SEQUENT CALCULUS, DIALOGUES, AND CUT-ELIMINATION

MORTEN HEINE SØRENSEN AND PAWEŁ URZYCZYN

Formalit, Byenden 32, 4660 Store Heddinge
e-mail address: mhs@formalit.dk

University of Warsaw, Banacha 2, 02-097 Warszawa, Poland
e-mail address: urzy@mimuw.edu.pl

Abstract. It is well-known that intuitionistic and classical provability can be characterized by the existence of winning proponent strategies in Lorenzen dialogues. In fact, one can bring the winning proponent strategies more or less into correspondence with sequent calculus proofs.

This paper elaborates on the correspondence. We study a variant of sequent calculus with only one right rule and one left rule. The rules do not concern any particular connective. They are similar to the definition of Lorenzen dialogues laying out the interaction between proponent and opponent. In the latter setting one additionally specifies, for each connective, the attacks and corresponding defenses. Similarly, our left and right rule are parameterized by such specifications.

The main result is cut-elimination for the system without specification of the actual connectives. Cut-elimination for any combination of the usual connectives follow as a special case. We also give a very compact proof that derivations in the system are isomorphic to winning proponent strategies. We focus on classical propositional logic. The results carry over to intuitionistic logic as well as first-order logic, and the equivalence of proofs and strategies also carry over to second-order logic, in all cases with some adjustments.

1. Lorenzen dialogues

We consider E-dialogues (see [4]) following the formulation in [5].

Definition 1.1. A *dialogue* over φ is an either empty, finite, or infinite sequence of *moves* $M_1, M_2, \dots$, where the odd (resp. even) moves are called *proponent* (resp. *opponent*) moves, such that the following conditions hold.

- $M_1 = (\mathbf{D}, \varphi, 0)$. (Proponent begins.)
- Each opponent move has form $M_i = (X, \psi, i{-}1)$. For $i > 1$, each proponent move has form $M_i = (X, \psi, j)$, where $j < i$ and M_j is an opponent move. (Opponent refers to the immediately preceding move, proponent refers to any preceding opponent move.)
- For each proponent move $M_i = (X, p, j)$ stating some variable p, there is an opponent move $M_k = (Y, p, l)$ with $k < i$. (Proponent may assert a variable, if it is already asserted by opponent.)

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- For all $i > 1$ with $M_i = (X, \psi, j)$ and $M_j = (Y, \rho, k)$, either $X : \psi$ is an attack on ρ (see left table[1]) or $M_k = (Z, \pi, l)$ and $Y : \rho$ is an attack on π and $X : \psi$ is a defense of π against Y (see right table). (One can attack asserted formulas or defend against a matching attack.)

Formula	Attacks
$\sigma \to \tau$	$\mathbf{A} : \sigma$
$\sigma \wedge \tau$	$\mathbf{A}_{\mathrm{L}} : \emptyset\ ,\ \mathbf{A}_{\mathrm{R}} : \emptyset$
$\sigma \vee \tau$	$\mathbf{A} : \emptyset$
$\neg\sigma$	$\mathbf{A} : \sigma$

Formula	Attack	Defenses
$\sigma \to \tau$	$\mathbf{A}$	$\mathbf{D} : \tau$
$\sigma \wedge \tau$	$\mathbf{A}_{\mathrm{L}}$	$\mathbf{D} : \sigma$
$\sigma \wedge \tau$	$\mathbf{A}_{\mathrm{R}}$	$\mathbf{D} : \tau$
$\sigma \vee \tau$	$\mathbf{A}$	$\mathbf{D}_{\mathrm{L}} : \sigma\ ,\ \mathbf{D}_{\mathrm{R}} : \tau$

If $\mathbf{P}$ and $\mathbf{P}, M$ are dialogues over φ, then M is a *possible* move after $\mathbf{P}$.

