fast-growing hierarchy depends on how we define $[\alpha]_n$ [3], which is why generally we don't talk about '*the* fast-growing hierarchy'. When we do, most often the Wainer hierarchy is meant.

Definition 2.11 (Wainer hierarchy, F_α)**.** The fast-growing hierarchy that has all $\alpha < \varepsilon_0$ and $[\alpha]_n$ as we defined it is called the *Wainer hierarchy* and we will write F_α for a function in this hierarchy.

Definition 2.12 (Hardy hierarchy, H_α)**.** The *Hardy hierarchy* is not a fast-growing hierarchy by the earlier definition, but it is related to it for $\alpha < \varepsilon_0$. It is defined as follows [11]

$$\begin{aligned} H_0(n) &= n \\ H_{\alpha+1} &= H_\alpha(n + 1) \\ H_\alpha(n) &= H_{[\alpha]_n}(n) \text{ if } \alpha \text{ is a limit.} \end{aligned}$$

For all $\alpha < \varepsilon_0$, the Wainer and Hardy hierarchies are related by $F_\alpha(n) = H_{\omega^\alpha}(n)$. [3] This means the Hardy hierarchy grows significantly slower than the Wainer hierarchy.

Definition 2.13. $F_{\varepsilon_0}(n) = F_{\tau_n}(n)$. Note that for every $\alpha < \varepsilon_0$, for high enough n, $F_{\varepsilon_0}(n) > F_\alpha(n)$.

Lemma 2.14 (Erickson et al. [7] Lemma 3.5.)**.** $H_{\tau_n}(2) > F_{\varepsilon_0}(n-2)$ *for every* $n \geq 3$.

Chapter 3

Fusible numbers $\mathcal{F}$ and their well-orderedness

3.1 Definition of $\mathcal{F}$

The fusible numbers are based on a riddle that uses fuses to measure time. Say we have an unlimited supply of fuses. We know they each burn for exactly one hour, but not uniformly, so we cannot tell how much time has passed by simply measuring how much of the fuse has burnt up. How do we measure 45 minutes with two of these fuses?

Well, lighting one fuse at both ends simultaneously means that it will burn up in 30 minutes. So if we light three of the four ends of two fuses at the same time then one fuse will burn up after 30 minutes, and the other still has exactly half an hour left. If at that exact moment we light the last end, that remaining half hour will be halved. So the last fuse burns another 15 minutes, and in total the two fuses will have burned 45 minutes.

We could use this method to measure many other amounts of time by burning a number of fuses. The set of fusible numbers $\mathcal{F} \in \mathbb{Q}$, introduced by Erickson et al. [7], is based on this idea. To properly define it we need a function 'fuse'

$$x \sim y = (x + y + 1)/2 \quad \text{for } x, y \in \mathbb{R}.$$

If $|y - x| < 1$ then $x \sim y$ is the time at which a fuse burns up if its ends are lit at times x and y.

$\mathcal{F}$ is then defined recursively:

- $0 \in \mathcal{F}$.
- For $x, y \in \mathcal{F}$ and $|y - x| < 1$ also $x \sim y \in \mathcal{F}$.

3.2 Proof of well-orderedness

Recall the definition of well-orderedness:

Definition 3.1. A set S is *well-ordered* is each nonempty subset of S has a smallest element.

First a few observations:

Observation 3.2. $a \sim b = b \sim a$.

Observation 3.3. $\forall a > b \ : \ a \sim c > b \sim c$. (Together with Observation 3.2 clearly also $\forall a > b \ : \ c \sim a > c \sim b$.)

Observation 3.4. $(a + c) \sim b = (a \sim b) + \frac{c}{2}$.

Observation 3.5. If $x \sim y = z$ and $y - x < 1 - 2\varepsilon$ then $z - y > \varepsilon$.

Observation 3.6 (Erickson et al. [7] Observation 2.1)**.** Suppose $x \leq y$ with $|y - x| < 1$, and let $z = x \sim y$. Then $x + \frac{1}{2} \leq z < x + 1$ and $y < z$.

Lemma 3.7 (Erickson et al. [7] Lemma 2.2)**.** *Suppose $z \in \mathcal{F}$, and let $0 < m \leq 1$. Then there exists a fusible number in the window $I = (z + 1 - 2m, z + 1 - m]$.*

Proof. We show $\forall n \ : \ z + 1 - 2^{-n} \in \mathcal{F}$ by induction over n:

$$\begin{aligned} n = 0 \ &: \ z + 1 - 1 = z \in \mathcal{F} \\ n > 0 \ &: \ z + 1 - 2^{-n} = (z + 1 - 2^{-(n-1)}) \sim z \in \mathcal{F} \end{aligned}$$

and clearly for some n it holds that $z + 1 - 2^{-n} \in I$. $\square$

Lemma 3.8 (Erickson et al. [7] Lemma 2.3)**.** *The set $\mathcal{F}$, ordered by the usual order "$<$" of real numbers, is well-ordered, meaning every nonempty subset of $\mathcal{F}$ has a smallest element.*

Proof. Say $\mathcal{F}$ is not well-ordered. Then there is a non-empty set $\mathcal{G}$ of non-empty subsets of $\mathcal{F}$ without a least element. Each $g \in \mathcal{G}$ contains at least one infinite descending sequence. We name the set of all such sequences S. Since $\mathcal{F}$ has a least element we know that each $s \in S$ converges to a limit $l_s = \lim s \in \mathbb{R}$.

