ON HENK BARENDREGT'S FAVORITE OPEN PROBLEM

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Abstract. $\mathcal{H}$ is the λ-theory extending β-conversion by identifying all closed unsolvables. A long-standing open problem of H. Barendregt states that the range property holds in $\mathcal{H}$. Here we discuss what we know about the problem. We also make some remarks on the λ-theory $\mathcal{H}\omega$ (the closure of $\mathcal{H}$ under the ω-rule and β-conversion).

1. Introduction

Among the many outstanding contributions of Henk Barendregt to the λ-calculus, a special place is occupied by the numerous problems originated with him.
First of all, we have to mention his famous Open Problems List, which ends the 1975 Conference on "λ-Calculus and Computer Science Theory", edited by C. Böhm [3]. This list contains several celebrated problems of Barendregt and others, such as the the word problem for combinator **S** (still open: see [6] Problem 97). Among the problems in this list, there is also the problem (due to Barendregt himself) of the logical complexity of the ω-rule in λ-calculus, which has been recently solved by the authors [7, 10].
A number of problems can also be found in Henk Barendregt's handbook *The Lambda Calculus. Its Syntax and Semantics* (see [1]), whose importance for the students of the field can be understood recalling its usual nickname: *"The Bible of Lambda Calculus"*.
Among others, two problems concern the theory $\mathcal{H}$. $\mathcal{H}$ is the theory extending β-conversion by identifying all closed unsolvables. Although not a constructive theory, $\mathcal{H}$ is a natural one. This depends on the general idea - also due to Barendregt - that in λ-calculus, we can identify unsolvable terms with the notion of "undefined" (see the discussion in [1] 2.2). Put in different terms, if we consider a λ-term as a computational process, the unsolvable terms can be considered as those processes which never return "a value". So, identifying all unsolvable seems a natural step in the understanding of the relationships between the λ-calculus and the recursion theory, where there is only one "undefined value".
The first problem about $\mathcal{H}$ actually concerns its extension with the so-called *ω-rule*. In [1] Conjecture 17.4.15, it is conjectured that the provable equations of this extended theory form a $\mathbf{\Pi}^1_1$-Complete set. This conjecture has been recently proved by the authors to be

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true (see [9]). It adds evidence to the intuition that $\mathcal{H}$ is a natural theory. We shall discuss this result in Section 3.
The second problem asks whether the range property holds in $\mathcal{H}$. The range property states that a closed term has (up to equality in $\mathcal{H}$) either an infinite range or a singleton range (that is, it is a constant function). In [1] Conjecture 20.2.8, it is conjectured that the answer is positive.
The range property is known to hold in the λ-calculus, w.r.t. β-convertibility (another result of Barendregt!), but the proof does not extend to $\mathcal{H}$. This problem, which is still open, is a real challenge. We discuss some of its aspects in Section 4.
The problem has been also included, by the second author of the present paper, in the *TLCA List of Open Problems* (see [11], Problem 3).
To stress the importance of the problem, Henk Barendregt restated it in his beautiful paper [2]. Moreover, some time ago, in a private communication to the authors, Barendregt stated that this one is the problem he would like the most to see solved. We hope that the remarks that we present here, will be of help in the solution of the problem.

2. The λ-Theory $\mathcal{H}$

$\mathcal{H}$ is the λ-theory extending $\lambda\beta$ by identifying all closed unsolvable terms, see [1] Definition 4.1.6. We recall that this λ-theory can be formulated by adding to $\lambda\beta$ all equations of the form $M = \mathbf{\Omega}$, where M is a closed unsolvable term, the combinator $\mathbf{\Omega}$ is defined as $\omega\omega$ and ω is $\lambda x.xx$. Moreover, we recall also that $\mathcal{H}$ is generated by the notion of reduction $\beta\mathbf{\Omega}$ (see [1] Lemma 16.1.2).
The notion of reduction $\beta\mathbf{\Omega}$ is defined by adding to the β-reduction rule, the (non constructive) reduction rule:
$M \longrightarrow \mathbf{\Omega}$, if M is an unsolvable, possibly open term and $M \not\equiv \mathbf{\Omega}$.
We remark for following use that $\beta\mathbf{\Omega}$ is Church-Rosser (see [1] 15.2.15).

If the intuition "undefined = unsolvable" is correct, then the identification of all the unsolvable terms with $\mathbf{\Omega}$ should not affect "more defined" terms. That this is indeed the case, it is shown by the following result ([1], Proposition 16.1.9).