An implication $\sigma \to \tau$ is attacked by stating σ and defended by stating τ. A conjunction $\sigma \wedge \tau$ can be attacked in two different ways (questioning σ and τ, respectively) and defended correspondingly by stating σ or τ, respectively. A disjunction $\sigma \vee \tau$ can be attacked in only one way, but defended in two different ways, depending on whether one claims σ or τ. A negation $\neg\sigma$ can be attacked by stating σ, but cannot be defended.

Example 1.2. The following illustrates classical negation and disjunction.

Move	Player	Attack/Defense	Formula	Ref. to prev. move
1	P	$\mathbf{D}$	$p \vee \neg p$	0
2	O	$\mathbf{A}$	$\emptyset$	1
3	P	$\mathbf{D}_{\mathrm{R}}$	$\neg p$	2
4	O	$\mathbf{A}$	p	3
5	P	$\mathbf{D}_{\mathrm{L}}$	p	2

Definition 1.3. Let $\mathbf{P} = M_1, \ldots, M_n$ be a dialogue over φ. A *proponent strategy after* $\mathbf{P}$ is a tree labeled with moves such that

- The initial part of the tree is a single path labeled $M_1, \ldots, M_n$.
- In every branch, each node at or after M_n labeled with a proponent move has one child for every possible opponent move.
- In each branch, each node at or after M_n labeled with an opponent move has one child, if there is a possible proponent move, and otherwise has no children.

A proponent strategy after $\mathbf{P}$ is *winning* if every path is finite and every leaf is a proponent move (all dialogues end with the opponent having no reply). If $\mathbf{P}$ is empty, we call it a *winning proponent strategy for* φ.

Example 1.4. A winning proponent strategy for $((p \to p) \to q) \to q$:

$$\begin{tikzcd}
& 1\,\mathrm{P}\,\mathbf{D}\,((p \to p) \to q) \to q\ \ 0 \arrow[d] & \\
& 2\,O\,\mathbf{A}\,(p \to p) \to q\ \ 1 \arrow[d] & \\
& 3\,\mathrm{P}\,\mathbf{A}\,p \to p\ \ 2 \arrow[dl] \arrow[dr] & \\
4a\,O\,\mathbf{D}\,q\ \ 3 \arrow[d] & & 4b\,O\,\mathbf{A}\,p\ \ 3 \arrow[d] \\
5a\,\mathrm{P}\,\mathbf{D}\,q\ \ 2 & & 5b\,\mathrm{P}\,\mathbf{D}\,p\ \ 4
\end{tikzcd}$$

[1]We use $\emptyset$ when there is no formula to accompany an attack; the "formula" $\emptyset$ cannot be attacked.

Remark 1.5. Dialogues can be viewed as parameterized over:

- The set Φ of formulas generated from a set of variables Υ;
- For each $\varphi \in \Phi - \Upsilon$, the set $\{A_1 : \tau_1, \ldots, A_n : \tau_n\}$ of attacks on φ;
- For each $\varphi \in \Phi - \Upsilon$ and attack A_i on φ, the set $\{D^i_1 : \sigma^i_1, \ldots, D^i_{m_i} : \sigma^i_{m_i}\}$ of defenses of φ against A_i.

We write this shorter as $\varphi \lhd (\tau_1 \vdash \Sigma_1) \ldots (\tau_n \vdash \Sigma_n)$ where $\Sigma_i = \sigma^i_1, \ldots, \sigma^i_{m_i}$, leaving out names of attacks and defenses. We omit $\emptyset$ to the left of $\vdash$, and write $\Phi^{\emptyset}$ for $\Phi \cup \{\emptyset\}$. Attacks τ_i are in $\Phi^{\emptyset}$, but defenses σ^i_j are in Φ.