$\{l_s \mid s \in S\}$ has a least element l_0. Either the set does not contain an infinite descending sequence and the least element is obvious, or it does contain an infinite descending sequence T. Consider the latter case. Call the elements of T t_i. Note that these elements are not necessarily in $\mathcal{F}$. However by construction we can find for each t_i a sequence $p_i \in S$ such that p_i converges to t_i.

We construct a set $\mathcal{G}_0 = (g_1, g_2, g_3, \dots) \subseteq \mathcal{F}$ in which $g_i \in p_i$. Since the limit t_1 exists, we can find for $\varepsilon = 1$ some δ_1 such that $\forall i \geq \delta_1 |t_1 - p_{1i}| < 1$ where p_{ij} is the j^{th} element of p_i. Now take $g_1 = p_{1\delta_1}$.

Similarly we find for $\varepsilon = \frac{1}{2}$ a δ_2 such that $\forall i \geq \delta_2 |t_2 - p_{2i}| < \frac{1}{2}$. Take $g_2 = p_{2\delta_2}$.

And generally, from p_n we find for $\varepsilon = \frac{1}{n}$ a δ_n such that $\forall i \geq \delta_n |t_n - p_{ni}| < \frac{1}{n}$. Take $g_n = p_{n\delta_n}$.

By construction $\mathcal{G}_0$ is a subset of $\mathcal{F}$ that converges to l_0. We can form a sequence $S_0 = (z_0, z_1, z_2, \dots)$ consisting of the elements of (if necessary, a subset of) $\mathcal{G}_0$.

Still under the assumption that $\mathcal{F}$ is not well-ordered, we have now found an element l_0 that is strictly less than all elements of all infinite descending sequences in $\mathcal{F}$, and we have found an infinite descending sequence $S_0 = (z_0, z_1, z_2, \dots)$ that converges to this l_0.

Since all the z_i are in $\mathcal{F}$ we can write $z_i = x_i \sim y_i$ with $x_i \leq y_i$. Now we can assume without loss of generality that $\{x_i\}$ is strictly ascending, strictly descending, or constant.

By existence of $x_i \sim y_i$ we know that $|x_i - y_i| < 1$. Then by Observation 3.6 it follows that $x_i + \frac{1}{2} \leq z_i < x_i + 1$.

Say $\{x_i\}$ is strictly descending. Take $\varepsilon = \frac{1}{4}$. Then it holds for certain δ that $z_\delta - \varepsilon = l_0$. But then $x_\delta \leq z_\delta - \frac{1}{2} < z_\delta - \varepsilon = l_0$. Then $\{x_i\}$ would be an infinite descending sequence in $\mathcal{F}$ with an element smaller that l_0. So $\{x_i\}$ cannot be descending.

So $\{x_i\}$ must be strictly ascending or constant. Then by Observation 3.3 $\{y_i\}$ must be strictly descending. We use this to find a sequence in $\mathcal{F}$ that contains an element smaller than l_0.

By definition of $\{x_i\}$ and $\{y_i\}$ and because $\{x_i\}$ is strictly ascending or constant it holds that $y_i \geq x_i \geq x_1$. Also, since $\{y_i\}$ is strictly descending, $\forall i y_i < y_1$.

It follows that $\forall i |x_1 - y_i| = y_i - x_1 < y_1 - x_1$ and by existence of z_1 also $y_1 - x_1 < 1$. So in particular it holds that $\forall i |x_1 - y_i| < 1$ and thus $\{y'_i\} = \{x_1 \sim y_i\}$ exists.

We show that $\{y_i'\}$ contains an element smaller than l_0. Take

$$\varepsilon = y_1 - \lim y_i \tag{3.1}$$

Note that $\lim y_i$ exists since $\{y_i\}$ is descending and is limited by 0 by construction of $\mathcal{F}$.

Note that

$$\exists \delta' \ : \ \forall n' \geq \delta' \ y_{n'} - \lim y_i < \frac{\varepsilon}{3} \tag{3.2}$$
$$\exists \delta' \ : \ \forall n'' \geq \delta'' \ z_{n''} - l_0 < \frac{\varepsilon}{3} \tag{3.3}$$

Take $\delta = \delta' + \delta''$. Now we find that

$$\begin{aligned} y_\delta - x_\delta &\leq y_\delta - x_1 && (3.4)\\ &= y_\delta - y_1 - (x_1 - y_1) && (3.5)\\ &< y_\delta - y_1 + 1 && (3.6)\\ &= (y_\delta - \lim y_i) - (y_1 - \lim y_i) + 1 && (3.7)\\ &< 1 - \frac{2\varepsilon}{3} && (3.8) \end{aligned}$$

(3.4) holds since $\{x_i\}$ is not decreasing, (3.6) holds since $x_1 \sim x_2$ exists, (3.8) holds by (3.2) and (3.3).

So, by Observation 3.5,

$$z_\delta - y_\delta > \frac{\varepsilon}{3}. \tag{3.9}$$

It follows that

$$\begin{aligned} y_\delta &< z_\delta - \frac{\varepsilon}{3} && (3.10)\\ &< l_0 && (3.11) \end{aligned}$$

(3.10) holds by (3.9), (3.11) holds by (3.3).

Then it must hold that the limit of $\{y_i\}$ is also smaller than l_0, which contradicts the definition of l_0. $\lightning$

So our assumption was untrue, and thus $\mathcal{F}$ is indeed well-ordered. $\square$

So, $\mathcal{F}$ is well-ordered, which proves the first part of Theorem 1.1.