Proposition 1. Let N be a term in β-normal form. If $N = M$ holds in $\mathcal{H}$ then N and M are β-convertible.

As a corollary, we also obtain that $\mathcal{H}$ is consistent.

3. The λ-Theory $\mathcal{H}\omega$

The ω-rule states that if for every closed term M, $PM = QM$ can be proven, then we can conclude that $P = Q$.
The ω-rule is related to the so called extensionality: as the intended meaning of terms are functions, if two terms are equal on every argument they must be equal. Observe that the well known η-rule states this for a free variable: if $Px = Qx$ for x free not occurring in P and Q, then conclude $P = Q$.
Now in the λ-calculus there exist combinators (closed terms) which can be considered as completely *uninformative* from a computational point of view. The most famous one is the combinator $\mathbf{\Omega}$, defined above. $\mathbf{\Omega}$ only reduces to itself. Therefore the natural conjecture

seems to be that the ω-rule holds in the λ-calculus with the η-rule. Indeed, one can think that if $P\mathbf{\Omega} = Q\mathbf{\Omega}$ holds, then one can replace, in the proof of this equality, $\mathbf{\Omega}$ with a new free variable and then apply the η-rule.
This kind of reasoning seems so clear that one is astonished in learning that it is actually *incorrect*. The subtle point is that a free variable cannot have been generated by the term P or Q, but on the contrary $\mathbf{\Omega}$ can.
Indeed Plotkin, by a very beautiful construction, showed that the ω-rule actually fails in the λ-calculus (see [12]). The counterexample is based on some special terms (the so called *Plotkin Terms*) which could be of interest also for the theory $\mathcal{H}$, see the next Section. (For more considerations on Plotkin Terms see [1] 17.3.26 and [8]).
The natural problem of the logical power of the theory obtained by adding the ω-rule to the β-convertibility has been recently solved by the authors of the present paper [7, 10].

Now we turn to the problem of validity of the ω-rule in theory $\mathcal{H}\eta$, that is the theory $\mathcal{H}$ with the addition of rule η. It turns out that also in $\mathcal{H}\eta$ the ω-rule does not hold. The counterexample is due to Barendregt (see 17.4.5 of [1]), and we want to sketch it here also to motivate some points of the following Section. As observed by Barendregt himself, the counterexample is much simpler than Plotkin's one!
We take advantage of the following Lemma.

Lemma 1. Let M be any term. Then there exists an n such that: $M\underbrace{\mathbf{\Omega}\cdots\mathbf{\Omega}}_{n-times} = \mathbf{\Omega}$ holds in $\mathcal{H}$ (and, of course, also in $\mathcal{H}\eta$).

Now define, with the fixed point construction, a term A such that $Ax = \langle\ A(x\mathbf{\Omega})\ \rangle$, where as usual $\langle\ M\ \rangle$ stands for $\lambda x.xM$.

We can prove now that:

Proposition 2. The two terms $\lambda z.A\mathbf{\Omega}$ and $\lambda x.Ax$ are such that in $\mathcal{H}\eta$ for every closed term M, the equality $(\lambda z.A\mathbf{\Omega})M = (\lambda x.Ax)M$ holds, but it cannot be proved that the two terms are equal.

Proof. (Sketch). Let M be any closed term. Then by the lemma there in an n such that $M\underbrace{\mathbf{\Omega}\cdots\mathbf{\Omega}}_{n-times} = \mathbf{\Omega}$ holds in $\mathcal{H}\eta$. It follows that:

$$(\lambda x.Ax)M = AM = \ldots = \underbrace{\langle\cdots\langle}_{n-times}\ A(M\underbrace{\mathbf{\Omega}\cdots\mathbf{\Omega}}_{n-times})\ \rangle\cdots\rangle =$$

$$= \underbrace{\langle\cdots\langle}_{n-times}\ A\mathbf{\Omega}\ \rangle\cdots\rangle.$$

But $A\mathbf{\Omega} = \langle\ A(\mathbf{\Omega}\mathbf{\Omega})\ \rangle = \langle\ A\mathbf{\Omega}\ \rangle$. It follows that:

$$\underbrace{\langle\cdots\langle}_{n-times}\ A\mathbf{\Omega}\ \rangle\cdots\rangle = A\mathbf{\Omega} = (\lambda z.A\mathbf{\Omega})M.$$