Example 1.6. The tables in Definition 1.1 correspond to this specification:

- $\varphi_1 \wedge \varphi_2 \lhd (\vdash \varphi_1)(\vdash \varphi_2)$.
- $\varphi_1 \vee \varphi_2 \lhd (\vdash \varphi_1, \varphi_2)$.
- $\varphi_1 \rightarrow \varphi_2 \lhd (\varphi_1 \vdash \varphi_2)$.
- $\neg\varphi \lhd (\varphi \vdash)$.

And in general we have $p \lhd ()$ (variables cannot be attacked) as well as $\emptyset \lhd ()$.

2. The system LKd

We introduce a dialogue-inspired variant of LK.

Definition 2.1. Given a specification as in Remark 1.5, the system LKd is defined in Figure 1. Sequents use sequences of formulas, so we include structural rules. In left rules the leftmost premiss is omitted when τ_i is $\emptyset$.

For each connective there is one right rule and as many left rules as there are attacks on that connective. LK is a special case.

Proposition 2.2. LKd *with the specification in Example 1.6 is equivalent to* LK.

Proof. The system LKd relative to this specification is a standard LK except that the usual two right rules for disjunction (indicated to the left below) are replaced by a single, equivalent rule (indicated to the right below).

$$\frac{\Gamma \vdash \varphi, \Delta}{\Gamma \vdash \varphi \vee \psi, \Delta} \quad \frac{\Gamma \vdash \psi, \Delta}{\Gamma \vdash \varphi \vee \psi, \Delta} \qquad\qquad \frac{\Gamma \vdash \varphi, \psi, \Delta}{\Gamma \vdash \varphi \vee \psi, \Delta}$$

Also, the more usual axiom $\varphi \vdash \varphi$ follows from (Ax) and logical rules. □

Definition 2.3. Define $\succ$ on formulas: when $\varphi \lhd (\tau_1 \vdash \Sigma_1) \ldots (\tau_n \vdash \Sigma_n)$ we stipulate $\varphi \succ \tau_i$ and $\varphi \succ \sigma$, for all $\sigma \in \Sigma_j$ and all i, j.

We next prove cut-elimination for LKd relative to any specification such that $\succ$ is well-founded.

Definition 2.4. The *degree* of a cut is the length of the longest $\succ$-sequence starting from φ, the formula eliminated in the cut. We write $\Gamma \vdash^d \Delta$ if there is a derivation of $\Gamma \vdash_{\mathrm{LKd}} \Delta$ in which all cuts have degree at most d.

Lemma 2.5. *Let* $d(\varphi) = d + 1$*,* $\varphi \in \Delta$*, and* $\varphi \in \Gamma'$*. If* $\Gamma \vdash^d \Delta$ *and* $\Gamma' \vdash^d \Delta'$*, then* $\Gamma, (\Gamma' - \varphi) \vdash^d (\Delta - \varphi), \Delta'$.[2]

[2]The notaion "$-\,\varphi$" means with all occurrences of φ removed.

Axiom:

$$p \vdash p \text{ (Ax)}$$

Structural Rules:

$$\frac{\Gamma \vdash \Theta}{\Gamma, \varphi \vdash \Theta}\ (\mathrm{LW}) \qquad \frac{\Gamma \vdash \Pi}{\Gamma \vdash \varphi, \Pi}\ (\mathrm{RW})$$

$$\frac{\Gamma, \varphi, \psi, \Gamma' \vdash \Theta}{\Gamma, \psi, \varphi, \Gamma' \vdash \Theta}\ (\mathrm{LX}) \qquad \frac{\Gamma \vdash \Delta, \varphi, \psi, \Delta'}{\Gamma \vdash \Delta, \psi, \varphi, \Delta'}\ (\mathrm{RX})$$

$$\frac{\Gamma, \varphi, \varphi \vdash \Theta}{\Gamma, \varphi \vdash \Theta}\ (\mathrm{LC}) \qquad \frac{\Gamma \vdash \varphi, \varphi, \Delta}{\Gamma \vdash \varphi, \Delta}\ (\mathrm{RC})$$