Chapter 4

Tame fusible numbers $\mathcal{F}'$

The tame fusible numbers $\mathcal{F}'$ form a subset of $\mathcal{F}$ that has more of a defined structure to it. We use it to prove the lower bound part of Theorem 1.1, and to give a derivation for M. In the next chapter we also use it to find a proof for Theorem 1.2.

4.1 Definition of $\mathcal{F}'$

The set of tame fusible numbers $\mathcal{F}'$ is defined in such a way that we could prove with transfinite induction that it is well-defined. In other words, we define it by transfinite recursion. We will define $\mathcal{F}'_\alpha$ where α is an ordinal number such that $\bigcup \mathcal{F}'_\alpha = \mathcal{F}'$.

Define $\mathcal{F}'_0 = \{1 - 2^{-n} \mid n \in \mathbb{N}\}$ as a base case.

$\mathcal{F}'_\beta = \bigcup_{i \in \mathbb{N}} \mathcal{F}'_{\alpha_i}$ with $\beta = \lim_{i \to \infty} \alpha_i$ (so there is an infinite sequence of α_i that converges to limit ordinal β). This forms the limit step in our recursion.

Now all we need is the successor step. We define $\mathcal{F}'_{\alpha+1}$. Note that $\alpha + 1$ is always a successor ordinal, but α can still be both.

We look at the supremum of $\mathcal{F}'_\alpha$ and we name it $\sup(\mathcal{F}'_\alpha)$. We are allowed to do this since $\forall \alpha \in Ord\ \mathcal{F}'_\alpha \subset \mathcal{F}$ so we know that it is well-ordered (because we already know $\mathcal{F}$ is) and thus we can take a supremum. Let $x_\alpha = \sup(\mathcal{F}'_\alpha) - 1$. We show that this x_α is really an element of $\mathcal{F}'_\alpha$ in Corollary 4.4.

There exists a successor of x_α in $\mathcal{F}'_\alpha$, or otherwise x_α itself should have been the supremum. Call this successor y_α. The gap between these two is $m_\alpha = y_\alpha - x_\alpha$.

We look at an interval the size of this gap, but at the very end of $\mathcal{F}'_\alpha$, and we call it $I_{\alpha,0}$. To be precise,

$$I_{\alpha,0} = \mathcal{F}'_\alpha \cap [x_\alpha + 1 - m, x_\alpha + 1) = \mathcal{F}'_\alpha \cap [y_\alpha + 1 - 2m, y_\alpha + 1 - m)$$

We define

$$I_{\alpha,n+1} = \{y_\alpha \sim z \,|\, z \in I_{\alpha,n}\}$$

Note that $\forall z \in I_{\alpha,0} \, : \, z \sim y_\alpha \geq x_\alpha + 1$. So, for $i \geq 1$, all elements of $I_{\alpha,i}$ are outside the range of $\mathcal{F}'_\alpha$.

We see that $\forall n$ if $z \in \left[y_\alpha + 1 - \frac{2m}{2^n}, y_\alpha + 1 - \frac{m}{2^n}\right)$ then $y_\alpha \sim z \in \left[y_\alpha + 1 - \frac{2m}{2^{n+1}}, y_\alpha + 1 - \frac{m}{2^{n+1}}\right)$. From this it follows that the $I_{\alpha,i}$ are distinct, and that $\bigcup_{n\geq 1} I_{\alpha,n} \subset [x_\alpha + 1, y_\alpha + 1)$.

Now to finish this definition step:

$$\mathcal{F}'_{\alpha+1} := \mathcal{F}'_\alpha \cup \bigcup_{n\geq 1} I_{\alpha,n}$$

This concludes the (final) successor step of our recursion and we can define the tame fusible numbers:

$$\mathcal{F}' = \bigcup_{\alpha \in Ord} \mathcal{F}'_\alpha$$

We show that our assumption that $x_\alpha = \sup(\mathcal{F}'_\alpha) - 1 \in \mathcal{F}'_\alpha$ for all α indeed holds in Corollary 4.4. First follow a few lemmas that we need for that conclusion.

Lemma 4.1. $\forall \alpha \in \text{Ord}$: *if for every infinite and strictly increasing sequence* $\{a_i\} \in \mathcal{F}'_\alpha$ *with limit* $b < \sup(\mathcal{F}'_\alpha)$ *it holds that* $b \in \mathcal{F}'_\alpha$*, then it follows that* $x_\alpha = \sup(\mathcal{F}'_\alpha) - 1 \in \mathcal{F}'_\alpha$

Proof. By transfinite induction on α.

1. Base case, let $\alpha = 0$: $\sup(\mathcal{F}'_0) - 1 = 0$ and $0 \in \mathcal{F}'_0$ so the base case is finished.
2. Successor case, let $\alpha = \gamma + 1$:

$$\begin{aligned}
\sup(\mathcal{F}'_{\alpha+1}) - 1 &= \sup\left(\mathcal{F}'_\alpha \cup \bigcup_{n\geq 1} I_{\alpha,n}\right) - 1 \\
&= \{\sup\left(I_{\alpha,n}\right) \mid n \geq 1\} - 1 \\
&= \lim_{n\to\infty} \sup\left(I_{\alpha,n}\right) - 1 \\
&= y_\alpha + 1 - 1 = y_\alpha
\end{aligned}$$

and we know that $y_\gamma \in \mathcal{F}'_\gamma \subset \mathcal{F}'_{\gamma+1}$, which concludes this step of our proof.