On the other hand the equality of $\lambda x.Ax$ and $\lambda z.A\mathbf{\Omega}$ cannot be proved in $\mathcal{H}\eta$. Here we use the fact that $\mathcal{H}\eta$ is the theory corresponding to the notion of reduction $\beta\eta\mathbf{\Omega}$ (see above), which is Church-Rosser (see [1] 15.2.15). Indeed, if we apply both terms to the fresh variable y, there cannot exist a common reduct as y always occurs in any reduct of the former term and in no reduct of the latter. This ends the proof. □

Actually, if we add the ω-rule to $\mathcal{H}$, the resulting theory $\mathcal{H}\omega$ is a much more powerful one. The problem of the exact logical power of this extended theory has been solved by

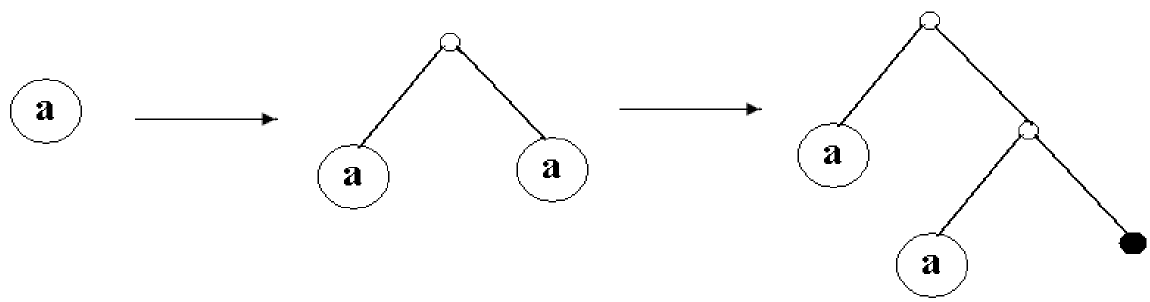

Figure 1: The Construction of the Tree (Constant **a** Case)

the authors, improving on previous results of Barendregt. $\mathcal{H}\omega$ turns out to be $\mathbf{\Pi^1_1}$-Complete.

We want now to give an intuitive idea of (some aspects of) the proof. The starting point is the idea of reducing the well foundness of a recursive tree (of sequences of natural numbers) to equality of two terms in $\mathcal{H}\omega$. This will settle the question by the following well known theorem (see [5] Ch.16 Th.20).

Theorem 1. The set of (indices of) well founded recursive trees is $\mathbf{\Pi^1_1}$-complete.

The basic construction is due to Barendregt. We illustrate it as folllows.
Given a recursive tree $\mathbf{t}$, define terms $M^{\mathbf{a}}_{\mathbf{t}}$ and $M^{\mathbf{b}}_{\mathbf{t}}$, containing the constant $\mathbf{a}$ and, respectively, $\mathbf{b}$.
Terms $M^{\mathbf{a}}_{\mathbf{t}}$ and $M^{\mathbf{b}}_{\mathbf{t}}$ construct two copies of the tree $\mathbf{t}$ as follows: at each level they construct the internal nodes as well as the leaves, according to the structure of the tree; they reproduce themselves at nodes where the construction must be continued. Both terms represent in same way the nodes of the tree, so that the special constants no more occur in the internal nodes and in the leaves.
The construction is represented, for the term containing the constant $\mathbf{a}$, in Figure 1, where for simplicity we have pictured the construction of a binary tree, whilst sequence trees can have an infinite span. White nodes represent internal nodes and black nodes represent the leaves. The term $M^{\mathbf{a}}_{\mathbf{t}}$ is represented by a circled $\mathbf{a}$.

Now the point is that if $\mathbf{t}$ is well founded (and only if) then constants $\mathbf{a}$ and $\mathbf{b}$ are eventually eliminated along every path, and a proof (using the ω-rule) of the equality of $M^{\mathbf{a}}_{\mathbf{t}}$ and $M^{\mathbf{b}}_{\mathbf{t}}$ exists in $\mathcal{H}\omega$.
The "if" part was proved by Barendregt (see Section 17.4 of [1]) and the authors provided the proof-theoretic arguments needed for the "only if" part (see [9]).

4. The Range Property in $\mathcal{H}$

As it is well known the range property holds in the λ-calculus (see [2]). This result was first conjectured by C. Böhm and independently solved by Barendregt and Myhill.
Quoting from [2], in the λ-calculus "perhaps the range property is really a result in recursion theory". (This point of view is substantiated in [2], by results from Barendregt himself and the second author of the present paper which give a general formulation of the range

property in the Ershov-enumerations setting).