Logical Rules:

$$\frac{\Gamma \vdash \tau_i, \Delta \quad \Gamma, \sigma_1^i \vdash \Theta \ \cdots \ \Gamma, \sigma_{m_i}^i \vdash \Theta}{\Gamma, \varphi \vdash \Delta, \Theta}\ (\mathrm{L}_i) \qquad \frac{\Gamma, \tau_1 \vdash \Sigma_1, \Delta \ \cdots \ \Gamma, \tau_n \vdash \Sigma_n, \Delta}{\Gamma \vdash \varphi, \Delta}\ (\mathrm{R})$$

Side condition: $\varphi \lhd (\tau_1 \vdash \Sigma_1) \ldots (\tau_n \vdash \Sigma_n),\ \Sigma_i = \{\sigma_1^i, \ldots, \sigma_{m_i}^i\},\ 1 \leq i \leq n$

Cut Rule:

$$\frac{\Gamma \vdash \varphi, \Delta \quad \Gamma, \varphi \vdash \Theta}{\Gamma \vdash \Delta, \Theta}\ (\mathrm{Cut})$$

Figure 1: Classical Sequent Calculus LKd.

Proof. We proceed by induction on the sum of the heights of the derivations D and D' of $\Gamma \vdash^d \Delta$ and $\Gamma' \vdash^d \Delta'$, respectively. We consider the different shapes of D and D'. The only interesting case is when D ends with a right rule (R), and D' ends with a left rule (L_i), and both introduce φ, to the right and left, respectively:

$$\frac{\Gamma, \tau_1 \vdash \Sigma_1, \Delta_1 \ \cdots \ \Gamma, \tau_n \vdash \Sigma_n, \Delta_1}{\Gamma \vdash \varphi, \Delta_1} \qquad \frac{\Gamma_1' \vdash \tau_i, \Delta' \quad \Gamma_1', \sigma_1^i \vdash \Theta' \ \cdots \ \Gamma_1', \sigma_{m_i}^i \vdash \Theta'}{\Gamma_1', \varphi \vdash \Delta', \Theta'}$$

where $\Sigma_i = \{\sigma_1^i, \ldots, \sigma_m^i\}$ and $\varphi \lhd (\sigma_1 \vdash \Sigma_1) \ldots (\sigma_n \vdash \Sigma_n)$. By the induction hypothesis, applied above the line in one rule and below the line in the other, and using structural rules:

$$\Gamma, \tau_i, (\Gamma_1' - \varphi) \vdash \Sigma_i, (\Delta_1 - \varphi), \Delta', \Theta' \text{ for all } i \tag{2.1}$$

$$\Gamma, (\Gamma_1' - \varphi) \vdash (\Delta_1 - \varphi), \tau_i, \Delta' \text{ for some } i \tag{2.2}$$

$$\Gamma, (\Gamma_1' - \varphi), \sigma_j^i \vdash (\Delta_1 - \varphi), \Theta' \text{ for some } i \text{ and all } j \tag{2.3}$$

Combining (2.1) and (2.2) with (Cut), we have $\Gamma, (\Gamma_1' - \varphi) \vdash \Sigma_i, (\Delta_1 - \varphi), \Delta', \Theta'$. Combining this and (2.3) with (Cut), we get $\Gamma, (\Gamma_1' - \varphi) \vdash (\Delta_1 - \varphi), \Delta', \Theta'$. □

Theorem 2.6. *The system* LKd *without Cut is complete.*

Proof. We show for any d that $\Gamma \vdash^{d+1} \Delta$ implies $\Gamma \vdash^{d} \Delta$. The proof is by induction on the derivation of $\Gamma \vdash^{d+1} \Delta$ using Lemma 2.5. □

Example 2.7. Say we remove negation from the propositional language, and instead add $\top$ and $\bot$ by stating that there is no attack on $\top$, while $\bot$ can be attacked but not defended, i.e. $\top \lhd ()$ and $\bot \lhd (\vdash)$. By Theorem 2.6, we immediately know that cut-elimination holds for this instance of LKd.