3. Limit case, let $\alpha = \lim_{i\to\infty} \gamma_i$: $\mathcal{F}'_\alpha = \bigcup_{i\in\mathbb{N}} \mathcal{F}'_{\gamma_i}$. By induction it holds that for each of these γ_i, $x_{\gamma_i} \in \mathcal{F}'_\alpha$. We know that $\lim x_{\gamma_i} = x_\alpha$. Since $x_\alpha < \sup(\mathcal{F}'_\alpha)$ it then follows by the premise of this Lemma that $x_\alpha \in \mathcal{F}'_\alpha$.

□

Lemma 4.2. $\forall \alpha \in \mathrm{Ord} : \sup(\mathcal{F}'_\alpha) \in \mathcal{F}'_{\alpha+1}$.

Proof. By transfinite induction over α. Given our current induction hypothesis we make some observations:

$$\forall \beta < \alpha : y_\beta = x_{\beta+1} \tag{4.1}$$

$$\forall f \in \mathcal{F}'_\alpha : (f < x_\alpha) \to \exists \beta \in \mathrm{Ord} : f = x_\beta \wedge \beta < \alpha \tag{4.2}$$

1. Base case, let $\alpha = 0$: $\sup(\mathcal{F}'_0) = 1 = \frac{1}{2} \sim \frac{1}{2} = y_0 \sim \frac{1}{2} \in \mathcal{F}'_1$.
2. Successor case, let $\alpha = \gamma + 1$: We know that $\sup(\mathcal{F}'_{\gamma+1}) = x_{\gamma+1} + 1 = x_{\gamma+2} \sim (x_{\gamma+1} - m_{\gamma+1} + 1) = x_{\gamma+2} \sim (x_{\gamma+1} \sim (x_\gamma + 1)) = y_{\gamma+1} \sim (y_\gamma \sim (x_\gamma + 1))$.
   By induction we know $x_\gamma + 1 = \sup(\mathcal{F}'_\gamma) \in \mathcal{F}'_{\gamma+1}$. It is the left point of $I_{\gamma,1}$, so $x_{\gamma+1} - m_{\gamma+1} + 1 = y_\gamma \sim (x_\gamma + 1) \in I_{\gamma,2} \subset \mathcal{F}'_{\gamma+1}$. Then from the definition of $I_{\gamma+1,n}$ we know that $x_{\gamma+1} - m_{\gamma+1} + 1 \in I_{\gamma+1,0}$.
   From this it follows that indeed

$$\sup(\mathcal{F}'_{\gamma+1}) = y_{\gamma+1} \sim (x_{\gamma+1} - m_{\gamma+1} + 1) \in \mathcal{F}'_{\gamma+1}$$

   which completes this step of our proof.
3. Limit case, let α be a limit ordinal: We know that $\sup \mathcal{F}'_\alpha = y_\alpha \sim (x_\alpha - m_\alpha + 1)$. By induction we know that $x_\alpha, y_\alpha \in \mathcal{F}'_\alpha$. From Lemma 4.2 in the work of Erickson et al. [7] it follows that then also $x_\alpha - m_\alpha \in \mathcal{F}'_\alpha$. By (4.2) we can then find some ordinal $\beta < \alpha$ such that $x_\beta = x_\alpha - m_\alpha$. Since $x_\beta < x_\alpha$, induction now tells us that $x_\alpha - m_\alpha + 1 = \sup(\mathcal{F}'_\beta) \in \mathcal{F}'_{\beta+1} \subset \mathcal{F}'_\alpha$. Knowing this, it's clear that then

$$\sup(\mathcal{F}'_\alpha) = x_\alpha + 1 = y_\alpha \sim (x_\alpha - m_\alpha + 1) \in \mathcal{F}'_{\alpha+1}$$

   which completes this final step of our proof.

□

Lemma 4.3. $\forall \alpha \in \mathrm{Ord}$, *for every infinite and strictly increasing sequence* $\{a_i\} \in \mathcal{F}'_\alpha$ *with limit* $b < \sup(\mathcal{F}'_\alpha)$ *it holds that* $b \in \mathcal{F}'_\alpha$

Proof. By transfinite induction over α.