In $\mathcal{H}$ this reduction to a recursion theoretic argument is at least problematic since $\mathcal{H}$ is a Σ^2_0-complete theory. However, as the second author of the present paper points out in [11], the existence of a recursion theoretic argument cannot be totally ruled out. Put in different words, the possibility remains of a *relativized* argument using equality in $\mathcal{H}$ as an oracle, to make some variation of the usual proof going through.

But it seems as well possible that, if the range property holds in $\mathcal{H}$, this has to be proved by a direct analysis of the behavior of the terms. Along this direction we present a partial result. Observe that by a *$\mathcal{H}$-normal form* we mean either $\mathbf{\Omega}$ or a β (or $\beta\eta$)-normal form.

Proposition 3. In $\mathcal{H}$, if a closed term M has a $\mathcal{H}$-normal form in its range (that is for some closed N, MN has a $\mathcal{H}$-normal form), then the range of M is either infinite or a singleton.

Proof. Assume first that the normal form is $\mathbf{\Omega}$. If M is unsolvable then there is nothing to prove, so let M solvable with head normal form:
$\lambda x_1 \dots x_n . x_i M_1 \cdots M_t$.
If x_i is not x_1 then $\mathbf{\Omega}$ cannot be in the range of M. Therefore x_i is x_1 and the range of M is infinite.
Now assume that the normal form is a β-normal form. Then M cannot be unsolvable and has an head normal form $\lambda x_1 \dots x_n . x_i M_1 \cdots M_t$.
If x_1 does not occur in $x_i M_1 \cdots M_t$ then the range of M is a singleton.
If x_i is x_1 then the range of M is infinite.
Therefore assume that x_i is not x_1. Then x_1 occurs in some M_j, with $1 \leq j \leq t$. All such M_j must be solvable, otherwise the range of M cannot contain a β-normal form.
Now if in every M_j, where it occurs, x_1 is the head variable then M has an infinite range. Otherwise, repeat the previous argument inside each M_j where x_1 occurs.
Observe that the process must stop at some finite level, for otherwise the range of M cannot contain a β-normal form. So at the last level, x_1 is the head variable of all subterms in which it occurs. Then M has an infinite range. This ends the proof. □

A natural approach, when dealing with difficult problems, is to think how a possible *counterexample* could be. In this case, we need to partition the set of all terms in a finite number of pieces. Therefore, we need some mechanism to equalize terms. So a possible starting point for a counterexample, is to consider the term A used as a counterexample to the ω-rule in the theory $\mathcal{H}$ (see the previous Section). As we have seen, A is such that for all M, $AM = A\mathbf{\Omega}$. Nevertheless, A is solvable. The equality is obtained by iteratively applying M to $\mathbf{\Omega}$. So the natural question arises, whether we could obtain, by a suitable choice of the terms to which a generic M is applied, a finite set of possible outputs.
Let us call this idea a *striking mechanism* to *strike* a term M and transform it into, say, two possible outputs. The question is already considered in [1], where it is shown, by various examples, that this approach does not seem to work (see [1] 20.6.11).
However the question is very subtle, due to the variety of possible striking mechanisms. So, to prove that this approach is not viable at all, seems very difficult. As a partial result, we want to prove here that any *periodic* striking mechanism does not work. We make this precise as follows.

Of course we work in $\mathcal{H}$. By the letter $\mathcal{S}$ we denote a finite sequence of (possibly open) terms, that is $\mathcal{S} \equiv T_1, ..., T_k$ for some k. Given the sequence $\mathcal{S}$ and a term M, by $M\mathcal{S}$ we denote the application of M to the terms in $\mathcal{S}$, that is $M\mathcal{S} \equiv MT_1 \cdots T_k$.

Definition 1. Given m sequences $\mathcal{S}_1, ..., \mathcal{S}_m$ we define, for any closed term M, *the periodic application of* M *to* $\mathcal{S}_1, ..., \mathcal{S}_m$ as the following infinite sequence M_n of closed terms:

(1) $M_0 \equiv M$;
(2) if $n+1 \equiv i \ mod \ m$ and $i > 0$ then $M_{n+1} \equiv \lambda x_{i1}, \ldots, x_{it_i}.M_n\mathcal{S}_i$, where $x_{i1}, \ldots, x_{it_i}$ is an enumeration of all the free variables occurring in $\mathcal{S}_i$;
(3) if $n+1 \equiv i \ mod \ m$ and $i = 0$ then $M_{n+1} \equiv \lambda x_{m1}, \ldots, x_{mt_m}.M_n\mathcal{S}_m$, where $x_{m1}, \ldots, x_{mt_m}$ is an enumeration of all the free variables occurring in $\mathcal{S}_m$.