3. The system LKD

The system LKd suggests that in $\Gamma \vdash \Delta$, we can read Γ and Δ as utterances from a dialogue, and that derivations may be related to dialogues. Next we introduce the system LKD, where these suggestions can be substantiated more clearly than with LKd (larger "D" means closer to dialogues).

Definition 3.1. Given a specification as in Remark 1.5, the system LKD is defined in Figure 2. Sequents use sets of formulas.

$$\frac{\Gamma, \sigma_1 \vdash \Sigma_1, \Delta \quad \cdots \quad \Gamma, \sigma_n \vdash \Sigma_n, \Delta \qquad \Gamma, \rho_1 \vdash \Delta \quad \cdots \quad \Gamma, \rho_m \vdash \Delta}{\Gamma \vdash \Delta} \text{ (L)}$$

$$[\varphi \in \Gamma,\ \varphi \lhd \cdots (\sigma \vdash \rho_1, \ldots, \rho_m) \cdots,\ \sigma \in (\Phi^{\emptyset} - \Upsilon) \cup \Gamma,\ \sigma \lhd (\sigma_1 \vdash \Sigma_1) \ldots (\sigma_n \vdash \Sigma_n)]$$

$$\frac{\Gamma, \sigma_1 \vdash \Sigma_1, \Delta \quad \cdots \quad \Gamma, \sigma_n \vdash \Sigma_n, \Delta}{\Gamma \vdash \Delta} \text{ (R)}$$

$$[\varphi \in \Delta,\ \varphi \lhd (\sigma_1 \vdash \Sigma_1) \ldots (\sigma_n \vdash \Sigma_n),\ \varphi \in (\Phi - \Upsilon) \cup \Gamma\,]$$

$$\frac{\Gamma \vdash \varphi, \Delta \qquad \Gamma, \varphi \vdash \Theta}{\Gamma \vdash \Delta, \Theta} \text{ (Cut)}$$

Figure 2: Classical Sequent Calculus LKD.

By replacing sequences with sets, and by using G3-style (see [5]), the structural rules have been avoided. The right rule in LKD is similar to the one in LKd, but includes (Ax) as a special case; this is the condition $\varphi \in (\Phi - \Upsilon) \cup \Gamma$, which states that if φ is a variable, it must be in Γ. In this case we have $n = 0$, since $p \lhd ()$. The left rule in LKD is obtained from the one in LKd by insisting that the left-most premiss, if present, is inferred using a right rule.

LKd and LKD derive the same sequents.

Lemma 3.2. *Assume* $\psi \lhd \cdots (\sigma \vdash \rho_1 \ldots \rho_m) \cdots$ *and* $\psi \in \Gamma$. *If* $\Gamma \vdash_{\text{LKD}} \sigma, \Delta$ *(or* σ *is* $\emptyset$*) and* $\Gamma, \rho_1 \vdash_{\text{LKD}} \Delta \cdots \Gamma, \rho_m \vdash_{\text{LKD}} \Delta$, *then* $\Gamma \vdash_{\text{LKD}} \Delta$.

Proof. If σ is $\emptyset$, use (L). Otherwise we proceed by induction on the derivation of $\Gamma \vdash_{\text{LKD}} \sigma, \Delta$. Assume, for instance that the derivation ends in (R). First use the induction hypothesis on each premise of (R). If $\varphi \neq \sigma$ (using the same variable names as in Figure 2) use (R), otherwise use (L). $\square$

Proposition 3.3. *For a sequence* Γ, *let* Γ^* *denote the set of its members. For a set* Γ, *let* Γ^+ *denote the sequence (ordered somehow) of its elements.*

(i) *If* $\Gamma \vdash_{\text{LKd}} \Delta$ *then* $\Gamma^* \vdash_{\text{LKD}} \Delta^*$.
(ii) *If* $\Gamma \vdash_{\text{LKD}} \Delta$ *then* $\Gamma^+ \vdash_{\text{LKd}} \Delta^+$.