1. Base case: $\mathcal{F}'_0$ has only one such sequence. Its limit is $1 \not< \sup(\mathcal{F}'_0)$, so we are done.
2. Limit case: Let $\alpha = \lim\{\gamma_i\}$. Then as $\mathcal{F}'_\alpha = \bigcup \mathcal{F}'_{\gamma_i}$, we are done if for some i it holds that $b \in \mathcal{F}'_{\gamma_i}$, which holds by induction.
3. Successor case: Let $\alpha = \gamma + 1$. Then there are three cases that can hold:
   (a) $b < \sup(\mathcal{F}'_\gamma)$. Then by induction $b \in \mathcal{F}'_\gamma \subset \mathcal{F}'_{\gamma+1}$.
   (b) $b = \sup(\mathcal{F}'_\gamma)$. Then by Lemma 4.2 it holds that $b \in \mathcal{F}'_{\gamma+1}$.
   (c) $b > \sup(\mathcal{F}'_\gamma)$. Here we have two cases:
      i. $\exists n \geq 1$ for which $b = \sup(I_{\gamma,n})$.
      Since $\forall k, \beta : \sup(I_{\beta,k+1}) = y_\beta \sim \sup(I_{\beta,k})$, it follows by induction on n that $\sup(I_{\gamma,n}) \in I_{\gamma,n+1}$.
      So indeed $b \in \mathcal{F}'_{\gamma+1}$.
      ii. $\exists n \geq 0$ for which $b > \sup(I_{\gamma,n})$ and $b < \sup(I_{\gamma,n+1})$.
      Then there is some sequence $\{a_i\} \in I_{\gamma,n+1}$ with $\lim_{i\to\infty} a_i = b$. Recall $I_{\gamma,n+1} = \{y_\gamma \sim z \mid z \in I_{\gamma,n}\}$. Then we can find a sequence $\{z_i\}$ such that for all i it holds that $a_i = y_\gamma \sim z_i$.
      It holds that $b = \lim_{i\to\infty} a_i = y_\gamma \sim \lim_{i\to\infty} z_i$. By induction on n we know that $\lim_{i\to\infty} z_i \in I_{\gamma,n}$. (If instead it were $\sup(I_{\gamma,n}) \notin I_{\gamma,n}$ then $b = y_\gamma \sim \sup(I_{\gamma,n}) = \sup(I_{\gamma,n+1})$, so we would be in case 3.a.i.)
      Then indeed $b \in I_{\gamma,n+1} \subset \mathcal{F}'_\gamma$.

□

Corollary 4.4. $\forall \alpha \in \mathrm{Ord} : x_\alpha = \sup \mathcal{F}'_\alpha - 1 \in \mathcal{F}'_\alpha$.

Proof. This logically follows from Lemma 4.1 and Lemma 4.3. □

4.2 Algorithm M

In this section we take a closer look at algorithm M from Theorem 1.3. We show how it is derived from the construction of $\mathcal{F}'$ by expressing it in terms of TameSucc(r), which gives the smallest tame fusible number strictly larger than r. We use this to show that $M(r)$ terminates on all rational numbers. Both these algorithms are shown in Figure 4.1.

Lemma 4.5. *Procedure* TameSucc(r) *terminates for every input* $r \in \mathbb{Q}$.

Algorithm 4 Finds $\min(\mathcal{F}' \cap (r, \infty))$

procedure TameSucc(r)
  if $r < 0$ **then**
    return 0
  $x \leftarrow$TameSucc$(r-1)$
  $y \leftarrow$TameSucc$(2r - x - 1)$
  return $x \sim y$

Algorithm 5 Computes TameSucc$(r) - r$

procedure M(r)
  if $r < 0$ **then**
    return $-r$
  return $\mathrm{M}(r - \mathrm{M}(r-1))/2$

Figure 4.1: Algorithms TameSucc and M [7]

Proof. For $r < 0$ this is clear, so consider $r \geq 0$.
Let $x =$ TameSucc$(r-1)$. Then by induction this is defined, and x has the value of some tame fusible number.
Let $y =$ TameSucc$(2r - x - 1)$. Since always $x \geq r - 1$ this also terminates by induction, and y has the value of some tame fusible number.

Then for some r the procedure returns $x \sim y$, so since x and y are defined, this must terminate. □

Lemma 4.6. *Procedure $M(r)$ gives output* TameSucc$(r) - r$ *for every input* $r \in \mathbb{Q}$.

Proof. If $r < 0$ then $M(r) = -r = 0 - r =$ TameSucc$(r) - r$.

We continue with $r \geq 0$ by induction on r.

Let $r = 0$: Then $M(0) = \frac{M(-M(-1))}{2} = \frac{M(-1)}{2} = \frac{1}{2} =$ TameSucc$(0) - 0$.

Let $r > 0$:
Our induction hypothesis (IH) is $\forall k, k < r \to M(k) =$ TameSucc$(k) - k$.
For readability we write TameSucc(r) as $T(r)$. We see that

$$\begin{aligned}
2M(r) &= M(r - M(r-1)) \\
&\overset{\text{(IH)}}{=} M(r - (T(r-1) - (r-1))) \\
&= M(2r - T(r-1) - 1) \\
&\overset{\text{(IH)}}{=} T(2r - T(r-1) - 1) - (2r - T(r-1) - 1) \\
&= T(r-1) + T(2r - T(r-1) - 1) + 1 - 2r
\end{aligned}$$

So indeed

$$M(r) = \frac{T(r-1) + T(2r - T(r-1) - 1) + 1 - 2r}{2} = T(r) - r.$$

By Lemma 4.5 this equality also means that M terminates for all input in $\mathbb{Q}$. □

Erickson et al. [7] talk about M terminating on real inputs. However in terms of computability the notion of terminating on all real numbers does not make much sense. More on this in section 6.2.

4.3 Lower bound

We show that ε_0 is a lower bound for the order type of $\mathcal{F}$ by showing that it is exactly the order type of $\mathcal{F}'$.

Lemma 4.7 (Erickson et al. [7] Lemma 4.1)**.** *Given $z \in \mathcal{F}'$, let $\alpha = \mathrm{ord}(\mathcal{F}' \cap [0, z))$ and let $\beta = \mathrm{ord}(\mathcal{F}' \cap [0, z+1))$. Then $\beta = \omega^{1+\alpha}$. (Hence, for $z \geq 1$ we have $\beta = \omega^{\alpha}$.)*

$\mathrm{ord}(\mathcal{F}' \cap [0, z))$ is the amount of tame fusible numbers up until z, not including z.