Definition 2. Given sequences $\mathcal{S}_1, ..., \mathcal{S}_m$ we say that the sequences $\mathcal{S}_1, ..., \mathcal{S}_m$ *have finite character* if there exists a finite set of terms $N_1, ...N_t$ such that for any closed term M, there exists an n_0 such that for every $n > n_0$, the n-th term M_n of the periodic application of M to $\mathcal{S}_1, ..., \mathcal{S}_m$ is $\mathcal{H}$-equal to some N_i, with $1 \leq i \leq t$.

Proposition 4. Let the sequences $\mathcal{S}_1, ..., \mathcal{S}_m$ have finite character, then for any term M, every term P which appears in the the periodic application of M to $\mathcal{S}_1, ..., \mathcal{S}_m$ is, from some point on, $\mathcal{H}$-equal to $\mathbf{\Omega}$.

Proof. (Sketch) By cases.
First Case. In the sequences $\mathcal{S}_1, ..., \mathcal{S}_m$ all terms are equal to $\mathbf{\Omega}$. This case is obvious.
Second Case. In the sequences $\mathcal{S}_1, ..., \mathcal{S}_m$ there is at least one closed term different from $\mathbf{\Omega}$. Then use the *ant-lion paradigm* (see [4]), to construct an infinite family of terms able to select the closed term and use it to reproduce themselves. This is a contradiction with the hypothesis that the sequences have finite character.
Third Case. In the sequences $\mathcal{S}_1, ..., \mathcal{S}_m$ there are only open terms different from $\mathbf{\Omega}$. Use the range theorem for β-convertibility to get a contradiction with the hypothesis that the sequences have finite character.
Fourth Case. In the sequences $\mathcal{S}_1, ..., \mathcal{S}_m$ there are only open terms different from $\mathbf{\Omega}$ and $\mathbf{\Omega}$ itself. Observe that one can assume that every open term has the head variable free, for otherwise one can argue as in the second case. So all the terms are either open terms, with the first variable free, or $\mathbf{\Omega}$.
Assume that the periodic application of M to $\mathcal{S}_1, ..., \mathcal{S}_m$ is not eventually equal to $\mathbf{\Omega}$. Then from some step n on, when M_n is applied to the corresponding sequence $\mathcal{S}_j$, a open term is substituted for the head variable. Consider now a whole cycle, that is the application to $\mathcal{S}_1$, then to $\mathcal{S}_2$, to finish with $\mathcal{S}_m$. At each step a certain number of applications and a certain number of abstractions are performed. Let n_{App} and n_{Abs} be the total number of applications performed in the cycle and, respectively, the total number of abstractions performed in the cycle. Assume $n_{App} > n_{Abs}$, then the terms M_n have the form:
$\lambda x_1, \ldots, x_r.x_jQ_1 \cdots Q_{r_n}$
where the number r_n of the arguments of x_j is a growing function.
On the other hand, if $n_{App} < n_{Abs}$, we will have a growing number of abstractions.
Finally, assume $n_{App} = n_{Abs}$. In this case if we start with a term with k initial abstractions, these k abstractions are never eliminated.
So there are infinite many possible outputs. This ends the proof. □

Since this approach to the construction of a counterexample apparently does not work, we could consider another powerful mechanism which equalizes terms: the one used in the mentioned *Plotkin Terms*. Choose a suitable enumerator E such that, in the theory $\lambda\beta\eta$, $E_0 = \mathbf{\Omega}$ and E_{n+1} is equal to the n-th term with head normal form. As the second author of the present paper has remarked in [11] (Problem 3), it is possible to construct Plotkin Terms F and G such that, still in the theory $\lambda\beta\eta$, the following holds:

- the equality $F_0 G_0 E_{n+1} = F_0 G_0 E_{n+2}$ holds for every n; that is, $F_0 G_0 P = F_0 G_0 Q$ for every P and Q with head normal form;
- for every n, $F_0 G_0 E_{n+1} \neq F_0 G_0 \mathbf{\Omega}$.

Unfortunately, if we follow the original technique of Plotkin the terms turn out to be unsolvable, so that in $\mathcal{H}$ the whole construction is useless. Some suitable variation of the technique is needed, such that the equalities and the inequalities above still hold but the involved terms are *solvable*. It is far from obvious that such a construction exists at all.

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