Proof. Property (i) is by induction on the derivation of $\Gamma \vdash_{\text{LKd}} \Delta$, using Lemma 3.2, and (ii) is by induction on the derivation of $\Gamma \vdash_{\text{LKD}} \Delta$. $\square$

By noticing that cut-free proofs are taken to cut-free proofs, we obtain:

Corollary 3.4. *The system* LKD *without Cut is complete.*

The rules in LKD formalize winning proponent strategies. In $\Gamma \vdash \Delta$ we read Γ as the assertions that *have* been stated by the opponent and thus may be attacked by the proponent, and Δ as assertions that *may* be asserted by the proponent, as defenses or as the initial formula. Reading the rules upside-down, the right rule corresponds to a node where the proponent states φ in a defense, and the strategy has a branch for each possible opponent attack on φ. The left rule corresponds to a node where the proponent attacks a formula φ stated by the opponent. In this case the strategy has a branch for each possible opponent defense and for each opponent counter-attack.

Definition 3.5. Let P be a finite play over φ. Define $\text{Sit}(P) = \Gamma \vdash \Delta$, the *situation after* P, as follows. If P is the empty play, then $\text{Sit}(P) = \vdash \varphi$. Otherwise $P = Q, M$, for some $M = (X, \sigma, i)$, and $\text{Sit}(Q) = \Gamma \vdash \Delta$.

- If M is a proponent move, then $\text{Sit}(P) = \Gamma \vdash \Delta$.
- If M is an opponent defense, then $\text{Sit}(P) = \Gamma, \sigma \vdash \Delta$.
- If M is an opponent attack on the preceding proponent move (X, ρ, j), then $\text{Sit}(P) = \Gamma, \sigma \vdash \Sigma, \Delta$ where $\rho \lhd \cdots (\sigma \vdash \Sigma) \cdots$.

When translating from derivations to strategies, we would like to make sure that we do not make the proponent repeat the initial formula, since this is not necessarily permitted.

Definition 3.6. An instance of (R) is *redundant* if, for some i (using the same variable names as in Figure 2), $\Sigma_i \subseteq \Delta$ and either σ_i is $\emptyset$ or $\sigma_i \in \Gamma$.

Lemma 3.7. *If* $\Gamma \vdash_{\text{LKD}} \Delta$, *then there is a derivation of* $\Gamma \vdash_{\text{LKD}} \Delta$ *without redundant instances of (R).*

Proof. By induction on the derivation of $\Gamma \vdash_{\text{LKD}} \Delta$. The only interesting case is when the derivation ends in (R). If this instance is redundant, then $\Gamma, \sigma_i \vdash \Sigma_i, \Delta$ is $\Gamma \vdash \Delta$, so by the induction hypothesis, there is a derivation of this sequent without redundant instances of (R). $\square$

Theorem 3.8. *If* $\vdash_{\mathrm{LKD}} \psi$*, then* ψ *has a winning proponent strategy.*

Proof. We show that if P is a play (of length k) over ψ, not ending with a proponent move, resulting in the situation $\Gamma \vdash \Delta$, and there is a derivation of $\Gamma \vdash_{\mathrm{LKD}} \Delta$, then there is a winning proponent strategy for ψ after P. In particular, if $\vdash \psi$, then there is a winning proponent strategy for ψ. The proof is by induction on the derivation of $\Gamma \vdash_{\mathrm{LKD}} \Delta$. We assume it has no redundant occurrences of (R).