Lemma 4.8. *for every positive integer n we have $\mathrm{ord}(\mathcal{F}' \cap [0, n)) \geq \tau_n$, and* $\mathrm{ord}(\mathcal{F}') = \varepsilon_0$

Proof. By induction on n. Let $n = 1$. $\mathrm{ord}(1) = \omega \geq \tau_1$. Now for the induction step, assume $\mathrm{ord}(k) \geq \tau_n$ holds. Then it follows that $\mathrm{ord}(k+1) = \omega^{1+\mathrm{ord}(k)} \geq \omega^{1+\tau_k} \geq \omega^{\tau_k} = \tau_{k+1}$, which concludes our induction step. It follows that the statement holds for all positive integers n.

Then clearly as n goes to infinity $\mathrm{ord}(\mathcal{F}' \cap [0, n))$ nears $\mathcal{F}'$ and τ_n goes to ε_0, so clearly $\mathrm{ord}(\mathcal{F}') = \varepsilon_0$. □

Since $\mathcal{F}'$ is a (strict) subset of $\mathcal{F}$, we have ε_0 as a lower bound for the order type of $\mathcal{F}$, proving the lower bound part of Theorem 1.1.

Chapter 5

Finding a fast-growing function

In this section we will outline how Erickson et al. [7] find a fast-growing function to use for the final results on M, and clarify steps in their proof of Theorem 1.2 that follows from it. A large part of their proof consists of arithmetic, which would be redundant to repeat, so we will focus on the conclusions they draw..

5.1 Derivation of d from $\mathcal{F}'$

To build fast-growing function d from $\mathcal{F}'$, we will need some functions that describe properties of $\mathcal{F}'$:

- $\text{ord}'(z) = 1 + \text{ord}(\mathcal{F}' \cap [0, z))$ for $z \in \mathcal{F}'$. This can be read as the size of the tame fusible numbers up until z. The '1+' makes sure that we get $\text{ord}(0) = 1$. Note that for $z \geq 1$ the '1+' does not have any effect as in those cases $\text{ord}(\mathcal{F}' \cap [0, z)) \geq \omega$.
  For example: $\text{ord}'(1) = 1 + \text{ord}(\mathcal{F}' \cap [0, 1)) = 1 + \omega = \omega$. $\text{ord}'\left(\frac{5}{4}\right) = \omega 2$ and $\text{ord}'\left(\frac{3}{2}\right) = \omega^2$.
  Some examples where $z < 1$: $\text{ord}'\left(\frac{1}{2}\right) = 1 + \text{ord}\left(\mathcal{F}' \cap \left[0, \frac{1}{2}\right)\right) = 2$. Since $\frac{1}{2}$ is excluded, the only (tame) fusible number in this interval is 0. $\text{ord}'\left(\frac{3}{4}\right) = 1 + \text{ord}\left(\mathcal{F}' \cap \left[0, \frac{3}{4}\right)\right) = 3$.
- $\text{fus}'(\alpha)$ for $0 < \alpha < \varepsilon_0$ is the unique tame fusible number $z \in \mathcal{F}'$ such that $\text{ord}'(z) = \alpha$.
  For example: $\text{fus}'(\omega) = 1$, because $\text{ord}(1) = \omega$.
- $m(\alpha) = \text{fus}'(\alpha+1) - \text{fus}'(\alpha)$. From the definition of fus' we can interpret this as the distance between the fusible number corresponding to α and

its successor. This is almost exactly what algorithm M does, with the only difference being their domain and image.

With these we have a definition for d:

$$d(\alpha) = -\log_2 m(\alpha) = \log_2 \frac{1}{m(\alpha)}$$

You may wonder what the use is of having this function d rather than just looking at m. There are two main reasons for this. The first is that taking the log simplifies the calculations because due to the nature of $\mathcal{F}'$, m deals a lot with powers of two. The second is that m, which gives the size of the gaps between tame fusible numbers, *shrinks* rapidly rather than grows. To get to a fast-growing function we would have to take the inverse at some point. Doing it now reduces a lot of overhead as we can already relate it to some fast-growing hierarchy in intermediate proofs.

5.2 Growth rate of the density of $\mathcal{F}$ on $\mathbb{R}$

In this section we will find out the growth rate of d and how that translates to a property about $\mathcal{F}'$. We mostly rely on the proofs given by Erickson et al. [7].

5.2.1 Growth rate of d

Lemma 5.1 (Erickson et al. [7] Lemma 4.9)**.** *Define*

$$f_{\beta,\alpha}(n) = d\left(\omega^{\omega^{\omega^{\beta+\omega^{\alpha}n}}}\right).$$

Then $f_{\beta,\alpha}(n) \geq H_\alpha(n)$ for all β

Corollary 5.2 (Erickson et al. [7] Corollary 4.10)**.** *For every $n \geq 8$ we have $d(\tau_n) \geq F_{\varepsilon_0}(n-7)$.*

Erickson et al. [7] show this by the following equation, of which we will clarify the steps in Lemma 5.3 and Corollaries 5.4 and 5.5:

$$d(\tau_n) = d(\omega^{\tau_{n-1}}) \geq d(\omega^\gamma) = f_{0,\tau_{n-5}}(2) \geq H_{\tau_{n-5}}(2) \geq F_{\varepsilon_0}(n-7).$$

Lemma 5.3. *$\omega^{\tau_{n-1}} \geq \omega^\gamma$ for $\gamma = \omega^{\omega^\delta}, \delta = \tau_{n-4}2$.*

Proof. Let $\alpha_1 = \tau_{n-1}$ and $\beta = [\tau_{n-1}]_k$ with $x \geq 2$. Note that β is the first step in some canonical descent from τ_{n-1} to 0.

We see that $[\alpha_1]_2 \leq \beta < \alpha_1$, so either $[\alpha_1]_2 = \beta$ or by Lemma 2.10 it follows that every canonical descent from β contains $[\alpha_1]_2$.

Then there is some $\beta_2 = [\alpha_2]_l$ with $l \geq 2$ in the canonical descent of β. Then $[\alpha_2]_2 \leq \beta_2 < \alpha_2$. Again it follows that either $[\alpha_2]_2 = \beta_2$ or every canonical descent from β_2 contains $[\alpha_2]_2$.

It follows that $\tau_{n-1} \geq [\alpha_2]_2 = [[\tau_{n-1}]_2]_2 = \gamma$. Then clearly $\omega^{t_{n-1}} \geq \omega^\gamma$. $\square$

Corollary 5.4. $d(\omega^\gamma) = f_{0,t_{n-5}}(2)$.

Proof.

$$f_{0,t_{n-5}}(2) = d\left(\omega^{\omega^{\omega^{(\omega^{\tau_{n-5}})\cdot 2}}}\right) = d\left(\left(\omega^{\omega^{\omega^{\omega^{\tau_{n-5}}}}}\right)^2\right) = d({\tau_{n-1}}^2) = d(\omega^\gamma) \qquad \square$$

Corollary 5.5. $f_{0,t_{n-5}}(2) \geq H_{\tau_{n-5}}(2) \geq F_{\varepsilon_0}(n-7)$.

Proof. The first inequality follows by Lemma 5.1, the second by Lemma 2.14. $\square$

Corollary 5.2 confirms that d grows quickly on inputs of the form τ_n, but that is all we need. Next we will see how this leads to the proof for Theorem 1.2.

5.2.2 A function for the density of $\mathcal{F}$ on $\mathbb{R}$

We are looking for a function $g(n)$ that can show us how the density of the fusible numbers increases. Recall the function as in the formulation of Theorem 1.2,

Let $g(n)$ be the largest gap between consecutive fusible numbers in $\mathcal{F} \cap [n, \infty)$, for $n \in \mathbb{N}$.

We will derive $g'(n)$ from d that is similar to $g(n)$ except it considers the *tame* fusible numbers. Then from results about g' we will draw a conclusion about g.

Lemma 5.6 (Erickson et al. [7] Lemma 4.12)**.** *If $\alpha > \tau_n$ then $d(\alpha) > d(\tau_n)$.*

Corollary 5.7. *If $\alpha > \tau_n$ then $m(\alpha) < m(\tau_n)$.*

Proof. By Lemma 5.6 and the definition of d. $\square$

Observation 5.8. The least element of a set $\mathcal{F}' \cap [n, \infty)$ is $n = \mathrm{fus}'(\tau_n)$.

Corollary 5.9. *The size of the largest gap in* $\mathcal{F}' \cap [n, \infty)$ *is*

$$\sup(\{m(\beta) \mid \beta \geq \tau_n\}) = m(\tau_n).$$

Proof. This follows by Observation 5.8 and Corollary 5.7. □

Lemma 5.10. *Let* $g'(n)$ *be defined on all* $n \in \mathbb{N}$ *as the largest gap between consecutive tame fusible numbers in* $\mathcal{F}' \cap [n, \infty)$. *Then* $g'(n) = m(\tau_n)$ *and for every* $n \geq 8, g'(n)^{-1} \geq F_{\varepsilon_0}(n-7)$.

Proof. $g'(n) = m(\tau_n)$ holds by definition of g' and Corollary 5.9.

As for the second part: We see that

$$g'(n)^{-1} = \frac{1}{g'(n)} = \frac{1}{m(\tau_n)} \geq \log_2\left(\frac{1}{m(\tau_n)}\right) = d(\tau_n).$$

Then by Corollary 5.2: for every $n \geq 8$, $g'(n)^{-1} \geq F_{\varepsilon_0}(n-7)$. □

Since $\mathcal{F}' \subset \mathcal{F}$ clearly $(\mathcal{F}' \cap [n, \infty)) \subset (\mathcal{F} \cap [n, \infty))$ and thus $g(n) \leq g'(n)$ because we cannot get a larger gap by only adding elements.

In conclusion, $g(n)^{-1} \geq g'(n)^{-1} \geq F_{\varepsilon_0}(n-7)$, proving Theorem 1.2.

Chapter 6

M in Peano Aritmetic (PA)

6.1 What is PA?

Peano Arithmetic (PA) is a list of axioms in the language of logic, with a few added symbols:

- A constant symbol named 'zero': 0
- A relation symbol called 'equality': =
- The unary successor function symbol S
- Two binary function symbols + and ·

The axioms define how these symbols interact. If we interpret the symbol 0 as the natural number 0 then we can interpret $S0$ as the natural number 1, $SS0$ as 2, $SSS0$ as 3, and so on. The Peano axioms model basic arithmetic over all of $\mathbb{N}$.

We follow Benthem et al. [1] for our definitions of the axioms.