We consider the case where the derivation ends in (R). If P is the empty play, the situation is $\vdash \psi$, so $\psi = \varphi$ (using the same variable names as in Figure 2). In this case, $(\mathbf{D}, \varphi, 0)$ is a possible move. If P is not the empty play, then we must have $\psi \neq \varphi$. Indeed, assume $\psi = \varphi$. We have $P = M_1, M_2, \ldots$ and $M_2 = (A_i, \sigma_i, 1)$, for some i. Then $\sigma_i = \emptyset$ or $\sigma_i \in \Gamma$, and $\Sigma_i \subseteq \Delta$, a contradiction. Since $\Gamma \vdash \Delta$ is the situation after P, and $\varphi \in \Delta$ is not ψ, there must be a proponent move $M_l = (X, \rho, \ell)$ and and an opponent move $M_j = (A, \sigma, l)$ where $\rho \triangleleft \cdots (\sigma \vdash \cdots \varphi \cdots) \cdots$. But then (D, φ, j) is a possible move with this j and some D. Whether P is empty or not, we have a possible move (D, φ, j). With $\varphi \triangleleft (\sigma_1 \vdash \Sigma_1) \cdots (\sigma_1 \vdash \Sigma_1)$, each play $P_i = P, (D, \varphi, j), (A_i, \sigma_i, k+1)$ results in the situation $\Gamma, \sigma_i \vdash \Sigma_i, \Delta$, so there are winning proponent strategies S_i for $\Gamma, \sigma_i \vdash \Sigma_i, \Delta$. Then

$$
\begin{tikzcd}
 & (D,\varphi,j) \arrow[dl] \arrow[dr] & \\
(A_1,\sigma_1,k+1) \arrow[d] & \cdots & (A_n,\sigma_n,k+1) \arrow[d] \\
S_1 & & S_n
\end{tikzcd}
$$

is the desired strategy. $\square$

Theorem 3.9. *If* ψ *has a winning proponent strategy, then* $\vdash_{\mathrm{LKD}} \psi$*.*

Proof. Let T be a winning proponent strategy for ψ. We show that if P is a play in T not ending in an proponent move and $\Gamma \vdash \Delta$ is the situation after P, then $\Gamma \vdash_{\mathrm{LKD}} \Delta$. Taking P to be the empty play then yields $\vdash \psi$. The proof is by induction on the subtree below P inside T.

So, let $P = M_1, \ldots, M_n$ by a play in T not ending in a proponent move, and let the root of the subtree below P have label M. We consider the case where $M = (A, \sigma, i)$ is an attack. Then P has an opponent move $M' = (X, \varphi, i')$ with $\varphi \triangleleft \cdots (\sigma \vdash \rho_1 \ldots \rho_m) \cdots$. With $\sigma \triangleleft (\sigma_1 \vdash \Sigma_1) \ldots (\sigma_n \vdash \Sigma_n)$, the subtree below P looks as follows:

$$
\begin{tikzcd}
 & & & M \arrow[dlll] \arrow[dl] \arrow[dr] \arrow[drrr] & & & \\
A_1 \arrow[d] & \ldots & A_n \arrow[d] & & D_1 \arrow[d] & \ldots & D_m \arrow[d] \\
T_1 & & T_n & & S_1 & & S_m
\end{tikzcd}
$$

The situation at T_i and S_j are $\Gamma, \sigma_i \vdash \Sigma_i, \Delta$ and $\Gamma, \rho_j \vdash \Delta$, respectively. By the induction hypothesis, $\Gamma, \sigma_i \vdash_{\mathrm{LKD}} \Sigma_i, \Delta$ and $\Gamma, \rho_j \vdash_{\mathrm{LKD}} \Delta$. Since $\Gamma \vdash \Delta$ is the situation after P, we have $\varphi \in \Gamma$, and $\sigma \in (\Phi^{\emptyset} - \Upsilon) \cup \Gamma$, so $\Gamma \vdash_{\mathrm{LKD}} \Delta$. $\square$

4. Conclusion and related work

We have gone from LK to LKd by replacing the concrete connectives with a specification of attacks and defenses, and by implicitly replacing the traditional right rules for disjunction with a single rule, and by restricting (Ax) to variables. As we have seen, generating left and right rules from attacks and defenses is one way of ensuring that the system satisfies cut-elimination.

We have then gone from LKd to LKD by eliminating the structural rules, by building (Ax) into (R), and by adopting a left-right regime where the leftmost premiss of (L) must be inferred using (R). Derivations in the resulting system are practically identical to winning proponent strategies, if we agree to avoid redundant occurrences of (R).

Others have proved equivalence between derivations and dialogues, e.g. Felscher [1], who also comments on other such proofs [2]. See also Krabbe [4] for more references. The merits of the present proof are that it shows precisely which modest changes we have to make to LK to make derivations and strategies isomorphic. Moreover, it gives a condition for a sequent calculus to satisfy cut-elimination: that it can be obtained as an instance of Figure 1.

The similarity to some aspects of Girard's ludics [3] is obvious. The abstraction of concrete formulas to specifications of attacks and defenses yields the rules of LKd where there is no syntax for the formulas; these rules are similar to the positive and negative rules of ludics, used to build designs. Moreover, as inferences of LKD axiomatize winning prover strategies, so can designs in ludics be read as strategies, though with some differences.

The system LKD is one way of presenting classical sequent calculus for "arbitrary" formulas. Another is to recall that any formula φ in classical logic can be written as a conjunctive normal form $\wedge_i((\vee_j \neg p_{ij}) \vee (\vee_k q_{ik}))$. Applying left and right rules of LK upwards we get the following derived rules for the synthetic connective represented by φ, which also admit a dialogue interpretation:

$$\frac{\Gamma \vdash p_{i1}, \Delta \cdots \Gamma \vdash p_{ij_i}, \Delta \qquad \Gamma, q_{i1} \vdash \Delta \cdots \Gamma, q_{ik_i} \vdash \Delta}{\Gamma, \varphi \vdash \Delta} \text{ (L)}$$

$$\frac{\cdots \Gamma, p_{i1} \ldots, p_{ij_i} \vdash q_{i1} \ldots q_{ik_i}, \Delta \cdots}{\Gamma \vdash \varphi, \Delta} \text{ (R)}$$

where there is one (R) (with a premiss for each i) and one instance of (L) for each i.

By shifting formulas to the right and adding negations as appropriate, we can write

$$\frac{\neg p_{i1} \vdash \Delta \cdots \neg p_{ij_i} \vdash \Delta}{\vdash \vee_i \wedge_j p_{ij}, \Delta} \text{ (+)}$$

$$\frac{\cdots \vdash \neg p_{i1} \ldots \neg p_{ij_i}, \Delta \cdots}{\vee_i \wedge_j p_{ij} \vdash \Delta} \text{ (−)}$$

where there is one (−) (with a premiss for each i) and one instance of (+) for each i. Again these rules are similar to the positive and negative rules of ludics.

References

[1] W. Felscher. Dialogues, strategies and intuitionistic provability. *Annals of Pure and Applied Logic*, 28:217–254, 1985.

[2] W. Felscher. Dialogues as a foundation for intuitionistic logic. In D.M. Gabbay and F. Guenthner, editors, *Handbook of Philosophical Logic. Vol. III*, pages 341–372. Dordrecht: D. Reidel, 1986.

[3] Jean-Yves Girard. Locus solum: From the rules of logic to the logic of rules. *Mathematical Structures in Comp. Sci.*, 11(3):301–506, 2001.

[4] E. C. W. Krabbe. Dialogue logic. In D. Gabbay and J. Woods, editors, *Handbook of History of Logic, Vol. 7*, pages 665–704. Elsevier, 2006.

[5] M. H. Sørensen and P. Urzyczyn. *Lectures on the Curry-Howard Isomorphism*, volume 149 of *Studies in Logic and the Foundations of Mathematics*. Elsevier, 2006.