The axioms for PA are:

1. $\forall x \; \neg(0 = Sx)$ (0 is not a successor)
2. $\forall x \forall y \; (Sx = Sy \to x = y)$ (S is injective)
3. $\forall x \; x + 0 = x$
   $\forall x \forall y \; x + Sy = S(x + y)$ (definition of +)
4. $\forall x \; x \cdot 0 = 0$
   $\forall x \forall y \; x \cdot Sy = x \cdot y + x$ (definition of ·)
5. $([0/x]\varphi \wedge \forall x(\varphi \to [Sx/x]\varphi)) \to \forall x \; \varphi$ (induction)

In axiom 5, for a term t and a variable x, $[t/x]\varphi$ stands for the logical formula φ in which every free occurence of x is substituted by t.

6.2 Statements in PA

Any statement in PA is called a *PA-theorem*, because by definition they can be derived from the axioms of PA. This specifically means that PA being able to *express* a statement or for PA to *prove* it are the same thing.

PA is known to be *Turing complete*: Any function that can be computed by a Turing machine can be expressed in PA. By the Church-Turing thesis that means any computable function can be expressed by PA.

Coding bigger sets

PA describes natural numbers, so any set larger than that needs some kind of coding before PA can say anything about it. On a countable set we can use *Gödel numbering* for that. Take for example the set of rationals $\mathbb{Q}$, then the Gödel number for its elements is defined as follows:

Definition 6.1 (Downey and Hirschfieldt [6])**.** Let $r \in \mathbb{Q} \setminus \{0\}$ and write $r = (-1^{\delta})\frac{p}{q}$ with $p, q \in \mathbb{N}$ in lowest terms and $\delta = 0$ or 1. Then define the *Gödel number* of r, denoted by $\#(r)$, as $2^{\delta}3^{p}5^{q}$. Let the Gödel number of 0 be 0.

This gives an injection from $\mathbb{Q}$ to $\mathbb{N}$, and for any number in its image we can find the corresponding rational by prime factorisation.

A coding such as this is possible on any countable set. By coding the elements of a domain to $\mathbb{N}$, we enable PA to express it.

This specifically means that the function M on $\mathbb{Q}$ can be expressed in PA, and is therefore computable.

However we cannot properly code $\mathbb{R}$ to $\mathbb{N}$: There are only countably many finite sequences of natural numbers. Because $\mathbb{R}$ is uncountable, coding each real as a list of natural numbers would mean at least some of those lists would have to be infinite. Let us entertain for a second that this is a valid coding. Then for some total function $f(r)$ on $\mathbb{R}$, we could construct a Turing machine T that given any $r \in \mathbb{R}$ as input would give $f(r)$ in a finite number of steps. Now consider that we give T an input r that we had to code with an infinite sequence of natural numbers. Then even just reading the input would take an infinite amount of steps. (And if we somehow truncate the input while reading, there is no meaningful difference to limiting the input to the unique truncations, which would be a countable set).

Limits of induction

The set of fusible numbers is countable, and at least ε_0 big. Any obvious proof over the whole set would therefore require induction over ordinals up until (at least) ε_0. It would be tempting to try to use such a proof in PA, but we cannot do that.

Gentzen has shown that with induction over ordinals up to ε_0, PA can be proven consistent [9]. We know that PA is consistent [4], and that a consistent formal system cannot prove its own consistency [10]. So, there are things that can be proven with induction over ordinals up to ε_0 that cannot be proven by PA. The implication of this is that finding a proof that uses this type of induction tells us nothing about provability in PA.

6.3 The computability of M is unprovable in PA

In the previous section we have seen that M is computable. This raises the question whether PA can also prove that to be true? Is the statement "*M is computable*" computable?

Theorem 6.2 (Erickson et al. [7] Theorem 6.1, Buchholz and Wainer [3])**.** *Let T be a Turing machine that computes a function $g : \mathbb{N} \to \mathbb{N}$, terminating on every input. Suppose that PA can prove the statement "T terminates on every input." Then g cannot grow too fast: There exist $\alpha < \varepsilon_0$ and $n_0 \in \mathbb{N}$ such that $g(n) < F_\alpha(n)$ for every $n \geq n_0$.*

We show that by Theorem 6.2, M cannot be proven computable by PA.

Corollary 6.3. *Given $n \in \mathbb{N}$, $M(n) = m(\tau_n)$.*

Proof. This is clear by the definitions of M and m together with Observation 5.8. □

Lemma 6.4. *PA cannot prove the statement "for every $n \in \mathbb{N}$, algorithm $M(n)$ terminates."*

Proof. Let A be a Turing machine that, given n, outputs $\frac{1}{M(n)}$. Clearly if PA can prove that M terminates, then it can prove that A terminates as well.

By Lemma 5.10 and Observation 6.3, $M(n) = m(\tau_n) = g'(n)$, and thus for every $n \geq 8$, $\frac{1}{M(n)} \geq F_{\varepsilon_0}(n - 7)$. Then by Theorem 6.2 PA cannot prove A computable.

Then PA also cannot prove M computable. □

This proves Theorem 1.3.

Conclusion

We have seen how a mathematical riddle points us to a computable function M that is incredibly simple to implement, but very difficult to prove computable. We wanted to know whether the statement "M is computable" is itself computable. To figure this out we went looking for such a proof in PA, as any computable function can be expressed in this proof system. We defined the set of fusible numbers, constructed the tame fusible numbers, and figured out how M is derived from that. In this last step we found how ridiculously fast the density of the fusible numbers grows. Earlier work by Buchholz and Wainer then lead us to the conclusion that M cannot be proven computable by PA.

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